UNIT 3 BIPARTITE GRAPHS

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3.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- identify and illustrate bipartite graphs**,** complete bipartite graphs and star graphs .
- understand the role of cycles in the characterization of bipartite graphs.
- identify and distinguish between trees, forests, and spanning trees in graph theory.
- understand and apply Cayley's formula for enumerating spanning trees in complete graphs.

3.2 INTRODUCTION

In this unit, we will explore about bipartite graphs, including complete bipartite graphs and star graphs. We will also discuss the role of cycles in characterizing bipartite graphs. Further, we will explore trees, forests, and spanning trees, and finally understand Cayley's formula for counting spanning trees in complete labeled graphs.

3.3 BIPARTITE GRAPHS

A graph is *bipartite* if its vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X,Y) is called

a *bipartition* of the bipartite graph. The bipartite graph G with bipartition (X,Y) is denoted by G(X, Y).

A simple bipartite graph G(X, Y) is *complete* if each vertex of X is adjacent to all the vertices of Y. If G(X, Y) is complete with $|X| = p$ and $|Y| = q$ then G(X,Y) is denoted by $K_{p,q}$. A complete bipartite graph of the form $K_{1,q}$ is called a *star.*

Example 3.1:

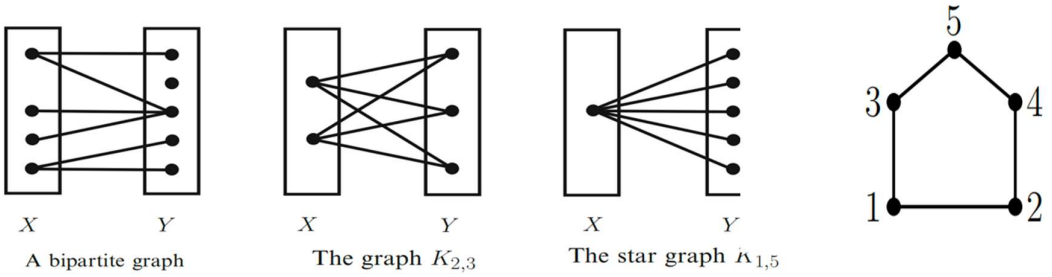


Figure 3.1: Bipartite graphs and a non-bipartite graph

We have already defined the concepts of walk and path in Chapter 4. For further details on walks, paths, and cycles, please refer to Chapter 4. Here, we define the concept of a cycle only in the context of characterizing bipartite graphs.

Cycle: A closed walk is called a *cycle*, if all its vertices are distinct except that $v_0 = v_k$. A cycle of length k is called an n-cycle and is denoted by C_k. A loop is 1-cycle and a pair of edges joining two vertices from a 2-cycle. An k-cycle is proper only if $k \geq 3$.

3.4 CHARACTERIZATION OF BIPARTITE GRAPHS

A cycle is *odd* or *even* depending on whether its length is odd or even. We now characterize bipartite graphs.

Theorem 3.4.1. *A graph is bipartite if and only if it contains no odd cycles.*

Proof. Suppose that G is a bipartite graph with the bipartition (X,Y).

Let $C = v_1 e_1 v_2 e_2 v_3 e_3 \ldots v_k e_k v_1$ be a cycle in G: Without loss of generality, we can suppose that $v_1 \in X$. As v_2 is adjacent to $v_1, v_2 \in Y$. Similarly, v_3 belongs to X, v_4 to Y, and so on. Thus, $v_i \in$ X or Y according as i is odd or even, $1 \leq i \leq k$. Since $v_k v_1$ is an edge of G and $v_1 \in X, v_k \in Y$. Accordingly, k is even and C is an even cycle.

Conversely, let us suppose that G contains no odd cycles. We first assume that G is connected. Let *u* be a vertex of G. Define $X = \{ v \in V \mid d(u,v)\ is\ even\}$ and

$Y = \{ v \in V \mid d(u,v)\ is\ odd\}$. We will prove that (X,Y) is a bipartition of G. To prove this we have only to show that no two vertices of X as well as no two vertices of Y are adjacent in G. Let *v*, *w* be two vertices of X. Then $p = d(u,v)$ and $q = d(u,v)$ are even. Further, as $d(u,u) = 0, u \in X$. Let P be a *u*-*v* shortest path of length *p* and Q, a *u*-*w* shortest path of length *q* (See Fig. 3.2.) Let w_1 be a vertex common to P and Q such that the w_1-*v* section of P and the w_1-*w* section of Q contain no vertices common to P and Q. Then the *u*-w_1 sections of both P and Q have the same length.

Hence, the lengths of the w_1-*v* section of P and the w_1-*w* section of Q are both even or both odd. Now if $e = vw$ is an edge of G, then the w_1-*v* section of P followed by the edge *vw* and the *w*-w_1 section of the *w*-*u* path Q^{-1} is an odd cycle in G, contradicting the hypothesis. This contradiction proves that no two vertices of X are adjacent in G. Similarly, no two vertices of Y are adjacent in G. This proves the result when G is connected.

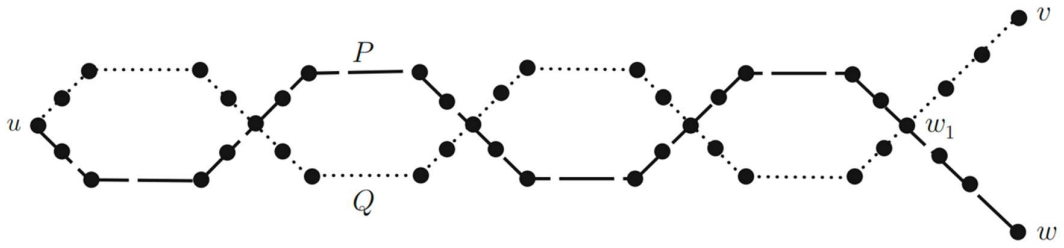


Figure 3.2: A shortest path P from *u* to *v* and a shortest path Q from *u* to *w*.

If G is not connected, let $G_1, G_2, \ldots . G_w$ be the components of G. By hypothesis, no component of G contains an odd cycle. Hence, by the previous paragraph, each component $G_i, 1 \leq i \leq w$ is bipartite. Let (X_i, Y_i) be the bipartition of G_i. Then (X,Y) where $X = \bigcup_{i=1}^{w} X_i$ and $Y = \bigcup_{i=1}^{w} Y_i$, is a bipartition of G, and G is a bipartite graph.

Trees are omnipresent in computer science and network analysis. The earliest mention of trees appear in the works of Kirchoffff (1847), Jordan (1869), Sylvester (1882) and Cayley (1957). Here, we provide the definitions of trees, forests, and spanning trees for Cayley's formula. For further details, refer to Chapter 6.

Tree: A connected acyclic graph is called a *tree.*

Forest: An acyclic graph is called a *forest*.

Thus, every component of a forest is a tree.

Spanning Tree: A spanning subgraph of a graph G, which is also a tree, is called a *spanning tree* of G.

Counting the number of spanning trees in a graph occurs as a natural problem in many branches of science. Spanning trees were used by Kirchoff to generate a "cycle

basis" for the cycles in the graphs of electrical networks. In this section, we consider the enumeration of spanning trees in graphs. The number of spanning trees of a connected labeled graph G will be denoted by $\tau(G)$. If G is disconnected, we take $\tau(G) = 0$.

3.5 CAYLEY'S FORMULA

Cayley was the first mathematician to obtain a formula for the number of spanning trees of a labeled complete graph.

Theorem 3.5.1 (Cayley).
$\tau(K_n) = n^{n-2}$, *where* K_n *is a labeled complete graph on n vertices,* $n \geq 2$.

Before we prove Theorem 3.5.1, we establish two lemmas.

Lemma 3.5.2. *Let* $(d_1, d_2, \ldots\ldots, d_n)$ *be a sequence of positive integers with* $\sum_{i=1}^{n} d_i = 2(n-1)$. *Then there exists a tree* T *with vertex set* $\{v_1, v_2, \ldots\ldots, v_n\}$ *and* $d(v_i) = d_i, 1 \leq i \leq n$.

Proof. It is easy to prove the result by induction on *n*.

Lemma 3.5.3 *Let* $\{v_1, v_2, \ldots\ldots, v_n\}, n \geq 2$ *be given and let* $\{d_1, d_2, \ldots\ldots, d_n\}$ *be a sequence of positive integers such that* $\sum_{i=1}^{n} d_i = 2(n-1)$. *Then the number of trees with* $\{v_1, v_2, \ldots\ldots v_n\}$ *as the vertex set in which* v_i *has degree* $d_i, 1 \leq i \leq n$ *is* $\frac{(n-2)!}{(d_1-1)!\ldots\ldots(d_n-1)!}$

Proof. We prove the result by induction on n. For $n = 2, 2(n-1) = 2$, so that $d_1 + d_2 = 2$. Since $d_1 \geq 1$ and $d_2 \geq 1$, it follows that $d_1 = d_2 = 1$. Hence, K_2 is the only tree in which v_i has degree $d_i, i = 1, 2$. Therefore, the result is true for $n = 2$. Now assume that the result is true for all positive integers up to $(n-1)$, where $n \geq 3$. Let $\{d_1, \ldots, d_h\}$ be a sequence of positive integers such that $\sum_{i=1}^{n} d_i = 2(n-1)$ and let $\{v_1, \ldots, v_n\}$ be any set of vertices. If $d_i \geq 2$ for every *i,* $1 \leq i \leq n$, then $\sum_{i=1}^{n} d_i \geq 2n$, which is impossible. Hence, there exists an *i,* $1 \leq i \leq n$, for which $d_1 = 1$. Without loss of generality, assume that $d_1 = 1$. By Lemma 3.5.2, there exists a tree T with $V(T) = \{v_1, \ldots, v_n\}$ such that the degree of v_1 is d_1. Let v_1 be the unique vertex of T adjacent to v_1. Delete v_1 from T. The resulting graph is a tree T′ with vertex set $\{v_1, \ldots, v_{n-1}\}$ and degree sequence $(d_1, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_{n-1})$.

In the opposite direction, given a tree T′ with $\{v_1, \ldots, v_{n-1}\}$ as its vertex set and $(d_1, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_{n-1})$ as its degree sequence, a tree T with vertex set

$\{v_1, \ldots, v_n\}$ and degree sequence $(d_1, \ldots, d_n)$, where $d_1 = 1$, can be obtained by introducing a new vertex v_1 and taking $T = T' + v_1v_1$.

Hence, the number of trees with vertex set $\{v_1, \ldots, v_n\}$ and degree sequence $(d_1, \ldots, d_n)$ with d_1 = degree of $v_1 = 1$ and v_1 adjacent to v_1 is the same as the number of trees with vertex set $\{v_1, \ldots, v_{n-1}\}$ and degree sequence $(d_1, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_{n-1})$.

By the induction hypothesis, the latter number is equal to

$$\frac{(n-3)!}{(d_1-1)!\ldots(d_{j-1}-1)!(d_j-2)!(d_{j+1}-1)!\ldots(d_{n-1}-1)!}$$

$$= \frac{(n-3)!(d_j-1)}{(d_1-1)!\ldots(d_{j-1}-1)!(d_j-1)!(d_{j+1}-1)!\ldots(d_{n-1}-1)!} \text{ Since } (d_1-1)! = (d_1-1)(d_1-2)!$$

Summing over j, the number of trees with $\{v_1, v_2, \ldots., v_n\}$ as its vertex set and$(d_1, d_2, \ldots.. d_n)$ as its degree sequence is

$$\sum_{j=1}^{n} \frac{(n-3)!\,(d_j-1)}{(d_1-1)!\ldots(d_{n-1}-1)!}$$

$$= \frac{(n-3)!}{(d_1-1)!\ldots(d_{n-1}-1)!} \sum_{j-1}^{n-1} (d_j - 1)$$

$$= \frac{(n-3)!}{(d_1-1)!\ldots(d_{n-1}-1)!} \left[\left(\sum_{j=1}^{n-1} d_j \right) - (n-1) \right]$$

$$= \frac{(n-3)!}{(d_1-1)!\ldots(d_{n-1}-1)!} [(2n-3) - (n-1)]$$

$$= \frac{(n-3)!}{(d_1-1)!\ldots(d_{n-1}-1)!} (n-2)$$

$$= \frac{(n-2)!}{(d_1-1)!\ldots(d_{n-1}-1)!}$$

$= \frac{(n-2)!}{(d_1-1)!\ldots(d_n-1)!}$ (recall that $d_n = 1$)

This completes the proof of Lemma 3.5.2.

Proof of theorem 3.5.1. The total number of trees Tn with vertex set $\{v_1, v_2, \ldots\ldots, v_n\}$ is obtained by summing over all possible sequences $(d_1, d_2, \ldots\ldots, d_n)$ with $\sum_{i=1}^{n} d_i = 2n-2$. Hence,

$\tau(K_n) = \sum_{d_i \geq 1} \frac{(n-2)!}{(d_1-1)!\ldots(d_n-1)!}$ With $\sum_{i=1}^{n} d_i = 2n-2$.

$= \sum_{k_i \geq 0} \frac{(n-2)!}{k_1!\ldots.k_n!}$ with $\sum_{i=1}^{n} k_i = n-2$, where $k_i = d_i - 1, 1 \leq i \leq n$.

Putting $x_1 = x_2 = \cdots\cdots = x_n = 1$ and $m = n-2$ in the multinomial expansion

$(x_1 + x_2 + \cdots\cdots + x_n)^m = \sum_{k_i \geq 0} \frac{x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}}{k_1! k_2! \ldots k_n!} m!$ with $k_1 + k_2 + \cdots + k_n = m$.

We get,

$$n^{n-2} = \sum_{k_i \geq 0} \frac{(n-2)!}{k_1! k_2! \ldots k_n!} \text{ with } k_1 + k_2 + \cdots + k_n = n-2.$$

Thus, $\tau(K_n) = n^{n-2}$.

W CHECK YOUR PROGRESS

Q1: Find the number of spanning trees in:

(a) K_3
(b) K_4

23.6 LET US SUM UP

- A bipartite graph is one whose vertices can be divided into two disjoint sets, with each edge joining a vertex from one set to the other. Eg: Complete bipartite graphs and star graphs.
- Bipartite graphs can be characterized by the absence of odd cycles.
- Trees are connected acyclic graphs, and forests are collections of trees.
- A spanning tree is a spanning subgraph of a graph that is also a tree.
- Kirchhoff and Cayley studied the enumeration of spanning trees, leading to Cayley's formula**.**
- Cayley's formula states that the number of spanning trees in a complete labeled graph K_n is n^{n-2}.

K 3.7 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1:
Using Cayley's formula $\tau(K_n) = n^{n-2}$.

(a) For $K_3, \tau(K_3) = 3$
(b) For $K_4, \tau(K_4) = 16$

O 3.8 FURTHER READINGS

1. F. Harary, Graph theory, Narosa Publishing House, New Delhi, 1988.
2. R. Balakrishnan and K. Renganathan, A textbook of Graph theory, Springer, 2000
3. B. Bollobas, Modern Graph Theory, Springer, 2002
4. G. Chartrand, L. Lesniak, Graphs & digraphs (Fourth edition), Chapman & Hall/CRC, 2005.
5. R. J. Wilson, Introduction to Graph Theory (5th Edition), Prentice Hall, 2010
6. S. Pirzada, An introduction to graph theory, Vol. 4, OrientBlackSwan, Hyderabad: Universities Press, 2012.
7. S. A. Choudum, Graph Theory, A NPTEL Course, Department of Mathematics, IIT Madras, Chennai, India 2012.

M 3.9 MODEL QUESTIONS

Q1. Define a bipartite graph. Give an example.
Q2. State and prove the characterization theorem of bipartite graphs in terms of odd cycles.
Q3. Explain with an example: complete bipartite graph and star graph.
Q4. What is a spanning tree? How is it different from a general tree?
Q5. Define forest. Show that every component of a forest is a tree.
Q6. State and explain Cayley's formula for the number of spanning trees in a complete graph.
Q7. Using Cayley's formula, find the number of spanning trees in K_6 and K_7.

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