UNIT 4: CONNECTIVITY

UNIT STRUCTURE

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4.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand the concepts of walks, trails, paths, and cycles in graphs
- distinguish between open and closed walks, trails, and cycles
- explain connected graphs, components, and the distance between vertices
- identify vertex cuts, edge cuts, cut-vertices, and cut-edges
- understand the notions of connectivity and edge-connectivity of a graph
- relate connectivity, edge-connectivity, and the minimum degree of a graph

4.2 INTRODUCTION

In this unit, we will first introduce the concepts of walks, paths, and cycles in graphs. We will also study connected graphs, components, and the notion of distance between vertices. Further, we shall discuss vertex cuts, edge cuts, cut-vertices, and cut-edges. Finally, we will learn about connectivity, edge-connectivity, and their relation with the minimum degree of a graph.

4.3 WALKS, PATHS AND CYCLES

A walk in a graph G is an alternating sequence $W: v_0e_1v_2e_2 \ldots\ldots\ldots e_kv_k$ of vertices and edges beginning and ending with vertices in which v_{i-1} and v_i are the ends of e_i; v_0 and v_k are called the origin and terminus of W. The walk W is said to join v_0 and v_k; it is also referred to as a $v_0 - v_k$ walk or $W(v_0, v_k)$. The number of edges in a walk is called length of walk. The walk is said to be open if v_0 and v_k are distinct, and closed if $v_0 = v_k$. A walk is called a *trail* if no edge is repeated (but vertices may get repeated). It is called a closed trail if $v_0 = v_k$. A walk is called a *path* if no vertex is repeated (and therefore no edge is repeated). By definition, every path is a trail and every trail is a walk. A closed walk is called a *cycle*, if all its vertices are distinct except that $v_0 = v_k$. A path of length k is called an k-path and is denoted by P_k. A cycle of length k is called an n-cycle and is denoted by C_k. A loop is 1-cycle and a pair of edges joining two vertices from a 2-cycle. An k-cycle is proper only if $k \geq 3$.

Example 4.1:

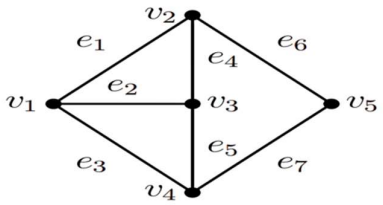


Figure 4.1

In the graph of figure 4.1, $v_1e_1v_2e_4v_3e_2v_1e_3v_4e_5v_3e_4v_2e_6v_5$ is a walk of length 7 but it is not a trail; $v_1e_1v_2e_4v_3e_2v_1e_3v_4e_7v_5$ is a trail of length 5 but it is not a path; $v_1e_1v_2e_4v_3e_5v_4e_7v_5$ is a path of length 4; $v_1e_1v_2e_4v_3e_2v_1e_3v_4e_5v_3e_2v_1$ is a closed walk of length 6 but it is not a cycle; $v_1e_1v_2e_4v_3e_5v_4e_3v_1$ is a cycle of length 4.

Theorem 4.3.1. Every $u - v$ walk contains a $u - v$ path.

Proof: We use mathematical induction on the length l of a $u - v$ walk W.

Base step ($l = 0$):

When there are no edges, the walk W contains only a single vertex, meaning $u - v$. This single vertex itself forms a a $u - v$ path of length 0.

Induction step ($l \geq 1$): Assume that the statement is true for all walks of length less than l.

If the walk W has no repeated vertices, then its vertices and edges naturally form a $u - v$ path.

If W does contain a repeated vertex w, we remove the edges and vertices between the two occurrences of w, keeping only one copy of w. This produces a shorter $u - v$ walk W′ that lies within W.

By the induction hypothesis, W′ contains a $u - v$ path P. Since W′⊆W, the path P is also contained in W.

Definition: A graph G is *connected* if it has a $u - v$ path whenever $u, v \in V(G)$ (Otherwise, G is *disconnected*). If G has a $u - v$ path, then u is connected to v in G. The relation "connected" is an equivalence relation on V(G). Let $V_1, V_2, \ldots \ldots \ldots, V_\omega$ be the equivalence classes. The subgraphs $G[V_1], G[V_2], \ldots \ldots G[V_\omega]$ are called the *components* of G. If $\omega = 1$. The graph G is connected; otherwise, the graph G is disconnected with $\omega \geq 2$ components.

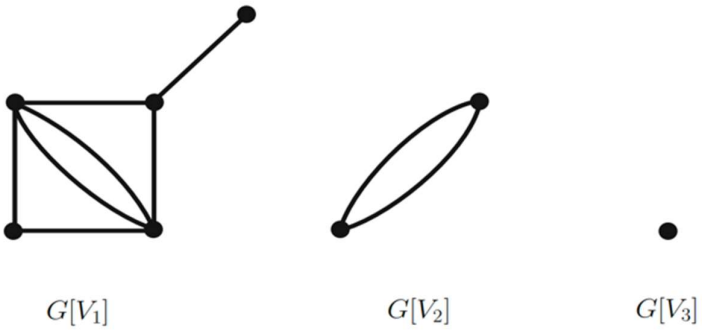


Figure 4.2: A graph G with three components

Remark 4.1: Components are pairwise disjoint; no two share a vertex. Adding an edge with endpoints in distinct components combines them into one component. Thus adding an edge decreases the number of components by 0 or 1, and deleting an edge increases the number of components by 0 or 1.

Definition: The components of G are clearly the maximal connected subgraphs of G. We denote the number of components of G by $\omega(G)$. Let u and v be two vertices of G. If u and v are in the same component of G; we define $d(u, v)$ to be the length of a shortest u-v path in G; otherwise, we define $d(u, v)$ to be ∞.

We now give some results relating to the connectedness of graphs.

Proposition 4.3.2: *If G is simple and* $\delta \geq \frac{n-1}{2}$*, then G is connected.*

Proof. We will prove this proposition using the method of contradiction. Assume that G contains at least two components, say G_1 and G_2. As $\delta \geq \frac{n-1}{2}, d(v) \geq \frac{n-1}{2}$ for any $v \in G_1$. All the vertices adjacent to v in G must belong to G_1. Hence G_1 contains at least $d(v) + \geq \frac{n-1}{2} + 1 = \frac{n+1}{2}$ vertices. Similarly, G_2 contains at least $\frac{n+1}{2}$ vertices. Hence, G has at least $n + 1$ $\left(\frac{n+1}{2} + \frac{n+1}{2}\right)$ vertices, which is a contradiction.

Theorem 4.3.3. *If a simple graph* G *is not connected, then* G^c *is connected.*

Proof. Let u and v be any two vertices of G^c (and hence also of G). If u and v lie in different components of G, then they are clearly non-adjacent in G, which means they must be adjacent in G^c. Therefore, u and v are connected in G^c.
If u and v belong to the same component of G, choose a vertex w in G that does not belong to this component. Since uw and vw are not edges in G, they must be edges in G^c. Consequently, the path u−w−v forms a u-v path in G^c. Hence, G^c is connected.

Proposition 4.3.4: Every graph with *n* vertices and *k* edges has at least *n-k* components.
Proof: An *n*-vertex graph with no edges has *n* components. By remark 4.1, each edge added reduces this by at most 1, so when *k* edges have been added the number of components is still at least *n-k.*

W CHECK YOUR PROGRESS

Q1. A walk in which no edge is repeated (but vertices may repeat) is called:
A) Path
B) Trail
C) Cycle
D) Closed walk

Q2. If a simple graph $G$is not connected, then its complement $G^c$is:
A) Always disconnected
B) Always connected
C) Connected only if Ghas one component
D) Connected only when n is even

Q3. A graph with $n$vertices and $k$edges has at least how many components?
A) $k - n$
B) $n - k$
C) $n + k$
D) k

Vertex Cuts and Edges Cuts

We now introduce the notions of vertex cuts, edge cuts, vertex connectivity, and edge connectivity.

Definition: A subset S of the vertex set V(G) of a connected graph G is *vertex cut* of G if $G - S$ is disconnected or $c(G - S) > c(G)$.
That is, A *vertex cut* is a subset of vertices in a graph whose removal, along with their incident edges, disconnects the graph or increases the number of its connected components. It is also known as a separating set. It is a *k-vertex cut* if $|S| = k$. A vertex v of G is a *cut vertex* of G $\{v\}$ is a vertex cut of G.

Let G be a non-trivial connected graph with the vertex set V(G) and let S be a nonempty subset of V(G). For $\bar{S} = V - S = \emptyset$. Let $[S, \bar{S}]$ denote the set of all edges of G that have end vertex in S and other in $\bar{S}$. A set of edges of G of the form $[S, \bar{S}]$ is called an *edge cut* of G. An edge e is called *cut edge* of G if $\{e\}$ is an edge cut of G. An edge cut of cardinality k is called *k-edge* cut of G.

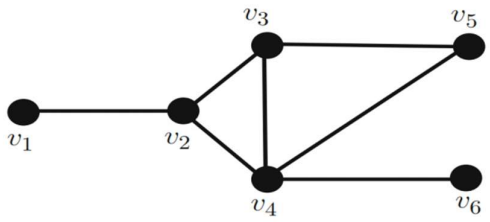


Figure 4.3

Example 4.3: For the graph of the Figure, $\{v_2\}$ and $\{v_3, v_4\}$ are vertex cuts. The edge subsets $\{v_3v_5, v_4v_5\}, \{v_1v_2\}$ and $\{v_4v_6\}$ are all edge cuts. v_2 is a cut vertex, and v_1v_2 and v_4v_6 are both cut edges. For the edge cut $\{v_3v_5, v_4v_5\}$, we may take $S = \{v_5\}$ so that $\bar{S} = \{v_1, v_2, v_3, v_4, v_6\}$.

***Remarks* 4.2.**

1. If *uv* is an edge of an edge cut S; then all the edges having *u* and *v* as their ends also belong to S.
2. No loop can belong to an edge cut.

Remarks 4.3. If G is connected and S is a set of edges whose deletion results in a disconnected graph, then S contains an edge cut of G: It is clear that if e is a cut edge of a connected graph G; then $G - e$ has exactly two components.

Remarks 4.4. Since the removal of a parallel edge of a connected graph does not result in a disconnected graph, such an edge cannot be a cut edge of the graph. A set of edges of a connected graph G whose deletion results in a disconnected graph is called a *separating set of edges.* In particular, any edge cut of a connected graph G is a separating set of edges of G.

Theorem4.3.5: ***A vertex v of a connected graph with at least three vertices is a cut-vertex if and only if there exist vertices x and y*** $(\neq v)$ ***such that every (x, y)-path contains v.***

Proof: *(1)* Suppose v is a cut-vertex. $G - v$ is disconnected so it contains at least two components, say C and D. Let $x \in V(C)$ and $y \in V(D)$. Since there is no (x, y)-path in $G - v$, it follows that every (x, y)-path in G contains v.

(2) Suppose there exist x and y$(\neq v)$ such that every (x, y)-path contains v. It follows that there is no (x, y)-path in $G - v$,. Hence $G - v$, is disconnected, that is v is a cut-vertex.

Theorem 4.3.6: ***An edge e = xy of a connected graph* G *is a cut edge of* G *if and only if e belongs to no cycle of* G.**

Proof: Let e be a cut edge of G and let $[S, \bar{S}] = \{e\}$ be the partition of V defined by $G - e$ so that one of x and y belongs to S; and the other to $\bar{S}$, say, $x \in S$ and $y \in \bar{S}$. If e belongs to a cycle of G; then $[S, \bar{S}]$ must contain at least one more edge, contradicting that Hence, $\{e\} = [S, \bar{S}]$ cannot belong to a cycle.
Conversely, assume that e is not a cut edge of G: Then $G - e$ is connected, and hence there exists an x-y path P in $G - e$. Then $P \cup \{e\}$ is a cycle in G containing e.

***Theorem 4.3.7: An edge $e = xy$ is a cut edge of a connected graph* G *if and only if there exist vertices u and v such that* e *belongs to every u-v path in* G.**
Proof: Let $e = xy$ be a cut edge of G: Then $G - e$ has two components, say,G_1 and G_2. Let $u \in V(G_1)$ and $v \in V(G_2)$. Then, clearly, every u-v path in G contains e. Conversely, suppose that there exist vertices u and v satisfying the condition of the theorem. Then there exists no u-v path in $G - e$ so that $G - e$ is disconnected.
Hence, e is a cut edge of G.

Theorem 4.3.8: ***Every connected graph G with* $n \geq 2$, *contains at least 2 vertices which are not cut-vertices.***
Proof: Let P be a path of maximum length in G; let P = P(x-y). Our claim is that neither x nor y is a cut-vertex of G. On the contrary, suppose x is a cut-vertex and consider G − x; see Figure 4.4. It contains at least 2 components say C and D where $y \in D$. Let v be any vertex in C. Then every (v-y) path in G contains x.
But then $Q = P_1(v\text{-}x),\ P_2(x\text{-}y))$ is a path of length greater than the length of P, a contradiction to the maximality of P.

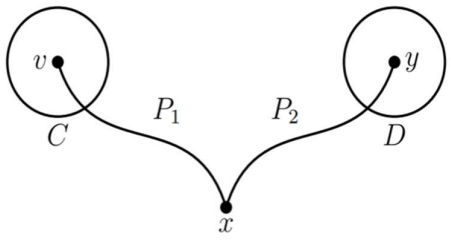


Figure 4.4: Existence of two paths P_1 and P_2 in G.

Proposition 4.3.9. ***A simple cubic (i.e., 3-regular) connected graph* G *has a cut vertex if and only if it has a cut edge.***

Proof: Let G have a cut vertex v_0. Let v_1, v_2, v_3 be the vertices of G that are adjacent to v_0 in G. Consider $G - v_0$ which has either two or three components. If $G - v_0$ has three components, no two of v_1, v_2 and v_3 can belong to the same component of $G - v_0$. In this case, each of v_0v_1, v_0v_2 and v_0v_3 is a cut edge of G. (See Fig. 4.5a.)
In the case when $G - v_0$ has only

two components, one of the vertices, say v_1 belongs to one component of $G - v_0$ and v_2 and v_3 belong to the other component. In this case, v_0v_1 is a cut edge. (See Fig. 4.5b.)

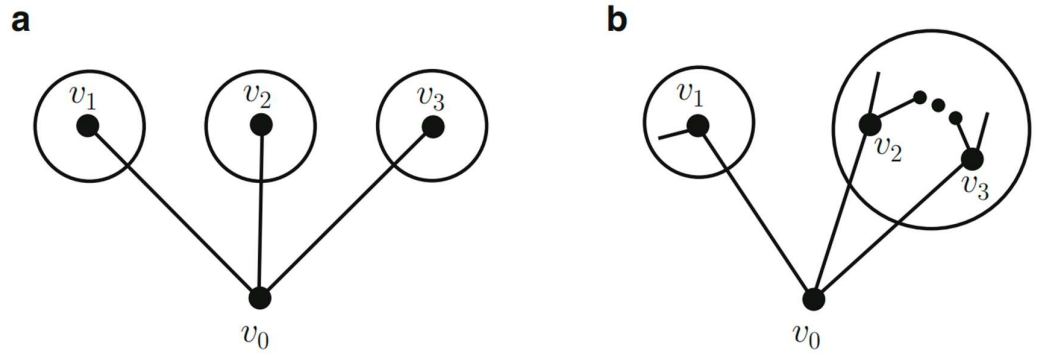
Conversely, suppose that $e = uv$ is a cut edge of G. Then the deletion of *u* results in the deletion of the edge *uv*. Since G is cubic, $G - u$ is disconnected. Accordingly, *u* is a cut vertex of G.

4.4 Edge and Vertex Connectivity

We now introduce two parameters of a graph that in a way measure the connectedness of the graph.

Definition For a nontrivial connected graph G having a pair of nonadjacent vertices, the minimum k for which there exists a k-vertex cut is called the *vertex connectivity* or simply the *connectivity* of G. It is denoted by $\kappa(G)$.

In other words, the integer



$\kappa(G) = \min\{|W| : W \subseteq V(G)$, W is a k-vertex cut $\}$ is called the *vertex-connectivity* of G.

If G is trivial or disconnected, $\kappa(G)$ is taken to be zero, whereas if G contains K_n as a spanning subgraph, $\kappa(G)$ is taken to be $n - 1$.

A set of vertices and/or edges of a connected graph G is said to *disconnect* G if its deletion results in a disconnected graph.

Remark 1: $\kappa(G) = 0$ if and only if *G* is disconnected or *G* is a single-vertex graph.
Remark 2: $\kappa(G) = 1$ if and only if *G* is connected and it has a cut-vertex.
Remark 3: $\kappa(G) = n - 1$ if G contains K_n as a spanning subgraph.

Definition: The *edge connectivity* of a connected graph G is the smallest k for which there exists a k-edge cut (i.e., an edge cut having k edges). The edge connectivity of G is denoted by $\lambda(G)$.

In other words, the integer

$\lambda(G) = \min\{|E| : E \subseteq V(G)$, E is a k-edge cut $\}$ is called the *edge-connectivity* of G.

Remark 1: $\lambda(G) = 0$ if and only if *G* is disconnected or *G* is a single-vertex graph.
Remark 2: $\lambda(G) = 1$ if and only if *G* is connected and it has a cut-edge.

Remark 3: $\lambda(G) \leq \delta(G)$; this follows since by deleting all the edges incident with a vertex of minimum degree, we disconnect the graph.

Definition: A graph G is *r-connected* if $\kappa(G) \geq r$. Also, G is r-*edge connected* if $\lambda(G) \geq r$.

An r-connected (respectively, r-edge-connected) graph is also l-connected (respectively,
l-edge connected) for each $l, 0 \leq l \leq r-1$.
We now derive inequalities connecting $\kappa(G), \lambda(G)$ and $\delta(G)$.

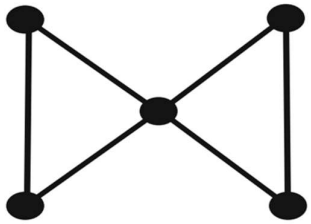


Figure 4.6: A 1-connected graph

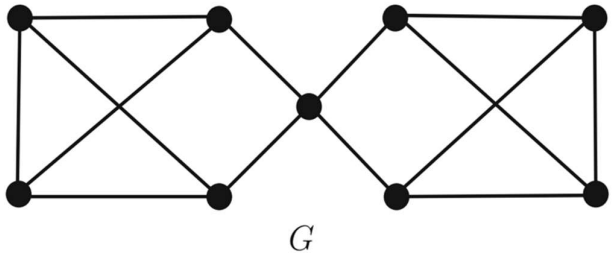


Figure 4.7: Graph G with
$\kappa = 1, \lambda = 2$ and $\delta = 3$

Theorem 4.5.1 ***For any loopless connected graph*** **G;** $\boldsymbol{\kappa(G) \leq \lambda(G) \leq \delta(G)}$.
Proof: We observe that $\kappa = 0$ if and only if $\lambda = 0$. Also, $\delta = 0$ implies that $\kappa = 0$ and $\lambda = 0$. Hence we may assume that κ, λ and δ are all at least 1: Let *E* be an edge cut of G with λ edges. Let *u* and *v* be the end vertices of an edge of *E.* For each edge of *E* that does not have both *u* and *v* as end vertices, remove an end vertex that is different from *u* and *v*. If there are *t* such edges, at most *t* vertices have been removed. If the resulting graph, say H, is disconnected,
then $k \leq t < \lambda$. Otherwise, there will remain a subset of edges of E having *u* and *v* as end vertices, the removal of which from *H* would disconnect G. Hence, in addition to the already removed vertices, the removal of one of *u* and *v* will result in either a disconnected graph or a trivial graph. In the process, a set of at most $t+1$ vertices has been removed and
$k \leq t+1 \leq \lambda$.

Finally, it is clear that $\lambda \leq \delta$. In fact, if v is a vertex of G with $d_G(v) = \delta$, then the set $[\{v\}, V\backslash\{v\}]$ of δ edges of G incident at v forms an edge cut of G. Thus $\lambda \leq \delta$.

Theorem 4.5.2: ***The connectivity and edge connectivity of a simple cubic graph*** **G** ***are equal.***

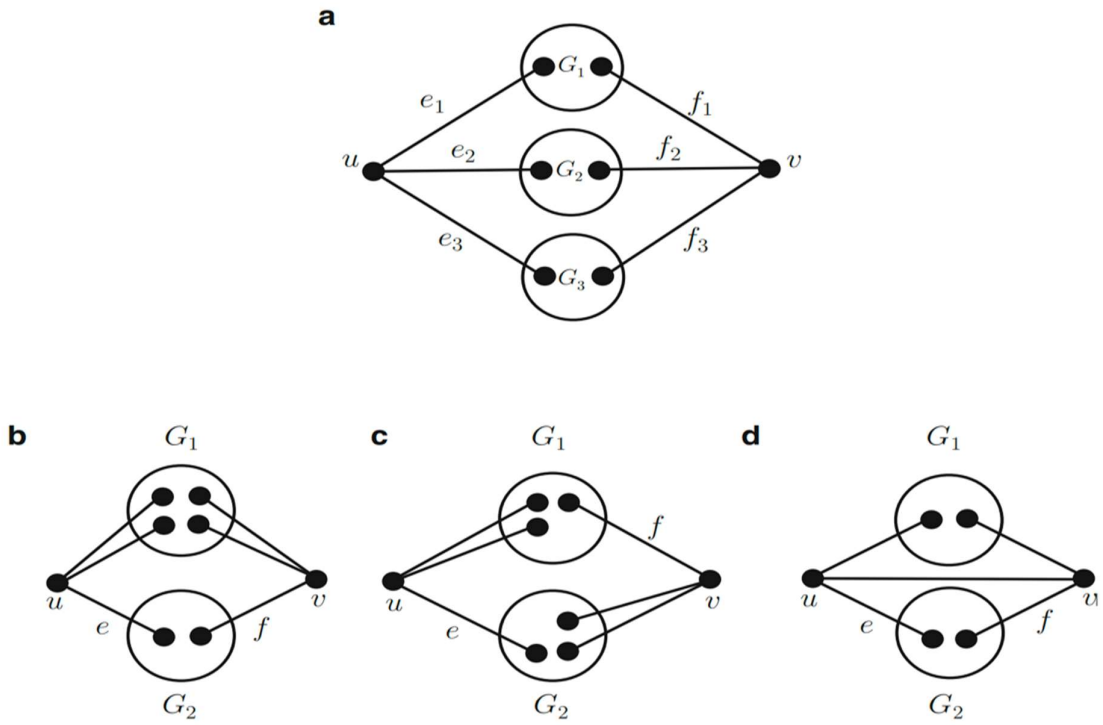


Figure 4.8

Proof: We need only consider the case of a connected cubic graph. Again, since $k \leq \lambda \leq \delta = 3$, we have only to consider the cases when $\kappa = 1, 2$ or 3. Now, **Proposition 4.3.9** implies that for a simple cubic graph G, $\kappa = 1$ if and only if $\lambda = 1$.
If $\kappa = 3$, then by Theorem 4.5.1, $3 = k \leq \lambda \leq \delta$ and hence $\delta = 3$.
We shall now prove that $\kappa = 2$ implies that $\lambda = 2$.
Suppose $\kappa = 2$ and $\{u, v\}$ is a 2-vertex cut of G. The deletion of $\{u, v\}$ results in a disconnected subgraph G' of G. Since each of u and v must be joined to each component of G' and since G is cubic, G' can have at most three components. If G' has three components, G_1, G_2 and G_3 and if e_i and $f_i, i = 1, 2, 3$ join, respectively, u and v with G_i then each pair $\{e_i, f_i\}$ is an edge cut of G (see Fig. 4.8a).

If G' has only two components, G_1 and G_2 then each of u and v is joined to one of G_1 and G_2 by a single edge, say, e and f; respectively, so that $\{e, f\}$ is an edge cut of G (see **Fig. 4.8b–d).**

Hence, in either case there exists an edge cut consisting of two edges. As such $\lambda \leq 2$. But by **Theorem 4.5.1**, $\lambda \geq \kappa = 2$. Hence $\lambda = 2$. Finally, the above arguments show that if $\lambda = 3$ then $\kappa = 3$ and if $\lambda = 2$, then $\kappa = 2$.

W CHECK YOUR PROGRESS

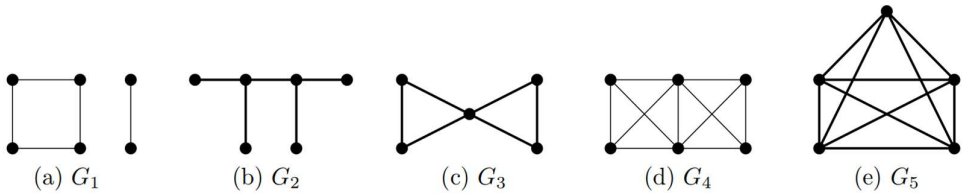
Q4. What is a cut edge of a graph?

Q5. Can a loop be part of an edge cut?

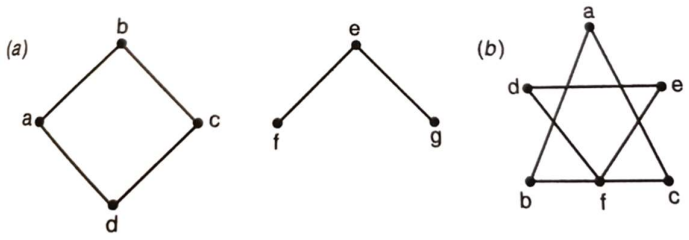
Q6. State the inequality connecting vertex connectivity, edge connectivity, and minimum degree.

Q7. What is the vertex connectivity $\kappa(G)$of a disconnected graph?

Q8. Find vertex connectivity and edge connectivity of the following graphs ?



Q9. Find the number of connected components of each of the following graphs.



24.5 LET US SUM UP

- A walk in a graph is a sequence of vertices and edges, which may be open or closed.
- A trail is a walk with no repeated edges, and a path is a walk with no repeated vertices.

- A closed walk in which all vertices are distinct except the first and last is called a cycle.
- Every u-v walk contains a u-v path.
- A graph is connected if there exists a path between every pair of vertices; otherwise, it is disconnected.
- Components of a graph are its maximal connected subgraphs.
- A vertex cut is a set of vertices whose removal disconnects a graph; a cut-vertex is a single such vertex.
- An edge cut is a set of edges whose removal disconnects a graph; a cut-edge is a single such edge.
- The connectivity $\kappa(G)$and edge-connectivity $\lambda(G)$measure the minimum number of vertices or edges, respectively, whose removal disconnects the graph.
- For any loopless connected graph G, the inequality $\kappa(G) \leq \lambda(G) \leq \delta(G)$holds, where $\delta(G)$is the minimum degree of G.

K 4.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q1: B) Trail

Ans. to Q2: B) Always connected

Ans. to Q3: B) $n - k$

Ans. to Q4: An edge $e$is a cut edge if its removal increases the number of components of the graph (i.e., $\{e\}$is an edge cut).

Ans. to Q5: No, a loop cannot belong to an edge cut.

Ans. to Q6: $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Ans. to Q7: $\kappa(G) = 0$.

Ans. to Q8:

	G_1	G_2	G_3	G_4	G_5
$\kappa(G)$	0	1	1	2	4
$\lambda(G)$	0	1	2	3	4

Ans. to Q9: (a) Two , (b) Two

O 4.7 FURTHER READINGS

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M 4.8 MODEL QUESTIONS

Q1. Define a walk, trail, path, and cycle in a graph. Give one example of each.

Q2. State and prove that every u-v walk in a graph contains a u-v path.

Q3. What is the difference between an open walk and a closed walk? Illustrate with an example.

Q4. Define connected and disconnected graphs. What are the components of a graph?

Q5. Show that if $G$is a simple graph with $\delta(G) \geq (n-1)/2$, then $G$is connected.

Q6. Prove that if a simple graph G is not connected, then its complement G^c is connected.

Q7. Define a cut-vertex and a cut-edge. Give an example of each.

Q8. State and prove the theorem: An edge $e = xy$ is a cut edge of Gif and only if $e$belongs to no cycle of G.

Q9. Define connectivity $\kappa(G)$and edge-connectivity $\lambda(G)$. State the inequality relating $\kappa(G)$, $\lambda(G)$, and $\delta(G)$.

Q10. Show that for a simple cubic connected graph, vertex-connectivity and edge-connectivity are equal.

Q11. Find vertex connectivity and edge connectivity of the following graphs ?

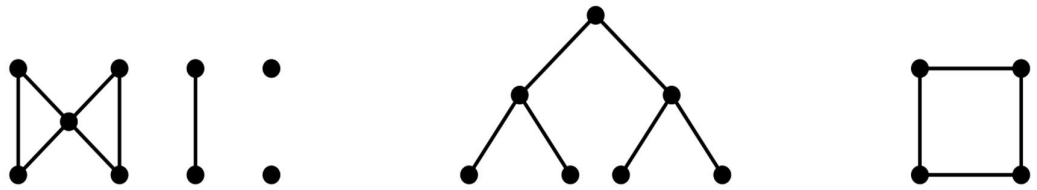
(i) Complete graph

(ii) Complete bipartite graph

(iii) Path graph

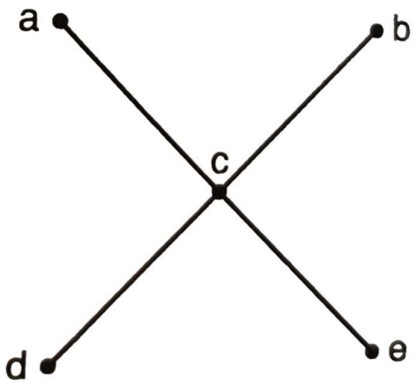
(iv) Cycle graph

Q12. Find all the cut-vertices of the given graphs



(a) (b) (c)

Q13. Find all the cut-edges of the following graphs



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