Unit 5: K-CONNECTED GRAPH

UNIT STRUCTURE

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5.1 LEARNING OBJECTIVES

After going through this unit, you will be able to-

- understand structural properties of graph in terms of connectivity
- understand the difference in K-connected and K-connected and R-edge connected
- know about complete graphs.

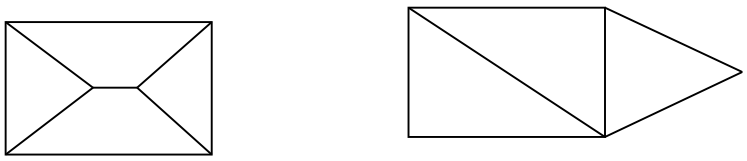
5.2 INTRODUCTION

In this unit, we introduce connected group, r-connected groups, R-connected edges, and complete groups and try to understand the properties exhibited by these groups.

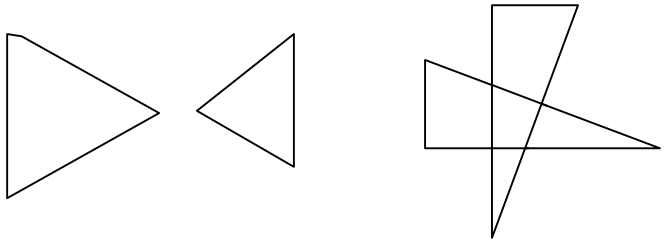
5.3 CONNECTED AND DISCONNECTED GRAPHS

A group is considered connected if we can reach any vertex from any other vertex by travelling along the edges. If no such path exists between any two vertices, then we consider it to be a disconnected graph.

Example:



(a) Connected graphs (b)



(c) Disconnected graphs (d)

Definition:

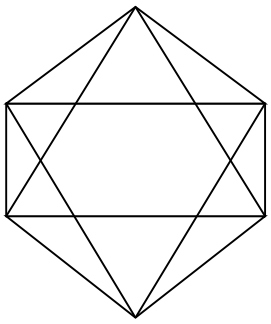
Connected: A graph G is said to be connected if there is at least one path between every pair of vertices in G.

Disconnected: A graph G is said to be disconnected if it is not connected, that is, it has at least one pair of vertices such that no path exists between them.

5.4 K-CONNECTED GRAPHS

K-vertex connected: A graph is R-vertex connected if at least K-vertices must be removed to disconnect the graph. This means G has more than R-vertices and remains connected whenever fewer than R-vertices are removed.

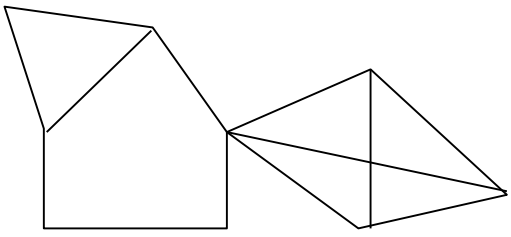
Example:



This is a 4-vertex connected graph. Removing vertices 1, 2, 3, 4, 5 makes it a disconnected graph.

K-edge connected graphs: A graph G is R-edge connected if at least K-edges must be removed to disconnect the graph. This means G has more than R-edges and remains connected whenever fewer than R-edges are removed.

Example:

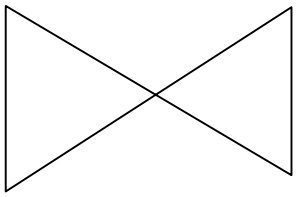


3- edge connected graph.

- Edge- connectivity and enumeration of K-edge connected graphs was studied by Camille Jordan in 1869.
- Connectivity measures fault tolerance of a network: how many connections can fail without cutting off communication.

Remark: K- vertex connected implies K-edge connected. But not conversely.

Example:



2- edge connected but 1-vertex connected.

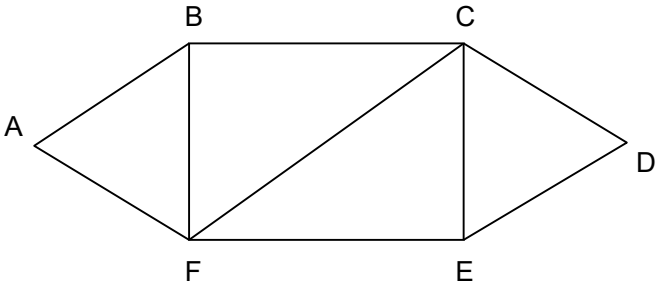
5.5 2-connected graphs

A 2-connected graph is a connected graph where removing any single vertex (and its incident edges) does not vertices need to be removed to break the connectivity of the graph.

2-connected graphs are also Known as bi-connected graphs.

A bi-connected graph is also called non-separable because removing a single vertex does not separate it into disconnected components.

Example:



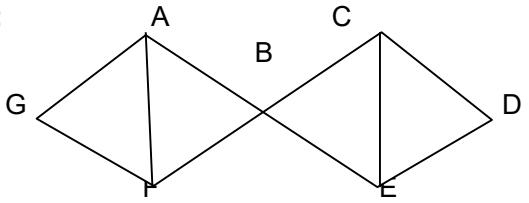
2- connected graph.

(Removing vertex C and F and its incident edges makes it disconnected)

Articulation Vertex (or cut vertex)

A vertex is a connected graph is an articulation vertex if removing it (and its incident edge) disconnects the graph. Clearly, a bi-connected graph has no cut vertex.

Example:



B is the cut vertex

W CHECK YOUR PROGRESS

Q1. Draw a 3- edge connected graph.
Q2. Draw a 2- connected graph.
Q3. Draw a graph such that it is R-edge connected but R-vertex connected (for any value of R)

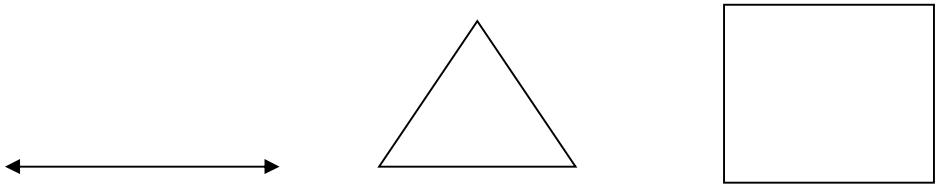
5.6 COMPLETE GRAPH

A complete graph is a simple, undirected graph where every pair of distinct vertices is connected by a unique edge.

A complete graph with 'n' vertices is denoted as K_n.

Note : In a complete graph, loops are not allowed as there has to be only one edge between any 2 vertices.

Example:



k_2 k_3 k_4

Properties of a complete graph

(i) In a complete graph with 'n' vertices, the number of edges is given by $\frac{n(n-1)}{2}$

(ii) In a complete graph with 'n' vertices, each vertex has a degree of (n-1) i.e it is connected to (n-1) other vertices.

5.7 THEOREMS AND PROPERTIES RELATED TO CONNECTIVITY

Notation: (i) Vertex connectivity is denoted by K(G)

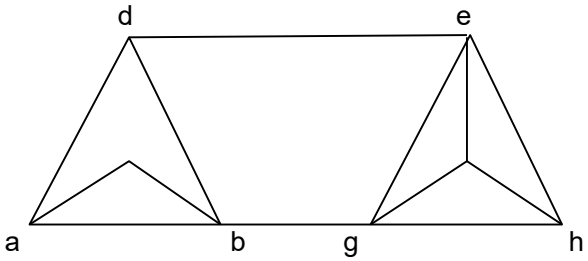
(iii) Edge connectivity is denoted by λ (G)

For any connected graph G.

I : For any connected graph G,

$K(G) \leq \lambda(G)$

Example: Calculate K(G) & λ (G) for the following graph.



We observe that deleting edges. {d, e} and {b, g} makes the graph disconnected.

$\therefore \lambda$ (G) = 2 …. (i)

Again, we observe that removing any single vertex does not disconnect the graph

$\therefore$ K(G) $\geq$ 2 …. (ii)

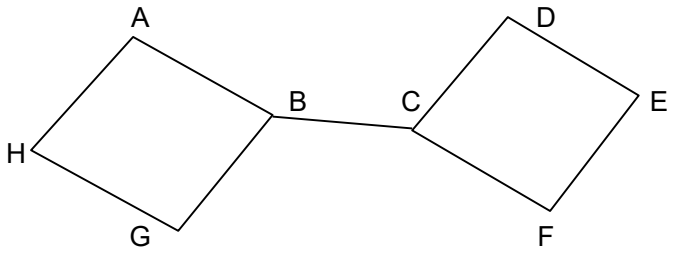
Also, we know K(G) $\leq \lambda$ (G)…… (iii)

….From (i), (ii) and (iii): K(G) = λ (G) = 2

Note: If G has a cut-vertex, then K(G)-1

Remark: Similar to cut vertex, we have cut edge. Yet G be a connected graph. An edge 'e' is called a cut edge if 'G-e' result in a disconnected graph.

Example:



{B, C} is the cut- edge

II : Yet G be 2- connected. If we remove one-edge to form a subgraph H, then H is also 2-connectec.

III : Every K-connected graph of order al least 2k has cycle of length at least 2k.

IV : A connected graph G is R-connected if and only if every pair of vertices in G is joined by k or more paths that do not intersect and at least one pair of vertices is joined by exactly k-non intersecting paths.

V : If G is k-connected, then min deg (ν) $\geq$ k v € VG

Where Vg is the set of vertices of G.

Proof : Suppose min deg (ν) ≥ R is not true. Then there exists some vertex say υ_1, such that degree of V, is less than R, that is, deg (υ_1)< (I, e vertices adjacent to υ_1)

Then clearly, ISI < k

By definition of R-connectivity, removing a set of less than R vertices should Keep the graph connected.

∴ G\S is connected.

However, there are no paths from ν_1, to any other vertices as all the adjacent vertices are removed.

Therefore, ν_1, Becomes isolated and the graph becomes disconnected. This is a contradiction to the R-connectivity of the graph. Thus, our assumption that deg (ν_1) < R is wrong.

∴ $\min_{v \in V_g} \deg(v)$ ≥ k is true.

VI : A graph G is 2-connected if and only if g, a cycle in G containing these two vertices.

Proof : Suppose there exists a cycle in G for any two vertices of G.

To prove : G is 2-connected.

Yet ν_1 and ν_2 be two vertices of G which lie on a common cycle. Then clearly there exists two paths connecting them having no common vertices except for the and vertices. Therefore, V, and … can never fall into distinct components by removing a single vertex.

Since ν_1 and ν_2 are any two arbitrary vertices of G, thus we can say that it holds for all the vertices of G and so G is 2-connected.

Conversely, let G be 2-connected and ν and ν' be two vertices of G we prove that there exists a cycle in G containing these two vertices.

The existence of a common cycle for v and v' shall be established for v and v' shall be established by induction on d g (ν , ν'), the distance of the vertices ν and ν'.

First, let dg (ν , ν') =1

By 2-connectivity of G, the graph G-2 is connected. Therefore, there exists a path from ν to ν' in the graph G- e and this path together with the edge e forms the required cycle containing both ν and ν'.

Next, suppose that any pair of vertices at distance less than k his on a common cycle for some k ≥ 2. This forms the inductive hypothesis.

Now, consider two vertices v and v' at distance k.

Let P = $v = (v_0, v_1, v_2, ...v_k = v')$ he a shorlest path from v to v'.

Since dg By the inductive hypothesis, there exists a cycle containing both v and v_{k-1}. This cycle consists of two paths, say P_1 and P_2 from v to v_{k-1}.

Now consider the graph G - v_{k-1} It is connected and hence it has a path P/ from v to v'. This path doesn't contain v_{k-1}. Suppose we go along this P/ from v to v' and consider the last vertex on this path which may either belong to one of the path P_1 and P_2. Yet us denote this vertex by 'U'

Without loss of generality, suppose 'u' is a vertex of P_1. Then the desired cycle containing v and v' is formed by the path P_2, by the portion of the path P_1 between v and u, by the portion of the path between U and V'.

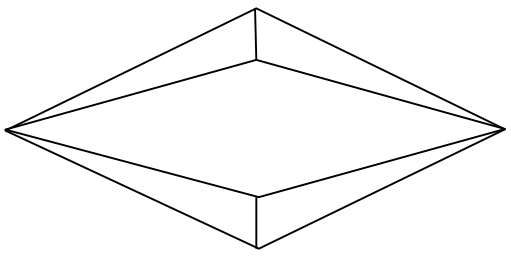
2 5.8 LET US SUM UP

This unit explores the concept of A graph is k-connectivity in graphs.

- A graph is k-connected if it remains connected wherever fewer then k vertices are removed.
- **Network Reliability:** k-connectivity is crucial in designing reliable networks, such as communication or transportation systems, where the failure of a few modes or edges should not distrupt the entire network.
- **Fault Tolerance:** A k-connected graph can tolerate up to (k-1) mode failures without losing connectivity, making it a vital concept in fault tolerant network design.

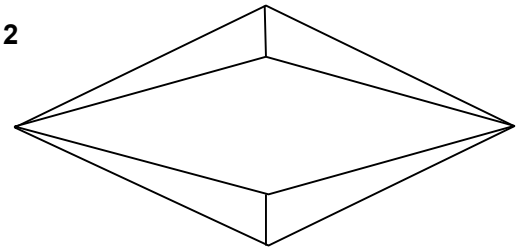
K 5.9 ANSWERS TO CHECK YOUR PROGRESS

Ans. To Q. No. 1



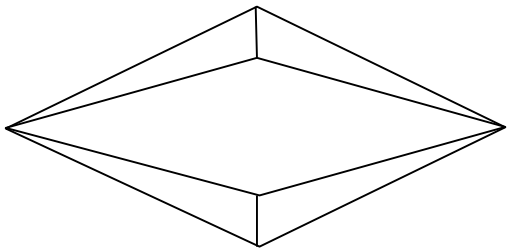
3-edge connected graph

Ans. To Q. No. 2



2-connected graph

Ans. To Q. No. 3



2-vertex connected and 3-edge connected graph

O 5.10 FUTHER READINGS

1. N. Deo, ‘‘Graph Theory with Applications to Engineering and Computer Science’’

2. S. Lipschutz, M. Lipson, ``Schaunis Outlines Discrete Mathematics’’.

M 5.11 MODEL QUESTIONS

1: Draw a 2-connected graph with 6 vertices.

2: Draw the complete graph with 5 vertices.

3: Given a graph with 6 vertices and 9 edges, determine its vertex connectivity and edge connectivity.

4: Yet G be a connected graph and let H be obtained from G by addling edges xy if and only if dist $dist_g(x, y) = 2$.

Prove that H is 2- connected.

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