Unit 6: TREE

UNIT STRUCTURE

6.1 Learning Objectives
6.2 Introduction
6.3 To know
6.4 Rooted Trees
6.5 Properties of Trees
6.6 Spanning Tree
6.7 Centre and Centroid of Tree
6.8 Binary tree
6.9 Independent Cycles
6.10 Block Cut Point Tree
6.11 Let Us Sum Up
6.12 Answers to Check Your Progress
6.13 Further Reading
6.14 Model Questions

6.1 LEARNING OBJECTIVES

After going through this unit, you will be able to-

- understand trees in graph theory and the various properties exhibited by trees.
- know about the different types of trees: binary trees, spanning trees etc.
- apply the concept of trees to real life situations.

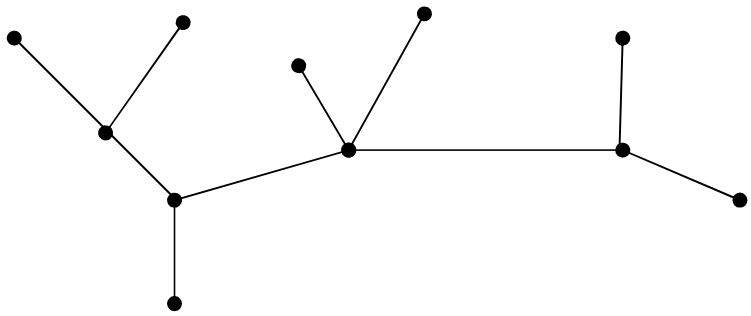
6.2 INTRODUCTION

Trees are a fundamental and widely studied topic in graph theory, characterized by their acyclic and connected structure.

Understanding trees is crucial as they form the basis for many data structures and algorithm data structures and algorithm used in computer science.

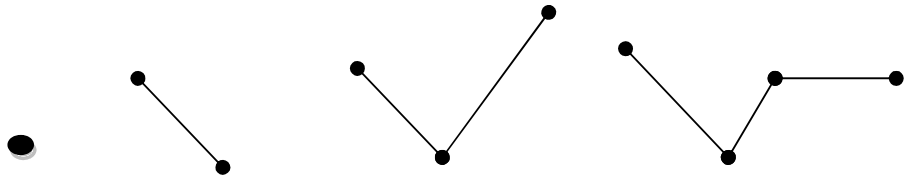
Definition: A tree is a connected graph without any circuits.

Example:



It is clear from the definition that a tree is a simple graph without any loops or parallel edges (as the presence of loops or parallel edges will result is a circuit)

Following are examples of trees with one, two three and four vertices:



(a) (b) (c) (d)

6.3 TO KNOW

- A graph without cycles is said to be cycle free or acyclic
- A forest G is a graph with no cycles, hence the connected components of a forest G are trees.
- A tree consisting of a single vertex with no edges is called the degenerate tree.
- A vertex of degree one in a tree is called a leaf or a terminal mode and a vertex of degree larger than one is called a branch mode or an internal mode.

Example:



Here a, b, c, f, g, h are leaves d and e are branch modes.

6.4 ROOTED TREES

A rooted tree is a tree where one specific mode is designated as the root and all other modes in the tree are connected through a path.

The root serves as the starting point or parent mode from which all other modes in the tree are connected.

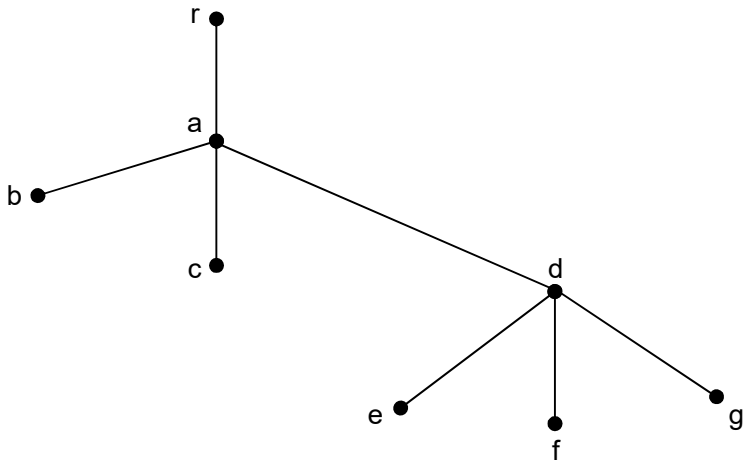
Example:



(a) (b)

Yet 'r' be the root of a tree and 's' be a branch mode. A vertex 't' is said to be a son of 's' if there is an edge from s to t. Also, s is said to be father of t. Two vertices are said to be brothers if they are sons of said to be a descendent of s if there is a directed path from s to m. Also, s is said to be an ancestor of m.

Example:



Here,

r $\rightarrow$ root

a $\rightarrow$ branch mode

b, c, d $\rightarrow$ sons of a (equivalently a $\rightarrow$ father of b, c, d)

b, c, d $\rightarrow$ brothers

e, f, g $\rightarrow$ descendants of a (equivalently a $\rightarrow$ ancestor of e, f, g)

6.5 PROPERTIES OF TREES

1: There is one and only one path between every pair of vertices in a tree T.

Proof: Since tree T is a connected graph (by definition), so there exist at least one path between every pair of vertices in T.

Now, suppose that between two vertices, say `a' and `b' of T, there are two distinct paths. Then the union of these two paths will contain a circuit which is a contradiction to the fact that T is a tree.

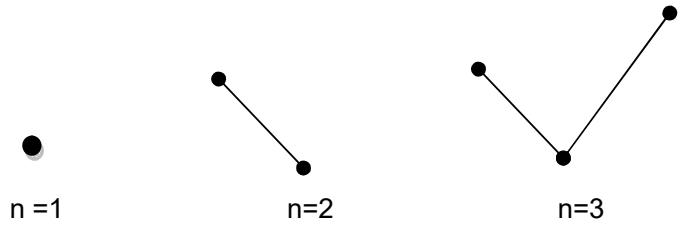
Thus, our assumption that between two vertices are two paths is wrong. Therefore, there is only one path between every pair of vertices in a tree T.

2. If in a graph G, there is one and only one path between every pair of vertices, then G is a tree.

Proof: The existence of a path between every pair of vertices assures that G is connected. A circuit in a graph (With two or more vertices) implies that there is at least one pair of vertices `a' and `b' such that there are two distinct paths between a and b. Since G has one and only one path between every pair of vertices, so H can have no circuit. Therefore, G is a tree.

3. A tree with n vertices has (n-1) edges.

Proof: We prove this by induction on the number of vertices.



Clearly, the theorem holds for n=1, 2, 3. Induction hypothesis: Assume that the theorem holds for all trees with fewer than n=vertices.

Let us now consider a tree T with n vertices. Yet ν_i And ν_j be two vertices in T and e be the connecting edge between them. From the previous property we can say that there is no other path between ν_i and ν_j except e_k. Therefore, deletion of .. from T will disconnect the graph.

Furthermore, T - e_k Consists of exactly two components, say T_1 and T_2, and since there were no circuits in T to begin with, so each of these components T_1 and T_2 Is itself a tree.

Clearly both these have fewer than n-vertices.
Let … have n vertices and T_2 have n_2 vertices. Than, $n_1+n_2 = n$.
Now, using induction hypothesis on T_1 and T_2 since they have fewer then n-vertices, we get that T_1 has $(n_1–1)$ edges and T_2 has $(n_2–1)$ number of edges. Therefore, $T_1 + T_2$ together has $(n_1–1) + (n_2–1)$

$= n_1+n_2 – 2$

$=(n–2)$ number of edges.

T_1 and T_2 are components of T formed by removing edge.

i.e $T_1+T_2 = T– e_k$

∴ If we join edge e_k well then we get T has (n-2) + 1 = (n-1) edges.

4. A graph with n vertices and …. Edges and no circuits is connected.

Proof: Suppose there exists a circuit less graph with n-vertices and … edges which is disconnected. In that case G will consist of two or more circuit less components. Without loss of generality, let G consist of two components G_1 and G_2 Let v_1 be a vertex in G_1 and v_2 be a vertex in G_2. We add an edge `e' between vertex $v_1$ and vertex $v_2$. Since there was no path between $v_1$ and $v_2$ in G, so adding `e' did not create a circuit. Thus, G Ue Is a circuitless, connected graph, i.e, G Ue is a tree. Since G had n vertices and (n-1) edges, so G Ue has n vertices and n edges. But this is a contradiction to the previous property that states that a tree with n-vertices has (n-1) edges. Therefore, our assumption that G is disconnected is wrong. Thus, G is connected.

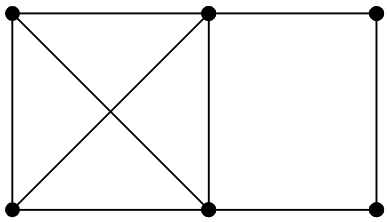
W CHECK YOUR PROGRESS

Q. 1. Prove that a connected graph with n vertices and (n-1) edges is a tree.

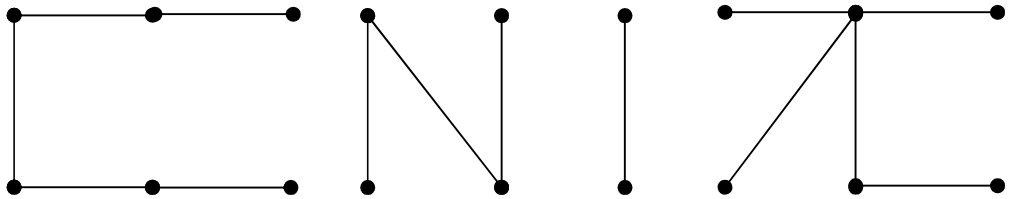
6.6 SPANNING TREE

A sub graph T of a connected graph G is called a spanning tree G if T is a tree and T includes all the vertices of G.

Example:



G



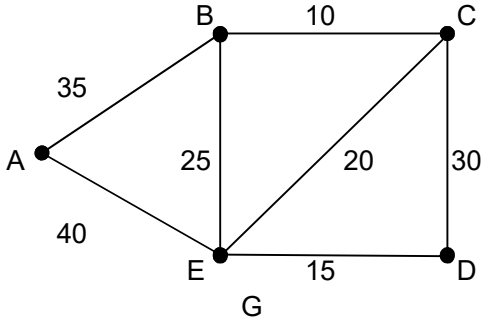
T_1 T_2 T_3

T_1, T_2, T_3, are spanning trees of G.

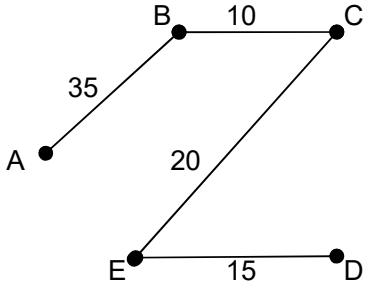
Minimum Spanning Tree

Suppose G is a connected weighted graph. That is, each edge of G is assigned a non-negative number called the weight of the edge. Then any spanning tree T of G is assigned a total weight obtained by adding the weights of the edges in T. A minimal spanning tree of G is a spanning tree whose total weight is as small as possible.

Example:



G



Minimal Spanning Tree of G

6.7 CENTRE AND CENTROID OF TREE

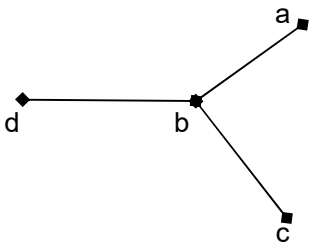
In a connected Graph G, the distance d (v_j, v_j) between two of its vertices v_i and v_j is the length of the shortest path (i.e, number of edges in the shortest path) between them.

Eccentricity: The eccentricity E (v) of a vertex v in a graph G is the distance from v to the vertex farthest from v in G, i. e.

$$E(v) = \max d(v_i, v_j)\ v_i \in G$$

Centre: A vertex with minimum eccentricity in a tree is called the eccentricity in a tree is called the centre of the tree.

Example:



E (a) = 2
E (b) = 1
E (c) = 2
E (d) = 2

. Vertex b is the centre of that tree.

Remark: Every tree has either one or two centres.

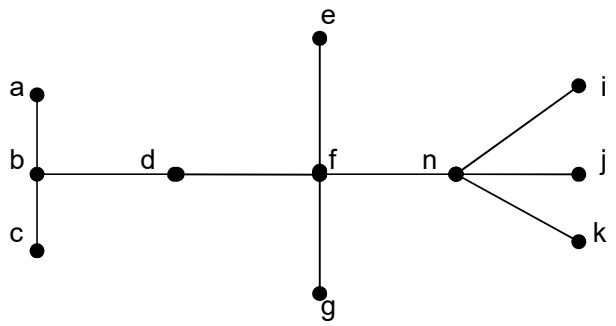
(i) If a tree has a single centre. It is called a centre it is called a central tree.

(ii) If a tree has two centres, it is called a bicentral tree. The two centers are always adjacent.

W CHECK YOUR PROGRESS

Q.2 Draw a tree with 2 centres.

Centroid of a tree:
A centroid of a tree is a mode that when removed result in sub trees with almost half the number of nodes of the original tree.
Every tree has at least one centroid and it can have either one or two centroids.
Example:

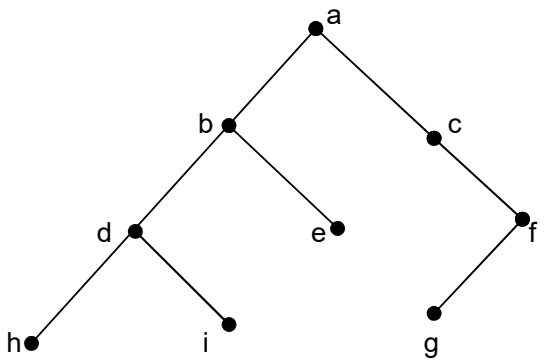


G

In the graph G, vertex f is the centroid.

6.8 BINARY TREE

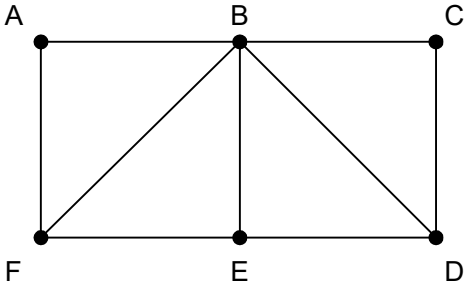
A binary tree is a rooted tree in which every internal mode have at most two children referred to as the left child and the right child.
Example:



6.9 INDEPENDENT CYCLES

A cycle in a graph is a path that starts and ends at the same vertex, containing no repeated vertices except the starting/ending vertex.
Independent cycles are cycles that do not share any edges.

Example:

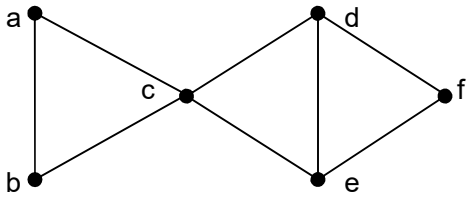


Observe that cycle ABFA and BDEB are independent. But cycles BEFB and BDEB are not independent as they share a common edge BE.

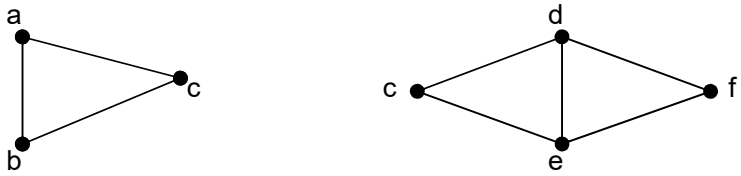
6.10 BLOCK CUT POINT TREE

Let G be a connected graph with tree or more vertices, G is said to be separable if it has at least one cut vertex. Otherwise, G is non- separable. A maximal non-separable connected sub graph of G is called a block of G.

Example:

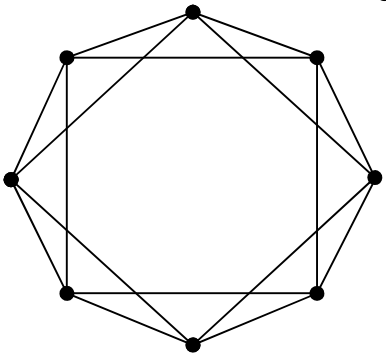


G_1



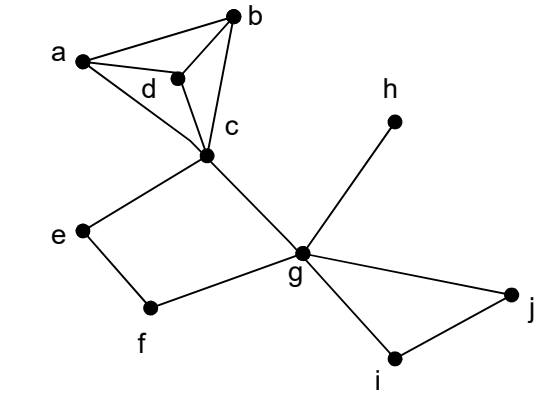
Blocks of graph G_1

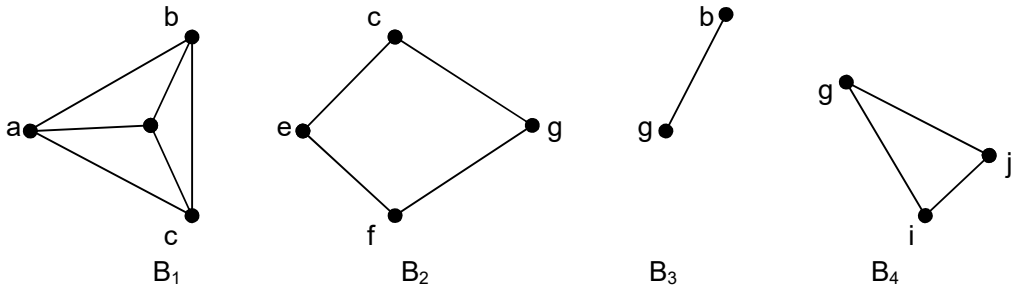
2)



G_2

Graph G_2 is non- separable, so it is a block in itself.





B_1 B_2 B_3 B_4

Graph G_3 has blocks: B_1, B_2, B_3 and B_4

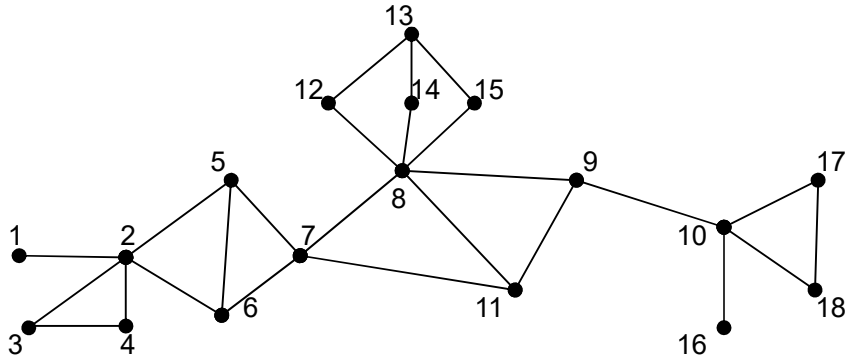
Properties of blocks:

i) An edge of a cycle cannot itself be a block since it is in a larger sub-graph with no cut vertex. Hence, an edge is a block if and only if is a cut edge
ii) The blocks of a tree are its edges.
iii) If a block has more than two vertices, then it is 2-connceted.
iv) Two blocks in a graph share at most one vertex.

Block Cut Tree

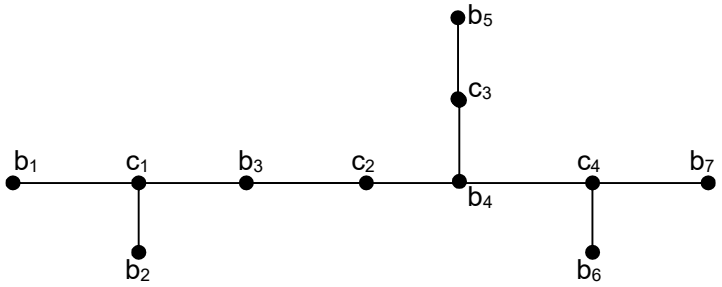
The structure of the blocks and cut points of a connected graph can be described by a tree called the block Cut tree. This tree has a vertex for each block and for each articulation point of the block cut tree for each pair of a block and an articulation point that belongs to that block.

Example:



G

The block cut tree of G is:



Blocks

b_1 = [1, 2]
b_2 = [2, 3, 4]
b_3 = [2, 5, 6, 7]
b_4 = [7, 8, 9, 10, 11]
b_5 = [8, 12, 13, 14, 15]
b_6 = [10, 16]
b_7 = [10, 17, 18]

Cut points

c_1 = 2
c_2 = 7
c_3 = 8
c_4 = 10

2 6.11 LET US SUM UP

- This Unit explores the fundamental concept of trees in Graph theory, including their definition, properties and various types such as spanning trees, rooted trees, binary trees etc.
- Significance: Understanding trees is essential as they have numerous applications in computer science, network design and decision making.

K 6.12 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q 1. A connected graph with n vertices and (n-1) edges is a tree.

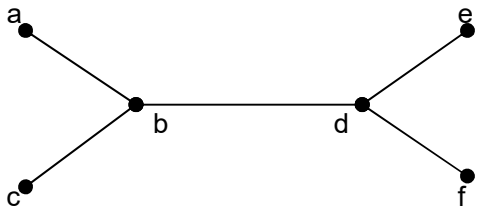
Proof: Let if possible, G is not a tree. Since G is already given to be connected, this implies that G must contain atleast one cycle, one cycle. If G contains one or more cycles, we can systematically remove one edge from each cycle. When an edge is removed from a cycle, the graph still remains connected because there are at least two paths connecting the endpoints of any edge within a cycle. After removing all necessary edges, we obtain a new graph $G^{/}$. $G^{/}$ is now connected and acyclic (by construction), making it a tree.

Since $G^{/}$ is tree with n vertices, it must have (n-1) edges.

However, to obtain $G^{/}$ from G, we removed at least one edge (because G was assumed to have a cycle).

This means $G^{/}$ would have fewer than (n-1) edges, which contradicts the property that a tree with n vertices must have exactly (n-1) Edges.

Ans. to Q 2.



Tree with two centers: b and d.

O6.13 FURTHER READING

1. N. Deo, ‘‘Graph Theory with Applications to Engineering and computer science.’’
2. S. Lipschutz, M Lipson, ‘‘Schaums Outlines Discrete Mathematics’’

M6.14 MODEL QUESTIONS

Q 1: What is the degree of a leaf mode in a tree?
Q 2: How does a rooted tree differ from an unrooted tree?
Q 3: Given a graph with 6 vertices, construct a tree that spans all vertices.

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