UNIT 7: CUT SETS AND CUT –VERTICES

UNIT STRUCTURE

7.1 Learning Objectives
7.2 Introduction
7.3 Edge and Vertex Connectivity
7.4 Cut Set
7.5 Fundamental Circuit
7.6 Some properties of a cut-set
7.7 Properties of Cut-Vertex
7.8 Fundamental circuit contains at least one cut sets
7.9 Let Us Sum Up
7.10 Answers to Check Your Progress
7.11 Further Reading
7.12 Model Questions

7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to-

- understand edge vertex connectivity
- define and identify cut sets and cut vertices
- determine various properties of cut sets.

7.2 INTRODUCTION

In graph theory, the concept of cut sets and cut vertices play a crucial role in understanding the connectivity and robustness of graphs. These concepts are essential in analyzing the vulnerability of networks to edge or vertex failures and have numerous applications in telecommunications and transportation systems.

This unit will explore the definitions and properties of cut sets, thereby providing insights into their significance in graph theory.

7.3 EDGE AND VERTEX CONNECTIVITY

Edge connectivity: The edge connectivity of a graph is the minimum number of edges that must be removed to disconnect the graph.

It is denoted by $\lambda(g)$

Vertex connectivity: The vertex connectivity of a graph is the minimum number of vertices that must be removed to disconnect the graph.

It is denoted by k(g)

Relationship between Edge Connectivity and Vertex Connectivity.

- $K(g) \leq \lambda(g) \leq \delta(g)$

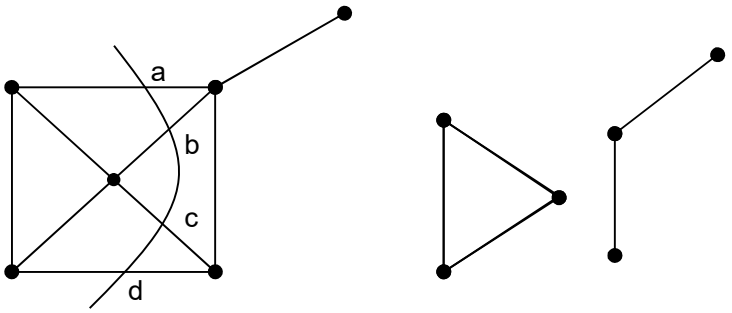
Where δ (g) Is the minimum degree of the graph G.

7.4 CUT SET

A cut set in graph theory is a set of edges where removal disconnects the connected graph.

Example:

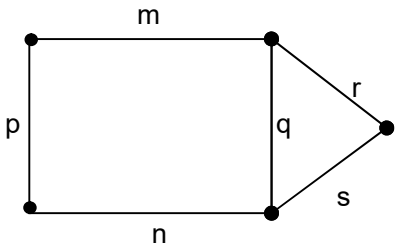
1)



G Disconnected

Cut Set of G: {a, b, c, d}

2)

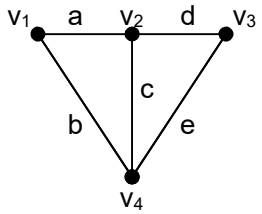


Cut Sets are: {m, n}
{P, n}
{r, s}
{m, q, r} etc.

Fundamental Cut Set

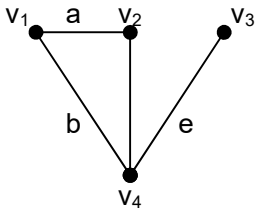
A cut set containing exactly one branch of spanning tree is called a fundamental cut set.

Example:



Cut sets are: {a, b} ; {d, e} ; {a, c, d} ; {a, c, e} ; {b, c, d} etc

Now, spanning tree of G is:



Here, a, b, e $\rightarrow$ branches of the tree. Now, observe the cut sects.

- {a, b} $\rightarrow$ both are branches of the spanning tree. But as per the definition, there should be exactly one branch of the spanning tree in the fundamental cut set.
- {a, b} cannot be a Fundamental Cut set.
- {d, e} $\rightarrow$ Contains exactly one branch which is `e'.
- {b, c, d} $\rightarrow$ exactly one branch which is `b'
- {b, c, d} is a Fundamental Cut set.
- {b, c, e} $\rightarrow$ Contains two branches `b' and `e'
- {b, c, e} cannot be a Fundamental Cut set. etc.

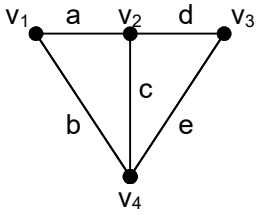
Note: The edges which are removed from the graph to create the spanning tree, are called chords.

In the above example, edges d and c have been removed from the main graph to from the spanning tree. Thus, d and c are the chords of G.

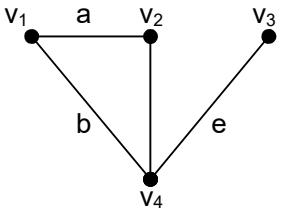
7.5 FUNDAMENTAL CIRCUIT

Let T be a spanning tree in a graph G. If adding any one chord to T creates exactly one circuit, then such a circuit is called Fundamental circuit.

Example:



G



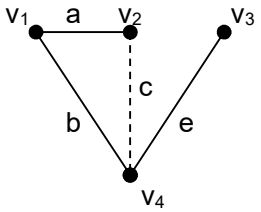
Spanning Tree T of G

Here, a, b, e $\rightarrow$ Branches

c, d $\rightarrow$ Chords.

Observe:

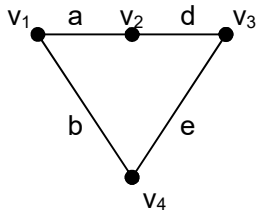
(i) Adding chord C



Clearly, a circuit is formed by adding exactly one chord, which is `c'

$\therefore$ The circuit formed, i.e, circuit $\{v_1\ v_2\ v_4\ v_1\}$ is a fundamental circuit

(ii) adding chord d.



Again, a circuit is formed by adding chord d.

$\therefore$ circuit $\{v_1\ v_2\ v_3 v_4\ v_1\}$ is a fundamental circuit.

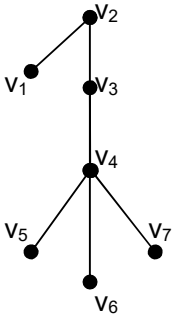
Cut Vertex

A vertex V of Graph G is said to be a cut vertex of G if G $-v$ is disconnected. In other words, a cut vertex is a vertex whose removal disconnects the graph.

Cut vertex is also Known as cut point or articulation point.

Example:

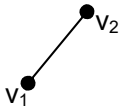
Cut vertex in G is …. The disconnected components are

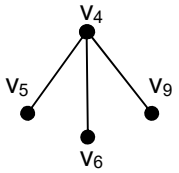


G

Cut vertex is G is 3

The disconnected components are

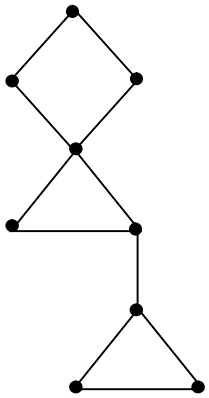




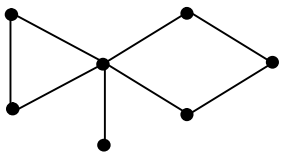
W CHECK YOUR PROGRESS

Q 1: Mention one cut-vertex and a cur-set for the following graphs

(i)



(ii)



7.6 SOME PROPERTIES OF A CUT-SET

(i) Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G.
(ii) In a connected graph G, any minimal set of edges containing are least one branch of every spanning of G is a cut set.
(iii) Every circuit has an even number of edges in common with any cut set.

Proof: Consider a cut set S in graph G. Let the removal of S partition the vertices of G into two (disjoint or mutually exclusive) subsets V_1 and V_2 Consider a circuit Γ in G. If all the vertices in Γ are entirely within vertex set V_1 (or V_2), then the number of edges common to S and Γ is zero, which is an even number.

If, on the other, some vertices in Γ are in V_1 and some in V_2, we transverse back and forth between the sets V_1 and V_2 as we traverse the circuit. Because of the closed nature of a circuit, the number of edges we traverse between V_1 and V_2 must be even. And since every edge in S has one end in V_1 and other in V_2 and no other edge in G has this property (of separating sets V_1 and V_2), the number of edges common to S and Γ is even.

(iv) Every cut set in a non-separable graph with more two vertices contains at least two edges.

Note: A connected graph in said to be separable if its vertex connectivity is one. All others connected graphs are called non-separable.

7.7 PROPERTIES OF CUT-VERTEX

(i) In a tree, every vertex with degree greater than one is a cut-vertex.
(ii) A vertex v in a connected graph G is a cut-vertex if and only if there exists two vertices x and y in G such that every path between x and y passes through v.

7.8 FUNDAMENTAL CIRCUIT CONTAINS AT LEAST ONE CUT SETS

1) Every fundamental circuit contains at least one cut set and every cut set contains at least one fundamental circuit.
2) The number of fundamental circuits is equal to the number of independent cycles in the graph.
3) With respect to a given spanning tree T, a chord c_i that determine a fundamental circuit Γ occurs in every fundamental cut-set associated with the branches in Γ and in no other.

Proof: Consider a spanning tree T in a given connected graph G. Let c_i be a chord with respect to T and let the fundamental circuit made by c_i be called Γ, consisting of k–branches $b_1, b_2, \ldots b_k$ in addition to the chord c_i that is,
$\Gamma = \{ c_i, b_1, b_2, \ldots b_k \}$ is a fundamental circuit with respect to T.

Every branch of any spanning tree has a fundamental cut set associated with it, Let S, be the fundamental cut set associated with b_1, consisting of q chords in addition to the branch b_1, that is, $S_1 = \{b_1, c_1, c_2, \ldots c_q\}$ is a fundamental cut set with respect to T.

By property (iii) of cut sets, we Know every circuit has an even number of edges in common with any cut set. Therefore, there must be an even number of edges common to Γ and S_1. Edge b_1 is in both Γ and S_1 and there is only one other edge in Γ (which is c_i) that can possibly also be in S_1. Therefore, we must have two edges b_1 and c_i common to S_1 and Γ. Thus, the chord c_i is one of the chords $c_1, c_2, \ldots c_q$.

Exactly the same argument holds for fundamental cut set associated with $b_1, b_3, \ldots$ and b_k. Therefore, the chord c_i is contained in every fundamental cut set associated with branches in Γ.

Suppose, if possible chord c_i be in some other fundamental cut set say $S^/$ (with respect to T) besides those associated with b_1, b_2, and b_k. Since move of the branches in Γ are in $S^/$, there would only be one edge c_i, common to $S^/$ and Γ. This is a contradiction to property (iii) as we need even number of edges in common.
Thus, chord c_i cannot be in any other fundamental cut set.

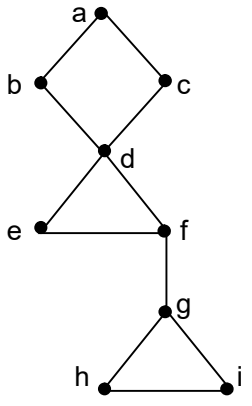
2 7.9 LET US SUM UP

- This unit explores the concepts of cut vertices, cut sets, fundamental cut set, fundamental cut set, fundamental circuit and various properties exhibited by them.
- **Network reliability:** Cut sets and cut vertices help identify critical components in networks.
    - **Graph decomposition:** Fundamental cut sets and circuit aid in decomposing graph into smaller sub-graphs.

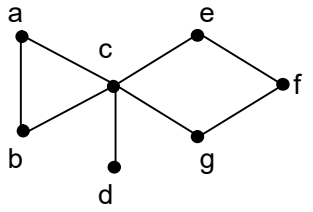
K 7.10 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1:

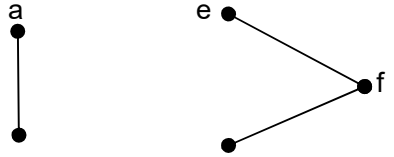
(i)



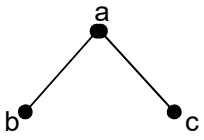
(ii)

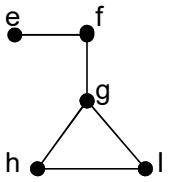


Cut vertex : c



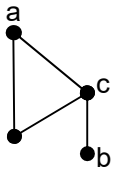
Cut vertex : d

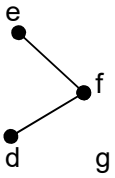




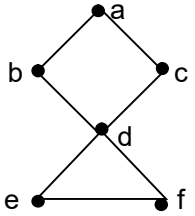
b g

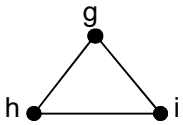
Cut set : {ce, cg}





Cut set : {fg}





O7.11 FURTHER READING

1) N. Deo, "Graph Theory with Applications to Engineering and Computer Science".

2) Lipschutz, M. Lipson, "Schaun's Outlines Discrete Mathematics".

M7.12 MODEL QUESTIONS

Q 1: Prove that every fundamental circuit contains exactly one chord.

Q 2: Show that every fundamental cut set contains exactly one branch of the spanning tree.

*** ***** ***