UNIT 1 INTRODUCTION TO GRAPH THEORY

UNIT STRUCTURE

1.1: Learning Objectives

1.2: Introduction

1.3: The Konigsberg Bridge problem

1.4: Definition of a Graph

1.5: The Degree of a Vertex

1.6: Isomorphism of Graphs

1.7: Directed Graph or Digraph

1.8 Let Us Sum Up

1.9 Answers to Check Your Progress

1.10 Further Reading

1.11 Model Questions

1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand the Königsberg Bridge Problem and its role in the origin of graph theory.
- learn key concepts such as graphs, vertex degrees, graph isomorphism, and their properties.
- explore the structure and representation of directed graphs with example.

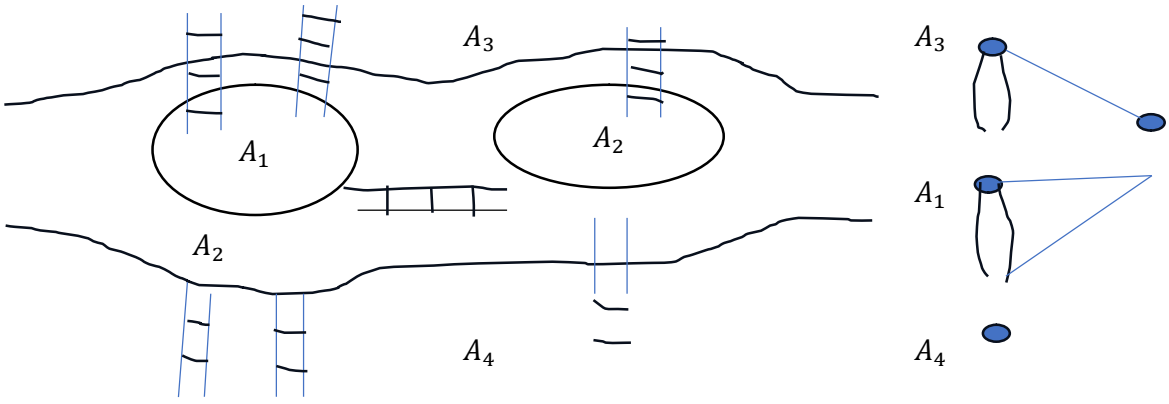
1.2 INTRODUCTION

In this unit, we will first discuss the famous Königsberg Bridge Problem, which marks the beginning of graph theory. We will also discuss the basic definition of a graph, which includes points called vertices and lines called edges. Next, we will learn how to find the degree of a vertex, which tells us how many edges are connected to it. We will also explore the concept of isomorphism to understand when two graphs are considered the same. Finally, we will study directed graphs, where edges have direction.

1.3 THE KONIGSBERG BRIDGE PROBLEM

Two Islands A_1 and A_2 formed by Pregal river (now Pregolya) in Konigsberg (then the capital of east Prussia, but now renamed Kaliningrad and in west Soviet Russia) were connected to each other and to banks A_3 and A_4 with seven bridges. The problem is to start at any of the four land areas, $A_1, A_2, A_3,$ or A_4 walk over each of the seven bridges exactly once and return to the starting point (Figure 1.1 (a)).

In 1735, mathematician Leonhard Euler mathematically proved that it was impossible to traverse the city by crossing each bridge exactly once. He did this by creating a simple model of the city, representing the four land areas by four vertices and the seven bridges by seven edges joining these vertices, as shown Figure 1.1 (b).



(a) (b)

Figure 1.1

1.4: DEFINITIONS

A graph G is an ordered pair (V,E), where V is a non-empty set whose elements are called the vertices of G and E is a set of unordered pairs of vertices, whose elements are

called edges of G. The vertices and edges of a graph are also called as points (nodes) and lines respectively.

A graph having finite number of vertices and edges is called a **finite graph**, otherwise it's an infinite graph.

The **order** of a graph G is the cardinality of the vertex set of and the **size** of the graph G is the cardinality of the edge set of G. That is, we define $|V(G)| = n$ to be the order of G and $|E(G)| = m$ to be the size of G. Such graph is called *a (n,m)* graph.

Two or more edges having same end vertices are called **multiple edges** and an edge with identical end vertices is called a **loop**. A graph G is **simple** if it has no multiple edges or loops.

The end vertices of an edge are said to the **incident** with the edge. Two vertices are **adjacent** if they are incident with a common edge and two **edges are adjacent** if they are incident to a common end vertex. Two adjacent vertices are said to be **neighbors** of each other. Any graph which contains some multiple edges is called a **multigraph**. In a multigraph, no loops are allowed. A graph with self-loops and multiple edges is called a **pseudograph**. A vertex is said to be **isolated** if it is not adjacent to any other vertex.

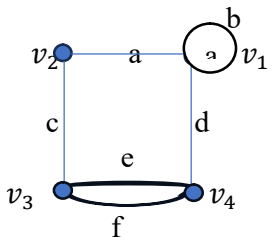


Figure 1.2

Example 1.4: *Consider the graph of Figure 1.2. Here,*

- v_1 and v_2 are adjacent vertices
- v_1 and v_3 are non-adjacent vertices.
- a joins v_1 and v_2 and therefore it is incident with v_1 and v_2.
- v_1 and v_2 are the end-vertices of a.

- e is not incident with v_1
- b is a loop
- e and f are multiple edges.
- It is not a simple graph
- It is a pseudograph

SUBGRAPHS

A graph $H = (V', E')$ is called a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ and each edge of H has the same endpoints in H as in G.

We next define various kinds of subgraphs.

Definitions: Let H be a subgraph of G.

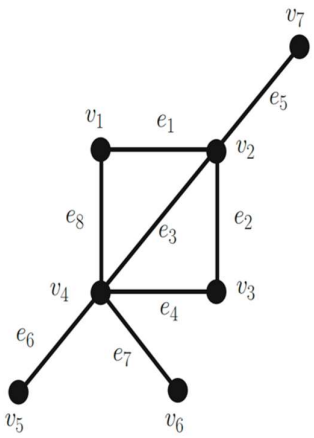
(i) H is called *proper subgroup* if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$.

(ii) H is called *spanning subgraph* if $V' = V$.

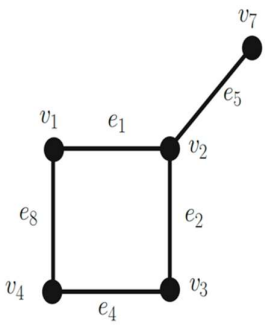
(iii) H is called an *induced subgraph* if G if each edge of G having its ends in $V(H)$ is
also an edge of H.

(iv) The induced subgraph of G with vertex set is called the *subgraph of* G *induced by* S and is denoted by $G[S]$.

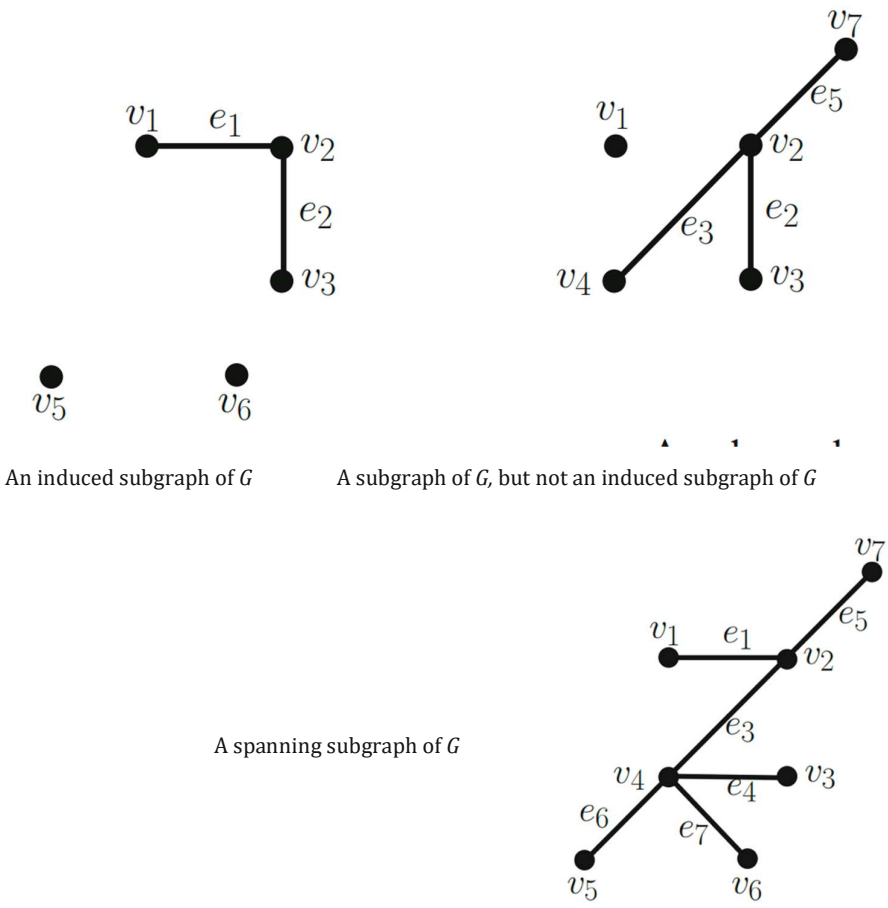
Example 1.5:



Graph G



A subgraph of G



An induced subgraph of G A subgraph of G, but not an induced subgraph of G

A spanning subgraph of G

Figure 1.3: Various subgraphs of G

1.5 THE DEGREE OF A VERTEX:

In a graph G, the degree of a vertex v is the number of edges of G which are incident to v and is denoted by $d(v)$ or $d(v|G)$ or $\deg(v)$. The degree of a vertex is also called the ***valency***

We have $d(v) = \{e \in E: e = uv, \forall\, u \in V\}$. The minimum degree and maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. In a graph G, a loop incident to a vertex v is counted as two edges incident to v. Therefore $d(v)$ is the number of non-loop edges incident to v plus twice the number of loops at v.

In the graph of Figure 1.4(a), $d(v_1) = 3, d(v_2) = d(v_5) = 2, d(v_4) = 1, d(v_6) = 0$ and $d(v_3) = 4$.

A graph is said to be **regular** if all its vertices are of same degree and k-regular if all its vertices are of degree k. A 3-regular graph is also called cubic graph. A vertex with degree zero is an isolated vertex, a vertex with degree one is a pendant vertex and the unique edge incident to a pendant vertex is a pendant edge.

A simple graph G is said to be **complete** if every vertex in G is connected with every other vertex. That is, if G contains exactly one edge between each pair of distinct vertices. A complete graph is usually dentoted by K_n. It should be noted that K_n has exactly $\frac{n(n-1)}{2}$ edges
(Figure 1.4(c)).

In Figure 1.4(a), v_6 is an isolated vertex and v_4 is a pendant vertex. In Figure 1.4(b), the graph is 2-regular.

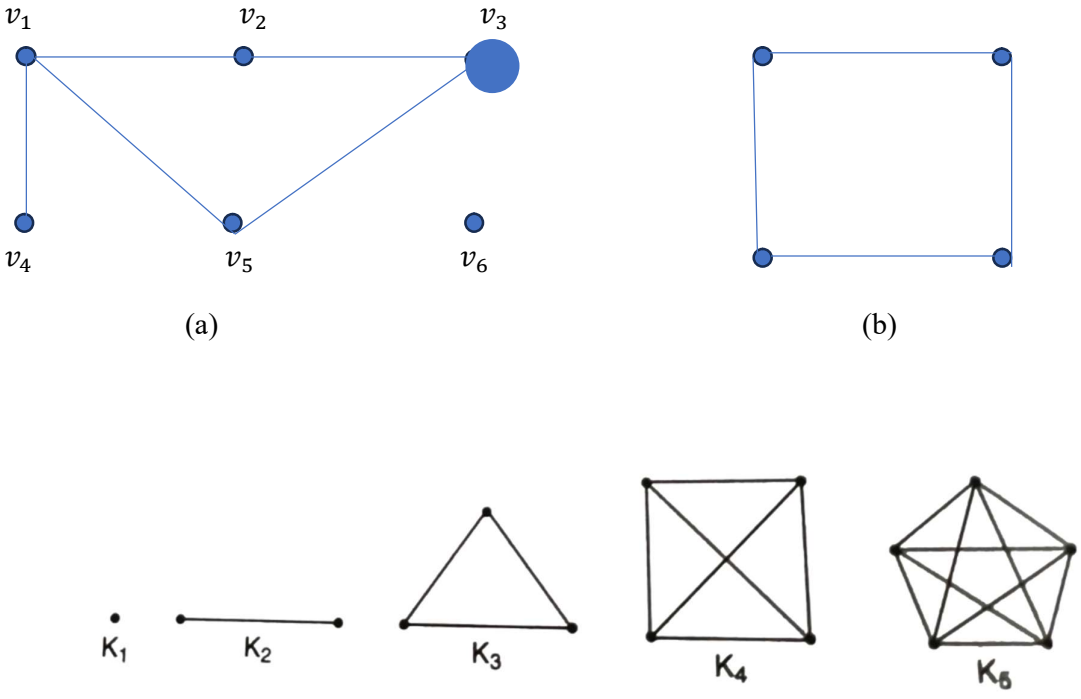


Figure 1.4

Theorem 1.5.1 *In any graph G,* $\sum_{v\in V} d(v) = 2m$. *That is, the sum of degrees of the vertices in a graph is even.*

Proof: Let m be the number of edges in a graph G. Since the degree of a vertex is the number of edges incident with that vertex , the sum of the degree counts the total number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices, each edge gets counted twice, one at each end. Thus, the sum of the degrees is equal twice the number of edges.

Theorem 1.5.2 *In any graph G, the number of vertices of odd degree is even.*

Proof: Let $G = (V, E)$ be a graph and $d(v)$ be the degree of a vertex for all $v \in V$. Suppose $|E| = m$. Then we have

$\sum_{v\in V} d(v) = 2m$ (see Theorem 1.5.1)

$$\underset{\substack{\text{odd degree}\\ \text{vertices}\\ \text{A}}}{\sum_{v_i\in V} d(v_i)} + \underset{\substack{\text{even degree}\\ \text{vertices}\\ \text{B}}}{\sum_{v_j\in V} d(v_j)} = 2m \qquad (1.5.1)$$

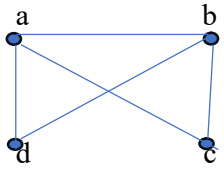
Since the R.H.S. of (1.5.1) is even, and (B) in (1.5.1) is also even, therefore (A) in (1.5.1) is even. Hence, $\sum_{v_i\in V} d(v_i)$ = even number.

Example 1.6 : *What is the size of an r-regular (n,m) graph ?*
Solution: Since G is r-regular graph. Then we have $d(v_i) = r,\ \forall\ v_i \in V(G)$.
By the theorem 1.5.1, $2m = \sum_{v\in V} d(v)$

$$2m = \sum r = nr \Rightarrow m = \frac{nr}{2}.$$

W CHECK YOUR PROGRESS

Q1. Determine the degree of each vertex for the following graph



Q2. Does there exists a 4-regular graph on 6 vertices? If so construct a graph.

1.6 ISOMORPHISM OF GRAPHS

Two graphs G(V,E) and H(W,F) are said to be **isomorphic** if there exist two bijections f : V → W and g : E → F such that an edge e joins u and v in G if and only if the edge g(e) joins f(u) and f(v) in H. If G and H are isomorphic, we write $G \simeq H$, and the pair (f, g) is called an **isomorphism** between G and H.

If G = H, then (f, g) is called an **automorphism**.

Clearly, two graphs G and H are isomorphic to each other if there is a one-one correspondence between their vertices, and between their edges such that incidence relationship is preserved.

For example, the graphs shown in Figure 1.5 are isomorphic.

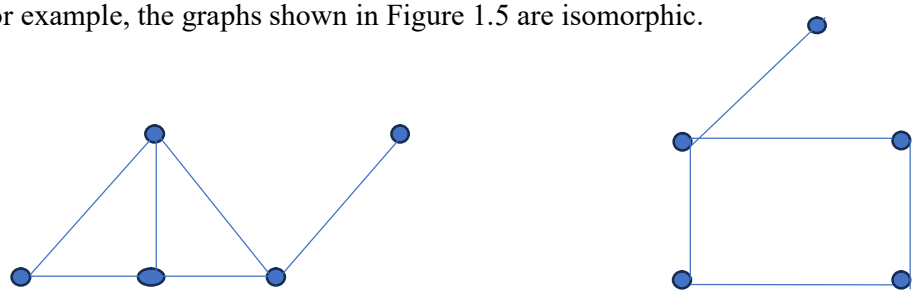


Figure 1.5: Isomorphic graphs

It follows from the definition of isomorphism that two isomorphic graphs have

i. The same number of vertices,
ii. The same number of edges, and
iii. An equal number of vertices with a given degree.

However, these conditions are not sufficient. To see this, consider two graphs given in figure 1.6.

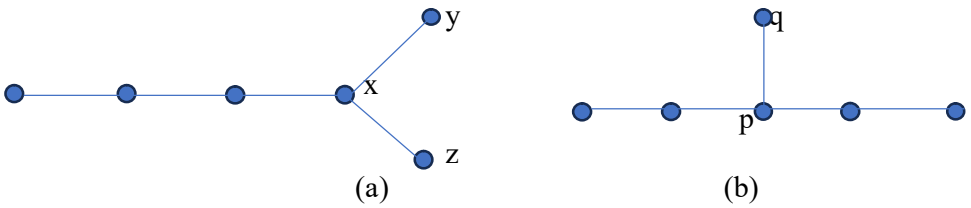


Figure 1.6: Non-isomorphic graphs

These graphs satisfy all the three conditions, but they are not isomorphic because the vertex x in (a) corresponds to vertex p in (b) as there are no other vertices of degree three, and in (b) there is only one pendant vertex q is adjacent to p, while in (a) there are two pendant vertices y and z adjacent to x.

1.7 DIRECTED GRAPHS

A directed graph D is a triple (V, A, I_D), where V and A are disjoint sets and $I_D: A \to V \times V$ is a function. An element of V is called a vertex and an element of A is called an arc.

If a is an arc of D, and $I_D(a) = (u, v), u$ is called the *tail* of a, and v is the *head* of a. The arc a is said to join v with u. u and v are called the *ends* of a. A directed graph is also called a *digraph*.

Example 1.7:

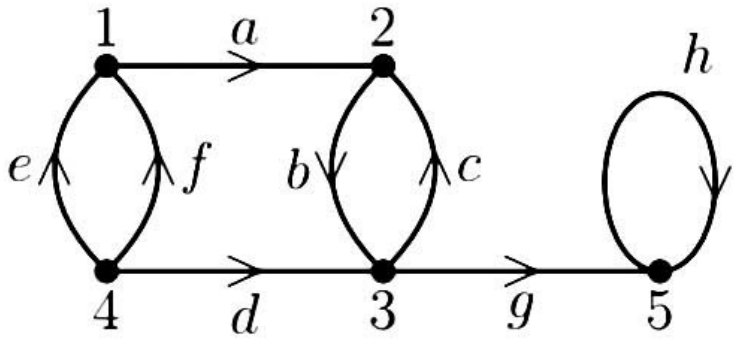


Figure 1.7: A digraph

Let $V = \{1,2,3,4,5\}$ and $A = \{a, b, c, d, e, f, g, h\}$ and $I_D: A \to V \times V$ is defined by $I_D(a) = (1,2), I_D(b) = (2,3), I_D(c) = (3,2)$ and so on. Then (V, A, I_D) is a digraph with 5 vertices and 8 arcs.

Underlying graph of a digraph

The underlying graph $G(D)$ of a digraph D is obtained by ignoring the direction of its arcs. Formally, $V(G) = V$, $E(G) = A$ and $I_G: A \rightarrow V^2$ is defined by $I_G(a) = (u, v)$ if $I_D(a) = (u, v)$ or (v, u). The underlying graph of a digraph is shown below.

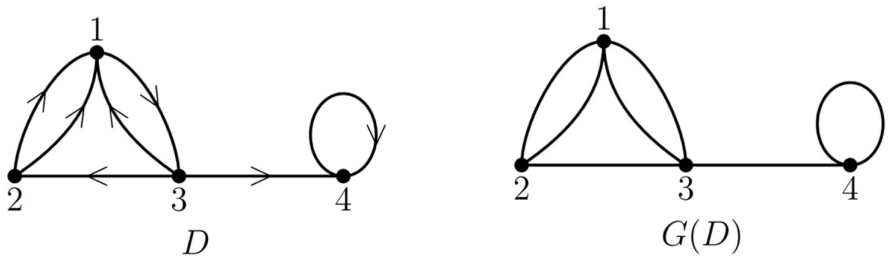


Figure 1.8: A digraph and its underlying graph.

Definition: If $a = (u, v)$ is an arc of D, a is said to be *incident out of u* and *incident into v*. v is called an *outneighbor* of u, and u is called an *inneighbor* of v. $N_D^+(u)$ denotes the set of outneighbors of u in D. Similarly, $N_D^-(u)$ denotes the set of inneighbors of u in D. When no explicit reference to D is needed, we denote these sets by $N^+(u)$ and $N^-(u)$ respectively. An arc *a is incident with u* if it is either incident into or incident out of u. An arc having the same ends is called a *loop* of D. The number of arcs incident out of a vertex v is the ***out-degree*** of v and is denoted by $d_D^+(v)$ or $d^+(v)$. The number of arcs incident into v is its ***in-degree*** and
is denoted by $d_D^-(v)$ or $d^-(v)$.

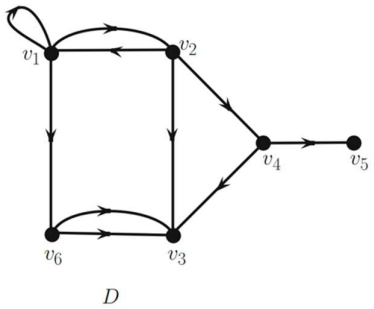


Figure 1.9

From Figure 1.9, we have $d^+(v_1) = 3 = d^+(v_2),\ d^+(v_3) = 0 = d^+(v_5),\ d^+(v_4) = 2 = d^+(v_6)$ and $d^-(v_1) = 2, d^-(v_3) = 4, d^-(v_2) = d^-(v_4) = d^-(v_5) = d^-(v_6) = 1$. (The loop at v_1 contributes 1 each to $d^+(v_1)$ an $d^-(v_1)$).

Theorem 1.7.1: *For every digraph D,*

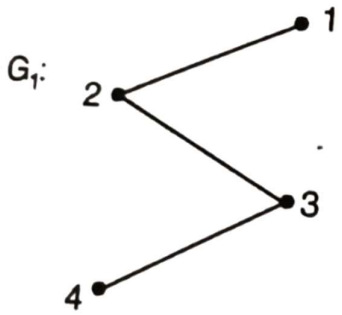
$$\sum_{v\in V(D)} d^+(v) = \sum_{v\in V(D)} d^-(v) = m(D),$$

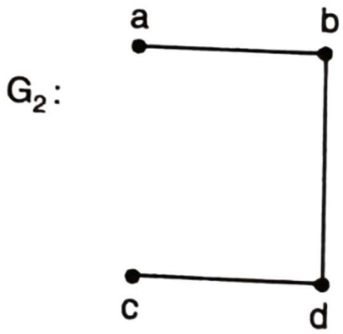
Where $m(D)$ is the number of arcs of D.

Proof: Every arc is counted once in $\sum d^+(v)$ and once in $\sum d^-(v)$.

W CHECK YOUR PROGRESS

Q3: Show that the given pair of graphs are isomorphic





2 1.8 LET US SUM UP

- The **Königsberg Bridge Problem**, solved by Euler in 1735, is regarded as the foundation of graph theory. Euler proved that it was impossible to traverse all seven bridges of the city exactly once and return to the starting point.

- A **graph** $G = (V, E)$consists of a set of vertices $V$and a set of edges E. Graphs may be finite or infinite. Special types include simple graphs, multigraphs, and pseudographs. Important terms include adjacency, incidence, isolated vertices, pendant vertices, loops, and multiple edges.

- A **subgraph** of a graph $G$is a graph formed from a subset of $V(G)$and $E(G)$. Special types include proper subgraphs, spanning subgraphs, and induced subgraphs.

- The **degree of a vertex** $d(v)$is the number of edges incident with it. A loop contributes two to the degree. Important results are:

  - The sum of the degrees of all vertices equals twice the number of edges.
  - The number of vertices of odd degree is always even.
  - A graph is called *regular* if all its vertices have the same degree.

- Two graphs are said to be **isomorphic** if there exists a one-to-one correspondence between their vertices and edges such that incidence is preserved. Isomorphic graphs have the same number of vertices, the same number of edges, and the same degree distribution. However, these conditions alone do not guarantee isomorphism; adjacency must also be preserved.

- A **directed graph (digraph)** consists of a set of vertices and a set of directed edges (arcs). Each arc has a tail and a head. The underlying graph of a digraph is obtained by ignoring directions. The indegree $d^-(v)$of a vertex counts arcs entering it, while the outdegree $d^+(v)$counts arcs leaving it.

  - The sum of all indegrees equals the sum of all outdegrees, both equal to the number of arcs.

K 1.9 ANSWERS TO CHECK YOUR PROGRESS

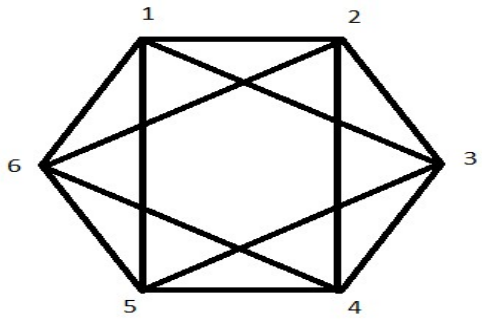
Ans. to Q. No. 1:

deg(a)= 5 as it has a loop (which counts as 2) and three other incident edges in the given graph.

Similarly, deg (b)=3, deg(c)=2, deg(d)=2.

Ans. to Q. No. 2: We have $m = \frac{nr}{2} = \frac{6\times4}{2} = 12.$

Since it is possible to construct a 4-regular graph on 6 vertices with 12 edges.



Ans. to Q. No. 3: We have $V(G_1) = \{1,2,3,4\}$, $V(G_2) = \{a, b, c, d\}$, $E(G_1) = \{\{1,2\},\{2,3\},\{3,4\}\}$ and $\{\{a, b\},\{b, d\},\{d, c\}\}$. Therefore $|V(G_1)| = |V(G_2)|$ and $|E(G_1)| = |E(G_2)|$.

In G_1, deg(1)=1=deg(4) and deg(2)=2=deg(3)

In G_2, deg(a)=1=deg(c) and deg(b)=2=deg(d)

Define a function $f: V(G_1) \to V(G_2)$ as $f(1) = a, f(2) = b, f(3) = d, f(4) = c$.

Since *f* is clearly one-one and onto. Also,

$\{1,2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a, b\} \in E(G_2)$

$\{2,3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b, d\} \in E(G_2)$

$\{3,4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d, c\} \in E(G_2)$

Hence f preserves adjacency of the vertices. Therefore, $G_1 \simeq G_2$.

1.10 FURTHER READINGS

1. F. Harary, Graph theory, Narosa Publishing House, New Delhi, 1988.
2. R. Balakrishnan and K. Renganathan, A textbook of Graph theory, Springer, 2000
3. B. Bollobas, Modern Graph Theory, Springer, 2002
4. G. Chartrand, L. Lesniak, Graphs & digraphs (Fourth edition), Chapman & Hall/CRC, 2005.
5. R. J. Wilson, Introduction to Graph Theory (5th Edition), Prentice Hall, 2010
6. S. Pirzada, *An introduction to graph theory,* Vol. 4,

OrientBlackSwan, Hyderabad: Universities Press, 2012.

7. S. A. Choudum, Graph Theory, A NPTEL Course, *Department of Mathematics, IIT Madras, Chennai, India* 2012.

M 1.11 MODEL QUESTIONS

Q1. Who introduced graph theory through the Königsberg Bridge Problem?

Q2. Define a graph with an example.

Q3. What is the degree of a loop at a vertex?

Q4. Differentiate between a multigraph and a pseudograph.

Q5. Define:(a) Isolated vertex (b) Pendant vertex (c) Regular graph

Q6. What is meant by a spanning subgraph? Give an example.

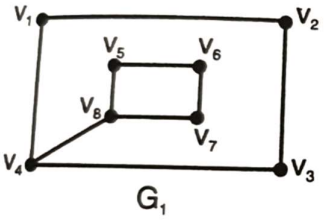
Q7. What are the necessary conditions for two graphs to be isomorphic? Give an example.

Q8. Consider a 4-regular graph on 6 vertices. Does such a graph exist? If yes, construct it.

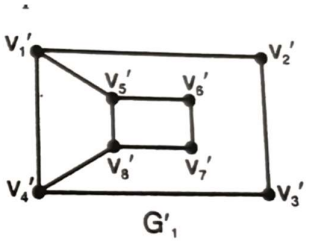
Q9. Explain the concepts of indegree and outdegree of a vertex in a digraph with a suitable example.

Q10. Determine whether the following graphs are isomorphic.

(a)

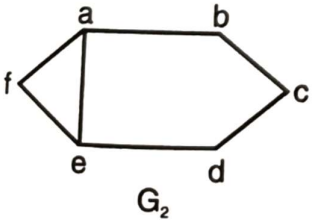


G_1

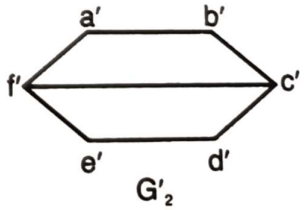


G'_1

(b)



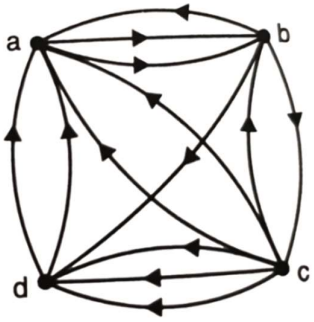
G_2



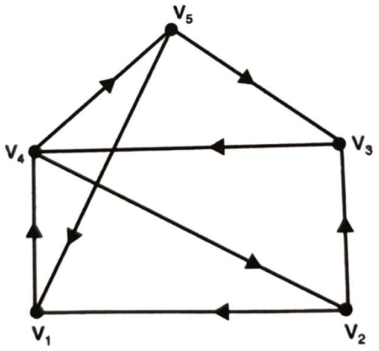
G'_2

Q11. Find the in-degree and out-degree of each vertex of the following directed graphs

(a)



(b)



*** *** ***