UNIT 2 GRAPH OPERATIONS

UNIT STRUCTURE

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2.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand the Königsberg Bridge Problem and its role in the origin of graph theory.
- learn key concepts such as graphs, vertex degrees, graph isomorphism, and their properties.
- explore the structure and representation of directed graphs with example.

2.2 INTRODUCTION

In the preceding unit, graphs were treated as static estructures - fined networks of vertiches and edges desigoud to model specific relationships. However, just as arithmetic operations like addition and multiplication are performed on numbers, or union and intersection on sets, mathematical operatione can also be applied to graphes to construct new structures and analyze complex networks. This unit introduces the foundational concepts of Graph Operations, providing the algebric and structural toolkit nucessary to actively manipulate, transform, and ungincer networks.

The material is organized systematically around three care algebric concepts. First, binary operations including Union, Intersection, Jain, and product are introduced to demonstrate how separate graphs combine to form larger, more intricate syatems. Second, the tent examines structural modifications through unary operations such as Vertex and Edge Deletion, Fusion, and Edge Contraction, which serve to streamline ar dissect existing networks. Finally, the concept of Homeomorphism is established to determine two topologically distinct graphs share the same underlying structural gramework.

2.3: UNION OF TWO GRAPHS

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. The union of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$.

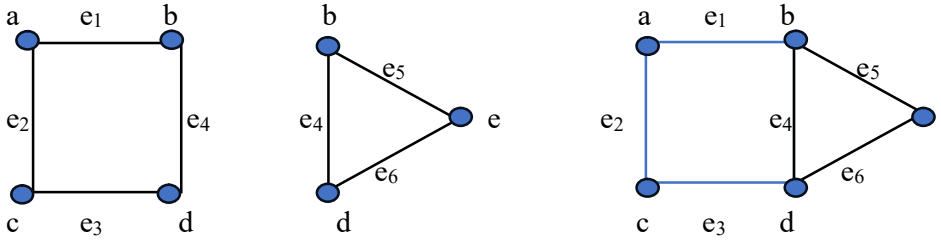


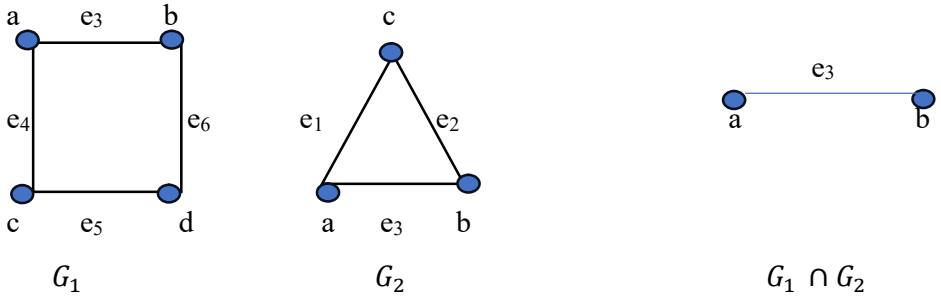
Figure 2.1

2.4 INTERSECTION OF TWO GRAPHS

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. The intersection of G_1 and G_2, denoted by

$G_1 \cap G_2$, is the graph consisting only of those vertices and edges that are both in G_1 and G_2.

Clearly, $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$.



G_1 G_2 $G_1 \cap G_2$

Figure 2.2

2.5 JOIN OF TWO GRAPHS

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. The join or sum of G_1 and G_2, denoted by $G_1 + G_2$, is defined as the graph whose vertex set is $V_1 + V_2$, and the edge set is consisting those edges, which are in G_1, and in G_2, and the edges obtained by joining each vertex of G_1, to each vertex of G_2.

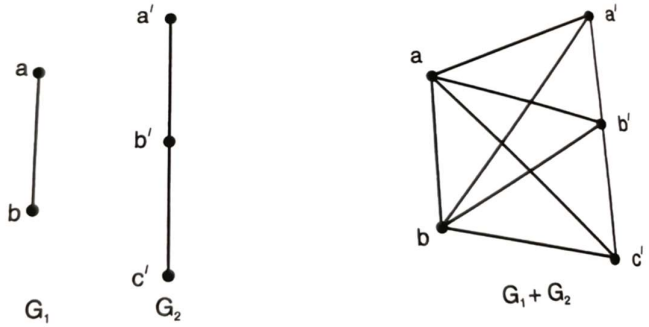


Figure 2.3

2.6 PRODUCT OF TWO GRAPHS

The product (cartesian) of G_1 and G_2, denoted by $G = G_1 \times G_2$, is the graph whose vertex set is V= $V_1 \times V_2$ and for any two vertices $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ in

$V; u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$, there is an edge $w_1 w_2 \in E(G)$ iff either (a) $u_1 = u_2$ and $v_1 v_2 \in E_2$, or (b) $v_1 = v_2$ and $u_1 u_2 \in E_1$.

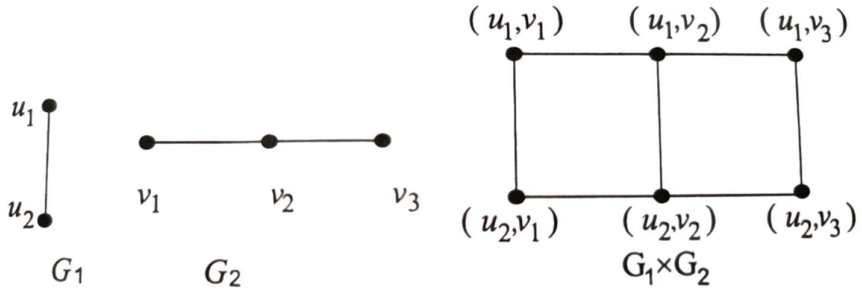


Figure 2.4

2.7 EDGE DELETION AND VERTEX DELETION IN GRAPHS

Let G be a graph and v be any vertex in G. Then vertex deletion, $G - v$ denotes the subgraph of G by deleting vertex v, and all edges of G which are incident with v.

If e is any edge of G, then edge deletion, $G - e$ is a subgraph of G obtained by deleting e from G.

Deletion of an edge does not imply deletion of its end vertices.

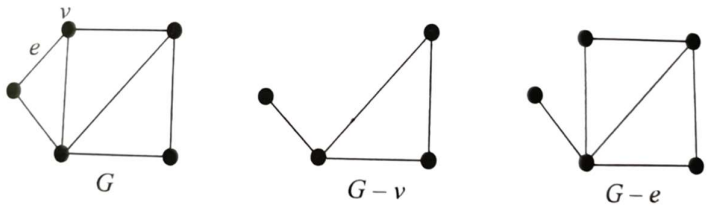
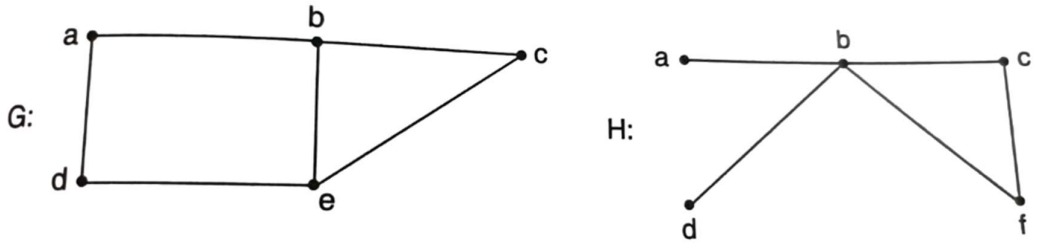


Figure 2.5

W CHECK YOUR PROGRESS

Q1. Find (i) $G \cup H$



Q2. A simple graph G on 6 vertices v_1, v_2, v_3, v_4, v_5 and v_6 has

(i) 7 edges in $G - v_1$ and $G - v_2$

(ii) 6 edges in $G - v_3$ and $G - v_4$

(iii) 5 edges in $G - v_5$ and $G - v_6$

2.8 FUSION OF VERTICES AND EDGE CONTRACTION IN GRAPHS

Fusion: A pair of vertices u and v in a graph are said to be fused (merged or identified) if u and v are replaced by a single new vertex such that every edge incident on u or v is incident on this new vertex. Therefore, fusion of vertices does not alter the number of edges, but reduces the number of vertices by one.

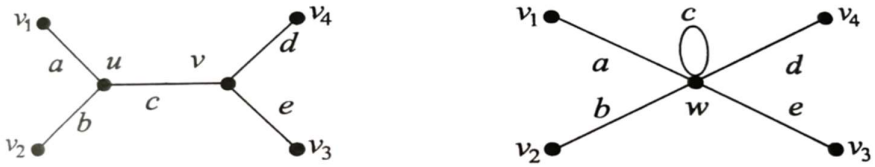


Figure 2.6

Contraction: An edge contraction, $G \circ e$ is operation occurs relative to a particular edge e. The edge e is removed and its two incident vertices u and v are merged into a new vertex w , where the edges incident to w each correspond to an edge incident to either u or v.

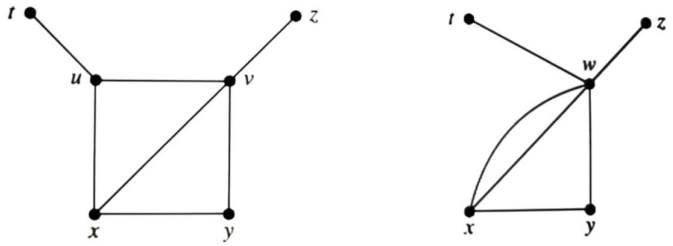


Figure 2.7: Edge contraction of $G \circ uv$

2.9 HOMEOMORPHISM OF GRAPHS

Definition: (Subdivision of an Edge) Let $e = uv$ be an arbitrary edge in G. The subdivision of the edge e yields a path of length 2 with end vertices u and v with a new internal vertex w (That is, the edge $e = uv$ is replaced by two new edges, uw and wv). This operation is also called an elementary subdivision of G.



Figure 2.8: Subdivision of an edge

Definition: A graph H obtained by a sequence of elementary subdivisions from a graph G is said to be a subdivision of G, or to be homeomorphic (or a homeomorph) of G.
Definition: (Homeomorphic Graphs) Two graphs are said to be homeomorphic if both can be obtained by the same graph by subdivisions of edges. In other words, Two graphs H_1 and H_2 which are homeomorphs of the same graph G are said to be homeomorphic to each other (or homeomorphically reducible to G).

In the following figure, the second and third graphs are homeomorphic, as they are obtained by subdividing the edges of the first graph in the figure.

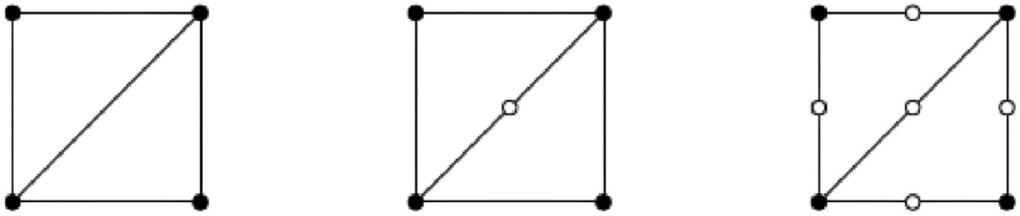


Figure 2.9

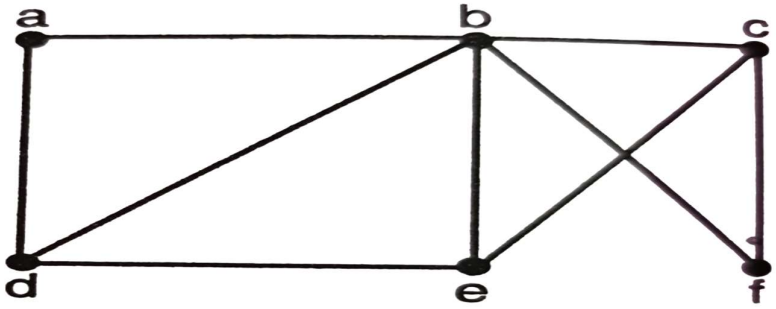
22.10 LET US SUM UP

In this unit, we studied various operations on graphs and their importance in graph theory. The key points are:

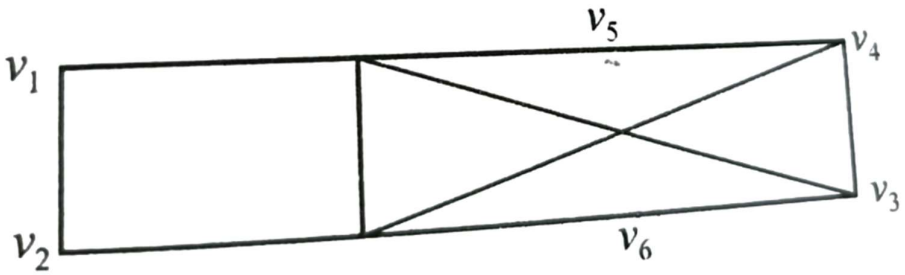
- **Union of two graphs**: The union of two graphs G_1 and G_2 combines their vertex sets and edge sets.
- **Intersection of two graphs**: The intersection of two graphs consists only of the common vertices and edges between them.
- **Join of two graphs**: The join of two graphs is obtained by taking their union and adding edges from every vertex of one graph to every vertex of the other.
- **Product of two graphs**: The Cartesian product of two graphs is formed by ordered pairs of their vertices, with adjacency defined under specific conditions.
- **Edge deletion and vertex deletion**: Deleting a vertex removes it along with all its incident edges, while deleting an edge removes only that edge but not its endpoints.
- **Fusion of vertices**: Two vertices can be merged into a single vertex while preserving all incident edges.
- **Edge contraction**: Contracting an edge removes it and merges its two end vertices into a new single vertex.
- **Homeomorphism of graphs**: A graph can be transformed by subdividing its edges. Two graphs are said to be homeomorphic if both can be obtained from the same graph by edge subdivisions.

K 2.11 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1:



Ans. to Q. No. 2:



O 2.12 FURTHER READINGS

1. F. Harary, Graph theory, Narosa Publishing House, New Delhi, 1988.
2. R. Balakrishnan and K. Renganathan, A textbook of Graph theory, Springer, 2000.
3. B. Bollobas, Modern Graph Theory, Springer, 2002.
4. G. Chartrand, L. Lesniak, Graphs & digraphs (Fourth edition), Chapman & Hall/CRC, 2005.
5. R. J. Wilson, Introduction to Graph Theory (5th Edition), Prentice Hall, 2010.
6. S. Pirzada, *An introduction to graph theory,* Vol. 4, OrientBlackSwan, Hyderabad: Universities Press, 2012.
7. 7. S. A. Choudum, Graph Theory, A NPTEL Course, Department of Mathematics, IIT Madras, Chennai, India 2012.

M 2.13 MODEL QUESTIONS

Q1. Define the union of two graphs with an example.

Q2. What is the intersection of two graphs with an example ?
Q3. Explain the *join of two graphs*. How does it differ from the union of two graphs?
Q4. Distinguish between edge deletion and vertex deletion in a graph. Give one example for each.
Q5. Define fusion of vertices. Show with a diagram how fusion changes the number of vertices but not the number of edges.
Q6. What is edge contraction? Explain the difference between fusion of vertices and edge contraction.
Q7. Define homeomorphic graphs. Give an example of two graphs that are homeomorphic.
Q8. A graph Ghas 5 vertices and 7 edges. If two vertices are fused, what will be the number of vertices and edges in the resulting graph? Justify your answer.
Q9. Write short notes on the following:
a) Cartesian product of two graphs
b) Subdivision of an edge
c) Join of two graphs
d) Edge deletion

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