Unit 13: PARTITIONS

UNIT STRUCTURE

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13.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define and Interpret Integer Partitions.
- construct and interpret Ferrers diagrams and Young tableaux to represent partitions graphically.
- comprehend Euler's classical result equating the number of partitions into distinct parts with those into odd parts.
- understand the role of generating functions in the study of partition identities.

13.2 INTRODUCTION

The theory of partitions, a branch of number theory, explores one of the simplest yet most profound questions in mathematics: In how many ways can a given integer be expressed as a sum of positive integers, irrespective of order? Starting with this introduction, the partition title of the unit itself says that we need to know what do we mean by partition of integers rather a positive integer. Then we will see their graphical representation, to visualize the abstract concept of partitions, mathematicians often turn to graphical representations, such as Ferrers diagrams and Young tableaux.

Which followed by an important subsection set as conjugate of a partition, which is thus followed by the very known and famous result in Number Theory, the Euler's partition theorem. It asserts a surprising equality: the number of partitions of an integer into distinct parts equals the number of partitions into odd parts. To proof the theorem we represent a topic generating function, which is nothing but to find a closed formula for a given sequence when written as a formal power series. These identities, ranging from the elementary to the arcane, often require a delicate interplay of algebra, generating functions, and bijective proofs. The pursuit of such identities continues to be a fertile ground for discovery and delight.

13.3 PARTITION OF A NUMBER

A partition of a positive integer is a representation of it as a sum of one or more positive integers where order is ignored. The individual summands of a partition are called parts and by convention, the parts of a partition are written in non-increasing order. For example, the partition of 4 are as follows: $4, 3+1, 2+2, 2+1+1, 1+1+1+1$. It should be noted that there are only 5 partitions of 4. And also $3 + 1$ is same as $1 + 3$ in case of partition since the order of the addends in a partition doesn't change the partition itself. The partition of a non-negative integer n can also be termed as integer partition of n. A composition of a non-negative integer is the way of writing it as the sum of one or more positive integer where the order is not ignored i.e., order of the summands matters here. That is the composition of 4 are $4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1$. It is clear that there are 8 compositions of the number 4 and also in this case $1 + 3$ is not same as $3 + 1$. It is also to be noted that the partition $2 + 1 + 1$ can also be written as a tuple $(2, 1, 1)$, just another way to represent the partition $2 + 1 + 1$ of 4.

Note: The only partition of zero is the sum having no parts, i.e., the empty sum.

Example 1. The number 6 can be partitioned in the following distinct ways:

6,

5 + 1,

4 + 2,

4 + 1 + 1,

3 + 3,

3 + 2 + 1,

3 + 1 + 1 + 1,

2 + 2 + 2,

2 + 2 + 1 + 1,

2 + 1 + 1 + 1 + 1,

1 + 1 + 1 + 1 + 1 + 1.

So it can be seen that there are 11 partitions of 6.

Theorem 13.3.1. *For a positive integer n, there are* 2^{n-1} *compositions.*

The above theorem is a well-known result, so the proof is not given. However if *p*(*n*) is denoted as the number of partitions of *n* then the function $p(n)$ is known as the partition function and by convention $p(0) = 0$. Also it can be easily found that $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5$ and so on.

13.4 GRAPHICAL REPRESENTATION

There are two graphical representation of integer partitions and they are Ferrers diagram and Youngs diagram representations which are named after Norman Macleod Ferrers and Alfred Young respectively. The Ferrers diagram is obtain by representing each part of the partition by row dots, whereas in Youngs diagram the dots are replaced by squares or box.

Illustration with example:

(i) **Ferrers Diagram**: The partition 4 + 3 + 2 + 1 of 10 can be represented as

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Similarly the partition 3 + 1 + 1 + 1 of 6 can be graphically (Ferrers diagram) represented as

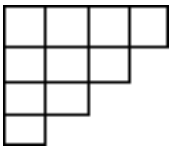
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(ii) **Youngs Diagram**: The partition 4 + 3 + 2 + 1 of 10 can be represented by using Youngs diagram as



Instead of using dots, as seen in case of Ferrers diagram representation, the Youngs diagram uses box or square to represent the partition of a number graphically.

CHECK YOUR PROGRESS

Q. 1: Write all the partitions of 10.

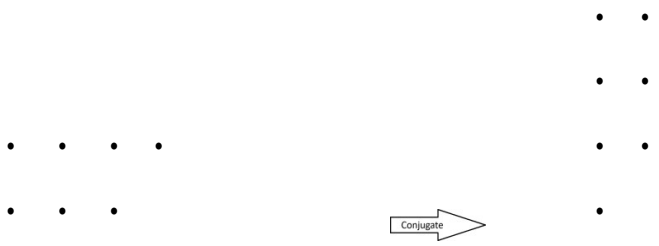
Q. 2: What is the difference between partition and composition of a positive integer? Give examples.

Q. 3: How can you represent the partition 3 + 3 + 2 of 8 graphically?

13.4.1 CONJUGATE PARTITION

The conjugate partition is such partition whose Ferrers diagram is obtained by flipping the original diagram across the main diagonal, thus transforming rows into columns and vice versa.

Illustrative example: The conjugate partition of 4 + 3 can be represented by using Ferrers diagram as below.



Clearly, we can see that the conjugate of the partition $4 + 3$ of 7 is the partition $2 + 2 + 2 + 1$. It should also be noted that a self conjugate partition is the partition whose conjugate is the partition itself. For example, the partition $4 + 3 + 2 + 1$ of 10 and its conjugate can be find as shown below.

13.5 EULER'S PARTITION THEOREM

Theorem 3.5.1. The number of integer partitions into odd parts is equal to the number of integer partition into distinct parts.
For the proof of the above theorem Euler use the idea of generating function.

13.5.1 GENERATING FUNCTION

In mathematics, a generating function is a formal power series representation whose coefficients are the infinite sequence of numbers. They are expressed in closed form rather than as a series, by some expression involving operations on the formal power series. Mathematically, a generating function for a sequence $b_0, b_1, b_2, \ldots$, is a formal power series given by:

$$G(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots = \sum_{k=0}^{\infty} b_k \, x^k$$

Here, $\{b_k k\}$ are the coefficients representing the sequence.

Example 2. Write the generating function generated by the sequence $1, 1, 1, \ldots$

Solution: The generating series generated by the sequence $1, 1, 1, \ldots$ is simply $1 + x + x^2 + x^3 + \cdots$. We now need to find the closed formula for the above series.

Let

$$S = 1 + x + x^2 + x^3 + \cdots \qquad (1)$$

Multiplying S by x we get

$$xS = x + x^2 + x^3 + x^4 + \cdots \qquad (2)$$

Thus subtracting (2) from (1) we get

$$(1 - x)S = 1$$
$$\Rightarrow S = \frac{1}{1 - x}$$

Therefore, the generating function for the sequence $1,1,1,\cdots$ is $\frac{1}{1-x}$. Thus we get

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots$$

If we replace x by $-x$ in the above series then we get

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots$$

which generates the sequence $1, -1,1, -1, \ldots$

Example 3. Find the generating function for the partition function.

Solution: Recall that a partition function $p(n)$ is the number of integer partitions of n. The function $p(n)$ have the values of $p(1) = 1, p(n) = 2, p(3) = 3, p(4) = 5, \ldots$.

We have

$$\frac{1}{1 - q} = 1 + q + q^2 + q^3 + \cdots.$$
$$\frac{1}{1 - q^2} = 1 + q^2 + q^4 + q^6 + \cdots.$$
$$\frac{1}{1 - q^3} = 1 + q^3 + q^6 + q^9 + \cdots.$$

… …

Now multiplying the above series we get

$$\frac{1}{1 - q} \cdot \frac{1}{1 - q^2} \cdot \frac{1}{1 - q^3}$$

$$= (1 + q + q^2 + q^3 + \cdots).(1 + q^2 + q^4 + q^6 + \cdots).(1 + q^3 + q^6 + q^9 + \cdots) \ldots$$
$$= 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots$$

It can be seen that the coefficient of the above infinite series are 1, 2, 3, 5, . . . which represents the values of partition function.

Thus the closed form generating function for the partition function is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\ldots}$$

W CHECK YOUR PROGRESS

Q. 1: Find the closed form generating function of the sequence 1, 0, 1, 0, . . .

Q. 2: What do you mean by conjugate partition? Define self-conjugate partition. Find self- conjugate partitions of 12, (if any).

Now we are going to proof the theorem 2 by using generating function.

Let $P_0(n)$ denotes the partition function of the number of integer partitions of n into odd parts and also let $P_d(n)$ denotes the partition of integer n into distinct parts. Thus the generating function for $P_0(n)$ and $P_d(n)$ are respectively given by

$$\sum_{n=0}^{\infty} P(n)q^n = \frac{1}{(1-q)(1-q^3)(1-q^5)\ldots}$$

and

$$\sum_{n=0}^{\infty} P_d(n)q^n = (1+q)(1+q^2)(1+q^3)\ldots$$

Therefore

$$\sum_{n=0}^{\infty} P_d(n)q^n = (1+q)(1+q^2)(1+q^3)\ldots$$
$$= \frac{(1+q)(1-q)(1+q^2)(1-q^2)\ldots}{(1-q)(1-q^2)(1-q^3)\ldots}$$
$$= \frac{(1-q^2)(1-q^4)(1-q^6)\ldots}{(1-q)(1-q^2)(1-q^3)\ldots}$$
$$= \frac{1}{(1-q)(1-q^3)(1-q^5)\ldots}$$

$$= \sum_{n=0}^{\infty} P_0(n) q^n$$

Thus equating both sides of the above we get, $P_0(n) = P_d(n)$ that is the number of integer partitions of n into odd parts is equal to the number of integer partitions of n into distinct parts and this proves the theorem.

Define the q-product notations

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1}), \quad n \geq 1$$

$$(a;q)_0 = 1,$$

And

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\dots$$

$$= \prod_{n=0}^{\infty}(1-aq^n)$$

Where $a, q \in C, |q| \leq 1$.

Thus by substituting $a = q$ we get

$$(q;q)_\infty = (1-q)(1-q^2)(1-q^3)\dots$$

$$= \prod_{n=0}^{\infty}(1-q^{n+1})$$

Thus, a few *q*-product notations are as follows.

$$(q;q^2)_\infty = (1-q)(1-q^3)(1-q^5)\dots$$

$$(-q;q) = (1+q)(1+q^2)(1+q^3)\dots$$

That means $(-q;q)_\infty$ denotes the number of integer partitions into distinct parts.

Thus the generating function for the partition function $P(n)$ is now can be written in the q-product notation as

$$\sum_{n=0}^{\infty} P(n) q^n = \frac{1}{(q;q)_\infty}$$

Also the partition function for Po(n) and Pd(n) are respectively can be written in the q-product notation as

$$\sum_{n=0}^{\infty} P_0(n) q^n = \frac{1}{(1-q)(1-q^3)(1-q^5)\dots}$$

$$= \frac{1}{(q;q^2)_\infty}$$

and

$$\sum_{n=0}^{\infty} P_d(n)q^n = (-q;q)_\infty$$

Therefore the Euler's partition theorem in to q-product notation can be written as

$$\frac{1}{(q;q^2)_\infty} = (-q;q)_\infty$$

13.6 SEARCHING FOR PARTITION IDENTITIES

In this section, we shall see a few of the results which are similar to the Euler's Partition Theorem. Our aim is not the rigor of proof, but the thrill of discovery.

To illustrate how we conduct the search, we forget Theorem 2 and attempt to discover it.

Definition 1.6.1. Let S be a set of positive integers. Then define $P_S(n)$ denote the number of partitions of n in which each summand is an element of S.

Problem-1 (Discovering Euler's Partition Theorem): If $A_1(n)$ denotes the number of partitions of *n* into distinct parts, can we find a set S_1 such that

$$P_{S_1}(n) = A_1(n)$$

for all n ?

Solution: We have $A_1(1) = 1$, we see that $1 \in S_1$ because otherwise $P_{S_1}(1) = 0 \neq 1 = A_1(1)$. Again $A_1(2) = 1$, we see that $2 \notin S_1$ because otherwise $p(S_1, 2) = 2 \neq 1 = A_1(2)$. Since $A_1(3) = 2$, we see that $3 \in S_1$ because otherwise $p(S_1, 3) = 1 \neq 2 = A_1(3)$. Since $A_1(4) = 2$, we see that $4 \notin S_1$ because otherwise $p(S_1, 4) = 3 \neq 2 = A_1(4)$. Now we put all these values in the following table

Table 1: Discovering elements of S_1

n	$A_1(n)$	$P_{S_1}(n)$ if $n \in S_1$	$P_{S_1}(n)$ if $n \notin S_1$	Conclusion
1	1	1	0	$1 \in S_1$
2	1	2	1	$2 \notin S_1$
3	2	2	1	$3 \in S_1$
4	2	3	2	$4 \notin S_1$

5	3	3	2	$5 \in S_1$
6	4	5	4	$6 \notin S_1$
7	5	5	4	$7 \in S_1$

From the above table it can be concluded that the set S_1 contains all the odd numbers. Thus we have the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts, which is nothing but the Euler's partition theorem.

Now using the same approach to "discover" a famous theorem of L. J. Rogers which was later also discovered independently by S. Ramanujan and I. J. Schur.

Problem 2 (Discovering the First Rogers–Ramanujan Identity): Let $A_2(n)$ denote the number of partitions of n in which any two summands differ by at least 2. Can we find a set of integers S_2 such that

$$P_{S_2}(n) = A_2(n)$$

for all n ?

Solution: We have $A_2(1) = 1$, we see that $1 \in S_2$ because otherwise $P_{S_2}(1) = 0 \neq 1 = A_2(1)$. Again $A_2(2) = 1$, we see that $2 \notin S_2$ because otherwise $p(S_2, 2) = 2 \neq 1 = A_2(2)$. Since $A_2(3) = 1$, we see that $3 \notin S_2$ because otherwise $p(S_2, 3) = 2 \neq 1 = A_2(3)$. Since $A_2(4) = 2$, we see that $4 \in S_2$ because otherwise $p(S_2, 4) = 1 \neq 2 = A_2(4)$. Since $A_2(5) = 2$, we see that $5 \notin S_2$ because otherwise $p(S_2, 5) = 3 \neq 2 = A_2(5)$. Since $A_2(6) = 3$, we see that $6 \in S_2$ because otherwise $p(S_2, 6) = 2 \neq 3 = A_2(6)$. Now we put all these values in the following table.

Table 2: Discovering elements of S_2

n	$A_2(n)$	$P_{S_2}(n)\ if\ n \in S_2$	$P_{S_2}(n)\ if\ n \notin S_2$	Conclusion
1	1	1	0	$1 \in S_2$
2	1	2	1	$2 \notin S_2$
3	1	2	1	$3 \notin S_2$
4	2	2	1	$4 \in S_2$
5	2	3	2	$5 \notin S_2$
6	3	3	2	$6 \in S_2$
7	3	4	3	$7 \notin S_2$
8	4	5	4	$8 \notin S_2$

9	5	5	4	$9 \in S_2$
10	6	7	6	$8 \notin S_2$
11	7	7	6	$11 \in S_2$
12	9	10	9	$12 \notin S_2$
13	10	11	10	$13 \notin S_2$
14	12	12	11	$14 \in S_2$

We can see that $1, 4, 6, 9, 11, 14$ are the first six elements of S_2. It can be observed that if we divide all these numbers by 5 then we will get a remainder of 1 or 4 for each case. Hence the set S_2 consist of the elements or positive integers which are congruent to 1 or 4 modulo 5.

Definition 1.6.2. Define $A_d(n)$ to denote the number of partitions of n in which any two sum- mands differ by at least d.

It can be observe that in the previous two problems $A_1(n)$ are $A_2(n)$ are two special cases of $A_d(n)$. We will now represent the problem for $D_3(n)$.

Problem 3: Can we find a set of integers S_3 such that

$$P_{S_3}(n) = A_3(n)?$$

Solution: Here $A_3(n)$ denotes the number of partitions of n in which any two summands differ by at least 3. So, we have $A_3(1) = 1$, we see that $1 \in S_3$ because otherwise $P_{S_3}(1) = 0 \neq 1 = A_3(1)$. Again $A_3(2) = 1$, we see that $2 \notin S_3$ because otherwise $p(S_3, 2) = 2 \neq 1 = A_3(2)$. Since $A_3(3) = 1$, we see that $3 \notin S_3$ because otherwise $p(S_3, 3) = 2 \neq 1 = A_3(3)$. Since $A_3(4) = 1$, we see that $4 \notin S_3$ because otherwise $p(S_3, 4) = 2 \neq 1 = A_3(4)$. Now we put all these values in the following table

Table 3: DISCOVERING ELEMENTS OF S_3

n	$A_3(n)$	$P_{S_3}(n)$ if $n \in S_3$	$P_{S_3}(n)$ if $n \notin S_3$	Conclusion
1	1	1	0	$1 \in S_3$
2	1	2	1	$2 \notin S_3$
3	1	2	1	$3 \notin S_3$

4	1	2	1	$4 \notin S_3$
5	2	2	1	$5 \in S_3$
6	2	3	2	$6 \notin S_3$
7	3	3	2	$7 \in S_3$
8	3	4	3	$8 \notin S_3$
9	4	4	3	$9 \in S_3$
10	4	6	4	$10 \notin S_3$

And it can be concluded that it is not able to find a relation from the above table. Thus we see that S_3 cannot exist. This problem can be seen as a theorem below.

Theorem 13.6.1. *There does not exist a set S_3, say, of integers such that*

$$\boldsymbol{P_{S_3}(n) = A_3(n)}.$$

The modification for the above theorem to find the set S_3 satisfying the above condition, with a slide change, was originally discovered by I. J. Schur, and is a part of Schur's result which indicates that in order to obtain valid theorems, we need to introduce new partition functions like $\widetilde{A_3}(n)$. So if S_3 denotes the set of integers congruent to 1 or 5 modulo 6, and if $\widetilde{D_3}(n)$ denotes the number of partitions of n in which the difference between any two parts is at least 3, and also no consecutive multiples of 3 appear, then $P_{S_3}(n) = \widetilde{A_3}(n)$.

Definition 1.6.3: Let $P_n = \frac{n(3n-1)}{2}$, where $n \geq 1$. Also let $P_n' = \frac{n(3n+1)}{2}$ where $n \geq 1$. Then first few terms of P_n are $1, 5, 12$ and so on. Similarly, first few terms of P_n' are $2,7,15$. So the set obtain by combining the terms of P_n and P_n' is called generalized pentagonal number.

Theorem 4. *Let $P_{DE}(n)$ denotes the number of partitions of n into even number of distinct parts and $P_{DO}(n)$ denotes the number of partitions of n into odd number of distinct parts. Then*

$$P_{DE}(n) - P_{DO}(n) = 0 \qquad if\ n \neq \frac{m(3m \pm 1)}{2}$$

$$(-1)^m \qquad if\ n = \frac{m(3m \pm 1)}{2}$$

This is known as Euler's Pentagonal number theorem.

The proof is not added here. One can refer to book [2] *George E. Andrews: Number Theory.*

W CHECK YOUR PROGRESS

Q.6: What do you mean by Euler's partition theorem? Define *q*-product notation for an infinite product. Also write the Euler's partition theorem in *q*-product notation.

Q.7: What does the *q*-product notations $(-q, q^2)_\infty, \frac{1}{\{q^2;q^3)_\infty}$ denotes?

Q.8: Define the term generating function. Write the closed form series for the Fibonacci sequence $1, 1, 2, 3, 5, \ldots$

Q.9: If $A(n)$ denotes the number of partitions of n in which no part appears more than twice then find a set of integers, say, B such that $A(n) = B(n)$.

2 13.7 LET US SUM UP

- The representation of a positive integer as a sum of one or more positive integers, irrespective of the order, is called Partition of a positive integer. The individual summands of a partition are called parts and the parts are written in non-increasing order.
- A partition of a positive integer can be represented graphically in two ways- one is Ferrers diagram representation and the other is Youngs diagram representation.
- A conjugate of a partition of a positive integer can be obtained from the Ferrers diagram whose diagram is obtained by flipping the original diagram across the main diagonal, thus transforming rows into columns and vice versa.
- A generating function is a representation of a formal power series whose coefficients are the infinite sequence of numbers. Mathematically, a generating function for a sequence b_0, b_1, b_2,. . ., is a formal power series given by:

$$G(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots = \sum_{k=0}^{infty} b_k x^k$$

Here, $\{b_k\}_k$ are the coefficients representing the sequence.

- Euler's partition theorem states that the number of integer partitions into odd parts is equal to the number of integer partition into distinct parts.

K 13.8 ANSWERS TO CHECK YOUR PROGRESS

Solution to the Q. 1: The integer partitions of 10 are

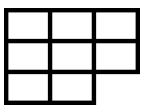
10, 9 + 1, 8 + 2, 8 + 1 + 1, 7 + 3, 7 + 2 + 1, 7 + 1 + 1 + 1, 6 + 4, 6 + 3 + 1, 6 + 2 + 2, 6 + 2 + 1 + 1, 6 + 1 + 1 + 1 + 1, 5 + 5, 5 + 4 + 1, 5 + 3 + 2, 5 + 3 + 1 + 1, 5 + 2 + 2 + 1, 5 + 2 + 1 + 1 + 1, 5 + 1 + 1 + 1 + 1 + 1, 4 + 4 + 2, 4 + 4 + 1 + 1, 4 + 3 + 3, 4 + 3 + 2 + 1, 4 + 3 + 1 + 1 + 1, 4 + 2 + 2 + 2, 4 + 2 + 2 + 1 + 1, 4 + 2 + 1 + 1 + 1 + 1, 4 + 1 + 1 + 1 + 1 + 1 + 1, 3 + 3 + 3 + 1, 3 + 3 + 2 + 2, 3 + 3 + 2 + 1 + 1, 3 + 3 + 1 + 1 + 1 + 1, 3 + 2 + 2 + 2 + 1, 3 + 2 + 2 + 1 + 1 + 1, 3 + 2 + 1 + 1 + 1 + 1 + 1, 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1, 2 + 2 + 2 + 2 + 2, 2 + 2 + 2 + 2 + 1 + 1, 2 + 2 + 2 + 1 + 1 + 1 + 1, 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1, 2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1.

Solution to the Q. 2: A partition of a positive integer is a representation of it as a sum of one or more positive integers where order is ignored, while in case of composition the order is not ignored, that is, order of the summands taken into consideration. For example, the partitions of 3 are: 3, 2 + 1, 1 + 1 + 1. While the composition of 3 are: 3, 2 + 1, 1 + 2, 1 + 1 + 1.

Solution to the Q. 3: Graphically, the partition 3 + 3 + 2 of 8 can be represented into two different ways. The first one is the Ferrers diagram representation which is given by

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And the second representation is Youngs diagram representation, which is given by



Solution to the Q. 4: The generating series generated by the sequence 1, 0, 1, 0, . . . is simply $1 + 0 \cdot x + x^2 + 0 \cdot x^3 + \cdots$. We now need to find the closed formula for the above series.

Let

$$S = 1 + x^2 + x^4 + \ldots \quad (3)$$

Multiplying S by x^2 we get

$$x^2 S = x^2 + x^4 + x^6 + \cdots \quad (4)$$

Therefore, the generating function for the sequence $1, 0, 1, 0, \ldots$ is $\frac{1}{1-x^2}$. Thus we get

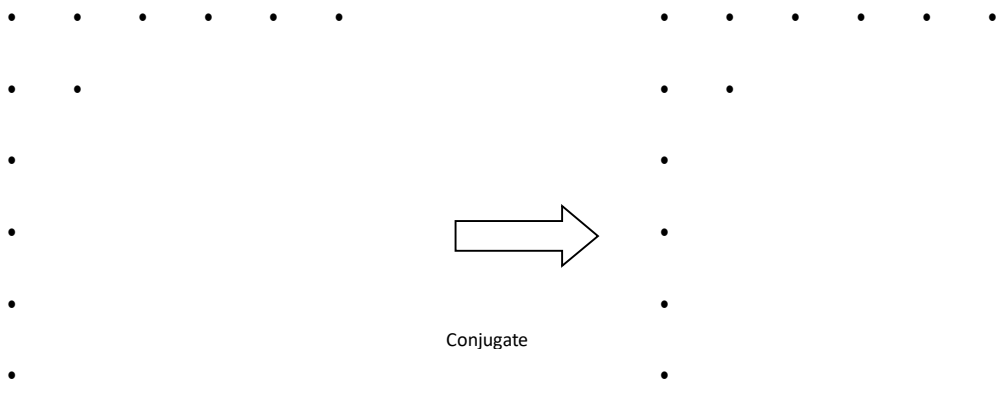
$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots$$

Solution to the Q. 5: A conjugate of a partition of a positive integer can be obtained from the Ferrers diagram whose diagram is obtained by flipping the original diagram across the main diagonal, thus transforming rows into columns and vice versa.

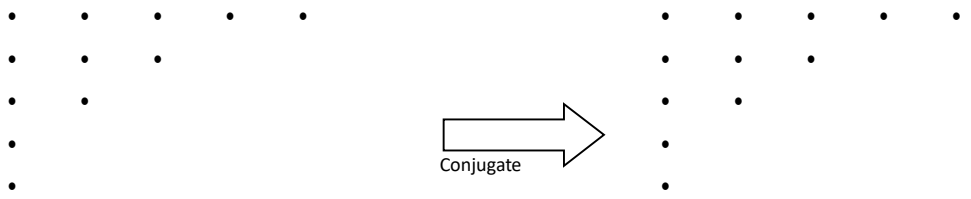
A self conjugate partition is the partition whose conjugate is the partition itself.

The self-conjugate partitions of 12 are: $6 + 2 + 1 + 1 + 1 + 1, 5 + 3 + 2 + 1 + 1$, and $4 + 4 + 2 + 2$. The Ferrers diagram of the self-conjugate partitions are as follows:

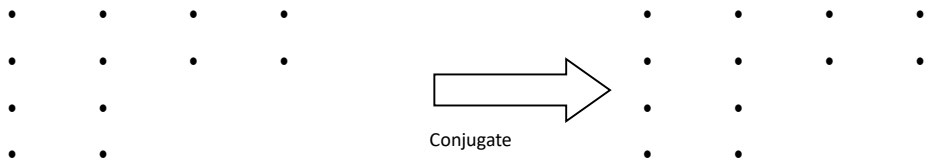
For $6 + 2 + 1 + 1 + 1 + 1$, we have



For 5 + 3 + 2 + 1 + 1



For the partition 4 + 4 + 2 + 2



Solution to the Q. 6: The number of integer partitions into odd parts is equal to the number of integer partition into distinct parts.

The *q*-product notation for an infinite product is given by

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$$
$$= \prod_{n=0}^{\infty}(1-aq^n)$$

where $a, q \in C, |q| \leq 1$.

The Euler's partition theorem, q-product notation is given by

$$\frac{1}{(q;q^2)_\infty} = (-q;q)_\infty$$

Where $\frac{1}{(q;q^2)_\infty}$ denotes denotes the generation function of the number of integer partitions into odd parts and $(-q;q)_\infty$ denotes the generation function of the number of integer partition into distinct parts.

Solution to the Q. 7: The q-product notation $(-q;q^2)$ denotes the generating function of the number of integer partition into distinct odd parts; and $\frac{1}{(q^2;q^3)_\infty}$ denotes the generating function of the number of integer partitions into parts of the form $3n+2$.

$$F_n = F_{n-1} + F_{n-2} \qquad n \geq 2 \qquad (5)$$

Where $F_0 = 1$ and $F_1 = 1$.

Let us consider the generating function for the given sequence be

$$F(x) = \sum_{n=0}^{\infty} F_n\, x^n$$

Now from (5), we have

$$\sum_{n=2}^{\infty} F_n\, x^n = \sum_{n=2}^{\infty} F_{n-1}\, x^n + \sum_{n=2}^{\infty} F_{n-2}\, x^n$$

$$\Rightarrow F_0 + F_1 x + \sum_{n=2}^{\infty} F_n\, x^n = F_0 + F_1 x + x \sum_{n=2}^{\infty} F_{n-1}\, x^{n-1} + x^2 \sum_{n=2}^{\infty} F_{n-2}\, x^{n-2}$$

$$\Rightarrow \sum_{n=0}^{\infty} F_n\, x^n = F_0 + F_0 x + x \sum_{n=2}^{\infty} F_{n-1}\, x^{n-1} + x^2 \sum_{n=2}^{\infty} F_n\, x^{n-2}$$

$$\Rightarrow F(x) = F_0 + x \sum_{n=0}^{\infty} F_n\, x^n + x^2 F(x)$$

$$\Rightarrow F(x) = 1 + xF(x) + x^2 F(x)$$

$$\Rightarrow F(x) = \frac{1}{1 - x - x^2}$$

which is the required generating function.

Solution to the Q. 9: Here $A(n)$ denotes the number of partitions of n in which no part appears more than twice. We have $A(1) = 1$, we see that $1 \in B$ because otherwise $B(1) = 0 \neq 1 = A(1)$. Again $A(2) = 2$, we see that $2 \in B$ because otherwise $B(2) = 1 \neq 2 = A(2)$. Since $A(3) = 2$, we see that $3 \notin B$ because otherwise $B(3) = 3 \neq 2 = A(3)$. Since $A(4) = 4$, we see that $4 \notin B$ because otherwise $B(4) = 3 \neq 2 = A(4)$. Since $A(5) = 5$, we see that $5 \notin B$ because otherwise $B(5) = 3 \neq 5 = A(5)$. Since $A(6) = 7$, we see that $6 \notin B$ because otherwise $B(6) = 3 \neq 7 = A(6)$. So from the above calculations it can be concluded that the set $B(n)$ is the number of partitions of n in which the parts are congruent to 1 and 2 modulo 3. That is, the parts are not divisible by 3.

Alt. Approach: Here $A(n)$ denotes the number of partitions of n in which no part appears more than twice. Then the generating function for $A(n)$ will be given by

$$\sum_{n=0}^{\infty} A(n)q^n = (1+q+q^2)(1+q^2+q^4)(1+q^3+q^6)\cdots$$

$$= \prod_{n=1}^{\infty}(1+q^n+q^{2n})$$

$$= \prod_{n=1}^{\infty}\frac{1-q^{3n}}{1-q^n}$$

$$= \frac{(q^3;q^3)_\infty}{(q;q)_\infty}$$

$$= \frac{(q^3;q^3)_\infty}{(q;q^3)_\infty(q^2;q^3)_\infty(q^2;q^3)_\infty}$$

$$= \frac{1}{(q;q^3)_\infty(q^2;q^3)_\infty}$$

$$= \sum_{n=0}^{\infty} B(n)q^n$$

where B(n) denotes the number of partitions of n in which the parts are congruent to 1, 2 modulo 3, that is, the parts are not divisible by 3

O 13.9 FURTHER READING

1. David Burton: Elementary Number Theory, Universal Book Stall, New Delhi.
2. George E. Andrews: Number Theory, Hindustan Publishing Corporation, New Delhi.
3. K.C. Chowdhury: A First Course in Theory of Numbers, Asian Books Pvt. Ltd.
4. Ramanujachary Kumanduri; Cristina Romero: Number Theory with Computer Applica- tions, Prentice Hall, New Jersey.
5. G.H. Hardy; E.M. Wright: An Introduction to the Theory of Numbers, OUP.
6. Thomas Koshy: Elementary Number Theory with Applications, Harcourt Science and Technology Company.
7. S.G. Telang (eds. M.G.Nadkarni & J.S.Dani) : Number Theory, Tata McGraw-Hill Publishing Company Limited .
8. Gareth A. Jones & J. Mary Jones, Elementary Number Theory, Springer.
9. Apostol, T. M. Introduction to Analytic Number Theory. Springer 2014.
10. Niven, I. and Zuckerman, H. S. Introduction to the Theory of Numbers. John Wiley & Sons, 2008.

11. Hardy, G. H. and Wright, E. M. Theory of Numbers. Oxford Science Publications, 2003.

M 13.10 MODEL QUESTIONS

Q. 1: Establish Euler's partition theorem by using q-product notation and generating function.

Q. 2: Write all the partitions of 9. Draw the graphical representations for each partition. From the graph, find out all the self-conjugate partitions of 9.

Q. 3: What is the importance of finding conjugate partition? Explain briefly.

Q. 4: If C(n) denotes the number of partitions of n in which all parts differ by at least 2 and no consecutive even numbers appear as summands, then find a set of integers, say, D such that C(n) = D(n).

Q. 5: If C1(n) denotes the number of partitions of n in which no part appears more than thrice, then find a set of integers, say, D_1 such that $C_1(n) = D_1(n)$.

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