UNIT 14: FAREY SEQUENCES AND CONTINUED FRACTIONS

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14.1 LEARNING OBJECTIVES

After going through this unit, the learner will be able to:

- Understand and construct Farey sequences and explore their properties and significance in number theory.
- Define finite and infinite continued fractions, and distinguish between their forms.
- Represent rational numbers as finite continued fractions and investigate the uniqueness and properties of such expansions.
- Analyze infinite continued fractions as representations of irrational numbers and study their uniqueness and approximation capabilities.
- Apply continued fractions to solve linear Diophantine equations with emphasis on Pell's equation.
- Explore the Fibonacci sequence from a structural and number-theoretic perspective.
- Prove and utilize certain classical identities involving Fibonacci numbers.

14.2 INTRODUCTION

The theory of continued fractions stands as one of the most elegant representations of numbers, offering profound insight into number theory, approximation, and algebraic equations. The beautiful structure of the Farey sequence and the golden harmony of the Fibonacci numbers, a rich mathematical tapestry emerges-one that reveals patterns, solves equations, and challenges our intuition. This unit embarks on a journey through both finite and infinite continued fractions, investigating their convergence properties, representation of rational and irrational numbers, and their role in approximations. These explorations naturally segue into practical applications such as solving Diophantine equations, most notably the celebrated Pell's equation. The Fibonacci sequence, beloved for its recursive charm, appears not only as a standalone wonder but also within the framework of continued fractions and identities. Through this integrated approach, we uncover certain elegant identities involving Fibonacci numbers, thus bridging abstract theory with concrete application.

14.3 FAREY SEQUENCE

Definition 14.3.1. A Farey sequence of order n is a set of rational numbers of the form $\frac{a}{b}$ with $0 \leq a \leq b \leq n$ and $gcd(a,b) = 1$ and it is denoted by F_n. It should be noted that the Farey sequences are written in increasing order. For example, a few values of F_n are as follows

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$

$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$

$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

$$F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$

$$F_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$$

It can be observed that for any n, if $\frac{a}{b} \in F_n$ then $\frac{a}{b} \in F_m$ for any $m \geq n$.

Theorem 14.3.1: If $\frac{a}{b} < \frac{c}{d}$ are consecutive fractions in the Farey sequence F_n, then $bc - ad = 1$.

Proof. Since $gcd(a, b) = 1$, the linear equation $bx - ay = 1$ has a solution $x = x_0, y = y_0$. Moreover, $x = x_0 + at, y = y_0 + bt$ will also be a solution for any integer t. Choose $t = t_0$ so that

$$0 \leq n - b < y_0 + bt_0 \leq n$$

and set $x = x_0 + bt_0, y = y_0 + bt_0$. Since $y \leq n$, $\frac{x}{y}$ will be a fraction in F_n. Also,

$$\frac{x}{y} = \frac{a}{b} + \frac{1}{by} > \frac{a}{b}$$

So that $\frac{x}{y}$ occurs later in the Farey sequence than $\frac{a}{b}$. If possible suppose that $\frac{x}{y} \neq \frac{c}{d}$, where $\frac{c}{d}$ is another fraction, then suppose $\frac{x}{y} > \frac{c}{d}$ and we get

$$\frac{x}{y} - \frac{c}{d} = \frac{dx - cy}{dy} \geq \frac{1}{dy}$$

as well as

$$\frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd} \geq \frac{1}{bd}$$

Adding the above two inequalities, we get

$$\frac{x}{y} - \frac{a}{b} \geq \frac{1}{dy} + \frac{1}{bd} = \frac{b + y}{bdy}$$

But since $b + y > n$ (recall that $n - b < y$) and $d \leq n$, we have

$$\frac{1}{by} = \frac{bx - ay}{by} = \frac{x}{y} - \frac{a}{b} > \frac{n}{bdy} \geq \frac{1}{by}$$

Which is a contradiction. Thus $\frac{x}{y} = \frac{c}{d}$ and the equation $bx - ay = 1$ has becomes $bc - ad = 1$. This completes the proof.

Definition 14.3.2 *If $\frac{a}{b} < \frac{c}{d}$ are two fractions in the Farey sequence F_n, then the fraction $\frac{a+c}{b+d}$ is called as their mediant fraction.*

From the above theorem it can be concluded that the mediant lies between the given fractions. For the relations

$$a(b + d) - b(a + c) = ad - bc < 0$$

$$(a+c)d-(b+d)c=ad-bc<0$$

Together imply that

$$\frac{a}{b}<\frac{a+c}{b+d}<\frac{c}{d}$$

Notice that if $\frac{a}{b}<\frac{c}{d}$ are consecutive fractions in F_n and $b+d\le n$, then the mediant would be a member of F_n lying between them, an obvious contradiction. Thus, for successive fractions, $b+d\ge n+1$.

It can be demonstrated that the fractions belonging to F_{n+1} but not to F_n are precisely the mediants of the fractions in F_n. For example, in progressing from F_4 to F_5, the newly introduced fractions are

$$\frac{1}{5}=\frac{0+\frac{1}{1+}}{4},\quad \frac{2}{5}=\frac{1+\frac{1}{3+}}{2},\quad \frac{3}{5}=\frac{1+\frac{1}{2+}}{3},\quad \frac{4}{5}=\frac{3+\frac{1}{4+}}{1}$$

This process allows one to construct the sequence F_{n+1} from F_n by systematically inserting by inserting mediants with the appropriate denominator.

Note: It is important to observe that when using the mediant of two fractions in F_n to generate a new element in F_{n+1}, the resulting three fractions need not be consecutive in F_{n+1} (for example, consider $\frac{1}{3}<\frac{3}{5}<\frac{2}{3}$ in F_5).

However, it can be stated that if $\frac{a}{b}<\frac{c}{d}<\frac{e}{f}$ are three consecutive fractions in any Farey sequence, then $\frac{c}{d}$ is the mediant of $\frac{a}{b}$ and $\frac{e}{f}$. The proof is given as follows. From the Theorem 1, we have

$$bc-ad=1 \qquad de-cf=1$$

Which lead to $(a+e)d=c(b+f)$. It follows that

$$\frac{c}{d}=\frac{a+e}{b+f}$$

Which is the mediant of $\frac{a}{b}$ and $\frac{e}{f}$.

Illustration: Consider the three functions $\frac{3}{8} < \frac{2}{5} < \frac{3}{7}$. Then they are consecutive in F_8 with $\frac{2}{5} = \frac{3+3}{8+7}$.

Let us now apply these concepts to demonstrate how an irrational number can be closely approximated by a rational number with remarkable accuracy.

Theorem 14.3.2. For any irrational number $0 < x < 1$ and integer $n > 0$, there exists a fraction $\frac{a}{b}$ in F_n such that $|x - \frac{a}{b}| < \frac{1}{b(n+1)}$.

Proof. In the Farey sequence F_n, there are consecutive fractions $\frac{a}{b} < \frac{c}{d}$ such that either

$$\frac{a}{b} < x < \frac{a+c}{b+d} \quad or \quad \frac{a+c}{b+d} < x < \frac{c}{d}$$

where $\frac{a+c}{b+d}$ is the mediant of the two fractions (we have applied the fact that there are infinitely many irrationals between two rational.) Now if $\frac{u}{v} = \frac{a}{b}$ and since $bc - ad = 1$ and $b + d \geq n + 1$, we can see that

$$x - \frac{u}{v} = x - \frac{a}{b} < \frac{a+c}{b+d} - \frac{a}{b} = \frac{bc-ad}{b(b+d)} \leq \frac{1}{b(n+1)} = \frac{1}{v(n+1)}$$

Again if $\frac{u}{v} = \frac{c}{d}$ then we get

$$\frac{u}{v} - x = \frac{c}{d} - x < \frac{c}{d} - \frac{a+c}{b+d} = \frac{bc-ad}{d(b+d)} < \frac{1}{d(n+1)} = \frac{1}{v(n+1)}$$

We can conclude from the above two inequalities that $\left|x - \frac{a}{b}\right| < \frac{1}{b(n+1)}$.

The above result can be extended beyond the unit interval with the following corollary.

Corollary: Given a positive irrational number x and an integer $n > 0$, there is a rational number $\frac{a}{b}$ with $0 < b \leq n$ such that $\left|x - \frac{a}{b}\right| < \frac{1}{b(n+1)}$.

Proof. By the definition if greatest integer function we can write $x = [x] + r$ where $0 < r < 1$. By using Theorem 2, there is a fraction $\frac{u}{v}$ for which

$$\left|r-\frac{u}{v}\right|<\frac{1}{v(n+1)}$$

Taking $a = [x]v + u$ and $b = v$, it follows that

$$\left|x-\frac{a}{b}\right|=\left|x-\frac{[x]v+u}{v}\right|=\left|r-\frac{u}{v}\right|<\frac{1}{v(n+1)}=\frac{1}{b(n+1)}$$

Hence everything is proved.

We finish with an example illustrating the corollary.

Example 1. *Let us determine a fraction* $\frac{a}{b}$ *with* $0 < b \leq 5$ *such that*

$$\left|\sqrt{7}-\frac{a}{b}\right|\leq\frac{1}{6b}.$$

Using the greatest integer function, we compute $\sqrt{7}-2=0.64755\ldots$. *Within the Farey sequence* F_5, *the value* 0.64755... *lies between the consecutive fractions* $\frac{3}{5}$ *and. The mediant of these two fractions is* $\frac{5}{8}$= 0.625, *and by using the Theorem 2, we have the following inequality*

$$\left|0.64755\ldots-\frac{2}{3}\right|<\frac{1}{6\cdot 3}.$$

The reasoning used in the corollary allows us to shift this inequality to

$$\left|\sqrt{7}-\frac{2}{3}\right|<\frac{1}{6\cdot 3},$$

hence $\frac{8}{3}$ *is the desired rational approximation.*

W CHECK YOUR PROGRESS

Q. 1: What do you mean by a Farey sequence?

Q. 2: List in ascending order the fractions that appear in the Farey sequences F8 and F9.

Q. 3: Show that the number of fractions in the Farey sequence F_n is 1 + ϕ(*l*) + ϕ(2) + ϕ(3) + ⋯ + ϕ(*n*), where ϕ is the Euler ϕ-function.

14.4 FINITE CONTINUED FRACTIONS

In the section of the Liber Abaci devoted to expressing fractions as sums of unit fractions, Fibonacci introduced an early form of what we now recognize as a "continued fraction." For example, he employed the symbol $\frac{111}{345}$ as an abbreviation for

$$\frac{1+\frac{1+\frac{1}{5}}{4}}{3}=\frac{1}{3}+\frac{1}{3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}$$

The modern practice is, however, to write continued fractions in a descending fashion, as with

$$2+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{3+\frac{1}{2}}}}$$

An expression of multiple-decked type is said to be a finite simple continued fraction.

Formally we have the following definition.

Definition 15.1. A *finite continued fraction* is a fraction of the form

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots+\cfrac{1}{a_{n-1}+\frac{1}{a_n}}}}}$$

where $a_0, a_1, \ldots, a_n$ are real numbers, all of which except possibly a_0 are positive. The numbers $a_1, a_2, \ldots, a_n$ are called the *partial denominators* of this fraction. When all of the a_i are integers then we called such a fraction a *simple* fraction.

While acknowledging Fibonacci's early contributions, most scholars concur that the formal development of continued fraction theory begins with Rafael Bombelli, regarded as the last great algebraist of Renaissance Italy. In his L'Algebra Opera (1572), Bombelli explored an ingenious and noval approach to extracting square roots using infinite continued fractions. In doing so, he effectively demonstrated that $\sqrt{13}$ could be represented as a continued fraction as follows.

$$\sqrt{13}=3+\cfrac{4}{6+\cfrac{4}{6+\cfrac{4}{6+\cdots}}}$$

Note: The value of any finite simple continued fraction will always be a rational number.

For example,

$$3+\frac{1}{4+\frac{1}{1+\frac{1}{4+\frac{1}{2}}}}=3+\frac{1}{4+\frac{1}{1+\frac{1}{\frac{9}{2}}}}$$

$$=3+\frac{1}{4+\frac{1}{1+\frac{2}{9}}}$$

$$=3+\frac{1}{4+\frac{9}{11}}$$

$$=3+\frac{11}{53}=\frac{170}{53}.$$

Theorem 14.4.1. *Any rational number can be written as a finite simple continued fraction.*

Proof. Let us consider an arbitrary rational number *a*/*b*, where *b* > 0. By using Euclid's algorithm for finding the greatest common divisor of *a* and *b*, we have

$$a = bq_0 + r_1 \qquad 0 < r_1 < b$$
$$b = r_1q_1 + r_2 \qquad 0 < r_2 < r_1$$
$$r_1 = r_2q_2 + r_3 \qquad 0 < r_3 < r_2$$
.
.
.
$$r_{n-2} = r_{n-1}q_{n-1} + r_n \qquad 0 < r_n < r_{n-1}$$
$$r_{n-1} = r_nq_n + 0.$$

Since all the r_i are positive, so are $a_1, a_2, \ldots, a_n$. Rewrite the equations of the algorithm in the following manner:

$$\frac{a}{b} = q_0 + \frac{r_1}{b} = q_0 + \frac{1}{\frac{b}{r_1}}$$

$$\frac{b}{r_1} = q_1 + \frac{r_2}{r_1} = q_1 + \frac{1}{\frac{r_1}{r_2}}$$

$$\frac{r_1}{r_2} = q_2 + \frac{r_3}{r_2} = q_2 + \frac{1}{\frac{r_2}{r_3}}$$

.

.

.

$$\frac{r_{n-1}}{r_n} = q_n.$$

Eliminating $\frac{b}{r_1}$ from the first of the above equations by using the second equation, we have

$$\frac{a}{b} = q_0 + \frac{1}{\frac{b}{r_1}} = q_0 + \frac{1}{a_1 + \frac{r_2}{r_1}} = q_0 + \frac{1}{q_1 + \frac{1}{\frac{r_1}{r_2}}}$$

Proceeding likewise, replacing r_1/r_2 in the above equation, by using the third equation, we have the following expression

$$\frac{a}{b} = q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \frac{1}{\frac{r_2}{r_3}}}}$$

In the similar way, continuing the process we get

$$\frac{a}{b} = q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \cdots + \frac{1}{q_{n-1} + \frac{1}{q_n}}}}}$$

which is the required proof.

Illustration: By using the Theorem 3, let us represent $\frac{19}{51}$ as a continued fraction. Using Euclid's algorithm to the integers 19 and 51, we get

$$51 = 2.19 + 13 \quad \text{or} \quad \frac{51}{19} = 2 + \frac{13}{19}$$

$$19 = 1.13 + 6 \quad \text{or} \quad \frac{19}{13} = 1 + \frac{6}{13}$$

$$13 = 2.6 + 1 \quad \text{or} \quad \frac{13}{6} = 2 + \frac{1}{6}$$

$$6 = 6.1 + 0 \quad \text{or} \quad \frac{6}{6} = 1.$$

Thus we have,

$$\frac{19}{51} = \frac{1}{\frac{51}{19}} = \frac{1}{2+\frac{13}{19}}$$
$$= \frac{1}{2+\frac{1}{\frac{19}{13}}}$$
$$= \frac{1}{2+\frac{1}{1+\frac{6}{13}}}$$
$$= \frac{1}{2+\frac{1}{1+\frac{1}{\frac{13}{6}}}}$$
$$= \frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{6}}}}$$
$$= \frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{6}}}}.$$

Which is the continued fraction expansion for $19/51$. Due to the cumbersome nature of writing or printing continued fractions in full, a compact notation is commonly adopted-namely, the use of a bracketed symbol that lists the partial quotients: $[a_0; a_1, \dots, a_n]$. That is

$$[a_0; a_1, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots \frac{1}{a_{n-1} + \frac{1}{a_n}}}}}$$

Using this convention, the continued fraction expansion of 19/51, as calculated above, is indicated by the symbol

[0; 2, 1, 2, 6]

Similarly, the notation for the continued fraction of 172/51 = 3 + 19/51 is given by

[3; 2, 1, 2, 6]

Note: It is to be noted that the initial integer in the symbol [a_0; a_1, . . . , a_n] is zero when the value of the fraction is positive but less than one. Also, the representation of a rational number as a finite simple continued fraction is not

unique. It has two representations as a simple continued fraction, one with an even number of partial denominators and other with an odd number (It turns out that these are the only two possible representations.) For example, we can have two representation of 19/51 as follows,

$$19/51 = [0; 2, 1, 2, 6]$$

Also since $6 = 5 + 1 = 5 + \frac{1}{1}$, so the above representation can be written as

$$19/51 = [0; 2, 1, 2, 6] = [0; 2, 1, 2, 5, 1].$$

Example 2: Consider the Fibonacci sequence and let us consider the quotient of two successive Fibonacci numbers, it is clearly a rational number, say u_{n+1}/u_n, and it can be written as a simple continued fraction in two different representations. Now the Euclidean Algorithm for the greatest common divisor of u_n and u_{n+1} produces the $n-1$ equations which are

$$u_{n+1} = 1.u_n + u_{n-1}$$
$$u_n = 1.u_{n-1} + u_{n-2}$$
.
.
.
$$u_4 = 1.u_3 + u_2$$
$$u_3 = 2.u_2 + 0.$$

So in the notation of simple continued fraction, we can have the two representations of the rational number u_{n+1}/u_n as follows.

$$\frac{u_{n+1}}{u_n} = [1; 1,1, \dots ,1,2] = [1; 1,1, \dots ,1,1].$$

where the integer 1 appears *n* times. Thus, the fraction $\frac{u_{n+1}}{u_n}$ has a continued fraction expansion that is very easy to describe: There are *n* − 1 partial denominators all equal to 1.

We now turn our attention to exploring how the theory of continued fractions can be applied to the solution of linear Diophantine equations. To proceed, it is essential to first introduce a few preliminary results, along with a key concept known as the convergent of a continued fraction. These foundational ideas will be developed in the sections below.

Definition 1.4.1. *The continued fraction made from* [a_0; a_1, . . . , a_n] *by cutting off the expansion after the kth partial denominator* a_k *is called the kth* ***convergent*** *of the given continued fraction and denoted by* C_k*; in symbols, we write,*

$$C_k = [a_0; a_1, \dots, a_k] \qquad for\ 1 \le k \le n.$$

By convention, the zeroth convergent C_0 *be equal to the number* a_0.

Note: It is to be noted that, $k < n$, if a_k can be written as $a_k + \frac{1}{a_{k+1}}$, then the convergent C_k becomes the convergent C_{k+1}; that is,

$$\left[a_0; a_1, \dots, a_{k-1}, a_k + \frac{1}{a_{k+1}}\right] = [a_0; a_1, \dots, a_{k-1}, a_k, a_{k+1}] = C_{k+1}.$$

It is also noted that the last convergent C_n is nothing but always equals the rational number represented by the original continued fraction.

For example, we have $19/51 = [0; 2,1,2,6]$, the successive convergent are as below.

$$C_0 = 0$$

$$C_1 = [0; 2] = 0 + \frac{1}{2} = \frac{1}{2}$$

$$C_2 = [0; 2,1] = 0 + \frac{1}{2 + \frac{1}{1}} = \frac{1}{3}$$

$$C_3 = [0; 2,1,2] = 0 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2}}} = \frac{3}{8}$$

$$C_4 = [0; 2,1,2,6]\,\frac{19}{51}.$$

Now we define numbers p_k and q_k $(k = 0,1, \dots, n)$ as follows:

$$p_0 = a_0 \qquad q_0 = 1$$

$$p_1 = a_1 a_0 + 1 \qquad q_1 = a_1 \quad for\ k = 2,3, \dots, n$$

$$p_k = a_k p_{k-1} + p_{k-2} \qquad q_k = a_k q_{k-1} + q_{k-2}.$$

Remarks: A direct computation shows that the first few convergent of [a_0; a_1, . . . , a_n] are

$$C_0 = a_0 = \frac{a_0}{1} = \frac{p_0}{q_0}$$

$$C_1 = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1} = \frac{p_1}{q_1}$$
$$C_2 = a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = \frac{a_2(a_1 a_0 + 1) + a_0}{a_2 a_1 + 1} = \frac{p_2}{q_2}.$$

Theorem 14.4.2. *The kth convergent of the simple continued fraction* $[a_0; a_1, \ldots, a_n]$ *has the value*

$$C_k = \frac{p_k}{q_k} \qquad 0 \le k \le n.$$

Proof. From the above remarks, it is observed that the theorem is true for $k = 0, 1, 2$. Suppose that it is true for $k = m$, where $2 \le m < n$; that is, we have the mth convergent C_m as,

$$C_m = \frac{p_m}{q_m} = \frac{a_m p_{m-1} + p_{m-2}}{a_m q_{m-1} + q_{m-2}} \qquad (1)$$

Since the integers $p_{m-1}, q_{m-1}, p_{m-2}, q_{m-2}$ are depending on the first $m-1$ partial denominators $a_1, a_2, \ldots, a_{m-1}$ and, so they are independent of a_m. Replacing a_m by the value $a_m + \frac{1}{a_{m+1}}$, we have

$$\begin{aligned} C_{m+1} &= \left[a_0; a_1, \ldots, a_{m-1}, a_m + \frac{1}{a_{m+1}}\right] \\ &= \frac{\left(a_m + \frac{1}{a_{m+1}}\right) p_{m-1} + p_{m-2}}{\left(a_m + \frac{1}{a_{m+1}}\right) q_{m-1} + q_{m-2}} \\ &= \frac{a_{m+1}(a_m p_{m-1} + p_{m-2}) + p_{m-1}}{a_{m+1}(a_m q_{m-1} + q_{m-2}) + q_{m-1}} \\ &= \frac{a_{m+1} p_m + p_{m-1}}{a_{m+1} q_m + q_{m-1}} \\ &= \frac{p_{m+1}}{q_{m+1}}. \end{aligned}$$

Thus we have the theorem is also true for $k = m + 1$. Hence by the principle of mathematical induction, the theorem is true for all k such that $0 \le k \le n$. This ends the proof.

Example 3.

Let us take the continued fraction of say, $19/51 = [0; 2,1,2,6]$. Then p_k and

q_k which were defined above are calculated as follows.

$$p_0 = 0 \qquad q_0 = 1$$
$$p_1 = 0.2 + 1 = 1 \qquad q_1 = 2$$
$$p_2 = 1.1 + 0 = 1 \qquad q_2 = 1.2 + 1 = 3$$
$$p_3 = 2.1 + 1 = 3 \qquad q_3 = 2.3 + 2 = 8$$
$$p_4 = 6.3 + 1 = 19 \qquad q_4 = 6.8 + 3 = 51$$

Now using these values, the convergent of $[0;\ 2, 1, 2, 6]$ are

$$C_0 = \frac{p_0}{q_0} = 0$$
$$C_1 = \frac{p_1}{q_1} = \frac{1}{2}$$
$$C_2 = \frac{p_2}{q_2} = \frac{1}{3}$$
$$C_3 = \frac{p_3}{q_3} = \frac{3}{8}$$
$$C_4 = \frac{p_4}{q_4} = \frac{19}{51}$$

And we have already calculated these values by using the definition of convergent.

Theorem 14.4.3: If $C_k = \frac{p_k}{q_k}$ is the kth convergent of the finite simple continued fraction

$[a_0; a_1, a_2, \dots, a_n]$ then

$$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1} \quad \text{for } 1 \le k \le n.$$

The proof is not added here. It is simply followed by the induction hypothesis.

Corollary. For $1 \le k \le n$, p_k and q_k are relatively prime.

Proof. *Suppose* $d = gcd(p_k, q_k)$ *then from the theorem 5, we have* $d \mid (-1)^{k-1}$. *We also know that the values of* $(-1)\hat{}k$ *are* -1 *and* 1 *only. Also since* $d > 0$, *we must have* $d = 1$. *This ends the proof.* ■

14.5 INFINITE CONTINUED FRACTIONS

Definition 15.5.1. An infinite continued fraction is an expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}}$$

where $a_0, a_1, a_2, \ldots$ *and* $b_1, b_2, b_3, \ldots$ *are real numbers.*

An early instance of such a fraction appears in the work of William Brouncker, who, in 1655, converted Wallis's renowned infinite product into a continued fraction representation, given as below.

$$\frac{4}{\pi} = \frac{3.3.5.5.7.7 \ldots}{2.4.4.6.6.8 \ldots}$$

Into the identity

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{\ddots}}}$$

Definition 15.5.2. *An infinite simple continued fractions have the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where $a_0, a_1, a_2, \ldots$ *is an infinite sequence of integers, all positive except possibly for* a_0.

In a compact notation likewise for simple finite continued fraction which was defined to be $C_n = [a_0; a_1, a_2, \ldots, a_n]$, for all $n \geq 0$, a notation for infinite continued fraction will be of the form, say, $C_n = [a_0; a_1, a_2, \ldots]$, which is set to be the limit of the sequence of rational numbers C_n. Thus we have the following definition.

Definition 15.5.3. *If* $a_0, a_1, a_2, \ldots$ *is an infinite sequence of integers, all positive except possibly* a_0*, then the infinite simple continued fraction* $[a_0; a_1, a_2, \ldots]$ *has the value*

$$\lim_{n\to\infty}[a_0; a_1, a_2, \dots, a_n].$$

Example 4.

Consider an infinite simple continued fraction [1; 1, 1, 1, . . .]. Previously, in example 2, it has been seen that the *n*th convergent C_n = [1; 1, . . . , 1], where the integer 1 appears *n* times, is equal to

$$C_n = \frac{u_{n+1}}{u_n} \qquad n \geq 0$$

a quotient of successive Fibonacci numbers. Suppose that y denotes the value of the infinite continued fraction $[1;\ 1, 1, 1, \dots]$, then

$$y = \lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{u_n + u_{n-1}}{u_n} = 1 + \lim_{n\to\infty} \frac{u_{n-1}}{u_n} = 1 + \frac{1}{y}.$$

Thus we get the following quadratic equation.

$$y^2 - y - 1 = 0.$$

On solving we get two roots and those are $y = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$. The only positive root is $y = \frac{1+\sqrt{5}}{2}$.

Hence we have

$$\frac{1+\sqrt{5}}{2} = [1; 1,1,1, \dots].$$

Remarks: For repeated partial denominators, say $b_1, b_2, \dots, b_n$ repeats indefinitely, then the infinite continued fraction is called periodic. Then we write such fractions in compact notation as

$$[a_0; a_1, \dots, a_m, b_1, \dots, b_n, b_1, \dots, b_n, \dots]$$

or

$$\left[a_0; a_1, a_2, \dots, a_m, \overline{b_1, b_2, \dots, b_n}\right]$$

where the overbar indicates that this block of integers repeats over and over. If the smallest block $b_1, b_2, \dots, b_n$ is constantly repeats, we say that $b_1, b_2, \dots, b_n$ is the period of the expansion and that the length of the period is n. Thus, for example, $[3; \overline{1,2,1,6}]$ would denote $[3;\ 1, 2, 1, 6, 1, 2, 1, 6, \dots]$, a continued

fraction whose period is $1, 2, 1, 6$ has length 4.

We, in the previous section, have the result that says that every finite continued fraction is represented by a rational number. Let us now know that which number can be written as an infinite continued fraction.

Theorem 14.5.1. *The value of any infinite continued fraction is an irrational number.*

Proof. Let x denotes the value of the infinite continued fraction $[a_0;\ a_1, a_2, \dots]$; that is, x is the sequence of convergent

$$C_n = [a_0; a_1, a_2, \dots, a_n] = \frac{p_n}{q_n}.$$

And if possible, suppose that x is a rational number such that $x = \frac{a}{b}$, where a and b are positive integers. Since x lies strictly between the successive convergent C_n and C_{n+1}, therefore, we have

$$0 < |x - C_n| = \left|\frac{a}{b} - C_n\right| < |C_{n+1} - C_n| == \left|\frac{\mathrm{p_{n+1}}}{\mathrm{p_{n+1}}} - \frac{\mathrm{p_n}}{\mathrm{p_n}}\right| = \frac{|q_n p_{n+1} - q_{n+1} p_n|}{q_n q_{n+1}}$$

Now since, $|q_n p_{n+1} - q_{n+1} p_n| = |(-1)^n| = 1$, so the above inequality reduces to,

$$0 < \left|\frac{a}{b} - \frac{p_n}{q_n}\right| < \frac{1}{q_{n+1}}.$$

Now, multiplying both side of the above by the positive number bq_n, we have

$$0 < |aq_n - bp_n| < \frac{b}{q_{n+1}}.$$

We know that the values of q_i increase without bound as i increases. So choosing n large enough such that $b < q_{n+1}$, the above inequality now becomes

$$0 < |aq_n - bp_n| < 1.$$

This says that there is a positive integer, namely $|aq_n - bp_n|$, between 0 and 1, which is not possible. Thus, we have x is an irrational number. That is, the

value of an infinite continued fraction is nothing but an irrational number. ■

Remarks: Let us see the following property of limit for an infinite continued fraction $[a_0; a_1, a_2, \ldots]$ as follows.

$$[a_0; a_1, a_2, \ldots] = \lim_{n\to\infty} [a_0; a_1, a_2, \ldots, a_n]$$
$$= \lim_{n\to\infty} \left(a_0 + \frac{1}{[a_0; a_1, a_2, \ldots, a_n]}\right)$$
$$= a_0 + \lim_{n\to\infty} \left(\frac{1}{[a_0; a_1, a_2, \ldots, a_n]}\right)$$
$$= a_0 + \frac{1}{[a_0; a_1, a_2, \ldots, a_n]}.$$

The next theorem tells us about the equality of two infinite continued fraction.

Theorem 14.5.2. *The infinite continued fractions* $[a_0; a_1, a_2, \ldots]$ *and* $[b_0; b_1, b_2, \ldots]$ *are equal, then* $a_n = b_n$ *for all* $n \geq 0$.

Proof. Let $x = [a_0; a_1, a_2, \ldots]$, then $C_0 < x < C_1$, which is the same as saying that $a_0 < x < a_0 + \frac{1}{a_1}$. As we know that the integer $a_1 \geq 1$, that is $\frac{1}{a_1} \leq 1$, so we have the inequality $a_0 < x < a_0 + 1$. Hence, $[x] = a_0$, where $[x]$ is the greatest integer function of x.

Now let us assume that $[a_0; a_1, a_2, \ldots] = x = [b_0; b_1, b_2, \ldots]$, Then we have, by using the above remarks,

$$a_0 + \frac{1}{[a_0; a_1, a_2, \ldots]} = x = b_0 + \frac{1}{[b_0; b_1, b_2, \ldots]}.$$

Since, $[x] = a_0$, we can similarly show that $[x] = b_0$, so that the above equation now reduces to $[a_1; a_2, \ldots] = [b_1; b_2, \ldots]$. Now continuing the same process for the next step, we get $a_1 = b_1$, and in the next step we get $a_2 = b_2$. Thus, in this way we finally arrived at a conclusion that $a_n = b_n$, for all $n \geq 0$.

Corollary. Two distinct infinite continued fractions represent two distinct irrational numbers.

The next example will tell us about the uniqueness of infinite continued fractions.

Example 5. Consider the infinite continued fraction $y = [3; 6, \overline{1,4}]$, let us write $y = [3; 6, x]$, where

$$x = [\overline{1,4}] = [1; 4, x].$$

Then we have

$$x = 1 + \frac{1}{4 + \frac{1}{x}} = \frac{5x + 1}{4x + 1}.$$

Which leads to the quadratic equation

$$4x^2 - 4x - 1 = 0.$$

On solving the above equation and consider the positive root only, we have

$$y = \frac{1 + \sqrt{2}}{2}.$$

From $y = [3; 6, x]$, we then find that

$$y = 3 + \frac{1}{6 + \frac{1}{x}} = \frac{25 + 19\sqrt{2}}{8 + 6\sqrt{2}}$$

$$= \frac{(25 + 19\sqrt{2})(8 - 6\sqrt{2})}{(8 + 6\sqrt{2})(8 - 6\sqrt{2})}$$

$$= \frac{14 - \sqrt{2}}{4}$$

That is,

$$[3; 6, \overline{1,4}] = \frac{14 - \sqrt{2}}{4}.$$

Theorem 8. *Every irrational number can be uniquely expressed as an infinite continued fraction, the representation though derived from the continued fraction algorithm.*

W CHECK YOUR PROGRESS

Q. 4: Express each of the rational numbers below as finite simple continued fractions:

(a) $-19/51$.

(b) $71/55$.

Q. 5: Compute the convergent of the following simple continued fractions:

(a) $[1;\ 2,3,3,2,1]$.

(b) $[0;\ 2,4,1,8,2]$.

Q. 6: Evaluate the value of the infinite continued fraction $[\overline{2,3}]$.

Q. 7: Determine the infinite continued fraction representation of $\sqrt{23}$.

14.6 APPLICATION TO EQUATIONS

Let us now consider the linear Diophantine equation as below.

$$ax + by = c$$

where a, b, c are given integers. Since we know that the solution of the above equation exists only if $d \nmid c$, where $d = gcd(a, b)$, so consider that $d \mid c$. Our main focus is that the coefficients are relatively prime to each other. If not, that is, if $d = gcd(a, b) > 1$, then dividing the equation d to produce

$$\frac{a}{d}x + \frac{b}{d}y = \frac{c}{d}.$$

Where we have $gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$. It is to be clear that both the above Diophantine equations will have same equations.

Observe that the solution of the equation

$$ax + by = c \qquad gcd(a, b) = 1$$

may be obtained by first solving the Diophantine equation

$$ax + by = 1 \qquad gcd(a, b) = 1.$$

and then the solution of the previous equation can be obtained by just multiplying c to the solutions of the latter equation.

Now writing the rational number a/b as a simple continued fraction, we have

$$\frac{a}{b} = [a_0; a_1, a_2, \dots, a_n]$$

Also, the last two convergent of this continued fraction are given by

$$C_{n-1} = \frac{p_{n-1}}{q_{n-1}} \quad \text{and} \quad C_n = \frac{p_n}{q_n} = \frac{a}{b}.$$

Because $gcd(p_n, q_n) = 1 = gcd(a, b)$, it may be concluded that

$$p - n = a \quad \text{and} \quad q_n = b.$$

By using the Theorem 5, we have

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}.$$

That is

$$aq_{n-1} - bp_{n-1} = (-1)^{n-1}.$$

Thus, if we take $x = q_{n-1}$ and $y = -p_{n-1}$, then we have

$$ax + by = (-1)^{n-1}.$$

If *n* is odd, then the equation $ax + by = 1$ has the particular solution $x_0 = q_{n-1}, y_0 = -p_{n-1}$; whereas if *n* is an even integer, then a solution is given by $x_0 = -q_{n-1}$, $y_0 = p_{n-1}$. And the general solution can be obtained by using the formula as given below.

$$x = x_0 + bt \qquad y = y_0 - at \quad \text{for} t = 0, \pm 1, \pm 2, \ldots$$

Example 6. As an example let us solve the following linear Diophantine equation

$$172x + 20y = 1000$$

by means of simple continued fractions. It is observed that $gcd(172,20) = 4$, and so by dividing the given equation by 4, we have

$$43x + 5y = 250.$$

At first, we solve the following linear Diophantine equation,

$$43x + 5y = 1.$$

Starting with the finding the simple continued fraction for 43/5. Applying the Euclidean Algorithm to the numbers 43 and 5, we have

$$43 = 8 \cdot 5 + 3$$
$$5 = 1 \cdot 3 + 2$$
$$3 = 1 \cdot 2 + 1$$
$$2 = 2 \cdot 1$$

Thus we have

$$\frac{43}{5} = [8; 1,1,2] = 8 + \frac{1}{1 + \frac{1}{1+\frac{1}{2}}}$$

The convergent of this continued fraction are

$$C_0 = \frac{8}{1} = \quad C_1 = \frac{9}{1} = \quad C_2 = \frac{17}{2} = \frac{p_2}{q_2} \quad C_3 = \frac{43}{5} = \frac{p_3}{q_3}$$

We can now write $p_2 = 17, q_2 = 2, p_3 = 43, q_3 = 5$. Using the Theorem 5, we have

$$p_3 q_2 - q_3 p_2 = (-1)^{3-1}$$

Substituting the values, the above equation reduces to

$$43 \,.\, 2 - 5 \,.\, 17 = 1.$$

Multiplying the above relation by 250, we obtain

$$43.500 + 5.(-4250) = 250.$$

Thus, a particular solution of the Diophantine equation $43x + 5y = 250$ is

$$x_0 = 500 \qquad y_0 = -4250.$$

And the general solution to the given linear Diaphantine equation is given by

$$x = 500 + 5t, \qquad y = -4250 - 437 \qquad t = 0, \pm 1, \pm 2, \ldots$$

14.7 PELL'S EQUATION

Definition 14.7.1. *Pell's equation, also known as the Pell-Fermat equation, is a Diophantine equation of the form $x^2 - dy^2 = 1$, where d is a given positive non-square integer, and integer solutions are sought for x and y.*

The characteristics of the Pell's equation $x^2 - dy^2 = 1$ is such that if the solution (positive) to the equation is x_0 and y_0, then $\frac{x_0}{y_0}$ gives an approximation for $\sqrt{d}$. In this section we will restrict the solution to the Pell's equation to only positive solutions and *d* to be non-squared positive integer. The next theorem will tell us about any pair of positive integers satisfying Pell's equation can be obtained from the continued fraction representing the irrational number $\sqrt{d}$.

Theorem 14.7.1: If p, q is a positive solution of $x^2 - dy^2 = 1$, then p/q is a convergent of the continued fraction expansion of $\sqrt{d}$.

Proof. **:** If p, q is a positive solution of $x^2 - dy^2 = 1$, then by substituting on the equation, we have $p^2 - dq^2 = 1$, so that

$$\left(p - q\sqrt{d}\right)\left(p + q\sqrt{d}\right) = 1$$

Implying that $p > q$, also

$$\frac{p}{q} - \sqrt{d} = \frac{p - \sqrt{d}q}{q} = \frac{1}{q\left(p + q\sqrt{d}\right)}.$$

Now, since $p > q$, and since d is a positive so we have, $\frac{1}{p\sqrt{d}} < \frac{1}{q\sqrt{d}}$. Thus we have

$$0 < \frac{p}{q} - \sqrt{d} = \frac{1}{q\left(p + q\sqrt{d}\right)} < \frac{\sqrt{d}}{q\left(q\sqrt{d} + q\sqrt{d}\right)} = \frac{\sqrt{d}}{2q^2\sqrt{d}} = \frac{1}{2q^2}.$$

Now, use a result, which states that, let x be an arbitrary irrational number. If the rational number a/b, where $b \geq 1$ and $gcd(a, b) = 1$, staisfies $\left|x - \frac{a}{b}\right| < \frac{1}{2b^2}$, then a/b is one of the convergents p_n/q_n in the continued fraction representation of x, we can directly conclude that p/q must be a convergent of $\sqrt{d}$.

Theorem 14.7.2. *If p/q is a convergent of the continued fraction expansion of $\sqrt{d}$, then $x = p, y = q$ is a solution of one of the equations*

$$x^2 - dy^2 = k$$

where $|k| < 1 + 2\sqrt{d}$.

Proof. If *p*/*q* is a convergent of $\sqrt{d}$, then by a similar result used in the proof of the previous theorem, which states that if p_n/q_n is the *n*th convergent to the irrational number x, then

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_{n+1}q_n} \leq \frac{1}{q_n^2}.$$

Thus in our case, we have

$$\left|\sqrt{d} - \frac{p}{q}\right| < \frac{1}{q^2}$$

And therefore,

$$\left|p - q\sqrt{d}\right| < \frac{1}{q}$$

Now,

$$\begin{aligned}\left|p + q\sqrt{d}\right| &= \left|\left(p - q\sqrt{d}\right) + 2q\sqrt{d}\right| \\ &\leq \left|p - q\sqrt{d}\right| + \left|2\sqrt{d}\right| \\ &< \frac{1}{q} + 2q\sqrt{d} \leq \left(1 + 2\sqrt{d}\right)q\end{aligned}$$

Now combining the above two inequalities, we have

$$|p^2 - dq^2| = |p - q\sqrt{d}||p + q\sqrt{d}|$$
$$< \frac{1}{q}(1 + 2\sqrt{q})q$$
$$= 1 + 2\sqrt{d}$$

Which implies that $|k| < 1 + 2\sqrt{d}$.

Illustration: To illustrate the above theorem, let *d* = 7. Now by using the continued fraction expansion $\sqrt{7} = [2; \overline{1,1,1,4}]$, the first few convergent of $\sqrt{7}$ are given by

$$2/1, 3/1, 5/2, 8/3, \ldots$$

Putting different values of p_n and q_n, we have the values of $p^2 - 7q^2$ as

$$2^2 - 7.1^2 = -3 \quad 3^2 - 7.1^2 = 2 \quad 5^2 - 7.2^2 = -3 \quad 8^2 - 7.3^2 = 1$$

Thus when $p_n = 8$ and $q_n = 3$ for some *n*, we have the solution to the equation $x^2 - 7y^2 = 1$ as $x = 8, y = 3$.

Theorem 14.7.2. *Let $\frac{p_k}{q_k}$ be the convergent of the continued fraction expansion of $\sqrt{d}$ and let n be the length of the expansion.*

(a) If n is even, then all positive solutions of $x^2 - dy^2 = 1$ are given by

$$x = p_{kn-1}, \quad y = q_{kn-1} \quad \text{for } k = 1,2,3,\ldots$$

(b) If n is odd, then all positive solutions of $x^2 - dy^2 = 1$ are given by

$$x = p_{2kn-1}, \quad y = q_{2kn-1} \quad \text{for } k = 1,2,3,\ldots$$

The proof of the above theorem is not enclosed here. As the above theorem is important in the sense of finding solution to Pell's equation, so one is suggested to go through the proof, given in the book **David Burton: Elementary Number Theory.**

W CHECK YOUR PROGRESS

Q. 8: Find all possible (positive) solutions of the Pell's equation $x^2 - 7y^2 = 1$.

Q. 9: Find the smallest positive solution to the Pell's equation $x^2 - 13y^2 = 1$.

14.8 FIBONACCI NUMBERS

Fibonacci proposed a problem which deals with the number of offspring generated by a pair of rabbits conjured up in the imagination:

A man put one pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year, if the nature of these rabbits is such that every month each pair bears a new pair which from the second month on becomes productive? Assuming that none of the rabbits dies, then a pair is born during the first month, so that there are two pairs present. During the second month, the original pair has produced another pair. One month later, both the original pair and the first-born pair have produced new pairs, so that three adult and two young pairs are present, and so on. The figures are tabulated in the
following chart.

Growth of Rabbit Colony

Months	Adult pairs	Young pairs	Total
1	1	1	2
2	2	1	3
3	3	2	5
4	5	3	8
5	8	5	13
6	13	8	21
7	21	13	34
8	34	21	55
9	55	34	89
10	89	55	144
11	144	89	233
12	233	144	377

Thus, the sequence generated is 1, 1, 2, 3, 5, 8, 13, . . . This sequence of numbers is known as Fibonacci sequence or Fibonacci numbers.

14.9 FIBONACCI SEQUENCE

Definition 14.9.1. *The sequence of numbers* 1, 1, 2, 3, 5, 8, 13, . . . *is called the Fibonacci sequence. In other words, the Fibonacci sequence is such that whose first two terms are* 1, 1 *and the following terms are obtained by adding the previous two terms. Mathematically let* $f_1 = 1, f_2 = 1$, *and* $f_n = f_{n-1} + f_{n-2}$ *where* $n \geq 3$.

Theorem 14.9.1. *For the Fibonacci sequence* (f_n), *we have* $gcd(f_n, f_{n+1}) = 1$ *for every* $n \geq 1$.

Proof. Let us suppose that $gcd(f_n, f_{n+1}) = d$, and if possible suppose that $d > 1$. Since d is also a common divisor, that is, d divides both the numbers f_n and f_{n+1}. Then their difference $f_{n+1} - f_n = f_{n-1}$ must also be divisible by d. Again the difference $f_n - f_{n-1} = f_{n-2}$, so that $d|f_{n-2}$. In the similar way (apply the same argument), it can be concluded that $d|f_{n-3}$, $d|f_{n-4}$,..., and finally at the last step we get $d|f_1$. But since $f_1 = 1$, which is not divisible by any $d > 1$. This is a contradiction. Thus we get $gcd(\ f_n, f_\{n+1\}) = d = 1$. ■

Theorem 14.9.2. *For a Fibonacci sequence* f_m, *we have* $f_{m+n} = f_{m-1}\, f_n + f_m\, f_{n+1}$, *for all* $m \geq 2$ *and* $n \geq 1$.

Proof. We will prove this result by using induction on n. So for $n = 1$ we have

$$f_{m+1} = f_{m-1}f_1 + f_m f_2$$

which follows from the definition of Fibonacci sequence as $f_2 = 1 = f_1$. Now let the result be true for $n = 1, 2, \ldots, k$. Now we will show the result is true for $n = k + 1$. Since the given result is true for $n = k - 1$ and $n = k$, so we have

$$f_{m+k} = f_{m-1}f_k + f_m f_{k+1}$$
$$f_{m+(k-1)} = f_{m-1}f_{k-1} + f_m f_k.$$

Now we have

$$\begin{aligned} f_{m+(k+1)} &= f_{m+k} + f_{m+(k-1)} \\ &= f_{m-1}f_k + f_m f_{k+1} + f_{m-1}f_{k-1} + f_m f_k \\ &= f_{m-1}(f_k + f_{k-1}) + f_m(f_k + f_{k+1}) \\ &= f_{m-1}f_{k+1} + f_m f_{k+2}. \end{aligned}$$

Thus the result is true for $n = k + 1$. So by the principle of mathematical induction the result is true for all $n \geq 1$ and $m \geq 2$. ■

Theorem 14.9.3. *For $m \geq 1, n \geq 1, f_{mn}$ is divisible by f_m.*

Proof. We will prove this result by induction on n. Since f_m is divisible by f_m, so the result is true for $n = 1$. For our induction hypothesis, let us assume that f_{mn} is divisible by f_m for $n = 1, 2, \ldots, k$. Now from the above Theorem 13, we have

$$f_{m(k+1)} = f_{mk+} \quad = f_{mk-1}f_m + f_{mk}f_{m+1}$$

since f_{mk} and f_{mk-1} are divisible by f_m, so is the right-hand side of the above expression and, hence, the left-hand side of the above expression must be divisible by f_m, And hence the result is proved by using induction. ■

We are heading towards to prove a beautiful result which states that the greatest common divisor of two Fibonacci numbers is again a Fibonacci number. For this we will proof the following lemma.

Lemma If $m = qn + r$, then $gcd(f_m, f_n) = gcd(f_r, f_n)$.

Proof. To begin with, Theorem 12 allows us to write

$$gcd(f_m, f_n) = gcd\left(f_{qn+r}, f_n\right)$$
$$= gcd\left(f_{qn-1}f_r + f_{qn}f_{r+1}, f_n\right)$$

By using Theorem 13 and the fact that $gcd(a + c, b) = gcd(a, b)$, whenever $b \mid c$, yields

$$gcd\left(f_{qn-1}f_r + f_{qn}f_{r+1}, f_n\right) = gcd\left(f_{qn-1}f_r, f_n\right)$$

Suppose $d = gcd\left(u_{qn-1}, u_n\right)$. Then the relations $d \mid f_n$ and $f_n \mid f_{qn}$ imply that $d \mid f_{qn}$, and therefore d is a (positive) common divisor of the successive Fibonacci numbers f_{qn-1} and f_{qn}. Since successive Fibonacci numbers are relatively prime, thus we get $d = gcd\left(u_{qn-1}, u_n\right) = 1$.

We know that when $gcd(a, c) = 1$, then $gcd(a, bc) = gcd(a, b)$. Thus we have

$$gcd(f_m, f_n) = gcd\left(f_{qn-1}f_r, f_n\right) = gcd(f_r, f_n).$$

This completes our proof. ■

Theorem 14.9.4. *The greatest common divisor of two Fibonacci numbers is again a Fibonacci number; that is,*

$$gcd(f_m, f_n) = f_d \qquad \text{where} \qquad d = gcd(m, n).$$

Proof. Assume that $m \geq n$. Applying the Euclidean Algorithm to *m* and *n*, we get the following system of equations:

$$\begin{aligned}
n &= q_2 r_1 + r_2 \qquad 0 < r_2 < r_1 \\
r_1 &= q_3 r_2 + r_3 \qquad 0 < r_3 < r_2 \\
&\vdots \\
r_{n-2} &= q_n r_{n-1} + r_n \qquad 0 < r_n < r_{n-1} \\
r_{n-1} &= q_{n+1} r_n + 0
\end{aligned}$$

In accordance with the previous lemma,

$$\begin{aligned}
\gcd(f_m, f_n) &= \gcd(f_{r_1}, f_n) \\
&= \gcd(f_{r_1}, f_{r_2}) \\
&= \cdots \\
&= \gcd(f_{r_{n-1}}, f_{r_n})
\end{aligned}$$

Since $r_n \mid r_{n-1}$, therefore from Theorem 13 it is clear that $f_{r_n} \mid f_{r_{n-1}}$, where $\gcd(f_{r_{n-1}}, f_{r_n}) = f_{r_n}$. But since r_n is the last non-zero remainder in the Euclidean Algorithm for m and n, and so

$r_n = \gcd(m, n)$. Hence, we get

$$gcd(f_m, f_n) = f_{gcd(m,n)}.$$

That is,

$$\gcd(f_m, f_n) = f_d$$

where $d = \gcd(m, n)$and this completes our proof. ■

Corollary In the Fibonacci sequence, $f_m \mid f_n$ if and only if $m \mid n$ for $n \geq m \geq 3$.

Proof. For the implication part, suppose that $f_m \mid f_n$. To show that $m \mid n$. Since $f_m \mid f_n$ so $\gcd(f_m, f_n) = f_m$, and also from Theorem 14, we have $\gcd(f_m, f_n) = f_{\gcd(m,n)}$, which implies that $\gcd(m, n) = m$, that means $m \mid n$. The converse part of this Corollary is just the Theorem 13. ■

Illustration: An illustration of Theorem 15 is provided by calculating

$$\gcd(f_{16}, f_{12}) = \gcd(987,144).$$

From the Euclidean Algorithm, we have

$$987 = 6 \cdot 144 + 123$$
$$144 = 1 \cdot 123 + 21$$
$$123 = 5 \cdot 21 + 18$$
$$21 = 1 \cdot 18 + 3$$
$$18 = 6 \cdot 3 + 0$$

and therefore $\gcd(987,144) = 3$. The net result is that

$$\gcd(f_{16}, f_{12}) = 3 = u_4 = f_{\gcd(16,12)}.$$

as asserted by Theorem 15.

W CHECK YOUR PROGRESS

Q. 10: Evaluate the following:

(a) $\gcd(f_9, f_{12})$

(b) $\gcd(f_{15}, f_{20})$

(c) $\gcd(f_{24}, f_{36})$

Q. 11: Show that the sum of the squares of the first nFibonacci numbers is given by the formula

$$f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 = f_n f_{n+1}.$$

Q. 12: Find the Fibonacci numbers that divide both $f_{24}$and f_{36}.

14.10 CERTAIN IDENTITIES INVOLVING FIBONACCI NUMBERS

In this section, we proceed to establish several fundamental identities involving Fibonacci numbers, which will be helpful in solving the exercises that follow this section.

Theorem 14.10.1. *Sum of the first n Fibonacci numbers is equal to $f_{n+2} - 1$.*

Proof. To prove the above result we will start writing the terms of the Fibonacci numbers as follows.

$$f_1 = f_3 - f_2$$

$$f_2 = f_4 - f_3$$
$$f_3 = f_5 - f_4$$
$$\vdots$$
$$f_{n-1} = f_{n+1} - f_n$$
$$f_n = f_{n+2} - f_{n+1}$$

Adding the above terms we get

$$f_1 + f_2 + f_3 + \cdots + f_n = f_{n+2} - 1$$

which completes the proof. ■

Theorem 14.10.2. *Prove that*

$$f_n^2 = f_{n+1}f_{n-1} + (-1)^{n-1}$$

Proof. Let us start with the following equation

$$\begin{aligned} f_n^2 - f_{n+1}f_{n-1} &= f_n(f_{n-1} + f_{n-2}) - f_{n+1}f_{n-1} \\ &= (f_n - f_{n+1})f_{n-1} + f_n f_{n-2}. \end{aligned}$$

Using the rule of formation of the Fibonacci sequence, we have $u_{n+1} = u_n + u_{n-1}$, and so the above expression reduces to

$$f_n^2 - f_{n+1}f_{n-1} = (-1)(f_{n-1}^2 - f_n f_{n-2})$$

Now comparing the left hand side and right hand side of the above expression and using the same, we get

$$(f_{n-1}^2 - f_n f_{n-2}) = (-1)(f_{n-2}^2 - f_{n-1}f_{n-3})$$

Thus we have

$$f_n^2 - f_{n+1}f_{n-1} = (-1)^2(f_{n-2}^2 - f_{n-1}f_{n-3})$$

Repeating the process after a finite number of times $(n-2)$ times, we get

$$\begin{aligned} f_n^2 - f_{n+1}f_{n-1} &= (-1)^{n-2}(f_2^2 - f_3 f_1) \\ &= (-1)^{n-2}(1^2 - 2 \cdot 1) \\ &= (-1)^{n-1} \end{aligned}$$

Corollary. By using the above theorem we can conclude that for $n = 2k$, we have

$$f_{2k}^2 = f_{2k+1}f_{2k-1} - 1$$

Theorem 14.10.3. *Any positive integer Ncan be expressed as a sum of distinct Fibonacci numbers, no two of which are consecutive, that is,*

$$N = u_{k_1} + u_{k_2} + \cdots + u_{k_r}$$

where $k_1 \geq 2$and $k_{i+1} \geq k_i + 2$ for $j = 1,2, \ldots, r-1$.

The proof is not added here. One can find the proof from the book titled as "*Elementary Number Theory*" by David M. Burton.

214.11 LET US SUM UP

- A set of rational numbers of n elements of the form $\frac{a}{b}$ with the condition $0 \leq a \leq b \leq n$ and $\gcd(a,b) = 1$ is called a Farey sequence of order n and it is denoted by F_n.
- Any rational number can be written as a finite simple continued fraction.
- The continued fraction made from $[a_0; a_1, \ldots, a_n]$ by cutting off the expansion after the kth partial denominator a_k is called the kth convergent of the given continued fraction and denoted by C_k; in symbols, we write,
- $C_k = [a_0; a_1, \ldots, a_k]$for $1 \leq k \leq n$.

  By convention, the zeroth convergent C_0be equal to the number a_0.
- The solution of the linear Diophantine equation can be solved by using the convergents of a continued fraction obtained from the equation itself.
- The infinite continued fraction is just the expansion of the finite continued fraction in the sense of limit taking to infinity.
- Mathematically, we define the Fibonacci numbers by assuming its first two term, $f_1 = 1$, $f_2 = 1$, and the further terms can be calculated by using the formula $f_n = f_{n-1} + f_{n-2}$, where $n \geq 3$.
- The greatest common divisor of two Fibonacci numbers is again a Fibonacci number; that is,

  $$\gcd(f_m, f_n) = f_d \quad \text{where } d = \gcd(m,n)$$

  Also the two consecutive Fibonacci numbers are always relatively prime. Mathematically, we have $\gcd(f_n, f_{n+1}) = 1$ for every $n \geq 1$.

K 14.12 ANSWERS TO CHECK YOUR PROGRESS

Answer to the Q. No. 1: A set of rational numbers of $n$elements of the form $\frac{a}{b}$with the condition $0 \leq a \leq b \leq n$and $\gcd(a,b) = 1$is called a Farey sequence of order $n$and it is denoted by F_n.

Answer to the Q. No. 2: The list of elements in the Farey sequence $F_8$and $F_9$are given below.

$$F_8 = \left\{\frac{0}{1}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{1}{1}\right\}$$

$$F_9 = \left\{\frac{0}{1}, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{7}{9}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{1}{1}\right\}$$

Answer to the Q. No. 3: To prove the formula

$$| F_n | = 1 + \phi(1) + \phi(2) + \cdots + \phi(n),$$

where $| F_n |$represents the number of elements in the Farey sequence F_n, for any positive integer $n \geq 0$, we will use the induction hypothesis.

For $n = 1$, we have

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}.$$

which shows that

$$| F_1 | = 2.$$

Also from the definition of Euler's ϕ-function, we have $\phi(1) = 1$, and so, we have that

$$| F_1 | = 2 = 1 + \phi(1).$$

That means the result is true for $n = 1$.

Now let us assume that the result be true for $n = k$, for some $k \in \mathbb{N}$, so that

$$| F_k | = 1 + \phi(1) + \phi(2) + \cdots + \phi(k).$$

We now prove the inductive step, that is we will show that the result is true for $n = k + 1$.

Observing the expression, we need to prove

$$| F_{k+1} | = 1 + \phi(1) + \phi(2) + \cdots + \phi(k) + \phi(k+1),$$

we see that the induction hypothesis transforms it as follows:

$$| F_{k+1} | = | F_k | + \phi(k+1).$$

We already know that we can obtain F_{k+1} from F_k by adding the mediants whose denominator is smaller than $k+1$. Since we already took all the fractions $\frac{r}{s}$ with $s < k+1$ in F_k, we just need to add those fractions $\frac{r}{k+1}$.

If a fraction $\frac{r}{k+1}$ occurs with $\gcd(r, k+1) > 1$, it means that the fraction can be reduced, and once we do that, we obtain a new fraction with a denominator smaller than $k+1$. All of those fractions were already taken in F_k, and we do not add them again.

Therefore, we just add those fractions with denominators relatively prime to $k+1$. The number of those fractions is obtained using the Euler's totient function ϕ. Hence, the number of mediants that are added to F_k to obtain F_{k+1} equals $\phi(k+1)$.

In other words,

$$| F_{k+1} | = | F_k | + \phi(k+1).$$

This proves that the result is also true for $n = k+1$. Hence by the mathematical induction principle, we proved the given formula.

Answer to the Q. No. 4(a): For $\frac{19}{-51}$, by using Euclid's division algorithm for the integers -19 and 51, we have

$$\begin{aligned} -19 &= -1 \cdot 51 + 32 \\ 51 &= 1 \cdot 32 + 19 \\ 32 &= 1 \cdot 19 + 13 \\ 19 &= 1 \cdot 13 + 6 \\ 13 &= 2 \cdot 6 + 1 \\ 6 &= 6 \cdot 1 + 0 \end{aligned}$$

Therefore we have

$$-\frac{19}{51} = [-1; 1,1,1,2,6].$$

Answer to the Q. No. 4(b): For $\frac{71}{55}$, by using Euclid's division algorithm for the integers $-19$and 51, we have

$$71 = 1 \cdot 55 + 16$$
$$55 = 3 \cdot 16 + 7$$
$$16 = 2 \cdot 7 + 2$$
$$7 = 3 \cdot 2 + 1$$
$$2 = 2 \cdot 1 + 0$$

Therefore we have

$$\frac{71}{55} = [1; 3,2,3,2].$$

Answer to the Q. No. 5(a): Given simple continued fraction is $[1; 2,3,3,2,1]$. We first calculate the values of p_k and q_k. Thus, we have

$p_0 = 1$ $q_0 = 1$

$p_1 = 2 \cdot 1 + 1 = 3$ $q_1 = 2$

$p_2 = 3 \cdot 3 + 1 = 10$ $q_2 = 3 \cdot 2 + 1 = 7$

$p_3 = 3 \cdot 10 + 13 = 33$ $q_3 = 3 \cdot 7 + 2 = 23$

$p_4 = 2 \cdot 33 + 10 = 76$ $q_4 = 2 \cdot 23 + 7 = 53$

$p_5 = 1 \cdot 76 + 33 = 109$ $q_5 = 1 \cdot 53 + 23 = 76$

Now using these values, the convergent of $[1; 2,3,3,2,1]$ are

$$C_0 = \frac{p_0}{q_0} = \frac{1}{1}$$
$$C_1 = \frac{p_1}{q_1} = \frac{3}{2}$$
$$C_2 = \frac{p_2}{q_2} = \frac{10}{7}$$
$$C_3 = \frac{p_3}{q_3} = \frac{33}{23}$$
$$C_4 = \frac{p_4}{q_4} = \frac{76}{53}$$
$$C_5 = \frac{p_5}{q_5} = \frac{109}{76}$$

Answer to the Q. No. 5(b): Given simple continued fraction is $[0;2,4,1,8,2]$. We first calculate the values of p_k and q_k. Thus, we have

$$\begin{aligned} p_0 &= 0 & q_0 &= 1 \\ p_1 &= 2\cdot 0+1=1 & q_1 &= 2 \\ p_2 &= 4\cdot 1+0=4 & q_2 &= 4\cdot 2+1=9 \\ p_3 &= 1\cdot 4+1=5 & q_3 &= 1\cdot 9+2=11 \\ p_4 &= 8\cdot 5+4=44 & q_4 &= 8\cdot 11+9=97 \\ p_5 &= 2\cdot 44+5=93 & q_5 &= 2\cdot 97+11=205. \end{aligned}$$

Now using these values, the convergent of $[0;2,4,1,8,2]$ are

$$C_0=\frac{p_0}{q_0}=0$$

$$C_1=\frac{p_1}{q_1}=\frac{1}{2}$$

$$C_2=\frac{p_2}{q_2}=\frac{4}{9}$$

$$C_3=\frac{p_3}{q_3}=\frac{5}{11}$$

$$C_4=\frac{p_4}{q_4}=\frac{44}{97}$$

$$C_5=\frac{p_5}{q_5}=\frac{93}{205}$$

Answer to the Q. No. 6: Let the given infinite continued fraction have the value of x, that is,

$$x=[2;\overline{2,3}].$$

Then we can write

$$x=[\overline{2;3}]=[2;3,x]$$

Then we have

$$x=2+\frac{1}{3+\frac{1}{x}}$$

$$x=2+\frac{x}{3x+1}=\frac{7x+2}{3x+1}.$$

From the above equation we get the quadratic equation $3x^2 - 6x - 2 = 0$, which on solving we get $x = \frac{3\pm\sqrt{15}}{3}$. The only positive root of the quadratic equation is $x = \frac{3+\sqrt{15}}{3}$. Thus the required value of the given infinite continued fraction is $[\overline{2;3}] = \frac{3+\sqrt{15}}{3}$.

Answer to the Q. No. 7: Here we have $\sqrt{23} \approx 4.8$. The successive irrational numbers x_k (and therefore the integers $a_k = [x_k]$) can be computed rather easily, the calculations are shown below:

$$x_0 = \sqrt{23} = 4 + \left(\sqrt{23} - 4\right) \qquad a_0 = 4$$

$$x_1 = \frac{1}{x_0 - [x_0]} = \frac{1}{\sqrt{23} - 4} = \frac{\sqrt{23} + 4}{7} = 1 + \frac{\sqrt{23} - 3}{7} \qquad a_1 = 1$$

$$x_2 = \frac{1}{x_1 - [x_1]} = \frac{7}{\sqrt{23} - 3} = \frac{\sqrt{23} + 3}{2} = 3 + \frac{\sqrt{23} - 3}{2} \qquad a_2 = 3$$

$$x_3 = \frac{1}{x_2 - [x_2]} = \frac{2}{\sqrt{23} - 3} = \frac{\sqrt{23} + 3}{7} = 1 + \frac{\sqrt{23} - 4}{7} \qquad a_3 = 1$$

$$x_4 = \frac{1}{x_3 - [x_3]} = \frac{7}{\sqrt{23} - 4} = \sqrt{23} + 4 = 8 + \left(\sqrt{23} - 4\right) \qquad a_4 = 8$$

Since, $x_5 = x_1$, also $x_6 = x_2$, $x_7 = x_3$, $x_8 = x_4$; thus we get $x_9 = x_5 = x_1$, and so on, which means that the block of integers 1,3,1,8repeats indefinitely. Thus, we find that the continued fraction expansion of $\sqrt{23}$ is periodic with the form

$$\sqrt{23} = [4; 1,3,1,8,1,3,1,8, \dots] = [4; \overline{1,3,1,8}].$$

Answer to the Q. No. 8: Given Pell's equation is $x^2 - 7y^2 = 1$. Now the continued fraction expansion of $\sqrt{7}$is given by $\sqrt{7} = [2; \overline{1,1,1,4}]$, and so the initial few convergent are

$$2, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \frac{37}{14}, \frac{45}{17}, \frac{82}{31}, \frac{127}{48}, \frac{590}{223}, \frac{717}{271}, \frac{1307}{494}, \frac{2024}{765}.$$

Since the continued fraction representation of $\sqrt{7}$ has a period of length 4, so by using Theorem 11, the numerator and denominator of any of the convergents $\frac{p_{4k-1}}{q_{4k-1}}$ form a solution of $x^2 - 7y^2 = 1$. Thus, substituting the values of $k = 1,2,3, \dots$, we have

$$\frac{p_3}{q_3} = \frac{8}{3}, \frac{p_7}{q_7} = \frac{127}{48}, \frac{p_{11}}{q_{11}} = \frac{2024}{765},$$

which gives the first three positive solutions and these solutions are

$$x_1 = 8, y_1 = 3; x_2 = 127, y_2 = 48; x_3 = 2024, y_3 = 765.$$

Answer to the Q. No. 9: Here we are concern to find the solution of $x^2 - 13y^2 = 1$ in the smallest positive integers, since $\sqrt{13} = [3; \overline{1,1,1,6}]$ and so the continued fraction representation of $\sqrt{13}$ has a period of length 5. The first 10 convergents of $\sqrt{13}$ are

$$\frac{3}{1}, \frac{4}{1}, \frac{7}{2}, \frac{11}{3}, \frac{18}{5}, \frac{119}{33}, \frac{137}{38}, \frac{256}{71}, \frac{393}{109}, \frac{649}{180}.$$

Since 5 is odd so by using the Theorem 11 part (b), we have, the least positive solution of $x^2 - 13y^2 = 1$ is obtained from the convergent $\frac{p_9}{q_9} = \frac{649}{180}$, and the solution is given $x_1 = 649, y_1 = 180$.

Answer to the Q. No. 10:

(a) We have $\gcd(f_9, f_{12}) = f_{\gcd(9,12)} = f_3 = 2$.

(b) We have $\gcd(f_{15}, f_{20}) = f_{\gcd(15,20)} = f_5 = 5$.

(c) We have

$$\gcd(f_{24}, f_{36}) = f_{\gcd(24,36)} = f_{12} = 144.$$

Answer to the Q. No. 11: It is obvious that the $\gcd(f_{24}, f_{36})$divides both $f_{24}$and f_{36}, and which can be obtained by the following formula,

$$\gcd(f_{24}, f_{36}) = f_{\gcd(24,36)} = f_{12} = 144.$$

Also the common divisors of 24 and 36 are $1,2,3,4,6,12$. So the Fibonacci numbers $f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_6 = 8$ and $f_{12} = 144$ also divides both of the numbers f_{24} and f_{36}.

Answer to the Q. No. 12: We will use induction hypothesis to prove the given result. So for $n = 1$, we have $f_1^2 = 1^2 = 1 \cdot 1 = f_1 f_2$. Thus, the result is true for $n = 1$. Let us consider that the result be true for $n = k$ for some k. That is, we have

$$f_1^2 + f_2^2 + \cdots + f_k^2 = f_k f_{k+1}.$$

It is remained to show that result is true for $n = k + 1$, that is we need to show that

$$f_1^2 + f_2^2 + \cdots + f_{k+1}^2 = f_{k+1} f_{k+2}.$$

Now taking the left-hand side of the above equation, we have

$$f_1^2 + f_2^2 + \cdots + f_{k+1}^2 = f_1^2 + f_2^2 + \cdots + f_k^2 + f_{k+1}^2$$

$$= f_k f_{k+1} + f_{k+1}^2$$
$$= f_{k+1}(f_k + f_{k+1})$$
$$= f_{k+1} f_{k+2}$$

which is the right-hand side of the above equation. We have used the definition of Fibonacci sequence which is $f_n = f_{n-1} + f_{n-2}$. Thus, by the principle of mathematical induction, the result is true for $n \geq 1$.

O 14.13 FURTHER READING

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M 14.14 MODEL QUESTIONS

Q. 1: Define finite continued fraction. What is the importance of finite continued fraction?

Q. 2: Determine the solution of each of the following Diophantine equations by using the continued fraction.

(a) $15x + 5y = 24$

(b) $19x + 51y = 1$

Q. 3: How can you relate finite and infinite continued fractions. What do you mean by period of an infinite continued fraction. Give examples.

Q. 4: Is the representation of two distinct infinite continued fraction unique? If so give examples, otherwise prove it.

Q. 4: Evaluate each of the following infinite continued fractions.

(a) $[4; \overline{1,3,1,8}]$

(b) $[1; \overline{2,1,2,12}]$

(c) $[1; 2,3, \overline{1}]$

Q. 5: What is the relationship between two consecutive Farey sequence? (Prove it.)

Q. 6: If $x_0, y_0$is a positive solution of the equation $x^2 - dy^2 = 1$, then prove that $x_0 > y_0$.

Q. 7: Find all possible positive solutions of the following Pell's equation for which $y < 250$:

(i) $x^2 - 2y^2 = 1$

(ii) $x^2 - 5y^2 = 1$

Q. 8: Find the smallest positive solution of the following Pell's equation:

(i) $x^2 - 41^2 = 1$

(ii) $x^2 - 33y^2 = 1$

Q. 9: For $n \geq 2$show that

$$f_{n+3} = 3f_{n+1} - f_{n-1}$$

Q. 10: Establish the result

$$f_{n+24} \equiv f_n (\mathrm{mod} 9)$$

Hence prove that $9 \mid f_{n+24}$if and only if $9 \mid f_n$.

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