UNIT 8 : NUMBER THEORETIC FUNCTIONS-II

UNIT STRUCTURE

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8.1 LEARNING OBJECTIVE

After reading this unit learner will gain ability to

* define and analyze the functions T(n), S(n), $\zeta(n)$ and $\overline{\phi}(n)$.
* compute values of the functions T(n), S(n), $\zeta(n)$ and $\overline{\phi}(n)$.
* understand the definition of the Riemann Zeta (ζ) function as a series $\zeta(s)=\sum_{n=1} \frac{1}{n^s}$ for Re (s) > 1 and its extension via analytic continuation to the entire complex plane.
* develop an understaning of functional equations and how they reflect the symmetry in the Zeta function.

8.2 INTRODUCTION

In number theory, function defined on the set of natural numbers that capture important arithmetic properties are known as number theoretic functions. One such function, denoted T(n), is introduced to deserибe a particular arithmetic behaviour or property related to the integer n. T(n) is defined as

$$T(n)=\zeta(1)+\zeta(2)+\ldots\ldots+\zeta(n)$$

where T(n) = the number of positive divisors of n.

Similarly $S(n)=\sigma(1)+\sigma(2)+.....+\sigma(n)$

Where $\sigma(n)$ = sum of positive divisors of n

and $\bar{\phi}(n)=\phi(1)+\phi(2)+.....+\phi(n)$

where ϕ is a Euler's ϕ-function.

The Riemann zeta function, devoted by $\zeta(s)$ is one of the most important and widely studied functions in number theory and complex analysis. It plays a central role in the distribution of prime numbers and is deeply connected to the structure of the natural numbers.

For complex number s with Re(s) > 1, the Riemann zeta function is defined by the infinite series

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

The series convergers absolutely in that region. For example

$$\zeta(2)=\frac{\pi^2}{6}$$

$$\zeta(3)\approx 1.202$$

A key feature of the zeta function is its connection to prime numbers via Euler's product formula

$$\zeta(s)=\prod_{p\text{ prime}}\left(1-\frac{1}{p^s}\right)^{-1} \quad \text{for Re (s) > 1}$$

This shows that the zeta function encodes information about the primes, as the product runs over all primes.

Although the series only converges for Re (s) > 1. The zeta function can be analytically continued to all complex numbers except for a simple pole at s = 1.

The zeta function is used to deseribe the asymptotic distribution of prime numbers. $\zeta(s)$ appears in quantum mechanics, statistical mechanies and random matrix theory.

8.3 THE FUNCTION T(N) : DEFINITION AND EXAMPLES

The function T(n) is defined as

$$T(n) = \sum_{k=1}^{n} \zeta(k)$$

$$= \zeta(1)+\zeta(2)+.....+\zeta(n)$$

Where $\zeta(n)$ = the number of positive divisors of n.

For example, $\zeta(5)=\zeta(1)+\zeta(2)+\zeta(3)+\zeta(4)+\zeta(5)$

$$= 1 + 2 + 2 + 3 + 2$$

$$= 10$$

$$\zeta(8)=\zeta(1)+\zeta(2)+\zeta(3)+\zeta(4)+\zeta(5)+\zeta(6)+\zeta(7)+\zeta(8)$$

$$= 1 + 2 + 2 + 3 + 2 + 4 + 2 + 4$$

$$= 20$$

8.4 PROPERTIES OF T (N)

Lemma 8.4.1 Let f and F be number-theoretic functions such that

$$F(n)=\sum_{d|n} f(d)$$

Then, for any positive integer N,

$$\sum_{n=1}^{N} F(n)=\sum_{n=1}^{N} F(k)\left[\frac{N}{k}\right]$$

Proof : Proof is given in unit – 7, section 7.10

Theorem 8.4.2. If N is a positive integer, then

$$T(N)=\sum_{n=1}^{N} \zeta(n)=\sum_{n=1}^{N}\left[\frac{N}{n}\right]$$

Proof : Note that $\zeta(n)=\sum_{d|n} 1$. Therefore if we wrute for F and take f be the constant function f (n) = 1 for all n in the above lemma, we get

$$T(N)=\sum_{n=1}^{N}\zeta(n)=\sum_{n=1}^{N}\left[\frac{N}{n}\right]$$

Example 1. Final T (8) using Theorem 8.4.2.

Solution : $T(8)=\sum_{n=1}^{8}\zeta(n)$

$$=\sum_{n=1}^{8}\left[\frac{8}{n}\right]$$

$$=\left[\frac{8}{1}\right]+\left[\frac{8}{2}\right]+\left[\frac{8}{3}\right]+\left[\frac{8}{4}\right]+\left[\frac{8}{5}\right]+\left[\frac{8}{6}\right]+\left[\frac{8}{7}\right]+\left[\frac{8}{8}\right]$$

$$= 8 + 4 + 2 + 2+ 1 + 1 + 1 + 1$$

$$= 20$$

W CHECK YOUR PROGRESS

Q.1. Find T (12) and T (15)

8.5. The function S (n) : Definition and Examples

The function $S:\mathbb{N}\to\mathbb{N}$ is defined as

$$S(n)=\sigma(1)+\sigma(2)+.....+\sigma(n)$$

$$=\sum_{k=1}^{n}\sigma(k)$$

Where $\sigma(n)$= sum of positive divisors of n.

For example,

$$S(5)=\sigma(1)+\sigma(2)+\sigma(3)+\sigma(4)+\sigma(5)$$

$$= 1 + (1 + 2) + (1 + 3) + (1 + 2 + 4) + (1 + 5)$$

$$= 1 + 3 + 4 + 7 + 6$$

$$= 21$$

$$S(8)=\sigma(1)+\sigma(2)+\sigma(3)+\sigma(4)+\sigma(5)+\sigma(6)+\sigma(7)+\sigma(8)$$

$= 1 + (1 + 2) + (1 + 3) + (1 + 2 + 4) + (1 + 5) + (1 + 2 + 3 + 6) + (1 + 7) + (1 + 2 + 4 + 8)$

$= 56$

Theorem 8.5.1. If N is a positive integer, then

$$S(N)=\sum_{n=1}^{N}\sigma(n)=\sum_{n=1}^{N} n\left[\frac{N}{n}\right]$$

Proof : We have $\sigma(n)=\sum_{d|n} d$. Therefore, if we write σ for F and take f to be the identity function f(n) = n for all n, in Lemma 8.4.1, we get

$$S(N)=\sum_{n=1}^{N}\sigma(n)=\sum_{n=1}^{N} n\left[\frac{N}{n}\right]$$

Example. Final S (8) using Theorem 8.5.1.

Solution. Here N = 8, Therefore

$$S(8)=\sum_{n=1}^{8} n\left[\frac{8}{n}\right]$$

$$=1.\left[\frac{8}{1}\right]+2.\left[\frac{8}{2}\right]+3.\left[\frac{8}{3}\right]+4.\left[\frac{8}{4}\right]+5.\left[\frac{8}{5}\right]+6.\left[\frac{8}{6}\right]+7.\left[\frac{8}{7}\right]+8.\left[\frac{8}{8}\right]$$

$= 8 + 8 + 6 + 8 + 5 + 6 + 7 + 8$

$= 56$

W CHECK YOUR PROGRESS

Q.2. Find S (10) and S (12).

8.5 Historical Background of Riemann ζ - function example of infinite series is

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}\ldots\ldots\ldots$$

Which is called harmonic series. Since number theory is mainly about the positive integers 1, 2, 3, …….., it is not surprising that this series is of interest to number theorists. Unfortunately, it diverges. To make the series converges, without losing its

important number-theoretic properties, we replace its general term $\frac{1}{n}$ with the smaller term $\frac{1}{n^s}$ where s > 1. This gives rise to the Riemann zeta function, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} \text{.....}$$

Although this function is named after Riemann, who wrote a fundamental paper on its properties in 1859, it was in fact introduced about 120 years earlier by Euler, who showed that it can be expanded as a product

$$\zeta(s) = \prod_{p} \left(\frac{1}{1-p^{-s}} \right)$$

Where p range over all prime. This formula, known as the Euler product, established a deep link between the zeta function and the distribution of prime numbers, marking a major break through in analytic number theory.

Riemann introduced a complex function that extended ζ (s) to the complex plane. He analytically continued the zeta function to all complex numbers except s = 1, where it has a simple pole, and established the function equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) |1-s\ \zeta(1-s)$$

Which relates the values of the function at s and s — 1.

Crucially, Riemann proposed that all the nontrivial zeros of the zeta function lie on the 'critical line' in the complex plane where the real part of s is $\frac{1}{2}$. This assection, known as the Riemann Hypothesis, remains one of the most famous and long-standing unsolved problems in mathematics.

Riemann's insights transform the zeta function into a central object of study. His hypothesis has profound implications for the distribution of prime numbers and underpins much of modern number theory. Following Riemann, mathematicians like Hadamard and De La Vallee Poussin in the late 19th century prove that ζ (s) has no zeros on the line Re (s) = 1, leading to the proof of the prime number theorem, which describes the asymptotic distribution of prime numbers. Throughout the 20th and 21th centuries, the Riemann Zeta function has continued to be a firtile ground for mathematical exploration. It connects areas such as random matrix theory, quantum physics and algebraic geometry.

8.7 THE ζ - FUNCTION : DEFINITION AND EXAMPLES :

The Riemann zeta-function, $\zeta(s)$ is defined as

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

The properties of ζ -function are intricately linked with the properties of prime numbers. The reason for this is the Euler factorization of the ζ -function.

$$\zeta(s)=\prod_{p}\frac{1}{1-p^{-s}}$$

where the product is over all the primes.

The ζ -function is viewed as a function of a complex variable s. Recall that if $s=\alpha+i\beta$ where $\alpha, \beta \in \mathbb{R}$, then

$$n^s=e^{\log n^s}=e^{s\log n}=e^{(\alpha+i\beta)\log n}$$

$$=e^{\alpha\log n}.e^{i\beta\log n}$$

$$=e^{\alpha\log n}\left[\cos(\beta\log n)+\sin(\beta\log n)\right]$$

Also, from above equation, the ζ -function is absolutely convergent for $\mathrm{Re}(s)=\alpha>1$. Also $\lim_{s\to 1+}\zeta(s)=+\infty$, which leads to the proof that there are infinitely many primes. The ζ -function can be defined on the entire complex plane, excluding the point s = 1, such that $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ for $\mathrm{Re}(s)>1$.

For example

$$\zeta(2)=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots=\frac{\pi^2}{6}$$

$$\zeta(3)=1+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots=\frac{\pi^4}{90}$$

8.8 PROPERTIES OF THE ζ - FUNCTION

Theorem 8.8.1 For $\mathrm{Re}(s)>1$

$$\zeta(s)=\left(1-\frac{2}{2^s}\right)^{-1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$$

Proof : We have

$$\zeta(s)=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\ldots\ldots\ldots \qquad \text{------- (1)}$$

Multiplying both sides by $\frac{1}{2^s}$, we obtain

$$\frac{1}{2^s}\zeta(s)=\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\ldots\ldots\ldots \qquad \text{--------- (2)}$$

Subtract (2) × 2 from (1), we get

$$\zeta(s)-\frac{2}{2^s}\zeta(s)=1-\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\ldots\ldots\ldots$$

$$\Rightarrow\left(1-\frac{2}{2^s}\right)\zeta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$$

$$\Rightarrow\zeta(s)=\left(1-\frac{2}{2^s}\right)^{-1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$$

Theorem 8.8.2. If $s > 1$, then $\zeta(s)=\prod_{p}\frac{1}{1-p^{-s}}$

Proof : Since $p \geq 2$ we have

$$\frac{1}{1-p^{-s}}=1+p^{-s}+p^{-2s}+\ldots\ldots$$

For $s > 1$. If we take $p = 2, 3, \ldots..$ P and multiplying the series together, the general term resulting is of the type $2^{-a_2 s}.\,3^{-a_3 s}\ldots\ldots\ldots P^{-a\,p^s}=n^{-s}$

Where $n = 2^{a_2}\ 3^{a_3}\ldots\ldots\ldots P^{a_p} \quad \left(a_2 \geq 0,\ a_3 \geq 0, \ldots\ldots\ldots, a_p \geq 0\right)$

A number n will occur if and only if it has no prime factors greater than P and then, by Fundamental Theorem of arithmetic, once only. Hence

$$\prod_{p\leq P}\frac{1}{1-p^{-s}}=\sum_{(p)}n^{-s}$$

the summation on the right-hand side extending over numbers formed from the prime up to P.

These mumbers include all numbers upto P, so that

$$o < \sum_{n=1}^{\infty} n^{-s} - \sum_{(P)} n^{-s} \sum_{P+1}^{\infty} n^{-s}$$

and the last sum tends to o when . Hence

$$\sum_{P+1}^{\infty} n^{-s} = \lim_{p\to\infty} \sum_{(P)} n^{-s} = \lim_{p\to\infty} \prod_{p\le P} \frac{1}{1-p^{-s}}$$

$$= \prod_{p} \frac{1}{1-p^{-s}}$$

Example. Show that $\zeta(n) = \frac{\pi^2}{6}$

Solution : We have

$$\sin z = z \prod_{n\neq 0} \left(1 - \frac{z}{n\pi}\right) = z \prod_{n\geq 1} \left(1 - \frac{z^2}{n^2\pi^2}\right) \quad \text{---------- (1)}$$

(proof of which can be found in books on complex variable theory, e.g. Jones and Singerman, Chapter 3, Section 8).

The first product in (1) is over all non zero integers n and the second product is obtained from the first by pairing the factors corresponding to $\pm n$. The first product regarding sinz as behaving like a polynomial with infinitely many zeros at $z = n\pi \ (n \in Z)$, so we have a factorization

$$\sin z = c\, z \prod_{n\neq 0} \left(1 - \frac{z}{n\pi}\right)$$

With $c = \lim_{z\to 0} \frac{\sin z}{z} = 1$

By expanding the second product in (1) and collecting powers of z, we obtain a power series for sin z which must coincide with its Taylon series expansion

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \quad \text{---------- (2)}$$

By compairing coefficients of z^3 in (1) and (2), we obtain

$$-\sum_{n\geq 1}\frac{1}{n^2\pi^2}=-\frac{1}{3!}$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

$$\Rightarrow\zeta(2)==\frac{\pi^2}{6}$$

W CHECK YOUR PROGRESS

Q.3. Evaluate for $\zeta(2k)\ k\geq 1$

Q.4. If n is a positive integer, then

$$\sum_{m=1}^{n}\frac{1}{m}=\log n+\upsilon+\frac{\theta}{n}$$

Where υ is a contant called Euler's constant and $0<\theta<1$.

Q.5. If f is multiplicative and $\sum_{n=1}^{\infty}f(n)$ is absolutely convergent, then prove that $\sum_{n=1}^{\infty}f(n)=\prod_{p}\left[1+f(p)+f(p^2)+.....\right]$. Also prove $\sum\frac{\mu(n)}{n^s}=\prod_{p}\left(1-\frac{1}{p^s}\right)=\frac{1}{\zeta(s)}$.

8.9. THE FUNCTION $\overline{\phi}(n)$: DEFINITION AND EXAMPLES :

Definition 8.9.1. The function $\overline{\phi}:\mathbb{N}\to\mathbb{N}$ is defined as

$$\overline{\phi}(n)=\phi(1)+\phi(2)+.....+\phi(n)$$

$$=\sum_{k=1}^{n}\phi(k)$$

The function $\overline{\phi}(n)$ is called the summatory function of $\phi(n)$.

$$\overline{\phi}(8)=\phi(1)+\phi(2)+\phi(3)+\phi(4)+\phi(5)+\phi(6)+\phi(7)+\phi(8)$$

$$=1+1+2+2+4+2+6+4$$

$= 22$

$$\bar{\phi}(12) = \phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(5) + \phi(6) + \phi(7) + \phi(8) + \phi(9) | \phi(10) + \phi(11) + \phi(12)$$

$$= 1 + 1 + 2 + 2 + 4 + 2 + 6 + 4 + 6 + 4 + 10 + 4$$

$$= 46$$

8.10 PROPERTIES OF $\bar{\phi}(n)$

Theorem 8.10.1 For $n \geq 1$, $\bar{\phi}(n) = \frac{1}{2} \sum_{k=1}^{n} \mu(k) \left[\left[\frac{n}{k} \right]^2 + \left[\frac{n}{k} \right] \right]$

Proof : We have

$$\bar{\phi}(n) = \phi(1) + \phi(2) + ...\phi(n)$$

$$= 1 \sum_{d|1} \frac{\mu(d)}{d} + 2 \sum_{d|2} \frac{\mu(d)}{d} + ...+ n \sum_{d|n} \frac{\mu(d)}{d}$$

The terms in the right hand sides can be rearranged as

$$\bar{\phi}(n) = a_1 \mu(1) + a_2 \mu(2) + ... + a_n \mu(n) \quad \text{--------- (1)}$$

$$= \sum_{k=1}^{n} a(k) \mu(k)$$

We see that appears once in the following terms of (1),

$$k \sum_{d|k} \frac{\mu(d)}{d}, \; 2k \sum_{d|2k} \frac{\mu(d)}{d}, ..., \left[\frac{n}{k} \right]^k \sum_{d|\left[\frac{n}{k}\right]^k} \frac{\mu(d)}{d}$$

Therefore, $a_k = k.\frac{1}{k} + 2k.\frac{1}{k} + ...+ \left[\frac{n}{k} \right] x. \frac{1}{x}$

$$= 1 + 2 + ...+ \left[\frac{n}{k} \right]$$

$$= \frac{1}{2} \left[\frac{n}{k} \right] \left(\left[\frac{n}{k} \right] + 1 \right)$$

$$= \frac{1}{2}\left[\left[\frac{n}{k}\right]^2 - \left[\frac{n}{k}\right]\right]$$

Thus, $\bar{\phi}(n) = \frac{1}{2}\sum_{k=1}^{n} \mu(k)\left[\left[\frac{n}{k}\right]^2 + \left[\frac{n}{k}\right]\right]$

Theorem 8.10.2 For $n \geq 1$, $\bar{\phi}(n) = \frac{3n^2}{\pi^2} + 0$ (nlogn)

Proof : From Theorem 8.10.1, for $n \geq 1$,

$$\bar{\phi}(n) = \frac{1}{2}\sum_{k=1}^{n} \mu(k)\left[\left[\frac{n}{k}\right]^2 + \left[\frac{n}{k}\right]\right] \qquad \text{---------- (1)}$$

We have $\left[\frac{n}{k}\right] = \frac{n}{k} + 0(1)$ ---------- (2)

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} = \frac{6}{\pi^2} + 0\left(\frac{1}{n}\right)$$

$$\sum_{k=1}^{n} \frac{1}{k} = 0(\log n) \qquad \text{---------- (4)}$$

From (1) and (2),

$$\bar{\phi}(n) = \frac{1}{2}\sum_{k=1}^{n} \mu(k)\left[\left(\frac{n}{k} + 0(1)\right)^2 + \left(\frac{n}{k} + 0(1)\right)\right]$$

$$= \frac{1}{2}\sum_{k=1}^{n} \mu(k)\left[\frac{n^2}{k^2} + 2\frac{n}{k}0(1) + 0(1)^2 + \frac{n}{x} + 0(1)\right]$$

$$= \frac{1}{2}\sum_{k=1}^{n} \mu(k)\left[\frac{n^2}{k^2} + 0\left(\frac{n}{x}\right) + 0(1)\right]$$

$$= \frac{1}{2}\sum_{k=1}^{n} \mu(k)\frac{n^2}{k^2} + \frac{1}{2}\sum_{k=1}^{n} \mu(k)\,0\left(\frac{n}{x}\right) + \frac{1}{2}\sum_{k=1}^{n} \mu(k)\,0(1)$$

$$= \frac{n^2}{2}\sum_{k=1}^{n} \frac{\mu(k)}{k^2} + 0\left(n\sum_{k=1}^{n} \frac{1}{k}\right) + 0\,(n)$$

$$= \frac{n^2}{2}\left[\frac{6}{\pi^2} + 0\left(\frac{1}{n}\right)\right] + n\,(n\log n) + 0(n) \qquad \text{[using (3) and (4)]}$$

$$= \frac{3n^2}{\pi^2} + 0(n) + 0(n\log n) + 0(n)$$

$$= \frac{3n^2}{\pi^2} + 0(n\log n) + 0(n) \qquad \text{[Combining the error terms]}$$

W CHECK YOUR PROGRESS

Q.5. Find $\bar{\phi}(8)$ and $\bar{\phi}(12)$ using Theorem 8.10.1

28.11 LET US SUM UP

* The function $T:\mathbb{N}\to\mathbb{N}$ is defined as

$$T(n) = \sum_{k=1}^{n} \zeta(k)$$

$$= \zeta(1) + \zeta(2) + \ldots\ldots + \zeta(n)$$

where $\zeta(n)$ = the number of positive divisors of n. This function is known as summatory function of function.

* If N is a positive integer, then

$$T(n) = \sum_{n=1}^{N} \zeta(n) = \sum_{n=1}^{N} \left[\frac{N}{n}\right]$$

* The function $S:\mathbb{N}\to\mathbb{N}$ is defined as

$$S(n) = \sigma(1) + \sigma(2) + \ldots\ldots + \sigma(n)$$

$$= \sum_{k=1}^{n} \sigma(k)$$

Where $\sigma(n)$ = sum of positive divisor of n.

This function is known as summatory function of σ - function.

* If N is a positive integer, then

$$S(N)=\sum_{n=1}^{N}\sigma(n)=\sum_{n=1}^{N} n\left[\frac{N}{n}\right]$$

* The Riemann zeta-function, $\zeta(s)$ is defined as

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad s>1$$

The Euler factorization of the ζ-function is

$$\zeta(s)=\prod_{P}\frac{1}{1-p^{-s}}$$

where the product is over all the primes.

* For $\mathrm{Re}(s)>1$, $\zeta(s)=\left(1-\frac{2}{2^s}\right)^{-1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$

* The function $\phi:\mathbb{N}\to\mathbb{N}$ is defined as

$$\overline{\phi}(n)=\phi(1)+\phi(2)+.....+\phi(n)$$

$$=\sum_{k=1}^{n}\phi(k)$$

where $\phi(n)=$ number of positive integers less than and relatively prime to n and $\phi(1)=1$. The function $\overline{\phi}(n)$ is called the summatory function of Euler's ϕ-function.

* For $n\geq 1$ $\quad\overline{\phi}(n)=\frac{1}{2}\sum_{k=1}^{n}\mu(k)\left[\left[\frac{n}{k}\right]^2+\left[\frac{n}{k}\right]\right]$

* For $n\geq 1$ $\quad\overline{\phi}(n)=\frac{3n^2}{\pi^2}+0\quad(n\log n)$

K 8.12 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q.1

$$T(12)=\sum_{n=1}^{12}\left[\frac{12}{n}\right]$$

$$=\left[\frac{12}{1}\right]+\left[\frac{12}{2}\right]+\left[\frac{12}{3}\right]+\left[\frac{12}{4}\right]+\left[\frac{12}{5}\right]+\left[\frac{12}{6}\right]+\left[\frac{12}{7}\right]+\left[\frac{12}{8}\right]+\left[\frac{12}{9}\right]+\left[\frac{12}{10}\right]+\left[\frac{12}{11}\right]+\left[\frac{12}{12}\right]$$

$$= 12 + 6 + 4 + 3 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1$$

$$= 35$$

Alternative method :

$$T(12)=\varsigma(1)+\varsigma(2)+\varsigma(3)+\varsigma(4)+\varsigma(5)+\varsigma(6)+\varsigma(7)+\varsigma(8)+\varsigma(9)+\varsigma(10)+\varsigma(11)+\varsigma(12)$$

$$= 1 + 2 + 2 + 3 + 2 + 4 + 2 + 4 + 3 + 4 + 2 + 6$$

$$= 35$$

$$T(15)=\sum_{n=1}^{15}\left[\frac{15}{n}\right]$$

$$=\left[\frac{15}{1}\right]+\left[\frac{15}{2}\right]+\left[\frac{15}{3}\right]+\left[\frac{15}{4}\right]+\left[\frac{15}{5}\right]+\left[\frac{15}{6}\right]+\left[\frac{15}{7}\right]+\left[\frac{15}{8}\right]$$

$$+\left[\frac{15}{9}\right]+\left[\frac{15}{10}\right]+\left[\frac{15}{11}\right]+\left[\frac{15}{12}\right]+\left[\frac{15}{13}\right]+\left[\frac{15}{14}\right]+\left[\frac{15}{15}\right]$$

$$= 15 + 7 + 5 + 3 + 3 + 2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$$

$$= 44$$

Ans. to Q.2

$$S(10)=\sum_{n=1}^{10} n\left[\frac{10}{n}\right]$$

$$=1.\left[\frac{10}{1}\right]+2.\left[\frac{10}{2}\right]+3.\left[\frac{10}{3}\right]+4.\left[\frac{10}{4}\right]+5.\left[\frac{10}{5}\right]+6.\left[\frac{10}{6}\right]$$

$$+7.\left[\frac{10}{7}\right]+8.\left[\frac{10}{8}\right]+9.\left[\frac{10}{9}\right]+10.\left[\frac{10}{10}\right]$$

$$= 1.10 + 2.5 + 3.3 + 4.2 + 5.2 + 6.1 + 7.1 + 8.1 + 9.1 + 10.1$$

$$= 10 + 10 + 9 + 8 + 10 + 6 + 7 + 8 + 9 + 10$$

$$= 87$$

Alternative method :

$$S(10) = \sigma(1) + \sigma(2) + \sigma(3) + \sigma(4) + \sigma(5) + \sigma(6) + \sigma(7) + \sigma(8) + \sigma(9) + \sigma(10)$$

$$= 1 + (1 + 2) + (1 + 3) + (1 + 2 + 4) + (1 + 5) + (1 + 2 + 3 + 4) + (1 + 7) + (1 + 2 + 4 + 8)$$

$$+ (1 + 3 + 9) + (1 + 2 + 5 + 10)$$

$$= 87$$

$$S(12) = \sigma(1) + \sigma(2) + \sigma(3) + \sigma(4) + \sigma(5) + \sigma(6) + \sigma(7) + \sigma(8) + \sigma(9) + \sigma(10) + \sigma(11) + \sigma(12)$$

$$= 1 + (1 + 2) + (1 + 3) + (1 + 2 + 4) + (1 + 5) + (1 + 2 + 3 + 6) + (1 + 7) + (1 + 2 + 4 + 8)$$

$$+ (1 + 3 + 9) + (1 + 2 + 5 + 10) + (1 + 11) + (1 + 2 + 3 + 4 + 6 + 12)$$

$$= 127$$

Ans. to Q. 3

We have

$$\sin z = z \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2 \pi^2}\right) \qquad \text{--------- (1)}$$

By taking logarithm in (1), we obtain

$$\log \sin z = \log z + \sum_{n \geq 1} \log \left(1 - \frac{z^2}{n^2 \pi^2}\right)$$

and differentiating term by term, we get

$$\cot z = \frac{1}{2} - \sum_{n \geq 1} \frac{2z}{n^2 \pi^2} \left(1 - \frac{z^2}{n^2 \pi^2}\right)^{-1} \qquad \text{---------- (2)}$$

If we use the geometric series to write,

$$\frac{2z}{n^2\pi^2} \left(1 - \frac{z^2}{n^2\pi^2}\right)^{-1} = \frac{2z}{n^2\pi^2} \sum_{k \geq 0} \left(\frac{z^2}{n^2\pi^2}\right)^k$$

$$= 2 \sum_{k \geq 0} \frac{z^{2k+1}}{n^{2k+2}\, \pi^{2k+2}}$$

$$= 2 \sum_{k \geq 1} \frac{z^{2k-1}}{n^{2k}\, \pi^{2k}} \qquad \text{-------- (3)}$$

From (2) and (3), we get

$$\cot z = \frac{1}{z} - 2 \sum_{k \geq 1} \sum_{n \geq 1} \frac{z^{2k-1}}{n^{2k}\, \pi^{2k}}$$

$$= \frac{1}{z} - 2 \sum_{k \geq 1} \frac{\zeta(2k) z^{k-1}}{\pi^{2k}} \qquad \text{-------- (4)}$$

which is the Laurent series for cot z.

We have

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + ...$$

$$\Rightarrow \frac{e^t - 1}{t} = 1 + \frac{t}{2!} + \frac{t^2}{3!} + ...$$

and the reciprocal of this has a Tayloy series expansion which can be written in the form

$$\frac{e^t - 1}{t} = \left(1 + \frac{t}{2!} + \frac{t^2}{3!} + ...\right)^{-1}$$

$$= \sum_{m \geq 0} \frac{B_m}{m!}\, t^m \qquad \text{---------- (5)}$$

for certain constants B_0, B_1 …….. known as the Bernoulli numbers. Now

$$\frac{e^t - 1}{t} = \frac{t}{2}\left(\frac{e^t + 1}{e^t - 1} - 1\right)$$

$$= \frac{t}{2}\left(\frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}} - 1\right)$$

$$= \frac{t}{2}\left(\coth\frac{t}{2} - 1\right)$$

$$= \frac{t}{2}\left(i\cot\frac{it}{2} - 1\right)$$

where $i = \sqrt{-1}$. Putting $z = \frac{it}{2}$, we get

$$\frac{e^t - 1}{t} = z\cot z - \frac{z}{i}$$

$$= z\cot z + iz$$

Dividing by z, we get

$$\frac{1}{z}\left(\frac{e^t - 1}{t}\right) = \cot z + i$$

$$\Rightarrow \cot z = -i + \frac{1}{2}\left(\frac{e^t - 1}{t}\right)$$

$$= -i + \frac{1}{2}\sum_{m\geq 0}\frac{Bm}{m!}t^m$$

$$= -i + \frac{1}{2}\sum_{m\geq 0}\frac{Bm}{m!}\left(\frac{2}{i}z\right)^m$$

$$= -i + \sum_{m\geq 0}\frac{Bm}{m!}\left(\frac{2}{i}\right)z^{m-1}$$

By compairing coefficients with those in (4), we see that if $m = 2k \geq 2$, then

$$-2\,\frac{\zeta(2k)}{\pi^{2k}} = \frac{B_{2k}}{(2k)!}\left(\frac{2}{i}\right)^{2k}$$

So that $$\zeta(2k) = \frac{(-1)^{k-1}\,2^{2k-1}\,\pi^{2k}\,B_{2k}}{(2k)!}$$

Ans. to Q.4.

Let $f(x)=\frac{1}{x}$, $x>0$ be a decreaing function. Then by the property of definite integral, we have

$$\frac{1}{m+1} < \int_{m}^{m+1} \frac{dx}{x} < \frac{1}{m},\ m>0$$

This gives $$0 < \frac{1}{m} - \int_{m}^{m+1} \frac{dx}{x} < \frac{1}{m} - \frac{1}{m+1} \qquad \text{--------- (1)}$$

Let $$F(m) = \frac{1}{m} - \int_{m}^{m+1} \frac{dx}{x} < \frac{1}{m} - \frac{1}{m+1} \qquad \text{---------- (2)}$$

Then (1) can be written as

$$0 < F(m) < \frac{1}{m} - \frac{1}{m+1} \qquad \text{----------(3)}$$

Let k be a fixed integer such that r > k. Putting m = k, k + 1, k + 2, ……, r in (3), we get

$$0 < F(k) < \frac{1}{k} - \frac{1}{k+1}$$

$$0 < F(k+1) < \frac{1}{k+1} - \frac{1}{k+2}$$

……..

……..

$$0 < F(r) < \frac{1}{r} - \frac{1}{r+1}$$

Adding above inequalities column-wise, we get

$$0 < \sum_{m=k}^{r} F(m) < \frac{1}{k} - \frac{1}{r+1} \qquad \text{---------- (4)}$$

Therefore the sum function $\sum_{m=k}^{r} F(m)$ is an increasing functions of r and it bounded above by $\frac{1}{k}$. Therefore the sum function tends to a finite limit as $r \to \infty$. Thus

$$\sum_{m=k}^{\infty} F(m) = \text{ a finite number } \le \frac{1}{k} \qquad \text{---------- (5)}$$

Let $\upsilon = \sum_{m=k}^{\infty} F(m)$ be a constant. Then

$$\upsilon = \sum_{m=k}^{\infty} F(m) + \sum_{m=n+1}^{\infty} F(m)$$

Since $\sum_{m=k}^{r} F(m) = \sum_{m=k}^{r} \left(\frac{1}{m} - \int_{m}^{m+1} F(m) \right)$

$$= \left(\frac{1}{k} - \int_{k}^{k+1} \frac{dx}{x} \right) + \left(\frac{1}{k+1} - \int_{k+1}^{k+2} \frac{dx}{x} \right) + \ldots\ldots + \left(\frac{1}{r} - \int_{r}^{r+1} \frac{dx}{x} \right)$$

$$= \sum_{m=k}^{r} \frac{1}{m} - \int_{k}^{r+1} \frac{dx}{x}, \text{ therefore}$$

$$\upsilon = \sum_{m=1}^{n} \frac{1}{m} - \int_{l}^{n+1} \frac{dx}{x} + \sum_{m=n+1}^{\infty} F(m)$$

$$= \sum_{m=1}^{n} \frac{1}{m} - \int_{l}^{n} \frac{dx}{x} - \int_{n}^{n+1} \frac{dx}{x} + \sum_{m=n+1}^{\infty} F(m)$$

$$= \sum_{m=1}^{n} \frac{1}{m} - \log n - \int_{n}^{n+1} \frac{dx}{x} - \sum_{m=n+1}^{\infty} F(m) \qquad \text{-------- (6)}$$

But $\frac{1}{n+1} < \int_{n}^{n+1} \frac{dx}{x} < \frac{1}{n}$ and From (5),

$$0 < \sum_{m=n+1}^{\infty} F(m) \le \frac{1}{n}$$

Consequently, $0 < \int_{n}^{n+1} \frac{dx}{x} - \sum_{m=n+1}^{\infty} F(m) \le \frac{1}{n}$

This gives $\int_{n}^{n+1} \frac{dx}{x} - \sum_{m=n+1}^{\infty} F(m) = \frac{\theta}{n}$

for some $\theta,\ 0 < \theta < 1$

From (6) and (7), we get

$$\upsilon = \sum_{m=1}^{n} \frac{1}{m} - \log n - \frac{\theta}{n}$$

$$\Rightarrow \sum_{m=1}^{n} \frac{1}{m} = r + \log n + \frac{\theta}{n}$$

Notes : (i) When $n \to \infty$, then

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}\right) = \upsilon + \lim_{n\to\infty} \log n + 0$$

$$\Rightarrow r = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}-\log n\right)$$

$$\approx 0.5772156649$$

This value of r is known as Euler's Constant.

(ii) when $n \to \infty$ then

$\sum_{m=1}^{\infty} \frac{1}{m}$ diverges. Therefore $\zeta(1)$ does not exist.

Ans. to Q.5

Let $p_1, p_2, \ldots\ldots\ldots p_k$ be the first k primes and

Let $p_k = \prod_{i=1}^{k} \left\{1 + f(p_i) + f\left(p_i^2\right) + ...\right\}$

The general term in the expansion of p_k is $f\left(p_1^{e1}\right).\ f\left(p_2^{e2} .....p_k^{ek}\right) = f\left(p_1^{e1} . p_2^{e2} .....p_k^{ek}\right)$ because f is multiplicative. Thus

$$P_k = \sum_{n\in A_k} f(n)$$

where $A_k = \left\{n \mid n = p_1^{e1}\ p_2^{e2} .....p_k^{ek},\ e_i \geq 0\right\}$. We have

$$\left| P_k - \sum_{n=1}^{\infty} f(n) \right| = \left| \sum_{n \notin A_k} f(n) \right|$$

$$\leq \sum_{n \notin A_k} f(n)$$

$$\leq \sum_{n > P_k} |f(n)|$$

Since $n > P_k$ for each $n \in A_k$. Now $\sum_{n=1}^{\infty} |f(n)|$ converges, so as $K \to \infty$ we have $\sum_{n > P_k} |f(n)| \to 0$ and hence $\left| P_k - \sum_{n=1}^{\infty} f(n) \right| \to 0$ thus $P_k \to \sum_{n=1}^{\infty} f(n)$ as $K \to \infty$

Hence $\sum_{n=1}^{\infty} f(n) = \prod_p \left\{ 1 + f(p) + f(p^2) + ... \right\}$

Second part : Since $\mu(n)$ and $\frac{1}{n^s}$ both are multiplicative function, their product is also multiplication.

Also $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ is absolutely convergent. Therefore

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left[1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + ... \right]$$

$$= \prod_p \left(1 - \frac{1}{p^s} \right) \quad \left[\because \mu(p) = -1, \mu(p^2) = \mu(p^3) = ... = 0 \right]$$

$$= \frac{1}{\zeta(s)} \quad \left[\because \zeta(s) = \prod_p \left(1 - \frac{1}{p^s} \right)^{-1} \right]$$

Ans. to Q. 6

$$\overline{\phi}(8) = \frac{1}{2} \sum_{k=1}^{8} \mu(k) \left\{ \left[\frac{8}{k} \right]^2 + \left[\frac{8}{k} \right] \right\}$$

$$= \frac{1}{2}\mu(1)\left\{\left[\frac{8}{1}\right]^2 + \left[\frac{8}{1}\right]\right\} + \frac{1}{2}\mu(2)\left\{\left[\frac{8}{2}\right]^2 + \left[\frac{8}{2}\right]\right\} + \frac{1}{2}\mu(3)\left\{\left[\frac{8}{3}\right]^2 + \left[\frac{8}{3}\right]\right\}$$

$$+ \frac{1}{2}\mu(4)\left\{\left[\frac{8}{4}\right]^2 + \left[\frac{8}{4}\right]\right\} + \frac{1}{2}\mu(5)\left\{\left[\frac{8}{5}\right]^2 + \left[\frac{8}{5}\right]\right\} + \frac{1}{2}\mu(6)\left\{\left[\frac{8}{6}\right]^2 + \left[\frac{8}{6}\right]\right\}$$

$$+ \frac{1}{2}\mu(7)\left\{\left[\frac{8}{7}\right]^2 + \left[\frac{8}{7}\right]\right\} + \frac{1}{2}\mu(8)\left\{\left[\frac{8}{8}\right]^2 + \left[\frac{8}{5}\right]\right\}$$

$$= \frac{1}{2}\times 1 \times (64+8) + \frac{1}{2}\times(-1)\times(16+4) + \frac{1}{2}\times(-1)\times(4+2) + 0 +$$

$$\frac{1}{2}\times(-1)\times(1^2+1) + \frac{1}{2}\times 1\times(1^2+1) + \frac{1}{2}\times(-1)\times(1^2+1) + 0$$

$[\because \mu(4)=\mu(8)=0,\ \mu(2)=\mu(3)=\mu(5)=\mu(7)=-1 \text{ and } \mu(6)=1]$

$$= 36 - 10 - 3 - 1 + 1 - 1$$

$$= 22$$

Verification :

$$\overline{\phi}(8) = \phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(5) + \phi(6) + \phi(7) + \phi(8)$$

$$= 1 + 1 + 2 + 2 + 4 + 2 + 6 + 4$$

$$= 22$$

$$\phi(12) = \frac{1}{2}\sum_{k=1}^{12}\mu(k)\left\{\left[\frac{12}{k}\right]^2 + \left[\frac{12}{k}\right]\right\}$$

$$= \frac{1}{2}\mu(1)\left\{\left[\frac{12}{1}\right]^2 + \left[\frac{12}{1}\right]\right\} + \frac{1}{2}\mu(2)\left\{\left[\frac{12}{2}\right]^2 + \left[\frac{12}{2}\right]\right\} + \ldots\ldots +$$

$$\frac{1}{2}\mu(12)\left\{\left[\frac{12}{12}\right]^2 + \left[\frac{12}{12}\right]\right\}$$

$$= \frac{1}{2}\times 1\times(144+12)+\frac{1}{2}\times(-1)\times(36+6)+\frac{1}{2}\times(-1)\times(16+4)+$$

$$\frac{1}{2}\times 0\times(9+3)+\frac{1}{2}\times(-1)(4+2)+\frac{1}{2}\times 1\times(4+2)+\frac{1}{2}\times(-1)\times(1^2+1)$$

$$\frac{1}{2}\times 0\times(1^2+1)+\frac{1}{2}\times 0\times(1^2+1)+\frac{1}{2}\times 1\times(1^2+1)+\frac{1}{2}\times(-1)\times(1^2+1)+\frac{1}{2}\times 0\ (1^2+1)$$

$$= 78 — 21 — 10 — 3 + 3 — 1 + 1 — 1$$

$$= 46$$

O 8.13 FURTHER READING

1. H. Kishan, Number Theory, Krishan Prakashan Media (P) Limited, Meerut – 250001 (UP), India, 2017.
2. David M. Burton, Elementary Number Theory (7th Edn.) Mc Graw Hill Education (India) Private Limited, Chennai – 60016, 2018.
3. Gareth A. Jones and J. Mary Jones, Elementary Number Theory, Springer Undergraduate Mathematics Series, Great Birtain, 2005.
4. Ramanujachary Kumanduri and Cristina Romeor, Number Theory wirh Computer Application, Prentice Hall, Upper Saddle River, New Jersery – 07458.

M 8.14 MODEL QUESTIONS

1. Find T (n) for n = 7, 8 and 10
2. Find S (n) for n = 10, 15, 20.
3. Find $\overline{\phi}(n)$ for n = 3, 6, 11.
4. If f is completely multiplication and $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent, then prove that

$$\sum_{n=1}^{\infty} f(n) = \prod_{P}\left[\frac{1}{1-f(p)}\right], \quad \text{where p range over all the primes.}$$

5. Show that $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=1+\frac{1}{2^s}+\frac{1}{3^s}+\ldots\ldots$ converges for all real s > 1 and diverges for all $s \le 1$.

6. Using Riemann Zeta-function, show that there are infinitely many primes.

7. If Pn devotes the n^{th} prime (in increasing order), then the series

$$\sum_{n=1}^{\infty}\frac{1}{p_n}=\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots\ldots \text{ diverges.}$$

8. Evaluat ζ (2), ζ (4) and ζ (6).

9. Suppose that converges absolutely. If f is multiplicative, then

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\prod_{P}\left\{1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\ldots\ldots\right\}$$ and if f is completely multiplicative, then

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\Pi\left\{\frac{1}{1-f(p)p^{-s}}\right\}.$$

*** *** ***