UNIT 10: Numerical solution of Ordinary Differential Equations

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10.1 LEARNING OBJECTIVES

- depending on the type and complexity of the differential equation. Understand the need for numerical methods in solving ordinary differential equations (ODEs), especially when analytical solutions are not available.
- Learn various techniques for solving initial and boundary value problems numerically.
- Acquire proficiency in applying methods such as Picard's, Taylor's Series, Euler's, Modified Euler's, Runge, Runge-Kutta, and Predictor-Corrector methods.
- Compare the accuracy and efficiency of different approaches

10.2 INTRODUCTION

A number of problems in science and technology can be formulated into differential equations. The analytical methods of solving differential equations are applicable only to a limited class of equations. Quite often differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realise that computing machines are now readily available which reduce numerical work considerably.

Solution of a differential equation: The solution of an ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x. Such a solution of a differential equation is known as the *closed or finite form of solution*. In the absence of such a solution, we have recourse to numerical methods of solution.

Let us consider the first order differential equation

$dy/dx = f(x, y)$ given $y(x_0 = y_0)$ (1)

to study the various numerical methods of solving such equations. In most of these methods, we replace the differential equation by a difference equation and then solve it. These methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or a set of values of x and y. The methods of Picard and Taylor series belong to the former class of solutions. In these methods, y in (1) is approximated by a truncated series, each term of which is a function of x. The information about the curve at one point is stilized and the solution is not iterated. As such, these are referred to as single-step methods.

The methods of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class of solutions. In these methods, the next point on the curve is evaluated in short steps ahead, by performing iterations till sufficient accuracy is achieved. As such, these methods are called **step-by-step methods.**

Euler and Runga-Kutta methods are used for computing y over a limited range of x values whereas Milne and Adams methods may be applied for finding y over a wider range of x-values. Therefore Milne and Adams methods require starting values which are found by Picard's Taylor series or Runge-Kutta methods.

Initial and boundary conditions: An ordinary differential equation of the nth order is of the form

$$F(x, y, dy/dx\,, d^2y/dx^2\,, \dots\dots, d^ny/dx^n) = 0 \qquad (2)$$

Its general solution contains a arbitrary constants and is of the form

$$\phi(x, y, c_1, c_2, \dots\dots, c_n) \qquad (3)$$

To obtain its particular solution, n conditions must be given so that the constants $c_1, c_2, \dots\dots, c_n$ can be determined.

If these conditions are prescribed at one point only (say: x_0), then the differential equation together with the conditions constitute an **initial value problem** of the nth order.

If the conditions are prescribed at two or more points, then the problem is termed as **boundary value problem**.

In this chapter, we shall first describe methods for solving initial value problems and then explain **finite difference method** and **shooting method** for solving boundary value problems

10.3 PICARD'S METHOD

Consider the first order equation $\frac{dy}{dx} = f(x, y)$

(1)

It is required to find that particular solution of (1) which assumes the value y_0 when $x = x_0$ Integrating (1) between limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx \text{ or } y = y_0 + \int_{x_0}^{x} f(x, y)\, dx$$

(2)

This is an integral equation equivalent to (1), for it contains the unknown y under the integral sign.

As a first approximation $y_1$to the solution we put $y = y_0$ in $f(x, y)$ and integrate (2) giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (2) giving

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

Similarly, a third approximation is

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

Continuing this process, we obtain $y_4, y_5, \ldots \ldots y_n$where

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

Hence this method gives a sequence of approximations $y_1, y_2, y_3, \ldots \ldots$ each giving a better result than the preceding one.

Example 1: Using Picard's process of successive approximations, obtain a solution upto the fifth approximation of the equation $dy/dx = y + x$ such that $y = 1$ when $x = 0$. Check your answer by finding the exact particular solution.

Solution: (i) We have $y = 1 + \int_0^x (y + x)\, dx$

First approximation, putting $y = 1$ in $y + x$ we get

$$y_1 = 1 + \int_0^x (1 + x)\, dx = 1 + x + x^2/2$$

Second approximation, putting $y = 1 + x + x^2/2$ in $y + x$ we get

$$y_2 = 1 + \int_0^x (1 + 2x + x^2/2)\, dx = 1 + x + x^2 + x^3/6.$$

Third approximation, putting $y = 1 + x + x^2 + x^3/6$ in $y + x$ we get

$$y_3 = 1 + \int_0^x (1 + 2x + x^2 + x^3/6)\, dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Fourth approximation, putting $y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$ in $y + x$ we get

$$y_4 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24})\, dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

Fifth approximation, putting $y = y_4$ in $y + x$ we get

$$y_5 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120})\, dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}. \quad (1)$$

(ii) Given equation:

$\frac{dy}{dx} - y = x$ is a Leibnitzs linear in x.

Its I.F. being e^{-x}, the solution is

$$ye^{-x} = \int xe^{-x}\, dx + c = -xe^{-x} - \int (-e^{-x})\, dx + c = -xe^{-x} - e^{-x} + c$$

$$\therefore \quad y = ce^x - x - 1.$$

Since $y = 1$, when $x = 0$, $\therefore c = 2$.

Thus the desired particular solution is

$$y = 2e^x - x - 1 \quad (2)$$

Or using the series: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \infty$

We get $$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} + \cdots \infty \quad (3)$$

Comparing (1) and (3), it is clear that (1), approximates to the exact particular solution (3) upto the term in x^5.

Example 2: Find the value of y for $x = 0.1$ by Picard's method, given that $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$

Solution: We have $y = 1 + \int_0^x \frac{y-x}{y+x} dx$

First approximation, putting $y = 1$ in the integrand, we get

$$y_1 = 1 + \int_0^x \frac{1-x}{1+x} dx = 1 + \int_0^x \left(-1 + \frac{2}{1+x}\right) dx$$

$$= 1 + [-x + 2\log(1+x)]_0^x = 1 - x + 2\log(1+x) \quad (1)$$

Second approximation, putting $y = 1 - x + 2\log(1+x)$ in in the integrand we get

$$y_2 = 1 + \int_0^x \frac{1-x+2\log(1+x)-x}{1-x+2\log(1+x)+x} dx$$

$$= 1 + \int_0^x \left(1 - \frac{2x}{1+2\log\,(1+x)}\right) dx$$ which is very difficult to integrate.

Hence we use the first approximation and taking $x = 0.1$ in (1) we get

$$y(0.1) = 1 - 0.1 + 2\log 1.1 = .9828$$

10.4 TAYLOR'S SERIES METHOD

Consider the first order equation $\frac{dy}{dx} = f(x, y)$ (1)

Differentiating (1) we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \; i.e. \; y'' = f_x + f_y f' \quad (2)$$

Differentiating this successively, we can get y''', y^{iv} etc. Putting $x = x_0$ and $y = 0$, the values of $(y')_0$, $(y'')_0$, $(y''')_0$ can be obtained. Hence the Taylor's series

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \cdots \quad (3)$$

Gives the values of y for every value of x for which (3) converges.

On finding the value y_1, for $x = x_1$, from (3), y', y" etc. can be evaluated at $x = x_1$, by means of (1), (2) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3).

Example 3. Solve $y' = x + y, y(0) = 1$ by Taylor's series method. Hence find the values of y at $x = 0.1$ and $x = 0.2$.

Solution: Differentiating successively, we get

$$y' = x + y \qquad y'(0) = 1 \qquad [y(0) = 1]$$
$$y'' = 1 + y' \qquad y''(0) = 2$$
$$y''' = y'' \qquad y'''(0) = 2$$
$$y^{iv} = y''' \qquad y^{iv}(0) = 2 \text{ etc.}$$

Taylor's series is

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \cdots$$

Here $x_0 = 0$, $y_0 = 1$

$$\therefore \qquad y = 1 + x(1) + \frac{x^2}{2}(2) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(4) + \cdots$$

Thus $y(0.1) = 1.1103$ and $y(0.2) = 1.2427$

Example 4. Find by Taylor's series method, the values of y at $x = 0.1$ and $x = 0.2$ to five places of decimals from $dy/dx = x^2y - 1, y(0) = 1$.

Solution: Differentiating successively, we get

$$y' = x^2y - 1, (y')^0 = -1 \qquad [\because y(0) = 1]$$
$$y'' = 2xy + x^2y', (y'')_0 = 0$$
$$y''' = 2y + 4xy' + x^2y'', (y''')_0 = 2$$
$$y^{iv} = 6y' + 6xy'' + x^2y''', \left(y^{iv}\right)^0 = -6 \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y = 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \ldots = 1 - x + (x^3/3) - (x^4/4) + \ldots$$

Hence $y(0.1) = 0.90003$ and $y(0.2) = 0.810227$.

Example 5. Employ Taylor's method to obtain approximate value of y at $x = 0.2$ for the differential equation $\frac{dy}{dx} = 2y + 3e$, $y(0) = 0$. Compare the numerical solution obtained with the exact solution.

Solution: (a) We have $y' = 2y + 3e$; $y(0) = 2y(0) + 3e^0 = 3$.

Differentiating successively and substituting $x = 0, y = 0$ we get

$$y'' = 2y' + 3e, \qquad (y'')_0 = 2y'(0) + 3 = 9$$

$$y''' = 2y'' + 3e, \qquad (y''')_0 = 2y''(0) + 3 = 21$$

$$y^{iv} = 2y''' + 3e, \qquad (y^{iv})_0 = 2y'''(0) + 3 = 45 \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + x\,y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(4)}(0) + \ldots$$

$$= 0 + 3x + (x^2/2) \times 9 + (x^3/6) \times 21 + (x^4/24) \times 45 + \ldots$$

$$= 3x + (9/2)x^2 + (7/2)x^3 + (15/8)x^4 + \ldots$$

Hence $y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.2)^4 + \ldots = 0.8110$ (i)

(b) Now $\frac{dy}{dx} - 2y = 3e$ is a Leibnitz's linear in x.

Its I.F. being e^{-2}, the solution is

$$y\,e^{-2} = \int 3\,e\,.e^{-2}\,dx + c = -3e^{-} + c \text{ or } y = -3e + ce^{2x}$$

Since $y = 0$ when $x = 0, \therefore c = 3$.

Thus the exact solution is $y = 3(e^{2x} - e)$

When $x = 0.2, y = 3(e^{04} - e^{02}) = 0.8112$ (ii)

Comparing (i) and (ii), it is clear that (i) approximates to the exact value upto 3 decimal places.

CHECK YOUR PROGRESS

Q1. Solve by Taylor series method of third order the equation $\frac{dy}{dx} = \frac{(x^3 + xy^2)}{e^x}$, $y(0) = 1$ for y at $x = 0.1$, $x = 0.2$ and $x = 0.3$.

Q2. Solve by Taylor's series method the equation $\frac{dy}{dx} = log(xy)$ for $y(1.1)$ and $y(1.2)$, given $y(1) = 2$.

10.5 EULER'S METHOD

Consider the equation $\frac{dy}{dx} = f(x, y)$ (1)

given that $y(x_0) = y_0$. Its curve of solution through P(x□, y□) is shown dotted in Fig. 10.1. Now we have to find the ordinate of any other point Q on this curve.

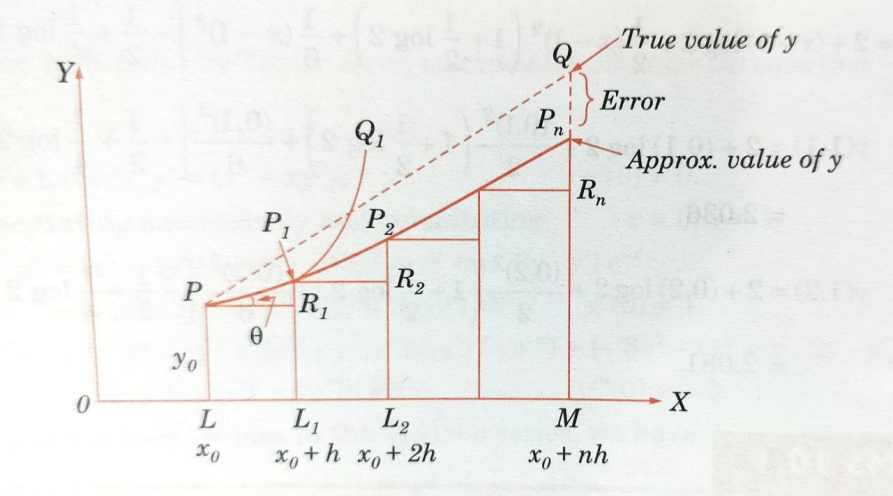


Fig. 10.1

Let us divide LM into n sub-intervals each of width h at $L_1, L_2, \ldots$, so that h is quite small.

In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$, then

$$y_1 = L_1P_1 = LP + P_1R_1 = y_0 + P_1R_1 \tan\theta$$

$$= y_0 + h\,(dy/dx)_P = y_0 + h\,f(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$. Then $y_2 = y_1 + h\,f(x_0 + h, y_1)$

Repeating these process n times, we finally reach on an approximate MP of MQ given by

$$y = y_{-1} + h\,f(x_0 + \overline{n-1}\,h, y_{-1})$$

This is Euler's method of finding an approximate solution of (1).

Example 6: Using Euler's method, find an approximate value of y corresponding to $x = 1$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution: We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows:

x	y	$dy/dx = x + y$	Old $y + 0.1\left(\frac{dy}{dx}\right) = new\ y$
0	1.00	1.00	1.00 + 0.1(1.00) = 1.10
0.1	1.10	1.20	1.10 + 0.1(1.20) = 1.22
0.2	1.22	1.42	1.22 + 0.1(1.42) = 1.36
0.3	1.36	1.66	1.36 + 0.1(1.66) = 1.53
0.4	1.53	1.93	1.53 + 0.1(1.93) = 1.72
0.5	1.72	2.22	1.72 + 0.1(2.22) = 1.94

0.6	1.94	2.54	1.94 + 0.1(2.54) = 2.19
0.7	2.19	2.89	2.19 + 0.1(2.89) = 2.48
0.8	2.48	3.29	2.48 + 0.1(3.29) = 2.81
0.9	2.81	3.71	2.81 + 0.1(3.71) = 3.18
1.0	3.18		

Thus the required approximate value of $y = 3.18$.

10.6 MODIFIED EULER'S METHOD

In the Euler's method, the curve of solution in the interval LL_1 is approximated by the tangent at P (Fig. 10.1) such that at P_1, we have $y_1 = y_0 + h f(x_0, y_0)$

(1)

Then the slope of the curve of solution through P_1 [i.e. $(dy/dx)P_1 = f(x_0 + h, y_1)$] is computed and the tangent at P_1 to P_1Q is drawn meeting the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$. Now we find a better approximation $y_1^{(1)}$ of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and P_1 i.e. $y_1^{(1)} = y_0 + \left(\frac{h}{2}\right)[f(x_0, y_0) + f(x_0 + h, y_1)]$ (2)

As the slope of the tangent at P_1 is not known, we take y_1 as found in (1) by Euler's method and insert it on R.H.S. of (2) to obtain the first modified value $y_1^{(1)}$.

Again (2) is applied and we find a still better value $y_1^{(2)}$ corresponding to L_1, as

$$y_1^{(2)} = y_0 + \left(\frac{h}{2}\right)[f(x_0, y_0) + f(x_0 + h, y_1(1))]$$

We repeat this step, till two consecutive values of y_1 agree. This is then taken as the starting point for the next interval L_1L_2.

Once y_1 is obtained to the desired degree of accuracy, y corresponding to L_2 is found from (1)

$$y_2 = y_1 + h f(x_0 + h, y_1)$$

and a better approximation $y_1^{(1)}$ is obtained from (2)

$$y_1^{(1)} = y_1 + \left(\frac{h}{2}\right)[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$$

We repeat this step until y_2 becomes stationary. Then we proceed to calculate y_3 as above and so on.

This is the modified Euler's method which gives great improvement in accuracy over the original method.

Example 7: Using modified Euler's method, find an approximate value of y when $x = 0.3$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution: The various calculations are arranged as follows taking h = 0.1:

x	$x + y = y'$	Mean Slope	$Old\ y + 0.1\ (mean\ slope) = new\ y$
0	0+1=1	--	1.00 + 0.1 (1.00) = 1.10
0.1	0.1+1.1=1.2	(1/2) (1 + 1.2)=1.1	1.00 + 0.1 (1.1) = 1.11
0.1	0.1+1.11=1.21	(1/2) (1 + 1.21)=1.105	1.00 + 0.1 (1.105) = 1.1105
0.1	0.1+1.1105=1.2105	(1/2) (1 + 1.2105)=1.1052	1.00 + 0.1 (1.1052) = 1.1105

Since last two values are equal, we take $y(0.1) = 1.1105$

x	$x + y = y'$	Mean Slope	$Old\ y + 0.1\ (mean\ slope) = new\ y$
0.1	1.2105	--	1.1105 + 0.1 (1.2105) = 1.2316
0.2	1.4316	(1/2) (1.2105 + 1.4316)=1.3211	1.1105 + 0.1 (1.3211) = 1.2426
0.2	1.4426	(1/2) (1.2105 + 1.4426)=1.3266	1.1105 + 0.1 (1.3268) = 1.2432
0.2	1.4432	(1/2) (1.2105 + 1.4432)=1.3268	1.1105 + 0.1 (1.3268) = 1.2432

Since last two values are equal, we take $y(0.2) = 1.2432$

x	$x + y = y'$	Mean Slope	$Old\ y + 0.1\ (mean\ slope) = new\ y$
0.2	1.4432	--	1.2432 + 0.1 (1.4432) = 1.3875
0.3	1.6875	(1/2) (1.4432 + 1.6875)=1.5654	1.2432 + 0.1 (1.5654) = 1.3997
0.3	1.6997	(1/2) (1.4432 + 1.6997)=1.5715	1.2432 + 0.1 (1.5715) = 1.4003
0.3	1.7003	(1/2) (1.4432 + 1.7003)=1.5718	1.2432 + 0.1 (1.5718) = 1.4004
0.3	1.7004	(1/2) (1.4432 + 1.7004)=1.5718	1.2432 + 0.1 (1.5718) = 1.4004

Since last two values are equal, we take $y(0.3) = 1.4004$.

Hence $y(0.3) = 1.4004$ approximately.

CHECK YOUR PROGRESS

Q3. Given $\frac{dy}{dx} = \frac{(y-x)}{(y+x)}$ with initial condition $y = 1$ at $x = 0$; find y for $x = 0.1$ by Euler's method.

Q4: Using modified Euler's method, find $y(0.2)$ and $y(0.4)$ given $y' = y + e^x, y(0) = 0$.

10.7 RUNGE'S METHOD

(Called after the German mathematician Carl Runge (1856–1927))

Consider the differential equation, $dy/dx = f(x,y), y(x_0) = y_0$ (1)

Clearly the slope of the curve through $P(x_0, y_0)$ is $f(x_0, y_0)$ (Fig. 10.2).

Integrating both sides of (1) from (x_0, y_0) to $(x_0 + h, y_0 + k)$, we have

$$\int_{y_0}^{y_0+k} dy = \int_{x_0}^{x_0+h} f(x,y)\,dx \quad (2)$$

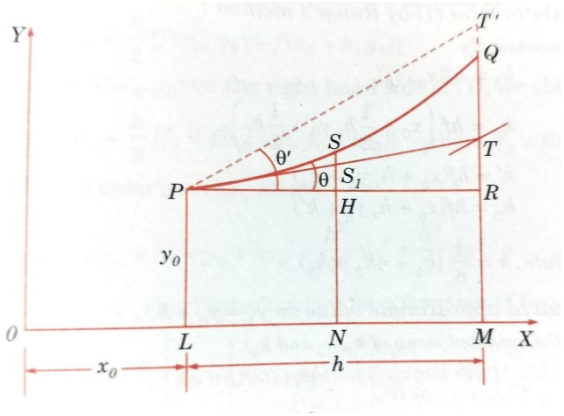


Fig. 10.2

To evaluate the integral on the right, we take N as the mid-point of LM and find the values of $f(x,y)$ (i.e. dy/dx) at the points $x_0, x_0 + h/2, x_0 + h$. For this purpose, we first determine the values of y at these points.

Let the ordinate through N cut the curve PQ in S and the tangent PT in S_1. The value of y_S is given by the point S_1

$$\therefore y_S = NS_1 = LP + HS_1 = y_0 + PH\,.\tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_P = y_0 + \left(\frac{h}{2}\right) f(x_0, y_0) \quad (3)$$

Also $y_T = MT = LP + RT = y_0 + PR\,.\tan\theta = y_0 + h\,f(x_0, y_0)$.

Now the value of y_Q at $x_0 + h$ is given by the point T' where the line through P drawn with slope at $T(x_0 + h, y_T)$ meets MQ.

$$\therefore \text{Slope at } T = \tan\theta' = f(x_0 + h, y_T) = f[x_0 + h, y_0 + h\,f(x_0, y_0)]$$

$$\therefore y_Q = MR + RT' = y_0 + PR\,.\tan\theta' = y_0 + h\,f[x_0 + h, y_0 + h\,f(x_0, y_0)] \quad (4)$$

Thus the value of $f(x,y)$ at $P = f(x_0, y_0)$, the value of $f(x,y)$ at $S = f(x_0 + h/2, y_S)$ and the value of $f(x,y)$ at $Q = f(x_0 + h, y_Q)$ where y_S and y_Q are given by (3) and (4).

Hence from (2), we obtain

$$k = \int_{x_0}^{x_0+h} f(x,y)\,dx = (\mathrm{h}/6)\,[\mathrm{f_P} + 4\mathrm{f_S} + \mathrm{f_Q}] \quad \text{by Simpson's rule}$$
$$= (h/6)\,[f(x_0,y_0) + 4f(x_0 + h/2, y_S) + f(x_0 + h, y_Q)] \tag{5}$$

which gives a sufficiently accurate value of k and also of $y = y_0 + k$

The repeated application of (5) gives the values of y for equi-spaced points.

Working rule to solve (1) by *Runge's method*:

Calculate successively

$$k_1 = h\,f(x_0, y_0),$$
$$k_2 = h\,f(x_0 + h/2\,, y_0 + (1/2)\,k_1)$$
$$k' = h\,f(x_0 + h, y_0 + k_1)$$

And $$k_3 = h\,f(x_0 + h, y_0 + k')$$

Finally compute, $k = (1/6)\,(k_1 + 4k_2 + k_3)$.

which gives the required approximate value as $y_1 = y_0 + k$. (Note that k is the weighted mean of $k_1, k_2\ and\ k_3$).

Example 8: Apply Runge's method to find an approximate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution. Here we have $x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$\therefore$ $$k_1 = h\,f(x_0, y_0) = 0.2\,(1) = 0.200$$
$$k_2 = h\,f(x_0 + h/2, y_0 + (1/2)\,k_1) = 0.2\,f(0.1, 1.1) = 0.240$$
$$k' = h\,f(x_0 + h, y_0 + k_1) = 0.2\,f(0.2, 1.2) = 0.280$$

and $$k_3 = h\,f(x_0 + h, y_0 + k') = 0.2\,f(0.1, 1.28) = 0.296$$

$\therefore$ $$k = (1/6)\,(k_1 + 4k_2 + k_3) = (1/6)\,(0.200 + 0.960 + 0.296) = 0.2426$$

Hence the required approximate value of y is 1.2426.

10.8 RUNGE-KUTTA METHOD

The Taylor's series method of solving differential equations numerically is restricted by the labour involved in finding the higher order derivatives. However there is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formulae possess the advantage of requiring only the function values at some selected points. These methods agree with Taylor's

series solution up to the term in h^r where r differs from method to method and is called the order of that method.

(i) First order R-K method. We have seen that Euler's method gives

$$y_1 = y_0 + h f(x_0, y_0) = y_0 + h y_0'$$

Expanding by Taylor's series

$$y_1 = y(x_0 + h) = y_0 + h y_0' + \frac{h^2}{2} y_0'' + ..$$

It follows that the Euler's method agrees with the Taylor's series solution up to the term in h.

Hence, Euler's method is the Runge-Kutta method of the first order.

(ii) Second order R-K method. The modified Euler's method gives

$$y_1 = y_0 + (h/2)\,[f(x_0, y_0) + f(x_0 + h, y_1)] \quad (1)$$

Substituting $y_1 = y_0 + h f(x_0, y_0)$ on the right hand side of (1), we obtain

$$y_1 = y_0 + (h/2)\,[f_0 + f(x_0 + h, y_0 + h f_0)] \; where \; f_0 = f(x_0, y_0) \quad (2)$$

Expanding L.H.S. by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + h y_0' + (h^2/2)\, y_0'' + (h^3/3!)\, y_0''' + \ldots.. \quad (3)$$

Expanding $f(x_0 + h, y_0 + h f_0)$ by Taylor's series for a function of two variables, (2) gives

$$y_1 = y_0 + (h/2)\,[f_0 + \{f(x_0, y_0) + h\,(\partial f/\partial x)_0 + h f_0\,(\partial f/\partial y)_0 + O(h^2)^*\}]$$

$$= y_0 + (h/2)\,[h f_0 + h f_0 + h^2\,\{(\partial f/\partial x)_0 + f_0\,(\partial f/\partial y)_0\} + O(h^3)]$$

$$= y_0 + h f_0 + (h^2/2)\, f_0' + O(h^3) \; [\because \frac{df(x,y)}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}]$$

$$= y_0 + h y_0' + (h^2/2!)\, y_0'' + O(h^3) \quad (4)$$

Comparing (3) and (4), it follows that the modified Euler's method agrees with the Taylor's series solution up to the term in h^2.

Hence the modified Euler's method is the Runge-Kutta method of the second order.

∴ The second order Runge-Kutta formula is

$$y_1 = y_0 + 1/2\ (k_1 + k_2)$$

where $k_1 = h f(x_0, y_0)$ and $k_2 = h f(x_0 + h, y_0 + k_1)$.

$^*O(h^2)$ means 'terms containing second and higher powers of h' and is read as order of h^2.

(iii) Third order R-K method. Similarly, it can be seen that Runge's method agrees with the Taylor's series solution up to the term in h^3.

As such, Runge's method is the Runge-Kutta method of the third order.

The third order Runge-Kutta formula is

$$y_1 = y_0 + (1/6)\,(k_1 + 4k_2 + k_3)$$

where $k_1 = h f(x_0, y_0), k_2 = h f(x_0 + (1/2)\,h, y_0 + (1/2)\,k_1)$

and $k_3 = h f(x_0 + h, y_0 + k'), where\ k' = h f(x_0 + h, y_0 + k_1)$.

(iv) Fourth order R-K method. This method is most commonly used and is often referred to as Runge-Kutta method only.

Working rule for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from

$$dy/dx = f(x,y), y(x_0) = y_0$$

is as follows:

Calculate successively

$$k_1 = h\,f(x_0, y_0),$$

$$k_2 = h\,f(x_0 + (1/2)\,h, y_0 + (1/2)\,k_1)$$

$$k_3 = h\,f(x_0 + (1/2)\,h, y_0 + (1/2)\,k_2)$$

And $$k_4 = h\,f(x_0 + h, y_0 + k_3)$$

Finally compute $$k = (1/6)\,(k_1 + 2k_2 + 2k_3 + k_4)$$

which gives the required approximate value as $y_1 = y_0 + k$. (Note that k is the weighted mean of k_1, k_2, k_3 and k_4).

Example 9: Apply Runge-Kutta fourth order method to find an approximate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution: Here $x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$\therefore$ $$k_1 = h\,f(x_0, y_0) = 0.2 \times 1 = 0.2000$$

$$k_2 = h\,f(x_0 + (1/2)\,h, y_0 + (1/2)\,k_1) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = h\,f(x_0 + (1/2)\,h, y_0 + (1/2)\,k_2) = 0.2 \times f(0.1, 1.12) = 0.2440$$

And $$k_4 = h\,f(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$$

$\therefore$ $$k = \left(\frac{1}{6}\right)(k^1 + 2k^2 + 2k^3 + k^4)$$

$$= (1/6)\,(0.2000 + 0.4800 + 0.4880 + 0.2888) = 0.2428$$

Hence the required approximate value of y is 1.2428.

CHECK YOUR PROGRESS

Q5: Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$.

Q6. Apply Runge-Kutta method to find approximate value of y for $x = 0.2$, in steps of 0.1, if $dy/dx = x + y^2$, given that $y = 1$ where $x = 0$.

10.9 PREDICTOR-CORRECTOR METHODS

If x_{i-1} and x_i be two consecutive mesh points, we have $x_i = x_{i-1} + h$. In the Euler's method we have

$$y_i = y_{i-1} + h\, f(x_{i-1}, y_{i-1});\ i = 1, 2, 3, \ldots \quad (1)$$

The modified Euler's method gives

$$y_i = y_{i-1} + (h/2)\,[f(x_{i-1}, y_{i-1}) + f(x_i, y_i)] \quad (2)$$

The side of (2), value of y_i is first estimated by using (1), then this value is inserted on the right find a still better approximation of y_i. This value of y_i is again substituted in (2) to agree. This technique of refining an initially crude estimate of y_i by means of a more accurate formula is known as predictor-corrector method. The equation (1) is therefore called the predictor while (2) serves as a corrector of y_i.

In the methods so far described to solve a differential equation over an interval, only the value of y at the beginning of the interval was required. In the predictor-corrector methods, the value of y at x_i from successive approximations to y_i. Though slightly complex, these four prior values are needed for finding the value of y at x_i from successive approximations methods have the advantage of giving an estimate of error.

10.10 ADAMS-BASHFORTH METHOD

Given $dy/dx = f(x, y)$ and $y_0 = y(x_0)$, we compute

$$y_{-1} = y(x_0 - h), y_{-2} = y(x_0 - 2h), y_{-3} = y(x_0 - 3h)$$

by Taylor's series or Euler's method or Runge-Kutta method.

Next we calculate

$$f_{-1} = f(x_0 - h, y_{-1}), f_{-2} = f(x_0 - 2h, y_{-2}), f_{-3} = f(x_0 - 3h, y_{-3}).$$

Then to find y_1, we substitute in Newton's backward interpolation formula

$$f(x, y) = f_0 + n\nabla f_0 + \left[\frac{n(n+1)}{2}\right]\nabla^2 f_0 + \left[\frac{n(n+1)(n+2)}{6}\right]\nabla^3 f_0 + \ldots$$

in $$y_1 = y_0 + \int_{x_0}^{x_0+h} f(x, y)\, dx \quad (1)$$

$$\therefore \quad y_1 = y_0 + \int_{x_0}^{x_1} \left(f_0 + n\nabla f_0 + \left[\frac{n(n+1)}{2}\right]\nabla^2 f_0 + \cdots\right) dx$$

Putting $x = x_0 + nh, dx = h\, dn$

$$y_1 = y_0 + h \int_0^1 (f_0 + n\nabla f_0 + \left[\frac{n(n+1)}{2}\right]\nabla^2 f_0 + \ldots)\, dn$$

$$= y_0 + h\,(f_0 + \frac{1}{2}\nabla f_0 + \frac{5}{12}\nabla^2 f_0 + \frac{3}{8}\nabla^3 f_0 + \ldots)$$

Neglecting fourth and higher order differences and expressing $\nabla f_0, \nabla^2 f_0$ and $\nabla^3 f_0$ in terms of function values, we get

$$y^{1()} = y^0 + \frac{h}{24}(55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}) \quad (2)$$

This is called Adams-Bashforth predictor formula.

Having found y_1, we find $f_1 = f(x_0 + h_1, y_1)$

Then to find a better value of y_1, we derive a corrector formula by substituting Newton's backward formula at f_1 i.e.

$$f(x,y) = f_1 + n\nabla f_1 + \left[\frac{n(n+1)}{2}\right]\nabla^2 f_1 + \left[\frac{n(n+1)(n+2)}{6}\right]\nabla^3 f_1 + \ldots \text{ in (1)}$$

$$\therefore \quad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_0 + n\nabla f_0 + \left[\frac{n(n+1)}{2}\right]\nabla^2 f_0 + \cdots\right) dx$$

Putting $x = x_1 + nh, dx = h\,dn$

$$y_1 = y_0 + h\int_0^1 (f_0 + n\nabla f_0 + \left[\frac{n(n+1)}{2}\right]\nabla^2 f_0 + \ldots)\, dn$$

$$= y_0 + h\,(f_1 - (1/2)\nabla f_1 - (1/12)\nabla^2 f_1 - (1/24)\nabla^3 f_1 - \ldots)$$

Neglecting fourth and higher order differences and expressing $\nabla f_1, \nabla^2 f_1$ and $\nabla^3 f_1$ in terms of function values, we obtain

$$y_1^{\,O} = y_0 + (h/24)(9f_1 + 19f_0 - 5f_{-1} + f_{-2}) \qquad (3)$$

which is called Adams-Moulton corrector formula.

Then an improved value of f is calculated and again the corrector (3) is applied to find a still better value y_1. This step is repeated till y_1 remains unchanged and then we proceed to calculate y_2 as above.

Example 10: Given $dy/dx = x^2(1 + y)$ and $y(1) = 1, y(1.1) = 1.233, y(1.2) = 1.548$, $y(1.3) = 1.979$, evaluate $y(1.4)$ by Adams-Bashforth method.

Solution: Here $f(x,y) = x^2(1 + y)$.

Starting values of the Adams-Bashforth method with $h = 0.1$ are

$$x = 1.0, y_{-3} = 1.000, \qquad f_{-3} = (1.0)^2(1 + 1.000) = 2.000$$

$$x = 1.1, y_{-2} = 1.233, \qquad f_{-2} = 2.702$$

$$x = 1.2, y_{-1} = 1.548, \qquad f_{-1} = 3.669$$

$$x = 1.3, y_0 = 1.979, \qquad f_0 = 5.035$$

Using the predictor,

$$y_1^{\,O} = y_0 + (h/24)(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$x = 1.4, \; y_1^{\,O} = 2.575, \; f_1 = 7.004$$

Using the corrector,

$$y_1^{\,O} = y_0 + (h/24)(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$y_1^{\,O} = 1.979 + (0.1/24)(9 \times 7.004 + 19 \times 5.035 - 5 \times 3.669 + 2.702) = 2.575$$

Hence $y(1.4) = 2.575$.

Example 11: If $dy/dx = 2ey$, $y(0) = 2$, find $y(4)$ using Adams predictor corrector formula by calculating $y(1), y(2)$ and $y(3)$ using Euler's modified formula.

Solution: We have $f(x,y) = 2ey$

To find 0.1:

x	$y' = 2ey$	Mean Slope	$old\ y + h\ (mean\ slope) = new\ y$
0	4	--	2 + 0.1(4) = 2.4
0.1	2e□·¹(2.4) = 5.305	(1/2)(4 + 5.305) = 4.6524	2 + 0.1(4.6524) = 2.4652
0.1	2e□·¹(2.4652) = 5.449	(1/2)(4 + 5.449) = 4.7244	2 + 0.1(4.7244) = 2.4724
0.1	2e□·¹(2.4724) = 5.465	(1/2)(4 + 5.465) = 4.7324	2 + 0.1(4.7324) = 2.4732
0.1	2e□·¹(2.4732) = 5.467	(1/2)(4 + 5.467) = 4.7335	2 + 0.1(4.7335) = 2.4733
0.1	5.467	--	2 + 0.1(5.467) = 3.0199
0.2	7.377	(1/2)(5.467 + 7.377) = 6.422	2.473 + 0.1(6.422) = 3.1155
0.2	7.611	(1/2)(5.467 + 7.611) = 6.539	2.473 + 0.1(6.539) = 3.1269
0.2	7.639	(1/2)(5.467 + 7.639) = 6.553	2.473 + 0.1(6.553) = 3.129
0.2	7.643	(1/2)(5.467 + 7.643) = 6.555	2.473 + 0.1(6.555) = 3.129
0.2	7.643	--	3.893
0.3	10.51	9.076	4.036
0.3	10.897	9.266	4.056

0.3	10.949	9.296	4.058
0.3	10.956	9.299	4.0586

To find $y(0.4)$ by Adam's method, the starting values with $h = 0.1$ are

$x = 0.0, \quad y_{-3} = 2.4, \quad f_{-3} = 4$

$x = 0.1, \quad y_{-2} = 2.473, \quad f_{-2} = 5.467$

$x = 0.2, \quad y_{-1} = 3.129, \quad f_{-1} = 7.643$

$x = 0.3, \quad y_0 = 4.059, \quad f_0 = 10.956$

Using the predictor formula

$$y_1^{(0)} = y_0 + (h/24)(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$= 4.059 + (0.1/24)(55 \times 10.957 - 59 \times 7.643 + 37 \times 5.467 - 9 \times 4)$$

$$= 5.383$$

Now $x = 0.4, \quad y_1 = 5.383, \quad f_1 = 2e^{0 \cdot 4}(5.383) = 16.061$

Using the corrector formula,

$$y_1^{(0)} = y_0 + (h/24)(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$= 4.0586 + (0.1/24)(9 \times 16.061 + 19 \times 10.956 - 5 \times 7.643 + 5.467) = 5.392$$

Hence $y(0.4) = 5.392$.

10.11 LET US SUM UP

- Many differential equations in science and engineering cannot be solved analytically, prompting the use of numerical methods.
- Picard's and Taylor's series methods offer single-step approximations using series expansion.
- Euler's and Modified Euler's methods provide step-by-step estimates using tangent approximations and iterative refinement.
- Runge's and Runge-Kutta methods achieve high accuracy without needing higher-order derivatives, making them widely used in practice.
- Predictor-Corrector methods, including Adams-Bashforth, refine solutions by forecasting and adjusting values using prior steps.

- Techniques are tailored to either initial value problems or boundary value problems, depending on the conditions given.
- Real-world problems often require combining multiple numerical methods to find reliable solutions over a wide range of inputs.

✓ 10.12 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1. We have $y' = (x^3 + xy^2)e^{-x}$ $y'(0) = 0$.

Differentiating successively and substituting $x = 0, y = 1$,

$$y'' = (x^3 + xy^2)(-e^{-x}) + (3x^2 + y^2 + 2xy\,y')\,e^{-x}$$

$$= (-x^3 - xy^2 + 3x^2 + y^2 + 2xy\,y')\,e^{-x};\ y''(0) = 1$$

$$y''' = (-x^3 - xy^2 + 3x^2 + y^2 + 2xy\,y')\,(-e^{-x}) +$$

$$[-3x^2 - (y^2 + 2xy\,y') + 6x + 2yy' + 2\{yy' + x(y'^2 + yy'')\}]\,e^{-x}$$

$y'''(0) = -2$

Substituting these values in the Taylor's series, we have

$$y(x) = y(0) + x\,y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \ldots$$

$$= 1 + 1(0) + \frac{x^2}{2}(1) + \frac{x^3}{6}(-2) + \ldots$$

$$= 1 + \frac{x^2}{2} - \frac{x^3}{3} + \ldots$$

Hence $y(0.1) = 1 + (1/2)(0.1)^2 - (1/3)(0.1)^3 = 1.005$

$y(0.2) = 1 + (1/2)(0.2)^2 - (1/3)(0.2)^3 = 1.017$

$y(0.3) = 1 + (1/2)(0.3)^2 - (1/3)(0.3)^3 = 1.036.$

Answer to Q2. We have $y' = \log x + \log y;\ y'(1) = \log 2$

Differentiating w.r.t. x and substituting $x = 1, y = 2$, we get

$$y'' = \frac{1}{x} + \frac{1}{y}y';\qquad y''(1) = 1 + \frac{1}{2}\log 2$$

$$y''' = -\frac{1}{x^2} + \frac{1}{y}y'' + \frac{1}{y}\left(-\frac{1}{y^2}\right)y';\ y'''(1) = -1 + \frac{1}{2}\left(1 + \frac{1}{2}\log 2\right) - \frac{1}{4}(\log 2)^2$$

Substituting these values in the Taylor's series about $x = 1$, we have

$$y(x) = y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2!}y''(1) + \frac{(x-1)^3}{3!}y'''(1) + \ldots$$

$$= 2 + (x-1)\log 2 + \frac{(x-1)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(x-1)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right]$$

$$\therefore\ y(1.1) = 2 + (0.1)\log 2 + \frac{(0.1)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(0.1)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] = 2.036$$

$$y(1.2) = 2 + (0.2)\log 2 + \frac{(0.2)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(0.2)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right]$$

$$= 2.081.$$

Answer to Q3.: We divide the interval $(0, 0.1)$ into five steps i.e. we take $n = 5$ and $h = 0.02$. The various calculations are arranged as follows:

x	y	dy/dx	Old $y + 0.02\left(\frac{dy}{dx}\right) = new\ y$
0	1.0000	1.000	1.0000 + 0.02(1.0000) = 1.0200
0.02	1.0200	0.9615	1.0200 + 0.02(0.9615) = 1.0392
0.04	1.0392	0.926	1.0392 + 0.02(0.926) = 1.0577
0.06	1.0577	0.893	1.0577 + 0.02(0.893) = 1.0756
0.08	1.0756	0.862	1.0756 + 0.02(0.862) = 1.0928
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

Answer to Q4.: We have $y' = y + e^x = f(x, y)$; $x = 0, y = 0$ and $h = 0.2$

To calculate $y(0.2)$:

x	$y' = y + e^x$	Mean Slope	$Old\ y + h\ (mean\ slope) = new\ y$
0.0	1	--	0 + 0.2 (1) = 0.2
0.2	$0.2 + e^{0.2} = 1.4214$	(1/2) (1 + 1.4214) = 1.2107	0 + 0.2 (1.2107) = 0.2421
0.2	$0.2421 + e^{0.2} = 1.4635$	(1/2) (1 + 1.4635) = 1.2317	0 + 0.2 (1.2317) = 0.2463
0.2	$0.2463 + e^{0.2} = 1.4677$	(1/2) (1 + 1.4677) = 1.2338	0 + 0.2 (1.2338) = 0.2468
0.2	$0.2468 + e^{0.2} = 1.4682$	(1/2) (1 + 1.4682) = 1.2341	0 + 0.2 (1.2341) = 0.2468

Since the last two values of y are equal, we take $y(0.2) = 0.2468$.

To calculate y(0.4):

x	$y' = y + e^x$	Mean Slope	$Old\ y + h\ (mean\ slope) = new\ y$
0.2	$0.2468 + e^{0.2} = 1.4682$	--	0.2468 + 0.2 (1.4682) = 0.5404
0.4	$0.5404 + e^{0.4} = 2.0322$	(1/2) (1.4682 + 2.0322) = 1.7502	0.2468 + 0.2 (1.7502) = 0.5968
0.4	$0.5968 + e^{0.4} = 2.0887$	(1/2) (1.4682 + 2.0887) = 1.7784	0.2468 + 0.2 (1.7784) = 0.6025
0.4	$0.6025 + e^{0.4} = 2.0943$	(1/2) (1.4682 + 2.0943) = 1.78125	0.2468 + 0.2 (1.78125) = 0.6030
0.4	$0.6030 + e^{0.4} = 2.0949$	(1/2) (1.4682 + 2.0949) = 1.7815	0.2468 + 0.2 (1.7815) = 0.6031
0.4	$0.6030 + e^{0.4} = 2.0949$	(1/2) (1.4682 + 2.0949) = 1.7816	0.2468 + 0.2 (1.7816) = 0.6031

Since the last two values of y are equal, we take $y(0.4) = 0.6031$.

Hence $y(0.2) = 0.2468$ and $y(0.4) = 0.6031$ approximately.

Answer to Q5.: We have $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$

To find $y(0.2)$:

Here $x_0 = 0, y_0 = 1, h = 0.2$

$$k_1 = h\,f(x_0, y_0) = 0.2\,f(0,1) = 0.2000$$

$$k_2 = h\,f(x_0 + (1/2)\,h, y_0 + (1/2)\,k_1) = 0.2\,f(0.1, 1.1) = 0.19672$$

$$k_3 = h\,f\,(\,x_0 + (1/2)h, y_0 + \left(\tfrac{1}{2}\right)k_2\,) = 0.2\,f(0.1, 1.09836) = 0.1967$$

$$k_4 = h\,f\,(\,x_0 + h, y_0 + k_3\,) = 0.2\,f(0.2, 1.1967) = 0.1891$$

$$k = (1/6)\,(\,k_1 + 2\,k_2 + 2\,k_3 + k_4\,)$$

$$= (1/6)\,[\,0.2 + 2(0.19672) + 2(0.1967) + 0.1891\,]$$

$$= 0.19599$$

Hence $y(0.2) = y_0 + k = 1.1959$.

To find (0.4) :

Here $x_1 = 0.2, y_1 = 1.1959, h = 0.2$.

$$k_1 = h\,f\,(\,x_1, y_1\,) = 0.1891$$

$$k_2 = h\,f\,(\,x_1 + (1/2)h, y_1 + \left(\tfrac{1}{2}\right)k_1\,) = 0.2\,f(0.3, 1.2906) = 0.1795$$

$$k_3 = h\,f\,(\,x_1 + (1/2)h, y_1 + \left(\tfrac{1}{2}\right)k_2\,) = 0.2\,f(0.3, 1.2858) = 0.1793$$

$$k_4 = h\,f\,(\,x_1 + h, y_1 + k_3\,) = 0.2\,f(0.4, 1.3753) = 0.1688$$

$$k = (1/6)\,(\,k_1 + 2\,k_2 + 2\,k_3 + k_4\,)$$

$$= (1/6)\,[\,0.1891 + 2(0.1795) + 2(0.1793) + 0.1688\,] = 0.1792$$

Hence $y(0.4) = y_1 + k = 1.1959 + 0.1792 = 1.3752$.

Answer to Q6.: Given $(\,x, y\,) = x + y^2$.

Here we take $h = 0.1$ and carry out the calculations in two steps.

Step I. $x_0 = 0, y_0 = 1, h = 0$.

$\therefore$ $$k_1 = h\,f\,(\,x_0, y_0\,) = 0.1\,f(0,1) = 0.1000$$

$$k_2 = h\,f\,(\,x_0 + (1/2)h, y_0 + \left(\tfrac{1}{2}\right)k_1\,) = 0.1\,f(0.05, 1.1) = 0.1152$$

$$k_3 = h\,f\,(\,x_0 + (1/2)h, y_0 + \left(\tfrac{1}{2}\right)k_2\,) = 0.1\,f(0.05, 1.1152) = 0.1168$$

$$k_4 = h\,f\,(\,x_0 + h, y_0 + k_3\,) = 0.1\,f(0.1, 1.1168) = 0.1347$$

$\therefore$ $$k = (1/6)\,(\,k_1 + 2\,k_2 + 2\,k_3 + k_4\,)$$

$$= (1/6)\,(\,0.1000 + 2.2304 + 0.2336 + 0.1347\,) = 0.11665$$

Giving $y(0.1) = y_0 + k = 1.11665$.

Step II. $x_1 = x_0 + h = 0.1,\ y_1 = y(0.1) = 1.11665, h = 0.1$

$\therefore$ $$k_1 = h\,f(x_1, y_1) = 0.1\,f(0.1, 1.11665) = 0.1347$$

$$k_2 = h\,f(x_1 + (1/2)h, y_1 + \left(\tfrac{1}{2}\right)k_1) = 0.1\,f(0.15, 1.1838) = 0.1551$$

$$k_3 = h\,f(x_1 + (1/2)h, y_1 + \left(\frac{1}{2}\right)k_2) = 0.1\,f(0.15, 1.194) = 0.1576$$

$$k_4 = h\,f(x_1 + h, y_2 + k_3) = 0.1\,f(0.2, 1.15776) = 0.1823$$

$$\therefore \quad k = (1/6)\,(k_1 + 2\,k_2 + 2\,k_3 + k_4) = 0.1571$$

Hence $y(0.2) = y_1 + k = 1.2736$.

10.13 FURTHER READING

1. Gupta, R. K. *Numerical Methods: Fundamentals and Applications*. 1st edition, Cambridge University Press, 2019.
2. Thangaraj, P. *Computer Oriented Numerical Methods*. PHI Learning Pvt. Ltd, 2013.
3. Jain, M. K., Iyengar, S. R. K. and Jain, R. K. *Numerical Methods for Scientific & Engineering Computation*. New Age International, 2012.

10.14 MODEL QUESTIONS

1. Using Picard's method, solve $\frac{dy}{dx} = -xy\ with\ x_0 = 0, y_0 = 1$ up to third approximation.
2. Employ Picard's method to obtain, correct to four places of decimal, solution of the differential equation $\frac{dy}{dx} = x^2 + y^2$ for $x = 0.4$, given that $y = 0$ when $x = 0$.
3. Find an approximate value of y when $x = 0.1$, if $\frac{dy}{dx} = x - y^2$ and $y = 1$ at $x = 0$, using
    a. (a) Picard's method, (b) Taylor's series.
4. Solve $y' = x + y$ given $y(1) = 0$. Find $y(1.1)$ and $y(1.2)$ by Taylor's method. Compare the result with its exact value.
5. Using Taylor's series method, compute $y(0.2)$ to three places of decimal from $dy/dx = 1 - 2xy$
    a. given that $y(0) = 0.6$
6. Using simple Euler's method solve for y at $= 0.1\ from\ dy/dx = x + y + xy$, $y(0) = 1$, taking step size $h = 0.025$.
7. Solve $y' = 1 - y, y(0) = 0$ by modified Euler's method and obtain y at $x = 0.1, 0.2, 0.3$.
8. Given that $dy/dx = x^2 + y$ and $y(0) = 1$. Find an approximate value of $y(0.1)$, taking $h = 0.05$ by modified Euler's method.
9. Given $y' = x + sin\,y, y(0) = 1$. Compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$ using Euler's modified method.

10. Use Runge's method to approximate y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and
11. $\frac{dy}{dx} = 3x + y^2$.
12. Using Runge-Kutta method of order 4, find $y(0.2)$ given that $\frac{dy}{dx} = 3x + (1/2)y$, $y(0) = 1$, taking $h = 0.1$.
13. Find $y(0.1)$ and $y(0.2)$ using Runge-Kutta 4th order formula, given that $y' = x^2 - y$ and $y(0) = 1$.
14. Using Adams-Bashforth method, obtain the solution of $\frac{dy}{dx} = x - y^2$ at $x = 0.8$, given the values

x:	0	0.2	0.4	0.6
y:	0	0.0200	0.0795	0.1762

15. Using Adams-Bashforth formulae, determine $y(0.4)$ given the differential equation $\frac{dy}{dx} = \frac{1}{2}xy$ and the data:

x:	0	0.1	0.2	0.3
y:	1	1.0025	1.0101	1.0228.

16. Given $y' = x^2 - y, y(0) = 1$ and the starting values $y(0.1) = 0.90516, y(0.2) = 0.82127, y(0.3) = 0.74918$, evaluate $y(0.4)$ using Adams-Bashforth method.
17. Using Adams-Bashforth method, find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1, y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$.

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