UNIT 11: SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS

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11.1 LEARNING OBJECTIVES

- To understand the concepts and techniques used to solve systems of linear simultaneous equations.
- To learn and apply various direct and iterative methods such as Gauss Elimination, Gauss-Jordan, LU Decomposition, Jacobi, and Gauss-Seidel methods.
- To recognize ill-conditioned systems and improve the accuracy of solutions using iterative refinement.
- To compare different solution techniques in terms of computational efficiency, accuracy, and stability.

11.2 INTRODUCTION

Systems of linear simultaneous equations are fundamental in mathematics and arise frequently in science and engineering—such as in circuit analysis, chemical process modelling, and network analysis. The solutions to these systems can be obtained through either direct methods (like Gauss Elimination or LU Decomposition) or iterative methods (such as Jacobi and Gauss-

Seidel). This unit introduces these methods and discusses their procedures, advantages, limitations, and applications.

11.3 SYSTEMS OF SIMULTANEOUS LINEAR EQUATIONS

Simultaneous linear equations occur quite often in engineering and science. The analysis of electronic circuits consisting of invariant elements, analysis of a network under sinusoidal steady-state conditions, determination of the output of a chemical plant, finding the cost of chemical reactions are some of the problems which depend on the solution of simultaneous linear algebraic equations. The solution of such equations can be obtained by Direct or Iterative methods. We describe below some such methods of solution.

11.4 GAUSS ELIMINATION METHOD

In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution. The method is quite general and is well-adapted for computer operations. Here we shall explain it by considering a system of three equations.

Consider the equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \tag{1}$$

Step I.: To eliminate x from second and third equations

Assuming $a_1 \neq 0$, we eliminate x from the second equation by subtracting (a_2/a_1) times the first equation from the second equation. Similarly we eliminate x from the third equation by eliminating (a_3/a_1) times the first equation from the third equation. We thus get the new system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ b_3'y + c_3'z &= d_3' \end{aligned} \tag{2}$$

Here the first equation is called the pivotal equation and a_1 is called the first pivot.

Step II. To eliminate y from third equation of (2)

Assuming $b_2' \neq 0$, we eliminate y from the third equation of (2), by subtracting (b_3'/b_2') times the second equation from the third equation. We thus, get the new system

$$a_1x + b_1y + c_1z = d_1$$

$$b_2'y + c_2'z = d_2' \qquad (3)$$

$$c_3''z = d_3''$$

Here the second equation is the pivotal equation and b_2' is the new pivot.

Step III. To evaluate the unknowns

The values of x, y, z are found from the reduced system (3) by back substitution.

Observation 1: On writing the given equations as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \qquad \text{i.e. } AX = D$$

this method consists in transforming the coefficient matrix A to upper triangular matrix by elementary row transformations only.

Observation 2: Clearly the method will fail if any one of the pivots $a_1, b_2', or\ c_3''$ becomes zero. In such cases, we rewrite the equations in a different order so that the pivots are non-zero.

Observation 3: Partial and complete pivoting. In the first step, the numerically largest coefficient of x is chosen from all the equations and brought as the first pivot by interchanging the first equation with the equation having the largest coefficient of x. In the second step, the numerically largest coefficient of y is chosen from the remaining equations (leaving the first equation) and brought as the second pivot by interchanging the second equation with the equation having the largest coefficient of y. This process is continued till we arrive at the equation with the single variable. This modified procedure is called partial pivoting.

If we are not keen about the elimination of x, y, z in a specified order, then we can choose at each stage the numerically largest coefficient of the entire matrix of coefficients. This requires not only an interchange of equations but also an interchange of the position of the variables. This method of elimination is called complete pivoting. It is more complicated and does not appreciably improve the accuracy.

Example 1: Apply Gauss elimination method to solve the equations $x + 4y - z = -5\,;\ x + y - 6z = -12\,;\ 3x - y - z = 4.$

Sol: We have

$$x + 4y - z = -5 \qquad \text{(i)}$$

$$x + y - 6z = -12 \qquad \text{(ii)}$$

$$3x - y - z = 4 \qquad \text{(iii)}$$

Step I. To eliminate x, operate (ii) - (i) and (iii) - 3(i)

$$-3y - 5z = -7 \quad \text{(iv)}$$

$$-13y + 2z = 19 \quad \text{(v)}$$

Step II. To eliminate y, operate (v) – $\frac{13}{3}$ (iv):

$$\frac{71}{3} z = \frac{148}{3} \quad \text{(vi)}$$

Step III. By back-substitution, we get

From (vi) : $z = \frac{148}{71} \approx 2.0845$

From (iv) : y = (7 - 5 · 148/71)/3 = -81/71 ≈ -1.1408

From (i) : x = -5 - 4 (-81/71) + 148/71 = 117/71 ≈ 1.6479

Hence, $x \approx 1.6479, y \approx -1.1408, z \approx 2.0845$

Note. A useful check is provided by noting the sum of the coefficients and terms on the right, operating on those numbers as on the equations and checking that the derived equations have the correct sum.

Otherwise: We have $\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$

Operate $R_2 - R_1$ and $R_3 - 3R_1$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 19 \end{bmatrix}$

Operate $R_3 - \frac{13}{3} R_2$ $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148/3 \end{bmatrix}$

Thus, we have z = 148/71 ≈ 2.0845,

$$3y = 7 - 5z = 7 - 10.4225 \approx -3.4225 \; i.e., y \approx -1.1408$$

$$x = -5 - 4y + z = -5 + 4\,(81/71) + 2.0845 \approx 1.6479$$

Hence $x \approx 1.6479, y \approx -1.1408, z \approx 2.0845$.

Example 2:. Using Gauss elimination method, solve the equations: $x + 2y + 3z - u = 10, 2x + 3y - 3z - u = 1, 2x - y + 2z + 3u = 7, 3x + 2y - 4z + 3u = 2$.

Sol. We have $\begin{bmatrix}1 & 2 & 3 & -1\\ 2 & 3 & -3 & -1\\ 2 & -1 & 2 & 3\\ 3 & 2 & -4 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ u\end{bmatrix}=\begin{bmatrix}10\\ 1\\ 7\\ 2\end{bmatrix}$

Operate $R_2 - 2R_1, R_3 - 2R_1, R_4 - 3R_1$

$$\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & -1 & -9 & 1\\ 0 & -5 & -4 & 5\\ 3 & -4 & -13 & 6\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ u\end{bmatrix}=\begin{bmatrix}10\\ -19\\ -13\\ -28\end{bmatrix}$$

Operate $R_3 - 5R_2, R_4 - 4R_2$ $\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & -1 & -9 & 1\\ 0 & 0 & 41 & 0\\ 0 & 0 & 23 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ u\end{bmatrix}=\begin{bmatrix}10\\ -19\\ 82\\ 48\end{bmatrix}$

Thus, we have $41z = 82\, i.e., z = 2,$

$$23z + 2u = 48\, i.e., 46 + 2u = 48, \quad \therefore u = 1$$

$$-y - 9z + u = -19\, i.e., -y - 18 + 1 = -19, \quad \therefore y = 2$$

$$x + 2y + 3z - u = 10\, i.e., x + 4 - 6 - 1 = 10, \quad \therefore x = 1$$

Hence $x = 1, y = 2, z = 2, u = 1.$

11.5 GAUSS-JORDAN METHOD

This is a modification of the Gauss elimination method. In this method, elimination of unknowns is performed not in the equations below but in the equations above also, ultimately reducing the system to a diagonal matrix form i.e. each equation involving only one unknown. From these equations, the unknowns x, y, z can be obtained readily.

Thus in this method, the labour of back-substitution for finding the unknowns is saved at the cost of additional calculations.

Note: For a system of 10 equations, the number of multiplications required for Gauss-Jordan method is about 500 whereas for Gauss elimination method we need only 333 multiplications. This shows that though Gauss-Jordan method appears to be easier but requires 50% more operations than the Gauss elimination method. As such Gauss elimination method is preferred for large systems.

Example 3: Apply Gauss-Jordan method to solve the equations $x + y + z = 9$; $2x - 3y + 4z = 13$; $3x + 4y + 5z = 40$.

Sol. We have $x + y + z = 9$ (i)

$$2x - 3y + 4z = 13 \quad \text{(ii)}$$

$$3x + 4y + 5z = 40 \quad \text{(iii)}$$

Step I. To eliminate x from (ii) and (iii), operate (ii) - 2(i) and (iii) - 3(i) :

$$x + y + z = 9 \quad \text{(iv)}$$

$$-5y + 2z = -5 \quad \text{(v)}$$

$$y + 2z = 13 \quad \text{(vi)}$$

Step II. To eliminate y from (iv) and (v), operate (iv) + 1/5 (v) and (vi) + 1/5 (v) :

$$x + 7/5\, z = 8 \quad \text{(vii)}$$

$$-5y + 2z = -5 \quad \text{(viii)}$$

$$12/5\, z = 12 \quad \text{(ix)}$$

Step III. To eliminate z from (vi) and (viii), operate (vi) - 7/12 (ix) and (viii) - 2/5 (ix) :

$$x = 1$$

$$-5y = -15$$

$$12/5\, z = 12$$

Hence the solution is $x = 1, y = 3, z = 5$.

Otherwise: Rewriting the equations as $\begin{bmatrix}1 & 1 & 1\\ 2 & -3 & 4\\ 3 & 4 & 5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}9\\ 13\\ 40\end{bmatrix}$

Operate $R_2 - 2R_1, R_3 - 3R_1$ $\begin{bmatrix}1 & 1 & 1\\ 0 & -5 & 2\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}9\\ -5\\ 13\end{bmatrix}$

Operate $R_3 + 1/5\, R_2$ $\begin{bmatrix}1 & 1 & 1\\ 0 & -5 & 2\\ 0 & 0 & 12/5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}9\\ -5\\ 12\end{bmatrix}$

Operate $-R_2, 5R_3$ $\begin{bmatrix}1 & 1 & 1\\ 0 & 5 & -2\\ 0 & 0 & 12\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}9\\ 5\\ 60\end{bmatrix}$

Operate $R_2 + 6/5\, R_3, 1/12\, R_3$ $\begin{bmatrix}1 & 1 & 1\\ 0 & 5 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}9\\ 15\\ 5\end{bmatrix}$

Operate $1/5\, R_2$ $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}9\\ 3\\ 5\end{bmatrix}$

Operate $R_1 - R_2 - R_3$ $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\3\\5\end{bmatrix}$

Hence $x = 1, y = 3, z = 5.$

Note: Here the process of elimination of variables amounts to reducing the given coefficient metric to a diagonal matrix by elementary row transformations only.

Example 4: Solve the equations $10x - 7y + 3z + 5u = 6$; $-6x + 8y - z - 4u = 5$; $3x + y + 4y + 11u = 2$; $5x - y - 2z + 4u = 7$ by Gauss-Jordan method.

Sol. We have $10x - 7y + 3z + 5u = 6$ (i)

$-6x + 8y - z - 4u = 5$ (ii)

$3x + y + 4y + 11u = 2$ (iii)

$5x - y - 2z + 4u = 7$ (iv)

Step I. To eliminate x, operate [(ii) - (-6/10) (i)], [(iii) - (3/10) (i)], [(iv) - (5/10) (i)] :

$10x - 7y + 3z + 5u = 6$ (v)

$3.8y + 0.8z - u = 8.6$ (vi)

$3.1y + 3.1z + 9.5u = 0.2$ (vii)

$-5.5y - 3.5z + 1.5u = 4$ (viii)

Step II. To eliminate y, operate [(v) - (7/3.8) (vi)], [(vii) - (3.1/3.8) (vi)], [(viii) - (5.5/3.8) (vi)]

$10x + 4.4736842z + 3.1578947u = -1.842105$ (ix)

$3.8y + 0.8z - u = 8.6$ (x)

$2.4473684z + 10.315789u = -6.8157895$ (xi)

$-2.3421053z + 0.0526315u = 16.447368$ (xii)

Step III. To eliminate z, operate [(ix) - (4.473684 / 2.4473684) (xi)], [(x) - (0.8 / 2.4473684) (xi)]

10x - 15.69923u = 34.301075

3.8y - 4.3720429u = 10.827957

2.4473684z + 10.315789u = -6.8157895

Step IV. From the last equation u = 1 nearly.

Substitution of u = 1 in the above three equations gives x = 5, y = 4, z = -7

11.6 LU DECOMPOSITION METHOD

This method is based on the fact that every square matrix can be expressed as the product of a lower triangular matrix and an upper triangular matrix, provided all the principal minors of A are non-singular, i.e. if $A = [a_{ij}]$, then

$$a_{11} \neq 0, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \neq 0, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \neq 0 \text{ etc.}$$

Also such a factorization if it exists, is unique.

Now consider the equations $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

which can be written as $AX = B$ (1)

where $= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Let $A = LU$ (2)

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Then (1) becomes $LU\,X = B$ (3)

Writing becomes $LX = V$ (4)

(3) becomes $LV = B$

which is equivalent to the equations $v_1 = b_1, l_{21}v_1 + v_2 = b_2, l_{31}v_1 + l_{32}v_2 + v_3 = b_3$

Solving these for v_1, v_2, v_3, we know V. Then (4) becomes

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = v_1, u_{22}x_2 + u_{23}x_3 = v_2, u_{33}x_3 = v_3$$

from which x_3, x_2, x_1 can be found by back-substitution.

To compute the matrices L and U, we write (2) as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the matrices on the left and equating corresponding elements from both sides, we obtain

(i) $u_{11} = a_{11}\ or\ u_{12} = a_{12}\ ;\ u_{13} = a_{13}$

(ii) $l_{21}u_{11} = a_{21}\ or\ l_{21} = a_{21}/a_{11}\ ;\ l_{31}u_{11} = a_{31}\ or\ l_{31} = a_{31}/a_{11}$

(iii) $l_{21}u_{12} + u_{22} = a_{22}\ or\ u_{22} = a_{22} - \frac{a_{21}}{a_{11}}a_{12}$

$l_{21}u_{13} + u_{23} = a_{23}\ or\ u_{23} = a_{23} - \frac{a_{21}}{a_{11}}a_{13}$

(iv) $l_{31}u_{12} + l_{32}u_{22} = a_{32}\ or\ l_{32} = 1/a_{11}\ [\ a_{32} - \frac{a_{31}}{a_{11}}a_{12}\ ]$

(v) $l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$ which gives u_{33}

Thus we compute the elements of L and U in the following set order :

(i) First row of U, (ii) First column of L,

(iii) Second row of U, (iv) Second column of L,

(v) Third row of U

This procedure can easily be generalised.

Observation: This method is superior to Gauss elimination method and is often used for the solution of linear systems and for finding the inverse of a matrix. The number of operations involved in terms of multiplications for a system of 10 equations by this method is about 110 as compared 333 operations of the Gauss method. Among the direct methods, factorization method is also preferred as the software for computers.

Example 5: Apply LU Decomposition method to solve the equations: $3x + 2y + 7z = 4\ ;\ 2x + 3y + 2 = 5\ ;\ 3x + 4y + z = 7.$

Sol. Let $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$

so that

$(i) R_1\ of\ U : u_{11} = 3, u_{12} = 2, u_{13} = 7.$

(ii) $C_1\ of\ L : l_{21}u_{11} = 2, \therefore\ l_{21} = \frac{2}{3},$

$l_{31}u_{11} = 3, \therefore l_{31} = 1,$

$(iii) R_2\ of\ U : l_{21}u_{12} + u_{22} = 3,$

$l_{21}u_{13} + u_{23} = 1,$

(iv) $C_2\ of\ L:\ l_{31}u_{12}\ +\ l_{32}u_{22}\ =\ 4,\ \ \therefore l_{32}\ =\frac{6}{5}$,

(v) $R_3\ of\ U:\ l_{31}u_{13}\ +\ l_{32}u_{23}\ +\ u_{33}\ =\ 1,:\ u_{33}\ =\ -\frac{8}{5}$.

Thus $A=\begin{bmatrix}1 & 0 & 0\\ 2/3 & 1 & 0\\ 1 & 6/5 & 1\end{bmatrix}\begin{bmatrix}3 & 2 & 7\\ 0 & 5/3 & -11/3\\ 0 & 0 & -8/5\end{bmatrix}$

Writing $UX\ =\ V$, the given system becomes

$$\begin{bmatrix}1 & 0 & 0\\ 2/3 & 1 & 0\\ 1 & 6/5 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}4\\ 5\\ 7\end{bmatrix}$$

Solving this system, we have $v_1\ =\ 4$,

$$2/3\ v_1\ +\ v_2\ =\ 5\ or\ v_2\ =\ 7/3$$

$$v_1\ +\ 6/5\ v_2\ =\ 7\ or\ v_3\ =\ 1/5$$

Hence the original system becomes

$$\begin{bmatrix}3 & 2 & 7\\ 0 & 5/3 & -11/3\\ 0 & 0 & -8/5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4\\ 7/3\\ 1/5\end{bmatrix}$$

i.e. $3x\ +\ 2y\ =\ 4,\frac{5}{3}\ y\ -\frac{11}{3}\ z\ =\frac{7}{3},-\frac{8}{5}\ z\ =\frac{1}{5}$

By back-substitution, we have

$$z\ =\ -\frac{1}{8},y\ =\frac{9}{8}\ and\ x\ =\frac{7}{8}$$

Example 6: Solve the equations $10x\ -\ 7y\ +\ 3z\ +\ 5u\ =\ 6$; $-6x\ +\ 8y\ -\ z\ -\ 4u\ =\ 5$; $3x\ +\ y\ +\ 4y\ +\ 11u\ =\ 2$; $5x\ -\ y\ -\ 2z\ +\ 4u\ =\ 7$ by LU Decomposition method.

Sol. We have $\begin{bmatrix}1 & 0 & 0 & 0\\ l_{21} & 1 & 0 & 0\\ l_{31} & l_{32} & 1 & 0\\ l_{41} & l_{42} & l_{43} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13} & u_{14}\\ 0 & u_{22} & u_{23} & u_{24}\\ 0 & 0 & u_{33} & u_{34}\\ 0 & 0 & 0 & u_{44}\end{bmatrix}=\begin{bmatrix}10 & -7 & 3 & 5\\ -6 & 8 & -1 & -4\\ 3 & 1 & 4 & 11\\ 5 & -9 & -2 & 4\end{bmatrix}$

so that

$(i)\ u_{11}\ =\ 10,u_{12}\ =\ -7,u_{13}\ =\ 3,u_{14}\ =\ 5$

$(ii)\quad l_{21}u_{11}\ =\ -6,:\ l_{21}\ =\ -0.6,$

$l_{31}u_{11}\ =\ 3,:\ l_{31}\ =\ 0.3,$

$l_{41}u_{11}\ =\ 5,:\ l_{41}\ =\ 0.5$

(iii) $l_{21}u_{12} + u_{22} = 8,$

$l_{21}u_{13} + u_{23} = -1,$

$l_{21}u_{14} + u_{24} = -4$

(iv) $l_{31}u_{12} + l_{32}u_{22} = 5, \therefore l_{32} = 1.44737,$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 0, \therefore u_{33} = -10.31579,$

$l_{31}u_{14} + l_{32}u_{24} + u_{34} = 11, \therefore u_{34} = 9.44737$

(v) $l_{41}u_{12} + l_{42}u_{22} = -1, \therefore l_{42} = -0.5699,$

$l_{41}u_{13} + l_{42}u_{23} + l_{32}u_{33} + u_{43} = -2, \therefore u_{43} = 0,$

$l_{41}u_{14} + l_{42}u_{24} + l_{32}u_{34} + l_{42}u_{44} = 4, \therefore u_{44} = 9.92474$

Thus $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81579 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix} \begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix}$

Writing $UX = V$, the given system becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81579 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

Solving this system, we get

$$v_1 = 6, v_2 = 8.6, v_3 = -10.31579, v_4 = 9.92474.$$

Hence the original system becomes

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ -6.81579 \\ 9.92474 \end{bmatrix}$$

i.e. $10x - 7y + 3z + 5u = 6,\ 3.8y + 0.8z - u = 8.6$

$2.44737z + 10.31579u = -6.81579,\ u = 1.$

By back-substitution yields: $u = 1, z = -7, y = 4, x = 5.$

CHECK YOUR PROGRESS

Q1. Using Gauss elimination method, solve the equations : $x + 2y + 3z - u = 10$, $2x + 3y - 3z - u = 1, 2x - y + 2z + 3u = 7, 3x + 2y - 4z + 3u = 2.$

Q2. Solve the equations using Gauss-Jordan Method: $2x + 5y + 7z = 52$; $2x + y - z = 0$; $x + y + z = 9$.

Q3. Solve the equations using LU Decomposition Method: $2x + 3y + z = 9$; $x + 2y + 3z = 6$; $3x + y + 2z = 8$.

11.7 ITERATIVE METHOD : JACOBI METHOD

ITERATIVE METHODS OF SOLUTION: The preceding methods of solving simultaneous linear equations are known as direct methods, as these methods yield the solution after a certain amount of fixed computation. On the other hand, an iterative method is that in which we start from an approximation to the true solution and obtain better and better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy. Thus an iterative method, the amount of computation depends on the degree of accuracy required. For large systems, iterative methods may be faster than the direct methods. Even the round-off errors in iterative methods are smaller. In fact, iteration is a self-correcting process and any error made at any stage of computation gets automatically corrected in the subsequent steps.

Simple iterative methods can be devised for systems in which the coefficients of the leading diagonal are large as compared to others. We now describe three such methods:

(I) **Jacobi's iteration method**: Consider the equations

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \qquad (1) \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned}$$

If a_1, b_2, c_3 are large as compared to other coefficients, solve for x, y, z respectively. Then the system can be written as

$$\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \\ y &= \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \qquad (2) \\ z &= \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \end{aligned}$$

Let us start with the initial approximations x_0, y_0, z_0 for the values of x, y, z respectively. Substituting these on the right sides of (2), the first approximations are given by

$$\begin{aligned} x_1 &= \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0) \\ y_1 &= \frac{1}{b_2}(d_2 - a_2 x_0 - c_2 z_0) \qquad (3) \end{aligned}$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0)$$

Substituting the values x_1, y_1, z_1 on the right sides of (2), the second approximations are given by

$$x_2 = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

This process is repeated till the difference between two consecutive approximations is negligible.

Note: In the absence of any better estimates for x_0, y_0, z_0, these may each be taken as zero.

Example 7: Solve, by Jacobi's iteration method, the equations $20x + y - 2z = 17$; $3x + 20y - z = -18$; $2x - 3y + 20z = 25$

Sol. We write the given equations in the form

$$x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z) \qquad \text{(i)}$$

$$z = \frac{1}{20}(25 - 2x + 3y)$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.

Substituting these on the right sides of the equations (i), we get

$$x_1 = \frac{17}{20} = 0.85,\ y_1 = \frac{-18}{20} = -0.9,\ z_1 = \frac{25}{20} = 1.25$$

Putting these values on the right sides of the equations (i), we obtain

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.02$$

$$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = -0.965$$

$$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.03$$

Substituting these values on the right sides of the equations (i), we have

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.00125$$

$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = -1.0016$$

$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.00325$$

Substituting these values, we get

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = 1.0004$$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = -1.000025$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 0.9965$$

Putting these values, we have

$$x_5 = \frac{1}{20}(-17 - y_4 + 2z_4) = 0.999966$$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = -1.000078$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 0.999956$$

Again substituting these values, we get

$$x_6 = \frac{1}{20}(17 - y_5 + 2z_5) = 1.0000$$

$$y_6 = \frac{1}{20}(-18 - 3x_5 + z_5) = -0.999997$$

$$z_6 = \frac{1}{20}(25 - 2x_5 + 3y_5) = 0.999992$$

The values in the 5th and 6th iterations being practically the same, we can stop. Hence the solution is

$x = 1, y = -1, z = 1.$

Example 8: Solve by Jacobi's iteration method, the equations $10x + y - z = 11.19$

$x + 10y + z = 28.08, -x + y + 10z = 35.61$, correct to two decimal places.

Sol. rewriting the given equations as

$$x = \frac{1}{10}(11.19 - y + z), y = \frac{1}{10}(28.08 - x - z), z = \frac{1}{10}(35.61 + x - y)$$

We start from an approximation $x_0 = y_0 = z_0 = 0$

First iteration

$x_1 = \frac{11.19}{10} = 1.119, y_1 = \frac{28.08}{10} = 2.808, z_1 = \frac{35.61}{10} = 3.561$

Second iteration

$x_2 = \frac{1}{10}(11.19 - y_1 + z_1) = 1.19$

$y_2 = \frac{1}{10}(28.08 - x_1 - z_1) = 2.34$

$z_2 = \frac{1}{10}(35.61 + x_1 - y_1) = 3.39$

Third iteration

$x_3 = \frac{1}{10}(11.19 - y_2 + z_2) = 1.22$

$y_3 = \frac{1}{10}(28.08 - x_2 - z_2) = 2.35$

$z_3 = \frac{1}{10}(35.61 + x_2 - y_2) = 3.45$

Fourth iteration

$x_4 = \frac{1}{10}(11.19 - y_3 + z_3) = 1.23$

$y_4 = \frac{1}{10}(28.08 - x_3 - z_3) = 2.34$

$z_4 = \frac{1}{10}(35.61 + x_3 - y_3) = 3.45$

Fifth iteration

$x_5 = \frac{1}{10}(11.19 - y_4 + z_4) = 1.23$

$y_5 = \frac{1}{10}(28.08 - x_4 - z_4) = 2.34$

$z_5 = \frac{1}{10}(35.61 + x_4 - y_4) = 3.45$

Hence $x = 1.23, y = 2.34, z = 3.45$

11.8 GAUSS-SEIDEL ITERATION METHOD

This is a modification of Jacobi's method. As before the system of equations :

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned} \qquad (1)$$

is written as

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \qquad (2)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

Here also we start with the initial approximations x_0, y_0, z_0 for x, y, z respectively which may each be taken as zero. Substituting $y = y_0, z = z_0$ in the first of the equations (2), we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

Then putting $x = x_1, z = z_0$ in the second of the equations (2), we have

$$y = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_0)$$

Next substituting $x = x_1, y = y_1$ in the third of the equations (2), we obtain

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

and so on i.e. as soon as a new approximation for an unknown is found, it is immediately used in the next step. This process of iteration is repeated till the values of x, y, z are obtained to desired degree of accuracy.

Note 1: Since the most recent approximations of the unknowns are used while proceeding to the next step, the convergence in the Gauss-Seidal method is twice as fast as in Jacobi's method.

Note 2: Jacobi and Gauss-Seidal methods converge for any choice of the initial approximations if in each equation of the system, the absolute value of the largest co-efficient is almost equal to or in at least one equation greater than the sum of the absolute values of all the remaining coefficients.

Example 9: Apply Gauss-Seidal iteration method to solve the equations $20x + y - 2z = 17; 3x + 20y - z = -18; 2x - 3y + 20z = 25$

Sol. We write the given equations in the form

$$x = \frac{1}{20}(17 - y + 2z) \qquad \text{(i)}$$

$$y = \frac{1}{20}(-18 - 3x + z) \qquad \text{(ii)}$$

$$z = \frac{1}{20}(25 - 2x + 3y) \qquad \text{(iii)}$$

First iteration

Putting $y = y_0, z = z_0$ in (i), we get

$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$

Putting $x = x_1, z = z_0$ in (ii), we have

$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$

Putting $x = x_1, y = y_1$ in (iii), we obtain

$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$ \)

Second iteration

Putting $y = y_1, z = z_1$ in (i), we get $x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$

Putting $x = x_2, z = z_1$ in (ii), we obtain $y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$

Putting $x = x_2, y = y_2$ in (iii), we get $z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$

Third iteration, we get

$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$

$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$

$z_3 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 1.0000$

The values in the 2nd and 3rd iterations being practically the same, we can stop. Hence the solution is $x = 1, y = -1, z = 1$.

Example 10: Solve the equations $27x + 3y + z = 85$; $x + y + 54z = 110$; $6x + 15y + 2z = 72$ by Gauss-Jacobi method and Gauss-Seidal method.

Sol. Rewriting the given equations as

$x = \frac{1}{27}(85 - 6y + z)$

$y = \frac{1}{15}(72 - 6x - 2z)$

$z = \frac{1}{54}(110 - 2x - 5y)$

(a) Gauss-Jacobi's method

We start from an approximation $x_0 = y_0 = z_0 = 0$

First iteration

$x_1 = \frac{85}{27} = 3.148,\ y_1 = \frac{72}{15} = 4.8, z_1 = \frac{110}{54} = 2.037$

Second iteration

$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.157$

$y_2 = \frac{1}{15}(72 - 6x_1 - 2z_1) = 3.269$

$z_2 = \frac{1}{54}(110 - 2x_1 - 5y_1) = 1.890$

Third iteration

$x_3 = \frac{1}{27}(85 - 6y_2 + z_2) = 2.492$

$y_3 = \frac{1}{15}(72 - 6x_2 - 2z_2) = 3.685$

$z_3 = \frac{1}{54}(110 - 2x_2 - 5y_2) = 1.937$

Fourth iteration

$x_4 = \frac{1}{27}(85 - 6y_3 + z_3) = 2.401$

$y_4 = \frac{1}{15}(72 - 6x_3 - 2z_3) = 3.545$

$z_4 = \frac{1}{54}(110 - x_3 - y_3) = 1.923$

Fifth iteration

$x_5 = \frac{1}{27}(85 - 6y_4 + z_4) = 2.432$

$y_5 = \frac{1}{15}(72 - 6x_4 - 2z_4) = 3.583$

$z_5 = \frac{1}{54}(110 - x_4 - y_4) = 1.927$

Repeating this process, the successive iterations are

$x_6 = 2.423, y_6 = 3.570, z_6 = 1.926$

$x_7 = 2.426, y_7 = 3.574, z_7 = 1.926$

$x_8 = 2.425, y_8 = 3.573, z_8 = 1.926$

Hence $x = 2.426, y = 3.573, z = 1.926$

(b) Gauss-Seidal method

First iteration

Putting $y = y_0 = 0, z = z_0 = 0$ in (i),

$x_1 = \frac{1}{27}(85 - 6y_0 + z_0) = 3.14$

Putting $x = x_1, z = z_0$ in (ii),

$y_1 = \frac{1}{15}(72 - 6x_1 - 2z_0) = 3.541$

Putting $x = x_1, y = y_1$ in (iii),

$z_1 = \frac{1}{54}(110 - x_1 - y_1) = 1.913$

Second iteration

$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.432$

$y_2 = \frac{1}{15}(72 - 6x_2 - 2z_1) = 3.572$

$z_2 = \frac{1}{54}(110 - x_2 - y_2) = 1.926$

Third iteration

$x_3 = \frac{1}{27}(85 - 6y_2 + z_2) = 2.426$

$y_3 = \frac{1}{15}(72 - 6x_3 - 2z_2) = 3.573$

$z_3 = \frac{1}{54}(110 - x_3 - y_3) = 1.926$

Fourth iteration

$x_4 = \frac{1}{27}(85 - 6y_3 + z_3) = 2.426$

$y_4 = \frac{1}{15}(72 - 6x_3 - 2z_3) = 3.573$

$z_4 = \frac{1}{54}(110 - x_3 - y_3) = 1.926$

Hence $x = 2.426, y = 3.573, z = 1.926$

Observation: We have seen that the convergence is quite fast in Gauss-Seidal method as compared to Gauss-Jacobi method.

11.9 ILL CONDITIONED EQUATIONS

A linear system is said to be ill-conditioned if small changes in the coefficients of the equations result in large changes in the values of the unknowns. On the contrary, a system is well-conditioned if small changes in the coefficients of the system also produce small changes in the solution. We often come across ill-conditioned systems in practical applications. Ill-conditioning of a system is usually expected when the determinant of the coefficient matrix is small. The coefficient matrix of an ill-conditioned system is called an ill-conditioned matrix.

While solving simultaneous equations, we also come across two forms of instabilities. Inherent and Induced. Inherent instability of a system is a property of the given problem and occurs due to the problem being ill-conditioned. It can be avoided by reformulation of the problem suitably. Induced instability occurs because of the incorrect choice of method.

Iterative method to improve accuracy of an ill-conditioned system: Consider the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2 \qquad (1)$$

$$a_3 x + b_3 y + c_3 z = d_3$$

Let x', y', z' be an approximate solution. Substituting these values on the left-hand sides, we get new values of d_1, d_2, d_3 as d_1', d_2', d_3' so that the new system is

$$a_1 x' + b_1 y' + c_1 z' = d_1'$$

$$a_2 x' + b_2 y' + c_2 z' = d_2' \qquad (2)$$

$$a_3 x' + b_3 y' + c_3 z' = d_3'$$

Subtracting each equation in (2) from the corresponding equations in (1), we obtain

$$a_1 x_e + b_1 y_e + c_1 z_e = k_1$$

$$a_2 x_e + b_2 y_e + c_2 z_e = k_2 \qquad (3)$$

$$a_3 x_e + b_3 y_e + c_3 z_e = k_3$$

where $x_e = x - x', y_e = y - y', z_e = z - z'$ and $k_i = d_i - d_i'$

We now solve the system (3) for x_e, y_e, z_e giving $x = x + x_e, y = y' + y_e$ and $z = z' + z_e$ which will be better approximations for x, y, z. We can repeat the procedure for improving the accuracy.

Example 11: Establish whether the system $1.01x + 2y = 2.01; x + 2y = 2$ is well-conditioned or not ?

Sol. Its solution is $x = 1$ and $y = 0.5$

Now consider the system $x + 2.01y = 2.01$ and $x + 2y = 2$

which has the solution $x = -6$ and $y = 4$.

Hence the system is ill-conditioned.

Example 12: An approximate solution of the system $2x + 2y - z = 6$; $x + y + 2z = 8$; $-x + 3y + 2z = 4$ is given by $x = 2.8, y = 1, z = 1.8$. Using the above iterative method, improve this solution.

Sol. Substituting the approximate values $x' = 2.8, y' = 1, z' = 1.8$ in the given equations, we get

$2(2.8) + 2(1) - 1.8 = 5.8$

$-2.8 + 1 + 2(1.8) = 7.4$ (i)

$-2.8 + 3(1) + 2(1.8) = 3.8$

Subtracting each equation in (i) from the corresponding given equations, we obtain

$2x_e + 2y_e - z_e = 0.2$

$-x_e + y_e + 2z_e = 0.6$ (ii)

$-x_e + 3y_e + 2z_e = 0.2$

where $x_e = x - 2.8, y_e = y - 1, z_e = z - 1.8$.

Solving the equations (ii), we get $x_e = 0.2, y_e = 0, z_e = 0.2$.

This gives the better solution $x = 3, y = 1, z = 2$, which incidentally is the exact solution.

CHECK YOUR PROGRESS

Q4. Solve by Jacobi's method $5x - y + z = 10$; $2x + 4y = 12$; $x + y + 5z = -1$, starting with the solution (2,3,0).

Q5. Solve using Gauss-Siedal method: $4x + 2y + z = 14$; $x + 5y - z = 10$; $x + y + 8z = 20$

COMPARISON OF VARIOUS METHODS: Direct and iterative methods have their advantages and disadvantages and a choice of method depends on a particular system of equations. The direct methods yield a solution in a finite number of steps for any non-singular set of equations, while in an iterative method the amount of computation depends on the accuracy desired. In general, it is preferable to use a direct method for the solution of a linear system. However for large systems, an iterative method yields the solution faster and should therefore, be preferred.

Gauss elimination method requires more of recording and is quite time consuming for operations. As such it is more expensive from the programming point of view. Among the direct methods, Crout's triangularisation method is used more often for the solution of a linear system and as software for computers.

The rounding off errors also get propagated in the elimination method whereas in the iteration techniques only the rounding off errors committed in the final iteration have any effect. In general, the iteration methods have smaller round-off errors for iteration is a self-correcting technique. Thus the use of an iterative method for ill-conditioned system is preferable.

On the other hand, an iterative method may not always converge. When it converges, the iterative method is definitely better than the direct methods.

We come across two types of instabilities while solving a linear system of equations i.e. Inherent instability and Induced instability. Inherent instability occurs due to the set of equations being ill-conditioned and as such is a property of the problem itself. It can, however, be avoided by a suitable reformulation of the problem. On the other hand, induced instability occurs due to an incorrect choice of the method of solution.

11.10 LET US SUM UP

- This unit presents a comprehensive study of methods for solving systems of linear simultaneous equations, which are central to many problems in science and engineering.
- It introduces both direct methods—such as Gauss Elimination, Gauss-Jordan, and LU Decomposition—and iterative methods like Jacobi and Gauss-Seidel.
- Direct methods provide exact solutions through a finite sequence of operations and are typically preferred for smaller systems due to their precision. On the other hand, iterative methods start with an initial guess and improve the solution through repeated approximations, making them more efficient for large-scale systems and better suited for ill-conditioned equations.
- The unit also discusses the concept of ill-conditioned systems, where small changes in coefficients can lead to significant changes in the solution, and emphasizes the role of iterative refinement in improving accuracy.
- Overall, the choice between direct and iterative methods depends on the nature of the system, desired accuracy, and computational resources.

11.11 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1.: We have $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & -3 & -1 \\ 2 & -1 & -4 & 5 \\ 3 & 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 7 \\ 2 \end{bmatrix}$

Operate $R_2 - 2R_1, R_3 - 2R_1, R_4 - 3R_1$

$$\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & -1 & -9 & 1\\ 0 & -5 & -4 & 5\\ 0 & -4 & -13 & 6\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ u\end{bmatrix}=\begin{bmatrix}10\\ -19\\ -13\\ -28\end{bmatrix}$$

Operate $R_3 - 5R_2, R_4 - 4R_2$ $\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & -1 & -9 & 1\\ 0 & 0 & 41 & 0\\ 0 & 0 & 23 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ u\end{bmatrix}=\begin{bmatrix}10\\ -19\\ 82\\ 48\end{bmatrix}$

Thus, we have $41z = 82\, i.e., z = 2,$

$$23z + 2u = 48\, i.e., 46 + 2u = 48, \qquad \therefore u = 1$$

$$-y - 9z + u = -19\, i.e., -y - 18 + 1 = -19, \quad \therefore y = 2$$

$$x + 2y + 3z - u = 10\, i.e., x + 4 - 6 - 1 = 10, \quad \therefore x = 1$$

Hence $x = 1, y = 2, z = 2, u = 1.$

Answer to Q2.: Given system of equations:

$$2x + 5y + 7z = 52$$

$$2x + y - z = 0$$

$$x + y + z = 9$$

Now $\begin{bmatrix}2 & 5 & 7\\ 2 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}52\\ 0\\ 9\end{bmatrix}$

Operate $R_1 \leftrightarrow R_3$

$$\begin{bmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ 2 & 5 & 7\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}9\\ 0\\ 52\end{bmatrix}$$

Operate $R_2 \to R_2 - \mathbf{2} \times R_1,\ R_3 \to R_3 - \mathbf{2} \times R_2$

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 3 & 5\\ 0 & -1 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}9\\ 34\\ -18\end{bmatrix}$$

Operate $R_2 \to \frac{1}{3}R_2$ $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 5/3\\ 0 & -1 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}9\\ 34/3\\ -18\end{bmatrix}$

Operate $R_1 \to R_1 - R_2\ \ R_3 \to R_3 + R_2$

$$\begin{bmatrix}1 & 0 & -2/3\\ 0 & 1 & 5/3\\ 0 & 0 & -4/3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-7/3\\ 34/3\\ -20/3\end{bmatrix}$$

Operate $R_3 \rightarrow \frac{-3}{4}R_3$

$$\begin{bmatrix}1 & 0 & -2/3\\ 0 & 1 & 5/3\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-7/3\\ 34/3\\ 5\end{bmatrix}$$

Operate $R_1 \rightarrow R_1 + \frac{2}{3}R_3$, $R_2 \rightarrow R_2 - \frac{5}{3}R_3$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 5\end{bmatrix}$$

Hence $x = 1, y = 1, z = 5$

Answer to Q3.: given system:

$$2x + 3y + z = 9$$

$$x + 2y + 3z = 6$$

$$3x + y + 2z = 8$$

This can be represented in matrix form as AX = B, where:

$$\boldsymbol{A} = \begin{bmatrix}2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2\end{bmatrix}, \boldsymbol{X} = \begin{bmatrix}x\\ y\\ z\end{bmatrix}, \boldsymbol{B} = \begin{bmatrix}9\\ 6\\ 8\end{bmatrix}$$

We decompose A into L and U, where: $L = \begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}, U = \begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$

$$\therefore u_{11} = 2, u_{12} = 3, u_{13} = 1$$

$$l_{21}u_{11} = 1, \therefore l_{21} = \frac{1}{2}$$

$$l_{31}u_{11} = 3, l_{31} = \frac{3}{2}$$

$$l_{21}u_{12} + u_{22} = 2, \therefore u_{22} = \frac{1}{2}$$

$$l_{21}u_{13} + u_{23} = 3, \therefore u_{23} = \frac{5}{2}$$

$$l_{31}u_{12} + l_{32}u_{22} = 1, \therefore l_{32} = -7$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2, u_{33} = 18.$$

So, the matrices are: $L = \begin{bmatrix}1 & 0 & 0\\ 1/2 & 1 & 0\\ 3/2 & -7 & 1\end{bmatrix}, U = \begin{bmatrix}2 & 3 & 1\\ 0 & 1/2 & 5/2\\ 0 & 0 & 18\end{bmatrix}$

Writing $UX = V$, the given system becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Solving this system, we have $v_1 = 9$,

$$\frac{1}{2} v_1 + v_2 = 6 \; or \; v_2 = 3/2$$

$$\frac{3}{2} v^1 - 7 v^2 + v_3 = 8 \; or \; v_3 = 5$$

Hence the original system becomes

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix},$$

By back-substitution, we have

$$z = \frac{5}{18}, y = \frac{29}{18} \; and \; x = \frac{35}{18}$$

The solution to the system is**:** $x = \frac{35}{18}, y = \frac{29}{18}, z = \frac{5}{18}$

Answer to Q4.: Given

$5x - y + z = 10$

$2x + 4y = 12$

$x + y + 5z = -1$

Rewriting each equation:

$x = (1/5)(10 + y - z)$

$y = (1/4)(12 - 2x)$

$z = (1/5)(-1 - x - y)$

Initial values: $x_0 = 2, y_0 = 3, z_0 = 0$

Iteration 1:

$x_1 = (1/5)(10 + 3 - 0) = 13/5 = 2.6$

$y_1 = (1/4)(12 - 2 \times 2) = 2.0$

$z_1 = (1/5)(-1 - 2 - 3) = -6/5 = -1.2$

Iteration 2:

$x_2 = (1/5)(10 + 2.0 - (-1.2)) = 13.2/5 = 2.64$

$y_2 = (1/4)(12 - 2 \times 2.6) = 6.8/4 = 1.7$

$z_2 = (1/5)(-1 - 2.6 - 2.0) = -5.6/5 = -1.12$

Iteration 3:

$x_3 = (1/5)(10 + 1.7 - (-1.12)) = 12.82/5 = 2.564$

$y_3 = (1/4)(12 - 2 \times 2.64) = 6.72/4 = 1.68$

$z_3 = (1/5)(-1 - 2.64 - 1.7) = -5.34/5 = -1.068$

Approximate solution after 3 iterations:

$x \approx 2.564, y \approx 1.68, z \approx -1.068$

Answer to Q5.: Given

$4x + 2y + z = 14$

$x + 5y - z = 10$

$x + y + 8z = 20$

Rewriting each equation:

$x = (1/4)(14 - 2y - z)$

$y = (1/5)(10 - x + z)$

$z = (1/8)(20 - x - y)$

Initial value: $x_0 = 0, y_0 = 0, z_0 = 0$

Iteration 1
$x_1 = (1/4)(14 - 0 - 0) = 14/4 = 3.5$
$y_1 = (1/5)(10 - 3.5 + 0) = 6.5/5 = 1.3$
$z_1 = (1/8)(20 - 3.5 - 1.3) = 15.2/8 = 1.9$

Iteration 2
$x_2 = (1/4)(14 - 2 \times 1.3 - 1.9) = (14 - 2.6 - 1.9)/4 = 9.5/4 = 2.375$
$y_2 = (1/5)(10 - 2.375 + 1.9) = 9.525/5 = 1.905$
$z_2 = (1/8)(20 - 2.375 - 1.905) = 15.72/8 \approx 1.965$

Iteration 3
$x_3 = (1/4)(14 - 2 \times 1.905 - 1.965) = (14 - 3.81 - 1.965)/4 = 8.225/4 = 2.056$

$y_3 = (1/5)(10 - 2.056 + 1.965) = 9.909/5 \approx 1.982$
$z_3 = (1/8)(20 - 2.056 - 1.982) = 15.962/8 \approx 1.995$

Approximate solution after 3 iterations:
$x \approx 2.056 \quad y \approx 1.982$ and $z \approx 1.995$

11.12 FURTHER READING

1. Gupta, R. K. *Numerical Methods: Fundamentals and Applications*. 1st edition, Cambridge University Press, 2019.
2. Thangaraj, P. *Computer Oriented Numerical Methods*. PHI Learning Pvt. Ltd, 2013.
3. Jain, M. K., Iyengar, S. R. K. and Jain, R. K. *Numerical Methods for Scientific & Engineering Computation*. New Age International, 2012.

11.13 MODEL QUESTIONS

1. An approximate solution of the equations $x + 4y + 7z = 5;\ 2x + 5y + 8z = 7;\ 3x + 6y + 9.1z = 9.1$ is given by $x = 1.8, y = -1.2, z = 1$. Improve this solution by using the iterative method.

2. Establish whether the system of equations

$10x + 8y + 9z + 6w = 33,$

$6x + 7y + 5z + 5w = 23,$

$8x + 10y + 7z + 7w = 32,$

$9x + 7y + 10z + 5w = 31$ is well-conditioned or not ?

3. Solve By Gauss-Siedal method:

(i) $28x + 4y - z = 32;\ x + 3y + 10z = 24;\ 2x + 17y + 4z = 35$
(ii) $10x + y + z = 12\,;\ 2x + 10y + z = 13\,;\ 2x + 2y + 10z = 14$

4. Solve the following equations by Gauss-Jordan method:

(i) $10x + y = 12\,;\ 2x + 10y - z = 13\,;\ x + y + 5z = 7$
(ii) $x + y + z = 9\,;\ 2x + y - z = 0\,;\ 2x + 5y + 7z = 52$

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