UNIT 12: NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATION

UNIT STRUCTURE

12.1 Learning Objectives

12.2 Introduction

12.3 Finite-difference approximations to partial derivative

12.4 Solution to Laplace equation

12.5 Solution of one-dimensional heat equation

12.6 Solution of wave equation

12.7 Let Us Sum Up

12.8 Further Reading

12.9 Answer to Check Your Progress

12.10 Model Questions

12.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Learn the concept of finite-difference approximations for partial derivatives.
- Solve the Laplace equation using the standard five-point formula and diagonal five-point formula.
- Implement numerical solutions to the one-dimensional heat equation using: Schmidt explicit method, Crank-Nicolson implicit method, Du Fort-Frankel three-level method
- Solve the wave equation using both explicit and implicit finite difference schemes.

12.2 INTRODUCTION

Analytical methods can only solve a limited number of partial differential equations and often require advanced mathematical techniques. The majority of the time, using numerical techniques to produce approximation solutions is simpler. The method of finite differences is the most often used numerical technique for solving partial differential equations. This approach substitutes the finite difference approximations for the derivatives that exist in the equation and the boundary conditions. After that, the

given problem is transformed into a system of linear equations that may be solved iteratively. In many boundary value situations, this method yields good solutions despite its slowness. This method has the extra benefit of allowing computation to be done using electronic computers. In this chapter, we will briefly discuss the numerical solutions to the Laplace, heat, and wave equations.

12.3 FINITE-DIFFERENCE APPROXIMATIONS TO PARTIAL DERIVATIVE

Consider a rectangular region R in the x, y plane. Divide this region into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$ as shown in the fig. 12.1. The points of intersection of the dividing lines are called mesh points, nodal points, or grid points.

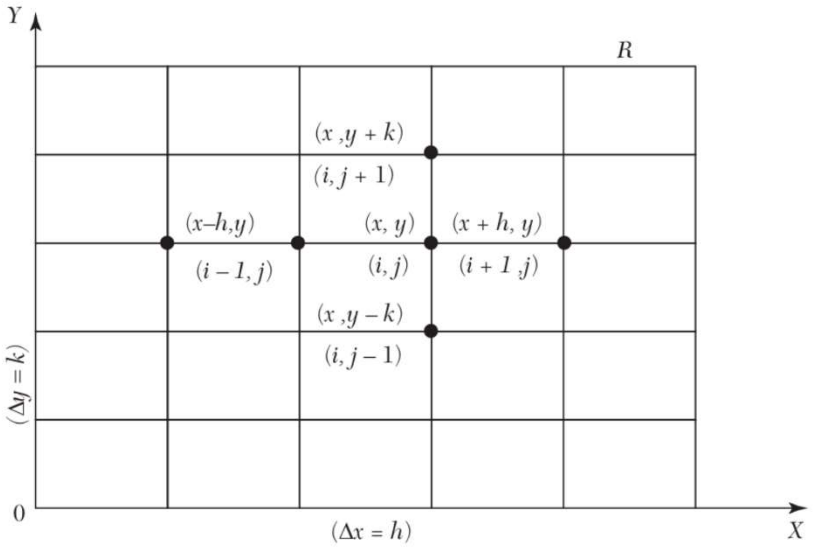


Fig. 12.1

Then we have the finite difference approximations for the partial derivatives in x direction

$$\frac{\partial u}{\partial x} = \frac{u(x+h,y) - u(x,y)}{h} + O(h)$$

$$= \frac{u(x,y) - u(x-h,y)}{h} + O(h)$$

$$= \frac{u(x+h,y) - u(x-h,y)}{2h} + O(h^2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x-h,y) - 2u(x,y) + u(x+h,y)}{h^2} + O(h^2)$$

Writing $u(x,y) = u(ih, jk)$ as simply $u_{i,j}$, the above approximations become

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h) \quad (1)$$

$$= \frac{u_{i,j} - u_{i-1,j}}{h} + O(h) \quad (2)$$

$$= \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2) \quad (3)$$

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2) \quad (4)$$

Similarly, we have the approximations for the derivatives w.r.t. y

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k) \quad (5)$$

$$= \frac{u_{i,j} - u_{i,j-1}}{k} + O(k) \quad (6)$$

$$= \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2) \quad (7)$$

$$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + O(k^2) \quad (8)$$

Replacing the derivatives in any partial differential equation by their corresponding difference approximations (1) to (8), we obtain the finite-difference analogues of the given equation.

12.4 SOLUTION TO LAPLACE EQUATION

The Laplace equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is an example of elliptic partial differential equations. The Laplace equation arises in steady-state flow and potential problems.

The solution of this equation is a function $u(x,y)$ which is satisfied at every point of a region R subject to certain boundary conditions specified on the closed curve C (Figure 12.2).

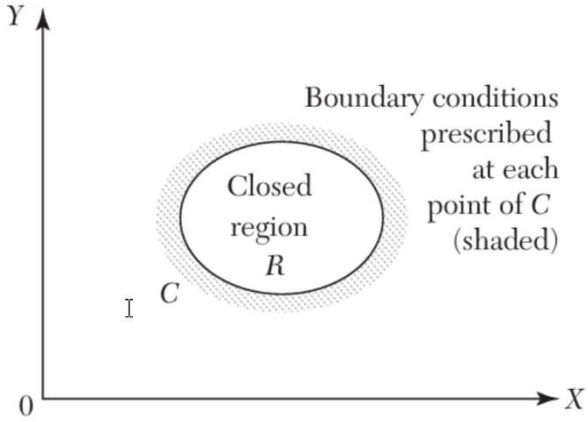


Fig. 12.2

In general, problems concerning steady viscous flow, equilibrium stresses in elastic structures etc., lead to elliptic type of equations.

We have,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

Consider a rectangular region R for which $u(x, y)$ is known at the boundary. Divide this region into a network of square mesh of side h, as shown in fig. 12.3 (assuming that an exact sub-division of R is possible). Replacing the derivatives in (1) by their difference approximations, we have

$$\left[\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right] + \left[\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}\right] = 0$$

$$\Rightarrow u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right] \qquad (2)$$

This shows that the value of u at any interior mesh point is the average of its values at four neighboring points to the left, right, above and below. (2) is called the standard 5-point formula which is exhibited in fig. 12.4.

Sometimes a formula similar to (2) is used which is given by

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}\right] \qquad (3)$$

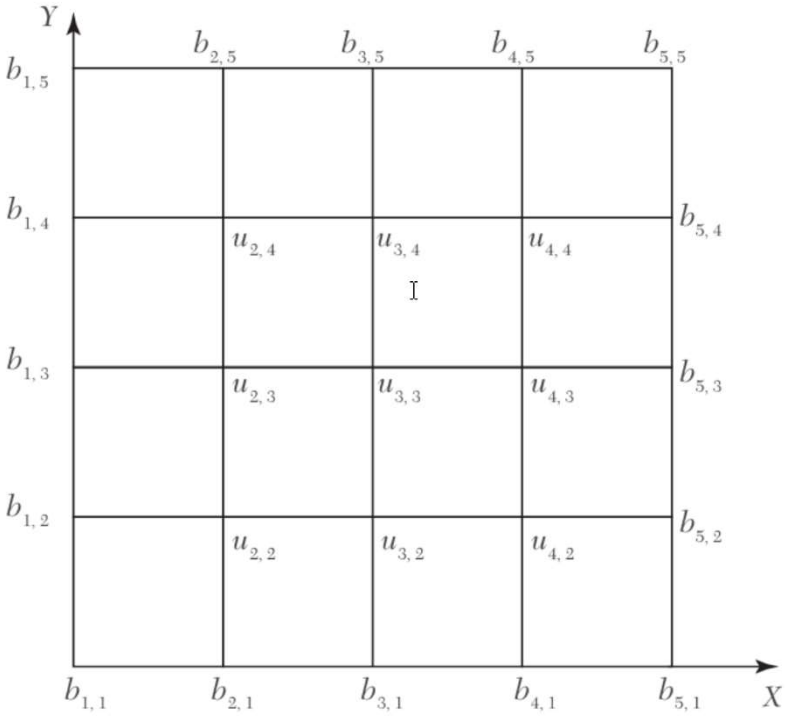


Fig. 12.3

This shows that the value of $u_{i,j}$ is the average of its values at the four neighboring diagonal mesh points. (3) is called the diagonal five-point formula which is represented in fig. 12.5. Although (3) is less accurate than (2), yet it serves as a reasonably good approximation for obtaining the starting values at the mesh points.

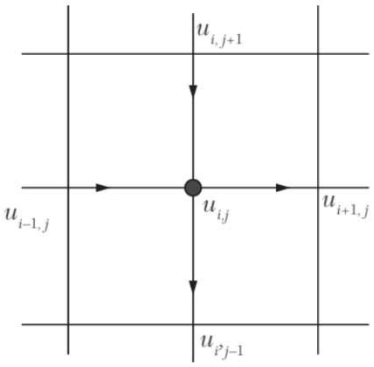


Fig. 12.4

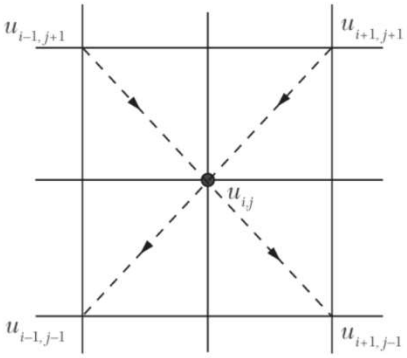


Fig 12.5

Now to find the initial values of u at the interior mesh points, we first use the diagonal five-point formula (3) and compute $u_{3,3}, u_{2,4}, u_{4,4}, u_{4,2}$, and $u_{2,2}$, in this order. Thus, we get,

$$u_{3,3} = \frac{1}{4}\left[b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1}\right]$$

$$u_{2,4} = \frac{1}{4}\left[b_{1,5} + b_{3,3} + b_{3,5} + b_{1,3}\right]$$

$$u_{4,4} = \frac{1}{4}\left[b_{3,5} + b_{5,3} + b_{5,5} + b_{3,3}\right]$$

$$u_{4,2} = \frac{1}{4}\left[b_{3,3} + b_{5,1} + b_{5,3} + b_{3,1}\right]$$

$$u_{2,2} = \frac{1}{4}\left[b_{1,3} + b_{3,1} + b_{3,3} + b_{1,1}\right]$$

The values at the remaining interior points, i.e., $u_{2,3}, u_{3,4}, u_{4,3}$ and $u_{3,2}$ are computed by the standard five-point formula (2). Thus, we obtain

$$u_{2,3} = \frac{1}{4}\left[b_{1,3} + b_{3,3} + b_{2,4} + b_{2,2}\right]$$

$$u_{3,4} = \frac{1}{4}\left[b_{2,4} + b_{4,4} + b_{3,5} + b_{3,3}\right]$$

$$u_{4,3} = \frac{1}{4}\left[b_{3,3} + b_{5,3} + b_{4,4} + b_{4,2}\right]$$

$$u_{3,2} = \frac{1}{4}\left[b_{2,2} + b_{4,2} + b_{3,3} + b_{3,1}\right]$$

Having found all the nine values of $u_{i,j}$ once, their accuracy is improved by either of the following iterative methods. In each case, the method is repeated until the difference between two consecutive iterates becomes negligible.

(i) **Jacobi's method**. Denoting the *n*th iterative value of $u_{i,j}$ by ${u^n}_{i,j}$ the iterative formula to solve (2) is

$${u^{n+1}}_{i,j} = \frac{1}{4}\left[{u^n}_{i-1,j} + {u^n}_{i+1,j} + {u^n}_{i,j+1} + {u^n}_{i,j-1}\right]$$

It gives improved values of $u_{i,j}$ at the interior mesh points and is called the point Jacobi's formula.

(ii) **Gauss-Seidel method.** In this method, the iteration formula is

$${u^{n+1}}_{i,j} = \frac{1}{4}\left[{u^{n+1}}_{i-1,j} + {u^n}_{i+1,j} + {u^{n+1}}_{i,j+1} + {u^n}_{i,j-1}\right]$$

It utilizes the latest iterative value available and scans the mesh points symmetrically from left to right along successive rows.

The Gauss-Seidel method is simple and can be adapted to computer calculations. Its convergence being slow, the working is somewhat lengthy. It can however, be shown that the Gauss-Seidel scheme converges twice as fast as Jacobi's scheme. The accuracy of calculations depends on the mesh-size, i.e., smaller the h, the better the

accuracy. But if h is too small, it may increase rounding-off errors and also increases the labor of computation.

Example 1: Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in Fig. 12.5.

Solution:

Let $u_1, u_2, \ldots, u_9$ be the values of u at the interior mesh-points. Since the boundary values of u are symmetrical about AB,

$$u_7 = u_1, \qquad u_8 = u_2, \qquad u_9 = u_3$$

Also, the values of u being symmetrical about CD.

$$u_3 = u_1, \qquad u_6 = u_4, \qquad u_9 = u_7$$

Thus, it is sufficient to find the values u_1, u_2, u_4 and u_5.

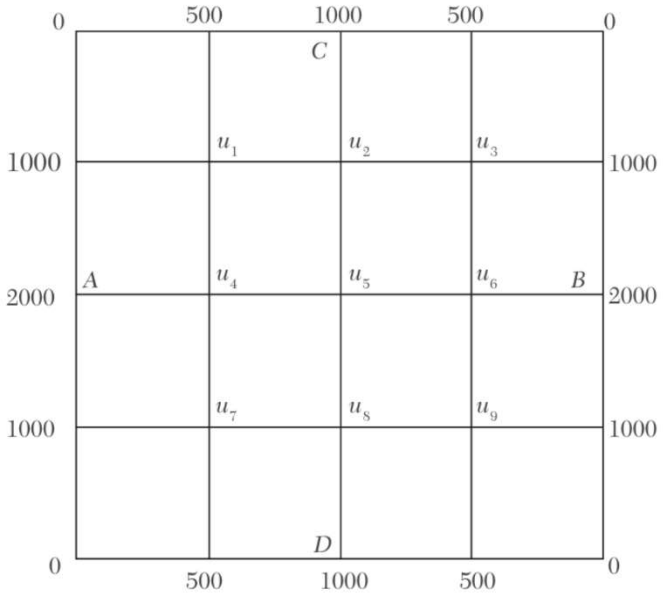


Fig. 12.5

Now we find their initial values in the following order:

$$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500 \qquad \text{(Std. formula)}$$

$$u_1 = \frac{1}{4}(0 + 1500 + 1000 + 2000) = 1125 \qquad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500) \approx 1188 \qquad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(2000 + 1500 + 1125 + 1125) \approx 1438 \qquad \text{(Std. formula)}$$

Now we carry out the iteration process using the standard formulae:

$$u_1^{n+1} = \frac{1}{4}[1000 + u_2^n + 500 + u_4^n]$$

$$u_2^{n+1} = \frac{1}{4}[u_1^{n+1} + u_1^n + 1000 + u_5^n]$$

$$u_4^{n+1} = \frac{1}{4}[2000 + u_5^n + u_1^{n+1} + u_1^n]$$

$$u_5^{n+1} = \frac{1}{4}[u_4^{n+1} + u_4^n + u_2^{n+1} + u_2^n]$$

First iteration: (put $n = 0$ in the above results)

$$u_1^1 = \frac{1}{4}[1000 + 1188 + 500 + 1438] \approx 1032$$

$$u_2^1 = \frac{1}{4}[1032 + 1125 + 1000 + 1500] = 1164$$

$$u_4^1 = \frac{1}{4}[2000 + 1500 + 1032 + 1125] = 1414$$

$$u_5^1 = \frac{1}{4}[1414 + 1438 + 1164 + 1188] = 1301$$

Second iteration: (put $n = 1$)

$$u_1^2 = \frac{1}{4}[1000 + 1164 + 500 + 1414] = 1020$$

$$u_2^2 = \frac{1}{4}[1020 + 1032 + 1000 + 1301] = 1088$$

$$u_4^2 = \frac{1}{4}[2000 + 1301 + 1020 + 1032] = 1338$$

$$u_5^2 = \frac{1}{4}[1338 + 1414 + 1088 + 1164] = 1251$$

Third iteration:

$$u_1^3 = \frac{1}{4}[1000 + 1088 + 500 + 1338] = 982$$

$$u_2^3 = \frac{1}{4}[982 + 1020 + 1000 + 1251] = 1063$$

$$u_4^3 = \frac{1}{4}[2000 + 1251 + 982 + 1020] = 1313$$

$$u_5^3 = \frac{1}{4}[1313 + 1338 + 1063 + 1088] = 1201$$

Fourth iteration:

$$u_1^4 = \frac{1}{4}[1000 + 1063 + 500 + 1313] = 969$$

$$u_2^4 = \frac{1}{4}[969 + 982 + 1000 + 1201] = 1038$$

$$u_4^4 = \frac{1}{4}[2000 + 1201 + 969 + 982] = 1288$$

$$u_5^4 = \frac{1}{4}[1288 + 1313 + 1038 + 1063] = 1176$$

Fifth iteration:

$$u_1^5 = \frac{1}{4}[1000 + 1038 + 500 + 1288] = 957$$

$$u_2^5 = \frac{1}{4}[957 + 969 + 1000 + 1176] \approx 1026$$

$$u_4^5 = \frac{1}{4}[2000 + 1176 + 957 + 969] \approx 1276$$

$$u_5^5 = \frac{1}{4}[1276 + 1288 + 1026 + 1038] = 1157$$

Similarly,

$$u_1^6 = 951, \quad u_2^6 = 1016, \quad u_4^6 = 1266, \quad u_5^6 = 1146$$
$$u_1^7 = 946, \quad u_2^7 = 1011, \quad u_4^7 = 1260, \quad u_5^7 = 1138$$
$$u_1^8 = 943, \quad u_2^8 = 1007, \quad u_4^8 = 1257, \quad u_5^8 = 1134$$
$$u_1^9 = 941, \quad u_2^9 = 1005, \quad u_4^9 = 1255, \quad u_5^9 = 1131$$
$$u_1^{10} = 940, \quad u_2^{10} = 1003, \quad u_4^{10} = 1253, \quad u_5^{10} = 1129$$
$$u_1^{11} = 939, \quad u_2^{11} = 1002, \quad u_4^{11} = 1252, \quad u_5^{11} = 1128$$
$$u_1^{12} \approx 939, \quad u_2^{12} \approx 1001, \quad u_4^{12} \approx 1251, \quad u_5^{12} = 1126$$

There is a negligible difference between the values obtained in the eleventh and twelfth iterations.

Hence $u_1 = 939, u_2 = 1001$, $u_4 = 1251$ and $u_5 = 1126$

Example 2: Given the values of $u(x, y)$ on the boundary of the square in the Figure 12.6, evaluate the function $u(x, y)$ satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of this figure by (a) Jacobi's method (b) Gauss-Seidel method

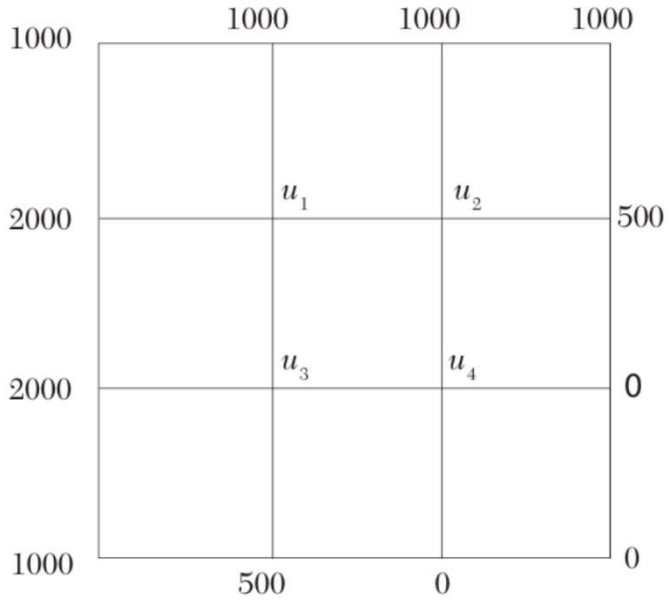


Fig. 12.6

Solution:

To get the initial values of u_1, u_2, u_3, u_4 we assume that $u_4 = 0$. Then

$$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000 \qquad \text{(Diag formula)}$$

$$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625 \qquad \text{(Std. formula)}$$

$$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875 \qquad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375 \qquad \text{(Std. formula)}$$

(a) We carry out the successive iterations, using Jacobi's formulae:

$$u_1^{n+1} = \frac{1}{4}[2000 + u_2^n + 1000 + u_3^n]$$

$$u_2^{n+1} = \frac{1}{4}[u_1^n + 500 + 1000 + u_4^n]$$

$$u_3^{n+1} = \frac{1}{4}[2000 + u_4^n + u_1^n + 500]$$

$$u_4^{n+1} = \frac{1}{4}[u_3^n + 0 + u_2^n + 0]$$

First iteration: (put $n = 0$ in the above results)

$$u_1^1 = \frac{1}{4}[2000 + 625 + 1000 + 875] = 1125$$

$$u_2^1 = \frac{1}{4}[1000 + 500 + 1000 + 375] \approx 719$$

$$u_4^1 = \frac{1}{4}[2000 + 375 + 1000 + 500] \approx 969$$

$$u_5^1 = \frac{1}{4}[875 + 0 + 625 + 0] \approx 375$$

Second iteration: (put $n = 1$)

$$u_1^2 = \frac{1}{4}[2000 + 719 + 1000 + 969] = 1172$$

$$u_2^2 = \frac{1}{4}[1125 + 500 + 1000 + 375] = 750$$

$$u_3^2 = \frac{1}{4}[2000 + 375 + 1125 + 500] = 1000$$

$$u_4^2 = \frac{1}{4}[969 + 0 + 719 + 0] = 422$$

Similarly,

$$u_1^3 \approx 1188, \quad u_2^3 \approx 774, \quad u_4^3 \approx 1024, \quad u_4^3 \approx 438$$
$$u_1^4 \approx 1200, \quad u_2^4 \approx 782, \quad u_3^4 \approx 1032, \quad u_4^4 \approx 450$$
$$u_1^5 \approx 1204, \quad u_2^5 \approx 788, \quad u_3^5 \approx 1038, \quad u_4^5 \approx 454$$
$$u_1^6 \approx 1206, \quad u_2^6 \approx 790, \quad u_3^6 \approx 1040, \quad u_4^6 \approx 456.5$$
$$u_1^7 \approx 1208, \quad u_2^7 \approx 791, \quad u_3^7 \approx 1041, \quad u_4^7 \approx 458$$
$$u_1^8 \approx 1208, \quad u_2^8 \approx 791.5, \quad u_3^8 \approx 1041.5, \quad u_4^8 \approx 458$$

There is no significant difference between the seventh and eighth iteration values.

Hence, $u_1 = 1208, u_2 = 792$, $u_3 = 1042$ and $u_4 = 458$

(b) We carry out the successive iterations, using Gauss-Seidel formulae

$$u_1^{n+1} = \frac{1}{4}[2000 + u_2^n + 1000 + u_3^n]$$

$$u_2^{n+1} = \frac{1}{4}[u_1^{n+1} + 500 + 1000 + u_4^n]$$

$$u_3^{n+1} = \frac{1}{4}[2000 + u_4^n + u_1^{n+1} + 500]$$

$$u_4^{n+1} = \frac{1}{4}[u_3^{n+1} + 0 + u_2^{n+1} + 0]$$

First iteration: (put $n = 0$ in the above results)

$$u_1^1 = \frac{1}{4}[2000 + 625 + 1000 + 875] = 1125$$

$$u_2^1 = \frac{1}{4}[1125 + 500 + 1000 + 375] = 750$$

$$u_4^1 = \frac{1}{4}[2000 + 375 + 1125 + 500] = 1000$$

$$u_5^1 = \frac{1}{4}[1000 + 0 + 750 + 0] \approx 438$$

Second iteration: (put $n = 1$)

$$u_1^2 = \frac{1}{4}[2000 + 750 + 1000 + 1000] \approx 1172$$

$$u_2^2 = \frac{1}{4}[1188 + 500 + 1000 + 438] \approx 782$$

$$u_3^2 = \frac{1}{4}[2000 + 438 + 1188 + 500] \approx 1032$$

$$u_4^2 = \frac{1}{4}[1032 + 0 + 782 + 0] \approx 454$$

Similarly,

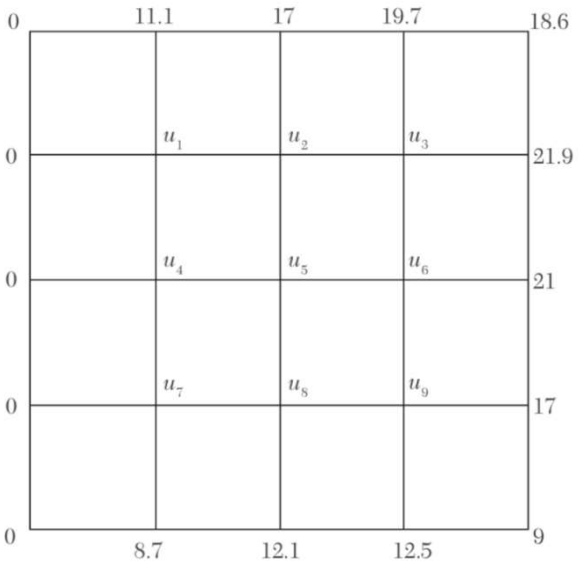
$$u_1^3 \approx 1204, \quad u_2^3 \approx 789, \quad u_4^3 \approx 1040, \quad u_4^3 \approx 458$$
$$u_1^4 \approx 1207, \quad u_2^4 \approx 791, \quad u_3^4 \approx 1041, \quad u_4^4 \approx 458$$
$$u_1^5 \approx 1208, \quad u_2^5 \approx 791.5, \quad u_3^5 \approx 1041.5, \quad u_4^5 \approx 458.25$$

There is no significant difference between the fourth and fifth iteration values.

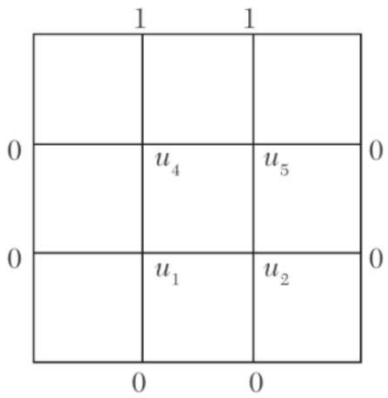
Hence, $u_1 = 1208, u_2 = 792, u_3 = 1042$ and $u_4 = 458$

CHECK YOUR PROGRESS 1

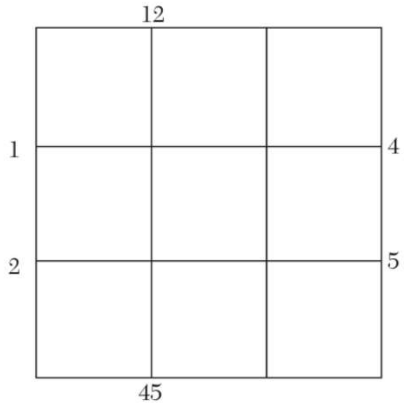
Q1. Solve the Laplace equation $u_{xx} + u_{yy} = 0$ given that



Q2. Solve the Laplace's equation $u_{xx} + u_{yy} = 0$ in the domain of given figure by (a) Jacobi's method, (b) Gauss-Seidel method.



Q3. Solve the equation $u_{xx} + u_{yy} = 0$ for the square mesh with the boundary values as shown in the below figure.



12.5 SOLUTION OF ONE-DIMENSIONAL HEAT EQUATION

The one-dimensional heat conduction equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is a well-known example of parabolic partial differential equations. The solution of this equation is a temperature function $u(x,t)$ which is defined for values of x from 0 to l and for values of time t from 0 to ∞. The solution is not defined in a closed domain but advances in an open-ended region from initial values, satisfying the prescribed boundary conditions (fig. 12.7).

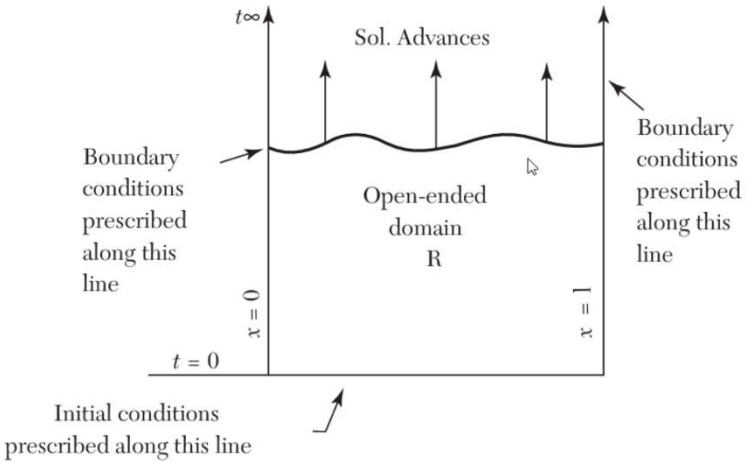


Fig. 12.7

In general, the study of pressure waves in a fluid, propagation of heat and unsteady state problems lead to parabolic type of equations.

We have,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

where $c^2 = \frac{k}{s\rho}$ is the diffusivity of the substance $(cm^2/sec.)$

Schmidt method. Consider a rectangular mesh in the $x - t$ plane with spacing h along x direction and k along time t direction. Denoting a mesh point $(x, t) = (ih, jk)$ as simply i, j, we have

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Substituting these in (1), we obtain

$$u_{i,j+1} - u_{i,j} = \frac{kc^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

$$u_{i,j+1} = \alpha u_{i-1,j} + (1 - 2\alpha)u_{i,j} + \alpha u_{i+1,j} \qquad (2)$$

where $\alpha = \frac{kc^2}{h^2}$ is the mesh ratio parameter.

This formula enables us to determine the value of u at the $(i, j+1)$th mesh point in terms of the known function values at the points x_{i-1}, x_i and x_{i+1} at the instant t_j. It is a relation between the function values at the two-time levels $j + 1$ and j and is therefore, called a two-level formula. In schematic form (2) is shown in fig. 12.8.

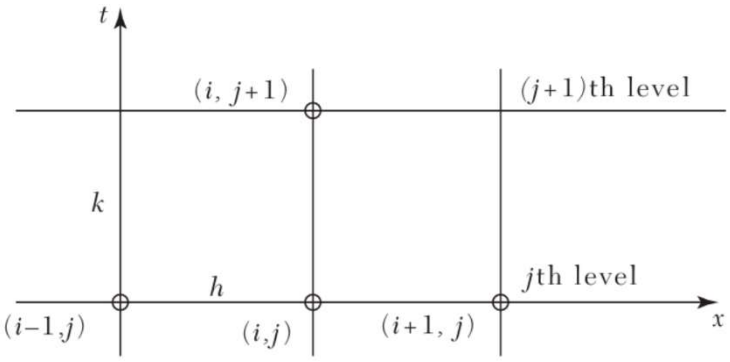


Fig. 12.8

Hence (2) is called the Schmidt explicit formula which is valid only for $0 \leq \alpha \leq 12$

In particular when $\alpha = \frac{1}{2}$, (2) reduces to

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right) \qquad (3)$$

which shows that the value of u at x_i at time t_{j+1} is the mean of the u-values at x_{i-1} and x_{i+1} at time t_j. This relation, known as Bendre-Schmidt recurrence relation, gives the values of u at the internal mesh points with the help of boundary conditions.

Crank-Nicolson method. We have seen that the Schmidt scheme is computationally simple and for convergent results $\alpha \leq 12$ i.e., $k \leq \frac{h^2}{2c^2}$. To obtain more accurate results, h should be small i.e. k is necessarily very small. This makes the computations exceptionally lengthy as more time levels would be required to cover the region. A method that does not restrict α and also reduces the volume of calculations was proposed by Crank and Nicolson in 1947.

According to this method, $\frac{\partial^2 u}{\partial x^2}$ is replaced by the average of its central-difference approximations on the jth and $(j+1)$th time rows. Thus (1) is reduced to

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{c^2}{2}\left[\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right] + \frac{c^2}{2}\left[\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right]$$

$$\Rightarrow -\alpha u_{i-1,j+1} + (2+2\alpha)u_{i,j+1} - \alpha u_{i+1,j+1}$$

$$= \alpha u_{i-1,j} + (2-2\alpha)u_{i,j} + \alpha u_{i+1,j} \qquad (4)$$

Where $\alpha = \frac{kc^2}{h^2}$

Clearly the left side of (4) contains three unknown values of u at the $(j+1)$th level while all the three values on the right are known values at the jth level. Thus (4) is a two-level implicit relation and is known as Crank-Nicolson formula. It is convergent for all finite values of α. Its computational model is given in fig 12.9.

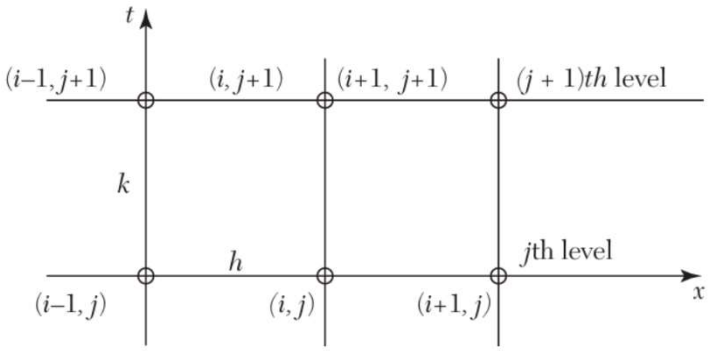


Fig. 12.9

If there are n internal mesh points on each row, then the relation (4) gives n simultaneous equations for the n unknown values in terms of the known boundary values. These equations can be solved to obtain the values at these mesh points. Similarly, the values at the internal mesh points on all rows can be found. A method such as this in which the calculation of an unknown mesh value necessitates the solution of a set of simultaneous equations, is known as an implicit scheme.

Iterative methods of solution for an implicit scheme

From (4), we have

$$(1+\alpha)u_{i,j+1} = \frac{\alpha}{2}\left(u_{i-1,j+1} + u_{i+1,j+1}\right) + u_{i,j} + \frac{\alpha}{2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) \quad (5)$$

Here only $u_{i,j+1}, u_{i-1,j+1}$ and $u_{i+1,j+1}$ are unknown while all others are known since these were already computed in the jth step.

$$\text{Put} \qquad b_i = u_{i,j} + \frac{\alpha}{2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right)$$

and dropping $j's$ (5) becomes

$$u_i = \frac{\alpha}{2(1+\alpha)}(u_{i-1} + u_{i+1}) + \frac{b_i}{1+\alpha}$$

This gives the iteration formula

$$u_i^{n+1} = \frac{\alpha}{2(1+\alpha)}(u_{i-1}^n + u_{i+1}^n) + \frac{b_i}{1+\alpha} \qquad (6)$$

which expresses the $(n+1)$th iterates in terms of the nth iterates only. This is known as the Jacobi's iteration formula.

As the latest value of u_{i-1} i.e., u_{i-1}^{n+1} is already available, the convergence of the iteration formula (6) can be improved by replacing u_{i-1}^{n} by u_{i-1}^{n+1}. Accordingly (6) may be written as

$$u_i^{n+1} = \frac{\alpha}{2(1+\alpha)}\left(u_{i-1}^{n+1} + u_{i+1}^{n+1}\right) + \frac{b_i}{1+\alpha} \qquad (7)$$

which is known as the Gauss-Seidel iteration formula.

Gauss-Seidel iteration scheme is valid for all finite values of α and converges twice as fast as Jacobi's scheme.

Du Fort and Frankel method. If we replace the derivatives in (1) by the central difference approximations

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Substituting these in (1), we obtain

$$u_{i,j+1} - u_{i,j-1} = \frac{2kc^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

$$\Rightarrow u_{i,j+1} = u_{i,j-1} + 2\alpha\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] \qquad (8)$$

Where $\alpha = \frac{kc^2}{h^2}$. This difference equation is called the Richardson scheme which is a three-level method.

If we replace $u_{i,j}$ by the mean of the values $u_{i,j-1}$ and $u_{i,j+1}$ i.e. $u_{i,j} = \frac{u_{i,j-1}+u_{i,j+1}}{2}$ in (8), then we get

$$u_{i,j+1} = u_{i,j-1} + 2\alpha\left[u_{i-1,j} - \left(u_{i,j-1} + u_{i,j+1}\right) + u_{i+1,j}\right]$$

On simplification, it can be written as

$$u_{i,j+1} = \frac{1-2\alpha}{1+2\alpha}u_{i,j-1} + \frac{2\alpha}{1+2\alpha}\left[u_{i-1,j} + u_{i+1,j}\right]$$

This difference scheme is called Du Fort-Frankel method which is a three-level explicit method. Its computational model is given in fig. 12.10.

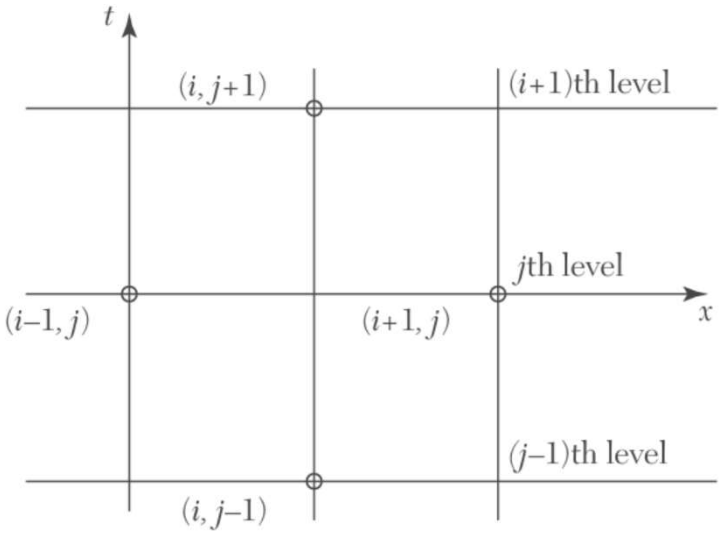


Fig. 12.10

Example 3: Solve $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ in $0 < x < 5,\ t \geq 0$ given that $u(x,0) = 20, u(0,t) = 0,$ $u(5,t) = 100$. Compute u for the time-step with $h = 1$ by the Crank-Nicholson method.

Solution:

Here $c^2 = 1$ and $h = 1$.

Taking $\alpha = \frac{kc^2}{h^2} = 1$, we get $k = 1$.

Also, we have

i / j	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

Then Crank-Nicholson formula becomes

$$4u_{i,j+1} = u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}$$

$\therefore \quad 4u_1 = 0 + 20 + 0 + u_2 \quad i.e.\ 4u_1 - u_2 = 20$ (1)

$4u_2 = 20 + 20 + u_1 + u_3 \quad i.e.\ u_1 - 4u_2 + u_3 = -40$ (2)

$4u_3 = 20 + 20 + u_2 + u_4 \quad i.e.\ u_2 - 4u_3 + u_4 = -40$ (3)

$4u_4 = 20 + 100 + u_3 + 100 \quad i.e.\ u_3 - 4u_4 = -220$ (4)

Now $(1) - 4(2)$ gives $15u_2 - 4u_3 = 180$ (5)

$4(3) + (4)$ gives $4u_2 - 15u_3 = -380$ (6)

Then $15(5) - 4(6)$ gives $209\, u_2 = 4220 \quad i.e.,\ u_2 = 20.2$

From (5), we get $4u_3 = 15 \times 20.2 - 180 \quad i.e.,\ u_3 = 30.75$

From (1), $4u_1 = 20 + 20.2 \quad i.e., \quad u_1 = 10.05$

From (4), $4u_4 = 220 + 30.75 \quad i.e., \quad u_4 = 62.69$

Thus, the required values are 10.05, 20.2, 30.75 and 62.68.

Example 4: Solve the equation $\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(x,0) = sin\,\pi x$, $0 \le x \le 1$; $u(0,t) = u(1,t) = 0$, using (a) Schmidt method, (b) Crank-Nicolson method, (c) Du Fort-Frankel method. Carryout computations for two levels, taking $h = \frac{1}{3},\ k = \frac{1}{36}$

Solution:

Here $c^2 = 1,\ h = \frac{1}{3},\ k = \frac{1}{36}$ so that $\alpha = \frac{kc^2}{h^2} = \frac{1}{4}$

Also $u_{1,0} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, u_{2,0} = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ and all boundary values are zero as shown in the fig. 12.11.

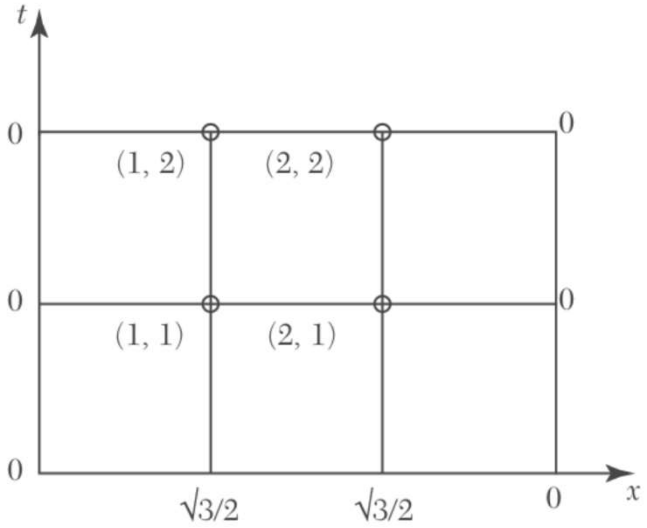


Fig. 12.11

(a) Schmidt's formula

$$u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha)u_{i,j} + \alpha u_{i+1,j}$$

$$\text{becomes} \quad u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}\right]$$

for $i = 1,2; j = 0$:

$$u_{1,1} = \frac{1}{4}\left[u_{0,0} + 2u_{1,0} + u_{2,0}\right] = \frac{1}{4}\left[0 + 2 \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right] = 0.65$$

$$u_{2,1} = \frac{1}{4}\left[u_{1,0} + 2u_{2,0} + u_{3,0}\right] = \frac{1}{4}\left[\frac{\sqrt{3}}{2} + 2 \times \frac{\sqrt{3}}{2} + 0\right] = 0.65$$

for $i = 1,2; j = 1$:

$$u_{1,2} = \frac{1}{4}\left[u_{0,1} + 2u_{1,1} + u_{2,1}\right] = 0.49$$

$$u_{2,2} = \frac{1}{4}\left[u_{1,1} + 2u_{2,1} + u_{3,1}\right] = 0.49$$

(b) Crank-Nicolson formula becomes

$$-\frac{1}{4}u_{i-1,j+1} + \frac{5}{2}u_{i,j+1} - \frac{1}{4}u_{i+1,j+1} = \frac{1}{4}u_{i-1,j} + \frac{3}{2}u_{i,j} + \frac{1}{4}u_{i+1,j}$$

for $i = 1,2; j = 0$:

$$-u_{0,1} + 10u_{1,1} - u_{2,1} = u_{0,0} + 6u_{1,0} + u_{2,0}$$

$$\Rightarrow 10u_{1,1} - u_{2,1} = \frac{7\sqrt{3}}{2}$$

$$-u_{1,1} + 10u_{2,1} - u_{3,1} = u_{1,0} + 6u_{2,0} + u_{3,0}$$

$$\Rightarrow -u_{1,1} + 10u_{2,1} = \frac{7\sqrt{3}}{2}$$

Solving these equations, we find

$$u_{1,1} = u_{2,1} = 0.67$$

for $i = 1,2; j = 1$:

$$-u_{0,2} + 10u_{1,2} - u_{2,2} = u_{0,1} + 6u_{1,1} + u_{2,1}$$

$$\Rightarrow 10u_{1,2} - u_{2,2} = 4.69$$

$$-u_{1,2} + 10u_{2,2} - u_{3,2} = u_{1,1} + 6u_{2,1} + u_{3,1}$$

$$\Rightarrow -u_{1,2} + 10u_{2,2} = 4.69$$

Solving these equations, we get $u_{1,2} = u_{2,2} = 0.52$

(c) Du Fort-Frankel formula becomes

$$u_{i,j+1} = \frac{1}{3}\left[u_{i,j-1} + u_{i-1,j} + u_{i+1,j}\right]$$

To start the calculations, we need $u_{1,1}$ and $u_{2,1}$.

We may take $u_{1,1} = u_{2,1} = 0.65$ from Schmidt method.

for $i = 1,2; j = 1$:

$$u_{1,2} = \frac{1}{4}\left[u_{1,0} + u_{0,1} + u_{2,1}\right] = \frac{1}{3}\left[\frac{\sqrt{3}}{2} + 0 + 0.65\right] = 0.5$$

$$u_{2,2} = \frac{1}{4}\left[u_{2,0} + u_{1,1} + u_{3,1}\right] = \frac{1}{3}\left[\frac{\sqrt{3}}{2} + 0.65 + 0\right] = 0.5$$

CHECK YOUR PROGRESS 2

Q1. Solve the boundary value problem $u_t = u_{xx}$ under the conditions $u(0,t) = u(1,t) = 0$ and $u(x,0) = sin\, px,\ 0 \le x \le 1$ using the Schmidt method (Take $h = 0.2$ and $\alpha = \frac{1}{2}$)

Q2. Find the values of $u(x,t)$ satisfying the parabolic equation $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0,t) = 0 = u(8,t)$ and $u(x,0) = 4x - \frac{1}{2}x^2$ at the points $x = \ i: i = 0, 1, 2, \ldots, 7$ and $t = \frac{1}{8}\ j: j = 0, 1, 2, \ldots, 5$

Q3. Find the solution of the parabolic equation $u_{xx} = 2u_t$ when $u(0,t) = u(4,t) = 0$ and $u(x,0)\ = x(4 - x)$, taking $h = 1$. Find the values up to $t = 5$.

12.6 SOLUTION OF WAVE EQUATION

The wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ is the simplest example of hyperbolic partial differential equations. Its solution is the displacement function $u(x,t)$ defined for values of x from 0 to l and for t from 0 to ∞, satisfying the initial and boundary conditions. The solution, as for parabolic equations, advances in an open-ended region. In the case of hyperbolic equations however, we have two initial conditions and two boundary conditions.

Such equations arise from convective type of problems in vibrations, wave mechanics, and gas dynamics.

We have,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

subject to the initial conditions: $u = f(x),\ \frac{\partial u}{\partial t} = g(x),\ 0 \le x \le 1$ at $t = 0$ (2)

and the boundary conditions: $u(0,t) = \phi(t),\ u(1,t) = \psi(t)$ (3)

Consider a rectangular mesh in the $x - t$ plane spacing h along x direction and k along time direction. Denoting a mesh point $(x,t) = (ih, jk)$ as simply i, j, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

Replacing the derivatives in (1) by their above approximations, we obtain

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = \frac{c^2k^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

$$\Rightarrow u_{i,j+1} = 2(1-\alpha^2c^2)u_{i,j} + \alpha^2c^2\left(u_{i-1,j} + u_{i+1,j}\right) - u_{i,j-1} \qquad (4)$$

Where $\alpha = \frac{k}{h}$

Now replacing the derivative in (2) by its central difference approximation, we get

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{\partial u}{\partial t} = g(x)$$

$$\Rightarrow u_{i,j+1} = u_{i,j-1} + 2k\, g(x) \quad \text{at} \quad t = 0$$

$$\therefore u_{i,1} = u_{i,-1} + 2k\, g(x) \quad \text{for} \quad j = 0 \qquad (5)$$

Also, initial condition $u = f(x)$ at $t = 0$ becomes $u_{i,-1} = f(x)$ (6)

Combining (5) and (6), we have

$$u_{i,1} = f(x) + 2k\, g(x)$$

Also (3) gives $u_{0,j} = \phi(t),\ u_{1,j} = \psi(t)$

Hence the explicit form (4) gives the values of $u_{i,j+1}$ at the $(j+1)$th level when the nodal values at $(j-1)$th and jth levels are known from (6) and (7) as shown in Fig. 12.12. Thus (4) gives an implicit scheme for the solution of the wave equation.

A special case. The coefficient of $u_{i,j}$ in (4) will vanish if $\alpha c = 1$ or $k = \frac{h}{c}$. Then (4) reduces to the simple form

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

Note 1: This provides an explicit scheme for the solution of the wave equation.

For $\alpha = \frac{1}{c}$, the solution of (4) is stable and coincides with the solution of (1).

For $\alpha < \frac{1}{c}$, the solution is stable but inaccurate.

For $\alpha > \frac{1}{c}$, the solution is unstable.

Note 2: The formula (4) converges for $\alpha \leq 1$ i.e., $k \leq h$

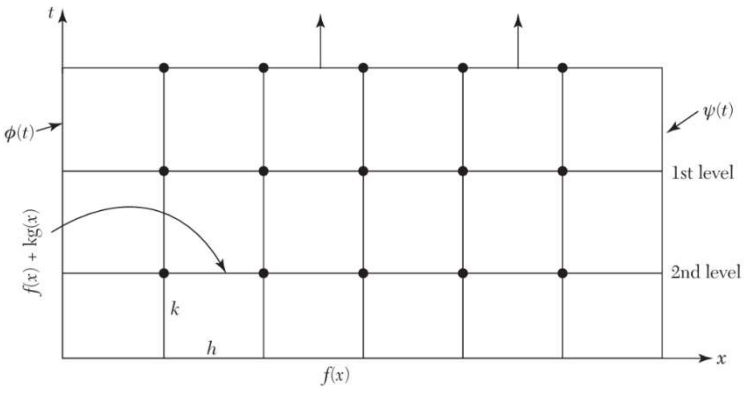


Fig. 12.12

Example 5: Evaluate the pivotal values of the equation $u_{tt} = 16\,u_{xx}$, taking $\Delta x \equiv 1$ up to $t = 1.25$. The boundary conditions are $u(0,t) = u(5,t) = 0$, $u_t(x,0) = 0$ and $u(x,0) = x^2(5 - x)$

Solution:

Here $c^2 = 16$

∴ The difference equation for the given equation is

$$u_{i,j+1} = 2(1 - 16\alpha^2)u_{i,j} + 16\alpha^2\left(u_{i-1,j} + u_{i+1,j}\right) - u_{i,j-1} \qquad (1)$$

Where $\alpha = \frac{k}{h}$

Taking $h = 1$ and choosing k so that the coefficient of $u_{i,j}$ vanishes, we have $16\alpha^2 = 1$, i.e., $k = \frac{h}{4} = \frac{1}{4}$.

∴ (1) reduces to

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad (2)$$

which gives a convergent solution (since $\frac{k}{h} < 1$). Its solution coincides with the solution of the given differential equation.

Now since $u(0,t) = u(5,t) = 0$

∴ $u_{0,j} = 0$ and $u_{5,j} = 0$ for all values of j

i.e., the entries in the first and last columns are zero.

Since $u_{x,0} = x^2(5 - x)$

∴ $u_{i,0} = i^2(5 - i) = 4, 12, 18, 16$ for $i = 1, 2, 3, 4$ at $t = 0$.

These are the entries for the first row.

Finally, since $u_{t\,(x,0)} = 0$ becomes $\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0$ when $j = 0$, giving $u_{i,1} = u_{i,-1}$

Thus, the entries of the second row are the same as those of the first row.

Putting $j = 0$ in (2),

$$u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1} = u_{i-1,0} + u_{i+1,0} - u_{i,1}$$

$$\Rightarrow u_{i,1} = \frac{u_{i-1,0} + u_{i+1,0}}{2}$$

Taking $i = 1, 2, 3, 4$ successively, we obtain

$$u_{1,1} = \frac{\left(u_{0,0} + u_{2,0}\right)}{2} = \frac{1}{2}(0 + 12) = 6$$

$$u_{2,1} = \frac{(u_{1,0} + u_{3,0})}{2} = \frac{1}{2}(4 + 18) = 11$$

$$u_{3,1} = \frac{(u_{2,0} + u_{4,0})}{2} = \frac{1}{2}(12 + 16) = 14$$

$$u_{4,1} = \frac{(u_{3,0} + u_{5,0})}{2} = \frac{1}{2}(18 + 0) = 9$$

These are the entries of the second row.

Putting $j = 1$ in (2), we get $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Taking $i = 1, 2, 3, 4$ successively, we obtain

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 11 - 4 = 7$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 6 + 14 - 12 = 8$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 11 + 9 - 18 = 2$$

$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 14 + 0 - 16 = -2$$

These are the entries of the third row.

Similarly putting $j = 2, 3, 4$ successively in (2), the entries of the fourth, fifth, and six throws are obtained.

Hence the values of $u_{i,j}$ are as shown in the table below:

i / ***j***	**0**	**1**	**2**	**3**	**4**	**5**
0	0	4	12	18	16	0
1	0	6	11	14	9	0
2	0	7	8	2	-2	0
3	0	2	-2	-8	-7	0
4	0	-9	-14	-11	-6	0
5	0	-16	-18	-12	-4	0

Example 6: Find the solution of the initial boundary value problem: $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\ 0 \le x \le 1$; subject to the initial conditions $u(x, 0) = sin\ \pi x, 0 \le x \le 1, \left(\frac{\partial u}{\partial t}\right)(x, 0) = 0,\ 0 \le x \le 1,$ and the boundary conditions $u(0, t) = 0, u(1, t) = 0, t > 0$; by using in the (a) the explicit scheme (b) the implicit scheme.

Solution:

(a) Explicit scheme

Since $c = 1$ and take $h = 0.2,\ k = \frac{h}{c} = 0.2$

We use the formula $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ (1)

Now since $u(0,t) = u(1,t) = 0$

$\therefore u_{0,j} = 0$ and $u_{1,j} = 0$ for all values of j

i.e., the entries in the first and last columns are zero.

Since $u_{x,0} = sin\,\pi x$, so $u_{i,0} = sin\,\pi x$

$$u_{1,0} = 0, \quad u_{2,0} = sin\,(0.2\pi) = 0.5878, \quad u_{3,0} = sin\,(0.4\pi) = 0.9511$$

$$u_{4,0} = sin\,(0.6\pi) = 0.5878$$

These are the entries of the first row.

Since $u_t\,(x,0) = 0$ we have $\frac{u_{i,j+1} - u_{i,j-1}}{2} = 0$ when $j = 0$ giving $u_{i,1} = u_{i,-1}$ (2)

Putting $j = 0$ in (1),

$$u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1} = u_{i-1,0} + u_{i+1,0} - u_{i,1}$$

$$\Rightarrow u_{i,1} = \frac{u_{i-1,0} + u_{i+1,0}}{2}$$

Taking $i = 1, 2, 3, 4$ successively, we obtain the entries of the second row.

$j = 1$ in (1), we get $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Taking $i = 1, 2, 3, 4$ successively, we get the entries of the third row.

Similarly taking $j = 2, j = 3, j = 4$ successively, we obtain the entries of the fourth, fifth, and sixth rows, respectively.

Hence the values of $u_{i,j}$ are as given in the table below:

i / *j*	**0**	**1**	**2**	**3**	**4**	**5**
0	0	0.5878	0.9511	0.9511	0.5878	0
1	0	0.4756	0.7695	0.9511	0.7695	0
2	0	0.1817	0.4756	0.5878	0.3633	0
3	0	0	0.0001	-0.1122	-0.1817	0
4	0	-0.1817	-0.5878	-0.7695	-0.4756	0
5	0	-0.5878	-0.9511	-0.9511	-0.5878	0

(b) Implicit scheme

We have the formula:

$$u_{i,j+1} = 2(1 - \alpha^2 c^2)u_{i,j} + \alpha^2 c^2\left(u_{i-1,j} + u_{i+1,j}\right) - u_{i,j-1} \qquad (1)$$

Where $\alpha = \frac{k}{h}$

Here $c^2 = 1$, Take $h = 0.25$ and $k = 0.5$ so that $\alpha = \frac{k}{h} = 2$

(1) reduces to

$$u_{i,j+1} = -6u_{i,j} + 4\left(u_{i-1,j} + u_{i+1,j}\right) - u_{i,j-1} \qquad (2)$$

Since $u_{i,0} = sin\,\pi x$

$$u_{1,0} = 0.7071, \qquad u_{2,0} = 0.5, \qquad u_{3,0} = 0.7071$$

There are the entries of the first row.

Since $u_t\,(x, 0) = 0$ we have $\frac{u_{i,j+1} - u_{i,j-1}}{2} = 0$ when $j = 0$ giving $u_{i,1} = u_{i,-1}$ (3)

Putting $j = 0$ and using (3), (2) reduces to

$$u_{i,1} = -3u_{i,0} + 2\left(u_{i-1,0} + u_{i+1,0}\right)$$

Taking $i = 1,2,3$ successively, we obtain

$$u_{1,1} = -3u_{1,0} + 2\left(u_{0,0} + u_{2,0}\right) = -\,0.1213$$

$$u_{2,1} = -3u_{2,0} + 2\left(u_{1,0} + u_{3,0}\right) = -\,0.1716$$

$$u_{3,1} = -3u_{3,0} + 2\left(u_{2,0} + u_{4,0}\right) = -\,0.1213$$

These are the entries of the second row.

Putting $j = 1$, (2) reduces to

$$u_{i,2} = -6u_{i,1} + 4\left(u_{i-1,1} + u_{i+1,1}\right)$$

Taking $i = 1,2,3$ successively, we obtain

$$u_{1,2} = -6u_{1,1} + 4\left(u_{0,1} + u_{2,1}\right) = 0.0414$$

$$u_{2,2} = -6u_{2,1} + 4\left(u_{1,1} + u_{3,1}\right) = 0.0592$$

$$u_{3,2} = -6u_{3,1} + 4\left(u_{2,1} + u_{4,1}\right) = 0.0414$$

Putting $j = 2$, (2) reduces to

$$u_{i,3} = -6u_{i,2} + 4\left(u_{i-1,2} + u_{i+1,2}\right) - u_{i,j-1}$$

Taking $i = 1,2,3$ successively, we obtain

$$u_{1,3} = -6u_{1,2} + 4\left(u_{0,2} + u_{2,2}\right) - u_{1,1} = 0.1097$$

$$u_{2,3} = -6u_{2,2} + 4\left(u_{1,2} + u_{3,2}\right) - u_{2,1} = 0.1476$$

$$u_{3,3} = -6u_{3,2} + 4\left(u_{2,2} + u_{4,2}\right) - u_{3,1} = 0.1097$$

These are the entries of the fourth row.

Hence the values of $u_{i,j}$ are as tabulated below:

i / *j*	**0**	**1**	**2**	**3**	**4**
0	0	0.7071	0.5	0.7071	0
1	0	-0.1213	-0.1716	-0.1213	0
2	0	0.0414	0.0592	0.0414	0

CHECK YOUR PROGRESS 3

Q1. Solve $y_{tt} = y_{xx}$ up to $t = 0.5$ with a spacing of 0.1 subject to $y(0,t) = 0,\ y(1,t) = 0, y_t\ (x,0) = 0$ and $y(x,0) = 10\ +\ x(1-x)$.

Q2. The transverse displacement u of a point at a distance x from one end and at any time t of a vibrating string satisfies the equation $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$, with boundary conditions $u\ =\ 0$ at $x\ =\ 0, t\ >\ 0$ and $u\ =\ 0$ at $x\ =\ 4, t\ >\ 0$ and initial conditions $u\ =\ x(4-x)$ and $\frac{\partial u}{\partial t} =\ 0, 0 \le x \le 4$. Solve this equation numerically for one-half period of vibration, taking $h\ =\ 1$ and $k\ = \frac{1}{2}$.

Q3. Solve the boundary value problem $u_{tt} = u_{xx}$ with the conditions $u(0,t) = u(1,t) = 0,\ u(x,0)\ = 1/2\ x(1-x)$ and $u_t(x,0) = 0$, taking $h = k = 0.1$ for $0 \le t \le\ 0.4$. Compare your solution with the exact solution at $x = 0.5$ and $t = 0.3$.

12.8 LET US SUM UP

A summary of the lessons we learnt from this unit is provided below.

- Analytical methods are often inadequate for solving many partial differential equations (PDEs), especially in practical applications.
- Numerical methods, particularly the finite-difference method, are used to approximate solutions by replacing derivatives with difference quotients.
- The Laplace equation (elliptic type) is solved using:
  - Standard five-point formula
  - Diagonal five-point formula
  - Iterative methods like Jacobi and Gauss-Seidel, with Gauss-Seidel converging faster.
- **4.** The one-dimensional heat equation (parabolic type) is handled using:

- Schmidt explicit method, simple but conditionally stable.
- Crank-Nicolson implicit method, more accurate and stable for any step size.
- Du Fort-Frankel method, a three-level explicit scheme providing stable results.

- **5.** The wave equation (hyperbolic type) is solved using:
  - Explicit finite difference schemes, simple and stable under specific conditions.
  - Implicit schemes, more stable and used for complex computations.
- **6.** The stability and convergence of each method depend on the choice of time and space step sizes.
- **7.** These numerical techniques are useful in solving real-world problems involving heat conduction, vibrations, fluid dynamics, and wave propagation.

✓ 12.9 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1

Answer to Q1.

We first find the initial values in the following order:

$$u_5 = \frac{1}{4}(0 + 17 + 21 + 12.1) = 12.5 \qquad \text{(Std. formula)}$$

$$u_1 = \frac{1}{4}(0 + 12.5 + 0 + 17) = 7.4 \qquad \text{(Diag. formula)}$$

$$u_3 = \frac{1}{4}(12.5 + 18.6 + 17 + 21) = 17.28 \qquad \text{(Diag. formula)}$$

$$u_7 = \frac{1}{4}(12.5 + 0 + 0 + 12.1) = 6.15 \qquad \text{(Diag. formula)}$$

$$u_9 = \frac{1}{4}(12.5 + 9 + 21 + 12.1) = 13.65 \qquad \text{(Diag. formula)}$$

$$u_2 = \frac{1}{4}(17 + 12.5 + 7.4 + 17.3) = 13.55 \qquad \text{(Std. formula)}$$

$$u_4 = \frac{1}{4}(7.4 + 6.2 + 0 + 12.5) = 6.52 \qquad \text{(Std. formula)}$$

$$u_6 = \frac{1}{4}(17.3 + 13.7 + 12.5 + 21) = 16.12 \qquad \text{(Std. formula)}$$

$$u_8 = \frac{1}{4}(12.5 + 12.1 + 6.2 + 13.7) = 11.12 \qquad \text{(Std. formula)}$$

(a) We carry out the successive iterations, using Jacobi's formulae:

$$u_1^{n+1} = \frac{1}{4}[0 + 11.1 + u_4^n + u_2^n]$$

$$u_2^{n+1} = \frac{1}{4}[u_1^{n+1} + 17 + u_5^n + u_3^n]$$

$$u_3^{n+1} = \frac{1}{4}[u_1^{n+1} + 19.7 + u_6^n + 219]$$

$$u_4^{n+1} = \frac{1}{4}[0 + u_1^{n+1} + u_7^n + u_5^n]$$

$$u_5^{n+1} = \frac{1}{4}[u_4^{n+1} + u_2^{n+1} + u_8^n + u_6^n]$$

$$u_6^{n+1} = \frac{1}{4}[u_5^{n+1} + u_3^{n+1} + u_9^n + 21]$$

$$u_7^{n+1} = \frac{1}{4}[0 + u_4^{n+1} + 8.7 + u_1^n]$$

$$u_8^{n+1} = \frac{1}{4}[u_7^{n+1} + u_5^{n+1} + 12.1 + u_9^n]$$

$$u_9^{n+1} = \frac{1}{4}[u_8^{n+1} + u_6^{n+1} + 12.8 + 17]$$

First iteration: (put $n = 0$ in the above results)

$$u_1^1 = \frac{1}{4}[0 + 11.1 + 6.52 + 13.55] = 7.79$$

$$u_2^1 = \frac{1}{4}[7.79 + 17 + 12.5 + 17.28] = 13.64$$

$$u_3^1 = \frac{1}{4}[13.64 + 19.7 + 16.12 + 21.9] = 12.84$$

$$u_4^1 = \frac{1}{4}[0 + 7.79 + 6.15 + 12.5] = 6.61$$

$$u_5^1 = \frac{1}{4}[6.61 + 13.64 + 11.12 + 16.12] = 11.88$$

$$u_6^1 = \frac{1}{4}[11.88 + 17.84 + 13.65 + 21] = 16.09$$

$$u_7^1 = \frac{1}{4}[0 + 6.61 + 8.7 + 11.12] = 6.61$$

$$u_8^1 = \frac{1}{4}[6.61 + 11.88 + 12.1 + 13.65] = 11.06$$

$$u_9^1 = \frac{1}{4}[11.06 + 16.09 + 12.8 + 17] = 12.238$$

Second iteration: (put $n = 1$)

$$u_1^2 = \frac{1}{4}[0 + 11.1 + 6.61 + 13.64] = 7.84$$

$$u_2^2 = \frac{1}{4}[7.84 + 17 + 11.88 + 17.84] = 16.64$$

$$u_3^2 = \frac{1}{4}[13.64 + 19.7 + 16.09 + 21.9] = 17.83$$

$$u_4^2 = \frac{1}{4}[0 + 7.84 + 6.61 + 11.88] = 6.58$$

$$u_5^2 = \frac{1}{4}[6.58 + 13.64 + 11.06 + 16.09] = 11.84$$

$$u_6^2 = \frac{1}{4}[11.84 + 17.83 + 14.24 + 21] = 16.23$$

$$u_7^2 = \frac{1}{4}[0 + 6.58 + 8.7 + 11.06] = 6.58$$

$$u_8^2 = \frac{1}{4}[6.58 + 11.84 + 12.1 + 14.24] = 11.19$$

$$u_9^2 = \frac{1}{4}[11.19 + 16.23 + 12.8 + 17] = 14.3$$

Third iteration:

$$u_1^3 = \frac{1}{4}[0 + 11.1 + 6.58 + 13.64] = 7.83$$

$$u_2^3 = \frac{1}{4}[7.83 + 17 + 11.84 + 17.83] = 13.637$$

$$u_3^3 = \frac{1}{4}[13.63 + 19.7 + 16.23 + 21.9] = 17.86$$

$$u_4^3 = \frac{1}{4}[0 + 7.83 + 6.58 + 11.84] = 6.56$$

$$u_5^3 = \frac{1}{4}[6.56 + 13.63 + 11.19 + 16.23] = 11.9$$

$$u_6^3 = \frac{1}{4}[11.9 + 17.86 + 14.3 + 21] = 16.27$$

$$u_7^3 = \frac{1}{4}[0 + 6.56 + 8.7 + 11.19] = 6.61$$

$$u_8^3 = \frac{1}{4}[6.61 + 11.9 + 12.1 + 14.3] = 11.23$$

$$u_9^3 = \frac{1}{4}[11.23 + 16.27 + 12.8 + 17] = 14.32$$

Similarly,

$$u_1^4 \approx 7.82, \quad u_2^4 \approx 13.65, \quad u_3^4 \approx 17.88, \quad u_4^4 \approx 6.58, \quad u_5^4 \approx 11.94,$$
$$u_6^4 \approx 16.28, \quad u_7^4 \approx 6.63, \quad u_8^4 \approx 11.25, \quad u_9^4 \approx 14.33$$

$u_1^5 \approx 7.83,\quad u_2^5 \approx 13.66,\quad u_3^5 \approx 17.89,\quad u_4^5 \approx 6.5,\quad u_5^5 \approx 11.95,\quad u_6^5 \approx 16.29,$
$u_7^5 \approx 6.64,\quad u_8^5 \approx 11.25,\quad u_9^5 \approx 14.34$

There is no significant difference between the fourth and fifth iteration values.

Hence, $u_1 = 7.83,\ u_2 = 13.66,\ u_3 = 17.89, u_4 = 6.5,\ u_5 = 11.95,\ u_6 = 16.29,\ u_7 = 6.64,\ u_8 = 11.25$ and $u_9 = 14.34$

Answer to Q2.

. Try Yourself

(a) $u_1 = 0.126,\ u_2 = 0.126,\ u_3 = 0.376, u_4 = 0.376,$

(b) $u_1 = 0.126,\ u_2 = 0.126,\ u_3 = 0.376,\ u_4 = 0.376$

Answer to Q3. Try Yourself

$u_1 = 7.9,\ u_2 = 13.7,\ u_3 = 17.9, u_4 = 6.6,\ u_5 = 11.9,\ u_6 = 16.3,\ u_7 = 6.6,\ u_8 = 11.2$ and $u_9 = 14.3$

Check your progress 2

Answer to Q1. Since $h = 0.2$ and $\alpha = \frac{1}{2}$

$\therefore \alpha = \frac{k}{h^2}$ gives $k = 0.02$

Since $\alpha = \frac{1}{2}$, we use the Bendre-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right) \qquad (1)$$

We have

$$u(0,0) = 0, \qquad u(0.2,0) = \sin\frac{\pi}{5} = 0.5875$$

$$u(0.4,0) = \sin\frac{2\pi}{5} = 0.9511, \quad u(0.6,0) = \sin\frac{3\pi}{5} = 0.9511$$

$$u(0.8,0) = \sin\frac{4\pi}{5} = 0.5875, \quad u(1,0) = \sin\pi = 0$$

The values of u at the mesh points can be obtained by using the recurrence relation (1) as shown in the table below:

$x \to 0$		0	0.2	0.4	0.6	0.8	1.0
t ↓ 0	i / j	0	1	2	3	4	5
	0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	1	0	0.4756	0.7695	0.7695	0.4756	0
0.04	2	0	0.3848	0.6225	0.6225	0.3848	0
0.06	3	0	0.3113	0.5036	0.5036	0.3113	0
0.08	4	0	0.2518	0.4074	0.4074	0.2518	0
0.1	5	0	0.2037	0.3296	0.3296	0.2037	0

Answer to Q2.

Here, $c^2 = 4, h = 1$ and $k = \frac{1}{8}$. Then $\alpha = \frac{kc^2}{h^2} = \frac{1}{2}$

We have Bendre-Schmidt's recurrence relation

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right) \qquad (1)$$

Now since $u(0,t) = 0 = u(8,t)$

$\therefore\ u_{0,i} = 0$ and $u_{8,j} = 0$ for all values of j, i.e., the entries in the first and last column are zero.

Since $u(x,0) = 4x - \frac{1}{2}x^2$

$\therefore u_{i,0} = 4i - \frac{1}{2}i^2$

$= 0, 3.5, 6, 7.5, 8, 7.5, 6, 3.5$ for $i = 0,1,2,3,4,5,6,7$ at $t = 0$

These are the entries of the first row.

Putting $j = 0$ in (1), we have

$$u_{i,1} = \frac{1}{2}\left(u_{i-1,0} + u_{i+1,0}\right)$$

Taking $i = 1, 2, \ldots, 7$ successively, we get

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{1,0}) = \frac{1}{2}(0 + 6) = 3$$

$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{2,0}) = \frac{1}{2}(3.5 + 7.5) = 5.5$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{3,0}) = \frac{1}{2}(6 + 8) = 7$$

$$u_{4,1} = 7.5, \qquad u_{5,1} = 7, \qquad u_{6,1} = 5.5, \qquad u_{7,1} = 3$$

These are the entries in the second row.

Putting $j = 1$ in (1), the entries of the third row are given by

$$u_{i,2} = \frac{1}{2}(u_{i-1,1} + u_{i+1,1})$$

Similarly putting $j = 2, 3, 4$ successively in (1), the entries of the fourth, fifth, and sixth rows are obtained.

Hence the values of $u_{i,j}$ are as given in the following table:

i / *j*	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0

Answer to Q3. Try Yourself

j / *i*	**0**	**1**	**2**	**3**	**4**
0	0	3	4	3	0

1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

Check your progress 3

Answer to Q1. As $c^2 = 1$, $h = 0.1$, $k = \frac{h}{c} = 0.1$; we use the formula

$$y_{i,j+1} = y_{i-1,j} + y_{i+1,j} - y_{i,j-1} \qquad (1)$$

Since $y(0,t) = y(1,t) = 0$

$\therefore y_{0,j} = 0$ and $y_{1,j} = 0$ for all values of j

i.e., the entries in the first and last columns are zero.

Since $y(x,0) = 10 + x(1-x)$, so $y_{i,0} = 10 + i(1-i)$

$$u_{0.1,0} = 10.09, \quad u_{0.2,0} = 10.16, \quad u_{0.3,0} = 10.21, \quad u_{0.4,0} = 10.24$$
$$u_{0.5,0} = 10.25, \quad u_{0.6,0} = 10.24, \quad u_{0.7,0} = 10.21, \quad u_{0.8,0} = 10.16$$
$$u_{0.9,0} = 10.09$$

These are the entries of the first row.

Since $y_t(x,0) = 0$ we have $\frac{y_{i,j+1} - y_{i,j-1}}{2} = 0$ when $j = 0$ giving $y_{i,1} = y_{i,-1}$

Putting $j = 0$ in (1),

$$y_{i,1} = y_{i-1,0} + y_{i+1,0} - u_{i,-1} = y_{i-1,0} + y_{i+1,0} - y_{i,1}$$
$$\Rightarrow y_{i,1} = \frac{y_{i-1,0} + y_{i+1,0}}{2}$$

Taking $i = 1, 2, 3, \ldots, 9$ successively, we obtain the entries of the second row.

$j = 1$ in (1), we get $y_{i,2} = y_{i-1,1} + y_{i+1,1} - y_{i,0}$

Taking $i = 1, 2, 3, \ldots, 9$ successively, we get the entries of the third row.

Similarly taking $j = 2, 3, \ldots, 7$ successively in (1), the entries of the fourth to ninth row are obtained. Hence the values of $y_{i,j}$ are as given in the table below:

i \ j	0	1	2	3	4	5	6	7	8	9	10
0	0	10.09	10.16	10.21	10.24	10.25	10.24	10.21	10.16	10.09	0
1	0	5.08	10.15	10.20	10.23	10.24	10.23	10.20	10.15	5.08	0
2	0	0.06	5.12	10.17	10.20	10.21	10.20	10.17	5.12	0.06	0
3	0	0.04	0.08	5.12	10.15	10.16	10.15	5.12	0.08	0.04	0
4	0	0.02	0.04	0.06	5.08	10.09	5.08	0.06	0.04	0.02	0
5	0	0	0	0	0	0	0	0	0		0

Answer to Q2. Try Yourself

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	0	0	0	0
3	0	-2	-3	-2	0
4	0	-3	-4	-3	0

Answer to Q3. Try Yourself

$t = 0.3, x =$	0.1	0.2	0.3	0.4	0.5
Num. sol. $u =$	0.02	0.04	0.06	0.075	0.08
Exact sol. $u =$	0.02	0.04	0.06	0.075	0.08

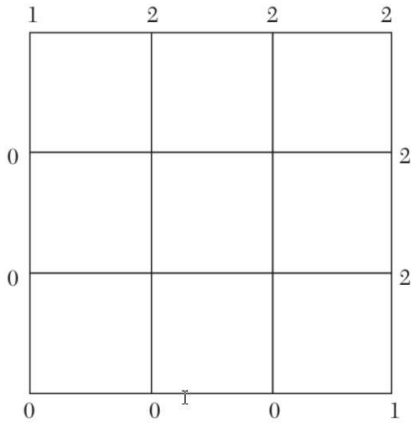
12.10 FURTHER READING

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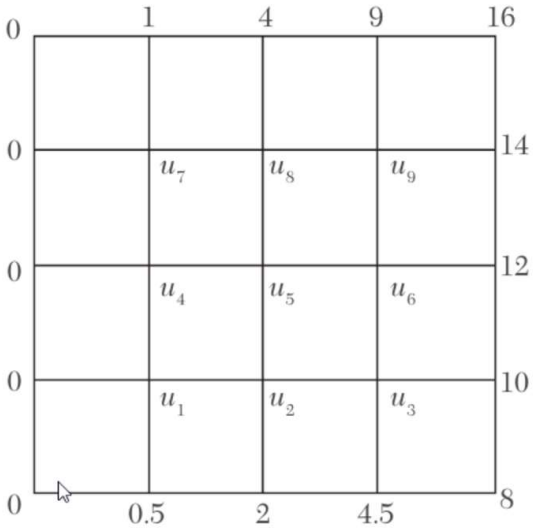
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12.11 MODEL QUESTIONS

1. Solve $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown in the figure. Iterate till the mesh values are correct to two decimal places.



2. Solve the Laplace's equation $\nabla^2 u = 0$ in the domain of the below figure



3. Solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ with the conditions $u(0,t) = 0,\ u(x,0) = x(1 - x)$, and $u(1,t) = 0$. Assume $h = 0.1$. Tabulate u for $t = k, 2k$ and $3k$ choosing an appropriate value of k.

4. Given $\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0$; $f(0,t) = f(5,t) = 0, f(x,0) = x^2(25 - x^2)$; find the values of f for $x = ih\ (i = 0, 1, \ldots, 5)$ and $t = jk\ (j = 0, 1, \ldots, 6)$ with $h = 1$ and $k = \frac{1}{2}$, using the explicit method.

5. Given $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ u(0,t) = 0,\ u(4,t) = 0$ and $u(x,0) = \frac{x}{3}(16 - x^2)$. Obtain $u_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2$ using Crank-Nicholson's method.

6. The function u satisfies the equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ and the conditions: $u(x,0) = \frac{1}{8}\ sin\ \pi x$, $u_t(x,0) = 0$ for $0 \le x \le 1, u(0,t) = u(1,t) = 0$ for $t \ge 0$. Use the explicit scheme to calculate u for $x = 0(0.1)1$ and $t = 0(0.1)0.5$.

7. Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\ \ 0 < x < 1,\ t > 0,$ given $u(x,0) = u_t(x,0) = u(0,1) = 0$ and $u\ (1,t) = 100 sin\ \pi t$. Compute u for four times with $h = 0.25$.

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