UNIT 14: NUMERICAL ALGORITHM IN C-II

UNIT STRUCTURE

14.1 Learning Objectives

14.2 Introduction

14.3 Trapezoidal method

14.4 Simpson's method

14.5 Gauss elimination method

14.6 Gauss-Jordan method

14.7 Gauss-Seidel method

14.8 Jacobi's method

14.9 Let Us Sum Up

14.10 Further Reading

14.11 Answer to Check Your Progress

14.12 Model Questions

14.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Understand and implement the Trapezoidal Rule in MATLAB.
- Apply Simpson's 1/3 and 3/8 Rules to evaluate definite integrals using MATLAB.
- Solve systems of linear equations using the Gauss Elimination Method in MATLAB.
- Understand the procedure and apply the Gauss-Jordan Method for solving linear systems in MATLAB.
- Implement and use the Jacobi Iteration Method for solving linear equations iteratively in MATLAB.
- Apply the Gauss-Seidel Iteration Method for more efficient iterative solutions of linear systems in MATLAB.

14.2 INTRODUCTION

Numerical methods play a vital role in solving complex mathematical problems that cannot be addressed analytically. In this unit, we explore key numerical algorithms implemented using the MATLAB, focusing particularly on integration techniques and systems of linear equations. The unit begins with fundamental numerical integration

methods such as the Trapezoidal Rule and Simpson's Rules (1/3 and 3/8), which provide approximate solutions to definite integrals. It then moves on to solving linear systems using direct methods like Gauss Elimination and Gauss-Jordan, followed by iterative techniques like the Jacobi and Gauss-Seidel methods. These algorithms are not only mathematically significant but also have practical importance in engineering and scientific computation. Through hands-on MATLAB implementations, learners will gain both conceptual understanding and practical skills in numerical problem-solving.

14.3 TRAPEZOIDAL METHOD

Problem: Compute $\int_a^b f(x)\,dx$.

Trapezoidal Rule Formula:

$$\int_a^b f(x)dx \approx \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right]$$

where:

$h = \frac{b-a}{n}$ is the width of each subinterval.

$x_i = a + ih$

Trapezoidal Rule in MATLAB

```
clear all;
clc;
close all;
f=input('Enter your function: ');
N=input('Enter number of sub interval: ');
a=input('Enter Lower limit: ');
b=input('Enter Upper limit: ');
h=(b-a)/N;
sum=0;
for i=1:N-1
    sum=sum+f(a+i*h);
end
Tr=(h/2)*(f(a)+2*sum+f(b));
fprintf('Result using trapezoidal rule is: %.6f\n', Tr)
```

Example 1: Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using Trapezoidal rule with $h = 1$

Solution:

Input
Enter your function: @(x) 1/(1+x^2)
Enter number of sub interval: 6
Enter Lower limit: 0
Enter Upper limit: 6
Output
Result using trapezoidal rule is: 1.410799

14.4 SIMPSON'S METHOD

- **Simpson's $\frac{1}{3}$ Rule:**

For an integral $\int_a^b f(x)\,dx$, the formula using Simpson's $\frac{1}{3}$ Rule with n subintervals (where n must be even) is:

Formula:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4\sum_{i=1,\text{odd}}^{n-1} f(x_i) + 2\sum_{i=2,\text{even}}^{n-2} f(x_i) + f(b)\right]$$

where:

$h = \frac{b-a}{n}$ is the width of each subinterval.

$x_i = a + ih$

Simpson's 1/3 Rule in MATLAB

```
clear all;
clc;
close all;
f=input('Enter your function: ');
N=input('Enter number of sub interval: ');
a=input('Enter Lower limit: ');
b=input('Enter Upper limit: ');
h=(b-a)/N;
oddsum=0;
for i=1:2:N-1
    oddsum=oddsum+f(a+i*h);
end
evensum=0;
for j=2:2:N-2
    evensum=evensum+f(a+j*h);
end
Sim=(h/3)*(f(a)+4*oddsum+2*evensum+f(b));
fprintf('Result using simpson 1/3rd rule is: %.6f\n', Sim)
```

Example 2: Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using Simpson's 1/3 rule with $h = 1$

Solution:

Input

Enter your function: @(x) 1/(1+x^2)

Enter number of sub interval: 6

Enter Lower limit: 0

Enter Upper limit: 6

Output

Result using simpson 1/3rd rule is: 1.366173

- **Simpson's $\frac{3}{8}$ Rule:**

For an integral $\int_a^b f(x)\,dx$, where $f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ for $x_0, x_1, x_2, \ldots, x_n$. The interval (a, b) divide into n sub-intervals of width h so that

$$x_0 = a, \quad x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad \ldots, \quad x_n = x_0 + nh = b$$

Then, the Simpson's $\frac{3}{8}$ Rule is:

Formula:

$$\int_a^b f(x)dx \approx \frac{3h}{8}[y_0 + 3(y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3}) + y_n]$$

```
clear all;
clc;
close all;
f=input('Enter your function: ');
N=input('Enter number of sub interval: ');
a=input('Enter Lower limit: ');
b=input('Enter Upper limit: ');
h=(b-a)/N;
sum=0;
for i=1:N-1
    sum=sum+f(a+i*h);
end
sum3=0;
for j=3:3:N-1
    sum3=sum3+f(a+j*h);
end
sumnot3=sum-sum3;
Sim=(3*h/8)*(f(a)+3*sumnot3+2*sum3+f(b));
fprintf('Result using simpson 3/8th rule is: %.6f\n', Sim)
```

Example 3: Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using Simpson's 3/8 rule with $h = 1$

Solution:

Input

Enter your function: @(x) 1/(1+x^2)

Enter number of sub interval: 6

Enter Lower limit: 0

Enter Upper limit: 6

Output

Result using simpson 3/8th rule is: 1.357081

CHECK YOUR PROGRESS 1

Q1. Use the Trapezoidal rule to estimate the integral $\int_0^2 e^{x^2} dx$ taking the number 10 intervals.

Q2. Evaluate the integral $\int_0^1 \frac{x^2}{1+x^3} dx$ using Simpson's 1/3 rule. Compare the error with the exact value.

Q3. Compute the value of $\int_{0.2}^{1.4} (\sin x - \log x + e^x) dx$ using Simpson's 3/8 rule.

14.5 GAUSS ELIMINATION METHOD

Gaussian Elimination is a systematic method for solving systems of linear equations. It transforms the system into an upper triangular form, making it easier to solve through back substitution. MATLAB provides tools to perform Gaussian elimination efficiently.

Steps of Gaussian Elimination

Consider a system of n linear equations with n unknowns represented in matrix form as:

$$Ax = b$$

where A is the coefficient matrix, x is the vector of unknowns, and b is the right-hand side vector.

1) Augmented Matrix Formation:

Combine the coefficient matrix A and the result vector b to form the augmented matrix $[A \mid b]$.

2) Forward Elimination:

Perform row operations to convert the augmented matrix into an upper triangular form.

3) Back Substitution:

Solve for the unknowns starting from the last equation.

Example

Solve the system of equations:

$$2x + y - z = 8$$
$$-3x - y + 2z = -11$$
$$-2x + y + 2z = -3$$

Step 1: Form the Augmented Matrix

$$\begin{bmatrix} 2 & 1 & -1 & | & 8 \\ -3 & -1 & 2 & | & -11 \\ -2 & 1 & 2 & | & -3 \end{bmatrix}$$

Step 2: Forward Elimination

I. Make the first pivot $a_{11} = 2$.

$$\text{Row 2} \rightarrow \text{Row 2} + \frac{3}{2}\text{ Row 1}, \qquad \text{Row 3} \rightarrow \text{Row 3} + \frac{1}{2}\text{Row 1}$$

Resulting matrix:

$$\begin{bmatrix} 2 & 1 & -1 & | & 8 \\ 0 & 0.5 & 0.5 & | & 1 \\ 0 & 1 & 1.5 & | & 1 \end{bmatrix}$$

II. Make the second pivot $a_{22} = 0.5$.

$$\text{Row 3} \rightarrow \text{Row 3} - 2 \times \text{Row 2}$$

Resulting matrix:

$$\begin{bmatrix} 2 & 1 & -1 & | & 8 \\ 0 & 0.5 & 0.5 & | & 1 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

Step 3: Back Substitution

Solve the equations from the upper triangular matrix:

I. From the last row: $z = -1$

II. From the second row: $0.5y + 0.5z = 1 \Rightarrow y = 2$

III. From the first row: $2x + y - z = 8 \quad \Rightarrow x = 3$

Final Solution:

$$x = 3, y = 2, z = -1$$

Gauss elimination in MATLAB

```
clear all;
clc;
close all;
A=input('Enter your coefficient matrix: ');
B=input('Enter source vector: ');
N=length(B);
X=zeros(N,1);
Aug=[A B];

for j=1:N-1
    for i=j+1:N
        m=Aug(i,j)/Aug(j,j);
        Aug(i,:)=Aug(i,:)-m*Aug(j,:);
    end
end
Aug
X(N)=Aug(N,N+1)/Aug(N,N);
for k=N-1:-1:1
    X(k)=(Aug(k,N+1)-Aug(k,k+1:N)*X(k+1:N))/Aug(k,k);
end
X
```

Example 4: Apply Gauss elimination method to solve the equations

$$2x + y - z = 8$$
$$-3x - y + 2z = -11$$
$$-2x + y + 2z = -3$$

Solution:

Input

Enter your coefficient matrix: [2 1 -1;-3 -1 2;-2 1 2]
Enter source vector: [8;-11;-3]

Output

```
Aug =
  2.    1.    -1    8
  0    0.5   0.5   1
  0     0    -1    1
X =
   2
   3
  -1
```

14.6 GAUSS-JORDAN METHOD

Steps to Implement Gauss-Jordan Method

1. Input the augmented matrix

Combine matrix A and vector b to form the augmented matrix $[A \mid b]$.

2. Convert to Reduced Row Echelon Form

Perform row operations to make the pivot of each row equal to 1 and all other elements in the pivot column 0.

3. Extract the solution

The last column of the reduced matrix is the solution vector.

```
clear all;
clc;
close all;
A=input('Enter your coefficient matrix: ');
B=input('Enter source vector: ');
N=length(B);
X=zeros(N,1);
Aug=[A B];

for j=1:N
    Aug(j,:)=Aug(j,:)/Aug(j,j);
    for i=1:N
        if i~=j
            m=Aug(i,j)
            Aug(i,:)=Aug(i,:)-m*Aug(j,:)
        end
    end
end
```

Example 4: Apply Gauss Jordan method to solve the equations

$$x + y + z = 9$$

$$2x - 3y + 4z = 13$$

$$3x + 4y + 5z = 40$$

Solution:

Input

Enter your coefficient matrix: [1 1 1;2 -3 4;3 4 5]
Enter source vector: [9;13;40]

Output

m = 2

Aug =

1	1	1	9
0	-5	2	-5
3	4	5	4

m = 3

Aug =

1	1	1	9
0	-5	2	-5
0	1	2	13

m = 1

Aug =

1	0	1.4	8
0	1	-0.4	1
0	1	2	13

m = 1

Aug =

1	0	1.4	8
0	1	-0.4	1
0	0	2.4	12

m = 1.4000

Aug =

1	0	0	1
0	1	-0.4	1
0	0	1	5

m = -0.4000

Aug =

1	0	0	1
0	1	0	3
0	0	1	5

$$\therefore x = 1, \quad y = 3, \quad z = 5$$

CHECK YOUR PROGRESS 2

Q1. Apply Gauss elimination method to solve the equations

$$x + 4y - z = -5$$
$$x + y - 6z = -12$$
$$3x - y - z = 4$$

Q2. Using the Gauss elimination method, solve the equations:

$$x + 2y + 3z - u = 10$$
$$2x + 3y - 3z - u = 1$$
$$2x - y + 2z + 3u = 7$$
$$3x + 2y - 4z + 3u = 2$$

Q3. Solve the equations by the Gauss-Jordan method.

$$10x - 7y + 3z + 5u = 6$$
$$-6x + 8y - z - 4u = 5$$
$$3x + y + 4z + 11u = 2$$
$$5x - 9y - 2z + 4u = 7$$

Q4. Solve the equations by the Gauss-Jordan method

$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$
$$x + y + z = 9$$

14.7 JACOBI'S METHOD

The Jacobi Method is an iterative algorithm for solving a system of linear equations $Ax = b$, where A is the coefficient matrix, x is the unknown vector, and b is the result vector. It is particularly useful for sparse or diagonally dominant matrices.

Steps to Implement Jacobi's Method in MATLAB

Step 1: Input the matrix A, source vector B, initial guess and tolerance for stopping criterion.

Step 2: Iterate Until Convergence or Maximum Iterations reached.

The iterative formula is:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)}\right)$$

Step 3: Output

Return the new vector X as the solution with iteration number.

```
clear all;
clc;
close all;
A=input('Enter your Coefficient Matrix: ');
B=input('Enter Source Vector: ');
X=input('Enter initial Guess Vector: ');
e=input('Enter your tolerance: ');
N=length(B);
P=X+1;
j=1;
while abs((X-P))>e
    P=X;
    for i=1:N
        X(i)=(B(i)/A(i,i))-(A(i,[1:i-1,i+1:N])*P([1:i-
1,i+1:N]))/A(i,i);
    end
    fprintf('Iteration no %d\n',j)
    X
    j=j+1;
end
```

Example 5: Solve, by Jacobi's iteration method, the equations

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Solution:

Input

Enter your Coefficient Matrix: [20 1 -2;3 20 -1;2 -3 20]

Enter Source Vector: [17;-18;25]

Enter initial Guess Vector: [0;0;0]

Enter your tolerance: 0.0001

Output

$$x_1 = 0.8500, \quad y_1 = -0.9000, \quad z_1 = 1.2500$$

$$x_2 = 1.0200, \quad y_2 = -0.9650, \quad z_2 = 1.0300$$

$$x_3 = 1.0013, \quad y_3 = -1.0015, \quad z_3 = 1.0033$$

$$x_4 = 1.0004, \quad y_4 = -1.0000, \quad z_4 = 0.9997$$

$$x_5 = 1.0000, \quad y_5 = -1.0001, \quad z_5 = 1.0000$$

Hence, $x = 1,\ y = -1,\ z = 1$

14.8 GAUSS-SEIDEL METHOD

The Gauss-Seidel method is an iterative algorithm for solving a system of linear equations $Ax = b$. It improves upon the Jacobi method by using updated values as soon as they are computed, leading to faster convergence.

Steps to Implement Gauss-Seidel Method in MATLAB

Step 1: Input the matrix A, source vector B, initial guess and tolerance for stopping criterion.

Step 2: Iterate Until Convergence or Maximum Iterations reached.

The iterative formula is:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

$x_i^{(k+1)}$: Updated value for the i-th variable at the $k+1$-th iteration.

a_{ii} : Diagonal element of the matrix A.

Step 3: Output

Return the new vector X as the solution with iteration number.

```
clear all;
clc;
close all;
A=input('Enter your Coefficient Matrix: ');
B=input('Enter Source Vector: ');
P=input('Enter initial Guess Vector: ');
e=input('Enter your tolerance: ');
N=length(B);
X=zeros(N,1);
Y=X+1;
j=1;
while abs(Y-X)>e
    Y=X;
    for i=1:N
        X(i)=(B(i)/A(i,i))-(A(i,[1:i-1,i+1:N])*P([1:i-
1,i+1:N]))/A(i,i);
        P(i)=X(i);
    end
    fprintf('Iteration no %d\n',j)
    X
    j=j+1;
```

Example 6: Apply the Gauss-Seidel iteration method to solve the equations

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 2$$

Solution:

Input

Enter your Coefficient Matrix: [20 1 -2;3 20 -1;2 -3 20]

Enter Source Vector: [17;-18;25]

Enter initial Guess Vector: [0;0;0]

Enter your tolerance: 0.0001

Output

$$x_1 = 0.8500, \quad y_1 = -1.0275, \quad z_1 = 1.0109$$

$$x_2 = 1.0025, \quad y_2 = -0.9998, \quad z_2 = 0.9998$$

$$x_3 = 1.0000, \quad y_3 = -1.0000, \quad z_3 = 1.0000$$

Hence, $x = 1,\ y = -1,\ z = 1$

CHECK YOUR PROGRESS 3

Q1. Solve by Jacobi's iteration method, the equations correct to two decimal places.

$$10x + y - z = 11.19$$
$$x + 10y + z = 28.08$$
$$-x + y + 10z = 35.61$$

Q2. Solve the equations by the Gauss-Jacobi method and the Gauss-Seidel method.

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

Q3. Apply the Gauss-Seidel iteration method to solve the equations:

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

14.9 LET US SUM UP

A summary of the lessons we learnt from this unit is provided below.

- **Trapezoidal Rule:**
  - A numerical integration method that approximates the area under a curve using trapezoids.
  - Suitable for simple, linear approximations over small intervals.
- **Simpson's 1/3 Rule:**
  - Applies parabolic fitting to sets of three points.

- Offers higher accuracy than the Trapezoidal Rule, especially when the number of intervals is even.

- **Simpson's 3/8 Rule:**
- Uses cubic interpolation over sets of four points.
- Often used when the number of intervals is a multiple of three.

- **Gauss Elimination Method:**
- A direct method to solve linear equations by converting the system into upper triangular form.
- Followed by back-substitution to find the unknowns.

- **Gauss-Jordan Method:**
- An extension of Gauss Elimination that reduces the system to reduced row echelon form.
- Provides a direct solution without back-substitution.

- **Jacobi's Method:**
- An iterative approach to solving linear systems.
- Each variable is updated based on the values from the previous iteration.

- **Gauss-Seidel Method:**
- Similar to Jacobi's method but uses the most recent updates immediately.
- Converges faster than Jacobi in most cases.

✓ 14.10 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1

Answer to Q 1.

Result using trapezoidal rule is: 17.170210

Answer to Q 2.

h=0.25, n=4

Result using simpson 1/3rd rule is: 0.23108

$$\int_0^1 \frac{x^2}{1+x^3} dx = 0.23105$$

Thus, the error $= 0.23108 - 0.23105 = -0.00003.$

Answer to Q 3.

$h = 0.2, n = 6$

Result using simpson 3/8th rule is: 4.052992

Check your progress 2

Answer to Q 1.

```
Aug =
   1    4    -1      -5
   0   -3    -5      -7
   0    0 23.67  49.33
X =
   1.6479
  -1.1408
   2.0845
```

Answer to Q 2.

```
Aug =
1     2     3   -1    10
0    -1    -9    1   -19
0     0    41    0    82
0     0     0    2     2

X =
    1
    2
    2
    1
```

Answer to Q 3.

$$x = 5, \quad y = 4, \quad z = -7, \quad u = 1$$

Answer to Q 4.

$$x = 1, \quad y = 3, \quad z = 5$$

Check your progress 3

Answer to Q 1.

$$x = 1.23, \quad y = 2.34, \quad z = 3.45$$

Answer to Q 2.

$$x = 2.426, \quad y = 3.573, \quad z = 1.926$$

Answer to Q 3.

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad x_4 = 0$$

14.11 FURTHER READING

1. Grewal, B. S.; Numerical Methods in Engineering and Science, 1st edition, David Pallai, 2019.
2. Sastry, S. S.; Introductory Methods of Numerical Analysis, Fifth Edition, Prentice-Hall, India, 2012.
3. Thangaraj, P.; Computer Oriented Numerical Methods. PHI Learning Pvt. Ltd, 2013.
4. Jain, M. K., Iyengar, S. R. K. and Jain, R. K.; Numerical Methods for Scientific & Engineering Computation. New Age International, 2012.
5. Chapra, S. C. and Canale, R. P.; Numerical Methods for Engineers. McGraw Hill, International Edition, 1998.

14.12 MODEL QUESTIONS

1. Use Simpson's 1/3rd rule to find $\int_0^{0.6} e^{-x^2} dx$ by taking seven ordinates.

2. Use trapezoidal rule to evaluate $\int_0^1 x^3 dx$ considering five sub-intervals.

3. Evaluate $\int_0^1 \frac{dx}{1+x}$ applying

(i) Trapezoidal rule

(ii) Simpson's 1/3 rule

(iii) Simpson's 3/8 rule.

4. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using

(i) Trapezoidal rule taking $h = \frac{1}{4}$.

(ii) Simpson's 1/3rd rule taking $h = \frac{1}{4}$.

(iii) Simpson's 3/8th rule taking $h = \frac{1}{6}$.

5. Solve the following equations by the Gauss elimination method:

(i)

$$2x + 2y + z = 12$$
$$3x + 2y + 2z = 8$$
$$5x + 10y - 8z = 10$$

(ii)

$$2x - y + 3z = 9$$
$$x + y + z = 6$$

$$x - y + z = 2$$

6. Solve the following equations by the Gauss-Jordan method:

(i)

$$2x - 3y + z = -1$$
$$x + 4y + 5z = 25$$
$$3x - 4y + z = 2$$

(ii)

$$x + 3y + 3z = 16$$
$$x + 4y + 3z = 18$$
$$x + 3y + 4z = 19$$

7. Solve the following equations by Jacobi's method:

(i)

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

(ii)

$$13x + 5y - 3z + u = 18$$
$$2x + 12y + z - 4u = 13$$
$$x - 4y + 10z + u = 29$$
$$2x + y - 3z + 9u = 31$$

8. Solve the following equations by Gauss-Seidel method:

(i)

$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

(ii)

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$-0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

*** *** ***