UNIT 2: INTERPOLATION

UNIT STRUCTURE

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2.1 LEARNING OBJECTIVES

After studying this unit, you will be able to:

- Understand the concept and significance of interpolation in numerical analysis.
- Apply different interpolation techniques such as Lagrange, Newton's divided differences, Gregory-Newton backward, and Hermite interpolation.
- Use forward, backward, and central difference operators in constructing interpolating polynomials.
- Analyze interpolation errors and understand their bounds.
- Select appropriate interpolation methods for various types of data distribution.

2.2 INTRODUCTION

In this chapter, we consider the problem of approximating a given function by a class of simpler functions, mainly polynomials. There are two main uses of interpolation. The first use are in reconstructing the function $f(x)$ when it is not given explicitly, and only the values of $f(x)$ and/or its certain order derivatives at a set of points, called nodes, tabular points, or arguments, are known. The second use is to replace the function $f(x)$ by an interpolating polynomial $P(x)$ so that many common operations such as determination of roots, differentiation, and integration, etc., which are intended for the function $f(x)$ may be performed using $P(x)$.

Definition: A polynomial $P(x)$ is called an interpolating polynomial if the value of $P(x)$ and/ or its certain derivatives coincide with those of $f(x)$ and/ or its same order derivatives at one or more tabular points.

Taylor series: If the polynomial $P(x)$ is written as Taylor's expansion, for the function $f(x)$ about a point x_0, $x_0 \in [a,b]$, in the form

$$P(x) = f(x_0) + (x-x_0) f'(x_0) + \frac{1}{2!}(x-x_0)^2 f''(x_0) + \ \ldots \ + \frac{1}{n!}(x-x_0)^n f^{(n)}(x_0) \tag{2.1}$$

then, $P(x)$ may be regarded as an interpolating polynomial of degree *n*, satisfying the conditions $P^{(k)}(x_0) = f^{(k)}(x_0)$, $k = 0,1,...,n$. (2.2)

The term $R_n = \frac{1}{(n+1)!}(x-x_0)^{n+1} f^{(n+1)}(\xi)$, $x_0 < \xi < x$, (2.3)

which has been neglected in (<2.1), is called the remainder or the truncation error.

The number of terms to be included in (<2.1) may be determined by the acceptable error. If this error is $\varepsilon > 0$ and the series is truncated at the term $f^{(n)}(x_0)$, then

$$\frac{1}{(n+1)!}|x-x_0|^{n+1}\left|f^{(n+1)}(x)\right| \le \varepsilon \Rightarrow \frac{1}{(n+1)!}|x-x_0|^{n+1} M_{n+1} \le \varepsilon, \tag{2.4}$$

where $M_{n+1} = \max_{a \le x \le b}\left|f^{(n+1)}(x)\right|$.

Assume that the value of M_{n+1} or its estimate is available.

For a given ε and x, (4) will determine n, and if n and x are prescribed, it will determine ε. When both n and ε are given, it will give an upper bound on $(x-x_0)$, that is it will give an interval about x_0 in which this Taylor's polynomials approximates $f(x)$ to the prescribed accuracy.

2.3 ILLUSTRATIVE EXAMPLES

Example 2.3.1: Obtain the Taylor series approximation about $x=1$, up to second degree terms for the function $f(x)=\frac{1}{1+x^2}$. Find a bound on the error if this approximation is to use in $[1,1.4]$.

Solution: We have

$$f(x)=\frac{1}{1+x^2}, f'(x)=-\frac{2x}{(1+x^2)^2}, f''(x)=-\frac{2(1-3x^2)}{(1+x^2)^3}, f'''(x)=-\frac{24x(x^2-1)}{(1+x^2)^4}$$

and $f(1)=\frac{1}{2}, f'(1)=-\frac{1}{2}, f''(1)=\frac{1}{2}$.

The Taylor series approximation is given by

$$\begin{aligned} f(x) &= f(1)+(x-1)f'(1)+\frac{(x-1)^2}{2}f''(1) \\ &= \frac{1}{2}+(x-1)\left(-\frac{1}{2}\right)+\frac{(x-1)^2}{2}.\frac{1}{2} \\ &= \frac{x^2}{2}-x+\frac{5}{4} \end{aligned}$$

The error bound is given by

$|R_2| \le \frac{1}{3!}(x-1)^3 M_3$, where $M_3 = \max_{1\le x\le 1.4}|f'''(x)| = 2.016$

Therefore, $|R_2| \le \frac{(x-1)^3}{6}\times 2.016 = 0.336(x-1)^3$

Hence, the maximum absolute error occurs at $x=1.4$ and this value is 0.0215.

CHECK YOUR PROGRESS

(a) Obtain a second-degree polynomial approximation to $f(x)=(1+x)^{1/2}$ over [0, 1] by means of the Taylor expansion about $x=0$. Use the first three terms of the expansion to approximate $f(0.05)$ and obtain a bound of the error.

Interpolation

In general, if there are $n+1$ distinct points $a \le x_0 < x_1 < x_2 < \ ... \ < x_n \le b,$ then the problem of interpolation is to obtain $P(x)$ satisfying the conditions

(i) $P(x_i)=f(x_i), i=0,1,2,...,n$

(2.5)

or

(ii) $P(x_i)=f(x_i)$

$P'(x_i)=f'(x_i),\ i=0,1,2,...,n$

(2.6)

The condition in (2.5) gives the Lagrange/ Newton interpolating polynomial and (2.6) gives the Hermite interpolating polynomial.

2.4 LAGRANGE AND NEWTON INTERPOLATIONS

Existence:

Suppose we are given an interval $[a,b]$ and a function $f(x)$ which is continuous on $[a,b]$. Also, we assume that we have $n+1$ distinct points $a \le x_0 < x_1 < x_2 < \ ... \ < x_{n-1} \le b$ of $[a,b]$ and that the values of a function $f(x)$ are known at this point. We try to find the polynomial

$$P(x)=a_0+a_1x+a_2x^2+\ ... \ +a_nx^n$$

(2.7)

which satisfies the conditions $P(x_i)=f(x_i), i=0,1,2,...,n$.

Thus, we obtain the following system of equations

$$a_0 + a_1 x_0 + a_2 x_0^2 + \ \ldots\ + a_n x_0^n = f\left(x_0\right)$$
$$a_0 + a_1 x_1 + a_2 x_1^2 + \ \ldots\ + a_n x_1^n = f\left(x_1\right)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_0 + a_1 x_n + a_2 x_n^2 + \ \ldots\ + a_n x_n^n = f\left(x_n\right)$$

This system of equations has a unique solution, or the polynomial $P(x)$ exists if the Vandermonde's determinant

$$V\left(x_0, x_1, ..., x_n\right) = \begin{vmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \ldots\ldots\ldots\ldots & & & & \\ \ldots\ldots\ldots\ldots & & & & \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{vmatrix} \neq 0$$

Let

$$V\left(x_0, x_1, ..., x_{n-1}, x\right) = \begin{vmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \ldots\ldots\ldots\ldots & & & & \\ \ldots\ldots\ldots\ldots & & & & \\ 1 & x_{n-1} & x_{n-1}^2 & \ldots & x_{n-1}^n \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{vmatrix}$$

Also, by the properties of determinants, we get

$V\left(x_0, x_1, \ \ldots\ , x_{n-1}, x\right) = \left(x - x_0\right)\left(x - x_1\right)\ \ldots\ \left(x - x_{n-1}\right) A$, where A is a constant. Comparing the coefficients of x^n, we have

$$A = \begin{vmatrix} 1 & x_0 & \ldots & x_0^n \\ 1 & x_1 & \ldots & x_1^n \\ \ldots\ldots\ldots\ldots & & & \\ 1 & x_{n-1} & \ldots & x_{n-1}^n \end{vmatrix} = V\left(x_0, x_1, ..., x_{n-1}\right).$$

Therefore, $V\left(x_0, x_1, \ \ldots\ , x_{n-1}, x\right) = \left(x - x_0\right)\left(x - x_1\right)\ \ldots\ \left(x - x_{n-1}\right) V\left(x_0, x_1, ..., x_{n-1}\right)$.

Substituting $x = x_n$, we have

$$V\left(x_0, x_1, \ \ldots\ , x_{n-1}, x\right) = V\left(x_0, x_1, ..., x_{n-1}\right) \prod_{i=0}^{n-1} \left(x_n - x_i\right).$$

Applying recursively, we get $V\left(x_0, x_1, \ \ldots\ , x_n\right) = \prod_{\substack{i,j=0 \\ i>j}}^{n-1} \left(x_i - x_j\right) \neq 0$, since x_i are distinct.

Uniqueness:

Now we have to show that the polynomial obtained above is unique. So, we assume that there is another polynomial $P^*(x)$ which satisfies $P^*(x_i) = f(x_i),\ i = 0,1,2,\ \ldots\ ,n$.

(2.8)

Consider the polynomial $Q(x) = P(x) - P^*(x)$

(2.9)

Since $P(x)$ and $P^*(x)$ are both polynomials of degree $\leq n$ which has $n+1$ distinct roots $x_0, x_1, x_2, ..., x_n$. Thus $Q(x) \equiv 0$, as a polynomial $Q(x)$ has exactly *n* roots, real or complex. So, $P(x) = P^*(x)$.

Therefore, the interpolating polynomials derived in two distinct ways may differ in form, but they are identical in all other respects. The polynomial is called either the Lagrange interpolating polynomial or the Newton divided differences interpolating polynomial, depending on its form.

2.4.1 Lagrange Interpolation

Linear Interpolation:

For $n = 1$, we have $P(x) = a_1 x + a_0$

(2.10)

where a_0 and a_1 are arbitrary constants that satisfy $f(x_i) = P(x_i),\ i = 0,1$.

We have

$$f(x_0) = P(x_0) = a_1 x_0 + a_0$$
$$f(x_1) = P(x_1) = a_1 x_1 + a_0$$

(2.11)

Eliminating a_0 and a_1 from (10) and (11), we obtain the required linear interpolating polynomial as

$$\begin{vmatrix} P(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0$$

(2.12)

which is shown graphically in Figure 2.1.

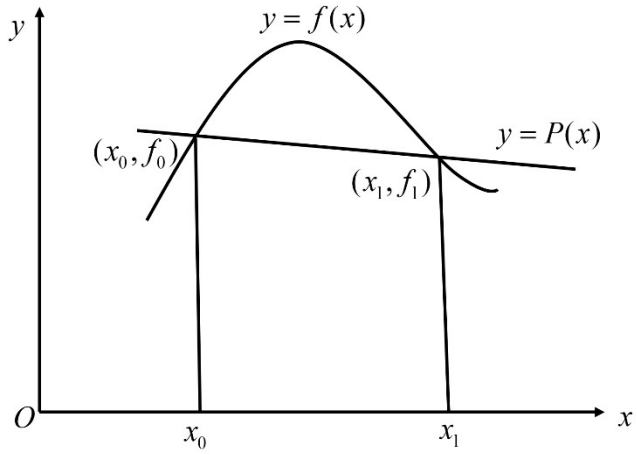


Figure 2.1: Linear interpolation.

Lagrange Interpolation:

From (12) we have

$$P(x)(x_0 - x_1) - f(x_0)(x - x_1) + f(x_1)(x - x_0) = 0$$

$$or, P(x) = f(x_0)\frac{(x - x_1)}{(x_0 - x_1)} + f(x_1)\frac{(x - x_0)}{(x_1 - x_0)} = f(x_0)l_0(x) + f(x_1)l_1(x) \tag{2.13}$$

where $l_0(x) = \frac{(x - x_1)}{(x_0 - x_1)}$, $l_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}$.

The functions $l_0(x)$ and $l_1(x)$ are called the Lagrange fundamental polynomials and it can be verified that they satisfy the conditions

$$l_0(x) + l_1(x) = 1$$
$$l_0(x_0) = 1, l_0(x_1) = 0$$
$$l_1(x_0) = 0, l_1(x_1) = 1$$

$$\text{or } l_i(x_i) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{2.14}$$

The equation (2.13) is the linear Lagrange interpolating polynomial.

Remark:

Equation (2.13) can be written as

$$P(x)=\frac{1}{x_1-x_0}\left[(x_1-x)f(x_0)-(x_0-x)f(x_1)\right]$$

$$=\frac{1}{(x_1-x_0)}\begin{vmatrix} I_0(x) & x_0-x \\ I_1(x) & x_1-x \end{vmatrix}=I_{0,1}(x)$$

(2.15)

where $I_0(x)=f(x_0)$ and $I_1(x)=f(x_1)$.

The equation (2.15) is called Aitken's or iterated linear interpolating polynomial.

2.4.2 Newton's divided difference Interpolation

Expanding the determinant (12), we get

$$P(x)=f(x_0)+(x-x_0)\frac{f(x_1)-f(x_0)}{x_1-x_0}=f(x_0)+(x-x_0)f\left[x_0,x_1\right],$$

(2.16)

where $f\left[x_0,x_1\right]=\frac{f(x_1)-f(x_0)}{x_1-x_0}$.

(2.17)

The ratio $f\left[x_0,x_1\right]$ is called the first divided difference of *f(x)* relative to x_0 and x_1.

The equation (2.16) can also be written as

$$\frac{P(x)-f(x_0)}{x-x_0}=f\left[x_0,x_1\right].$$

(2.18)

The equations (2.16) or (2.18) is the linear interpolating polynomial with divided differences.

2.5 ILLUSTRATIVE EXAMPLES

Example 2.5.1: Given $f(2)=4, f(2.5)=5.5$ find the linear interpolating polynomial using (a) Lagrange interpolation, (b) Aitken's interpolation, and (c) Newton's divided difference interpolation. Hence find an approximate value of $f(2.2)$.

Solution: We have $x_0 = 2, x_1 = 2.5$, so $f(x_0) = 4, f(x_1) = 5.5$.

(a) The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 2.5}{2 - 2.5} = \frac{x - 2.5}{-0.5},\ l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 2}{2.5 - 2} = \frac{x - 2}{0.5}$$

Therefore, $P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1) = -8(x - 2.5) + 11(x - 2) = 3x - 2$

Hence, $f(2.2) = P_1(2.2) = 3 \times 2.2 - 2 = 4.6$

(b) The Aitken's interpolating polynomials are given by

$$P_1(x) = \frac{1}{(x_1 - x_0)} \begin{vmatrix} I_0(x) & x_0 - x \\ I_1(x) & x_1 - x \end{vmatrix} = \frac{1}{(2.5 - 2)} \begin{vmatrix} 4 & 2 - x \\ 5.5 & 2.5 - x \end{vmatrix} = 3x - 2$$

Hence, $P_1(2.2) = 3 \times 2.2 - 2 = 4.6$.

(c) The Newton's divided difference interpolating polynomials are given by

$$\frac{P_1(x) - f(x_0)}{x - x_0} = f[x_0, x_1], \text{ where } f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So $f[x_0, x_1] = \frac{5.5 - 4}{2.5 - 2} = \frac{1.5}{0.5} = 3$

Therefore, $\frac{P_1(x) - 4}{x - 2} = 3 \Rightarrow P_1(x) = 3x - 2$

Hence, $P_1(2.2) = 3 \times 2.2 - 2 = 4.6$.

Truncation Error Bounds

The polynomial $P(x)$ coincides with the function $f(x)$ at x_0 and x_1, and it deviates at all other points, in the interval (x_0, x_1) as shown in figure 2.1. This deviation is called truncation error and written as $E_1(f; x) = f(x) - P(x)$.

Finite Difference Operators:

Let the tabular points $x_0, x_1, ..., x_n$ be equally spaced, that is $x_i = x_0 + ih, i = 0, 1, ..., n$. Now we define the following operators:

The shift operator: $Ef(x_i) = f(x_i + h)$

The forward difference operator: $\Delta f(x_i) = f(x_i + h) - f(x_i)$

The backward difference operator: $\nabla f(x_i) = f(x_i) - f(x_i - h)$

The central difference operator: $\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$

The average operator: $\mu f(x_i) = \frac{1}{2}\left[f\left(x_i + \frac{h}{2}\right) + f\left(x_i - \frac{h}{2}\right)\right]$.

It can be easily verified that

$\Delta f_i = \nabla f_{i+1} = \delta f_{i+1/2}, \Delta = E - 1, \nabla = 1 - E^{-1}, \delta = E^{1/2} - E^{-1/2}$, and

$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$.

Repeated application of the difference operators leads to the following higher order differences.

$E^n f(x_i) = f(x_i + nh)$

$\Delta^n f(x_i) = \Delta^{n-1} f_{i+1} - \Delta^{n-1} f_i = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i+n-k}$

$\nabla^n f(x_i) = \nabla^{n-1} f_i - \nabla^{n-1} f_{i-1} = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i-k}$

$\delta^n f(x_i) = \delta^{n-1} f_{i+1/2} - \delta^{n-1} f_{i-1/2} = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i+n/2-k}$, where $f_i = f(x_i)$.

Also, $\Delta^n f(x_i) = (E-1)^n f(x_i), \nabla^n f(x_i) = (1 - E^{-1})^n f(x_i)$.

From Tables 2.1, 2.2, 2.3 we can calculate the forward, backward, and central differences of various orders, respectively.

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	f_0			
		Δf_0		
x_1	f_1		$\Delta^2 f_0$	
		Δf_1		$\Delta^3 f_0$

x_2	f_2		$\Delta^2 f_1$	
		Δf_2		
x_3	f_3			

Table 2.1: Forward difference table

Table 2.2: Backward difference table

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
x_0	f_0			
		∇f_1		
x_1	f_1		$\nabla^2 f_2$	
		∇f_2		$\nabla^3 f_3$
x_2	f_2		$\nabla^2 f_3$	
		∇f_3		
x_3	f_3			

Table 2.3: Central difference table

x	$f(x)$	δf	$\delta^2 f$	$\delta^3 f$	$\delta^4 f$
x_0	f_0				
		$\delta f_{1/2}$			
x_1	f_1		$\delta^2 f_1$		
		$\delta f_{3/2}$		$\delta^3 f_{3/2}$	
x_2	f_2		$\delta^2 f_2$		$\delta^4 f_2$
		$\delta f_{5/2}$		$\delta^3 f_{5/2}$	
x_3	f_3		$\delta^2 f_3$		
		$\delta f_{7/2}$			
x_4	f_4				

The relations among different operators are mentioned in Table 2.4.

Table 2.4: Relationship between the operators

	E	Δ	∇	δ
E	E	$\Delta+1$	$(1-\nabla)^{-1}$	$1+\frac{1}{2}\delta^2+\delta\sqrt{\left(1+\frac{1}{4}\delta^2\right)}$
Δ	$E-1$	Δ	$(1-\nabla)^{-1}-1$	$\frac{1}{2}\delta^2+\delta\sqrt{\left(1+\frac{1}{4}\delta^2\right)}$
∇	$1-E^{-1}$	$1-(1+\Delta)^{-1}$	∇	$-\frac{1}{2}\delta^2+\delta\sqrt{\left(1+\frac{1}{4}\delta^2\right)}$
δ	$E^{1/2}-E^{-1/2}$	$\Delta(1+\Delta)^{-1/2}$	$\nabla(1-\nabla)^{-1/2}$	δ
μ	$\frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)$	$\left(1+\frac{1}{2}\Delta\right)(1+\Delta)^{1/2}$	$\left(1-\frac{1}{2}\nabla\right)(1-\nabla)^{-1/2}$	$\sqrt{\left(1+\frac{1}{4}\delta^2\right)}$

Example 2.5.2 Show that

(a) $\delta=\nabla(1-\nabla)^{-1/2}$

(b) $\mu=\left(1+\frac{\delta^2}{4}\right)^{1/2}$

Solution:

(a) We have

$$\nabla(1-\nabla)^{-1/2}=\left(1-E^{-1}\right)\left[1-\left(1-E^{-1}\right)\right]^{-1/2}=\left(1-E^{-1}\right)E^{1/2}=E^{1/2}-E^{-1/2}=\delta.$$

(b) $\delta^2=\left(E^{1/2}-E^{-1/2}\right)^2=E+E^{-1}-2$

$$\left(1+\frac{\delta^2}{4}\right)^{1/2}=\left[1+\frac{1}{4}\left(E+E^{-1}-2\right)\right]^{1/2}=\left[\frac{1}{4}\left(E+E^{-1}+2\right)\right]^{1/2}=\frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)=\mu.$$

CHECK YOUR PROGRESS

(b) Find the unique polynomial $P(x)$ of degree 2 or less such that $P(1)=1, P(3)=27,\ P(4)=64$, using each of the following methods:

(i) Lagrange interpolating formula

(ii) Newton divided difference formula

(iii) Aitken's iterated interpolation formula.Estimate $P(1.5)$.

2.6 GREGORY-NEWTON BACKWARD

From the backward difference Table 2.2, it can be observed that the Newton interpolation with divided differences in terms of backward differences should be in terms of the differences at the end point x_n. Thus, we may write

$$f(x)=f\left(x_n+\frac{x-x_n}{h}h\right)=f(x_n+hu)=E^u f(x_n)=(1-\nabla)^{-u} f(x_n)$$

$$=f(x_n)+u\nabla f(x_n)+\frac{u(u+1)}{2!}\nabla^2 f(x_n)+\ldots+\frac{u(u+1)\ldots(u+n-1)}{n!}\nabla^n f(x_n)+\ldots,$$

where $u=\dfrac{x-x_n}{h}$.

Neglecting the difference $\nabla^{n+1} f(x_n)$ and higher order differences, we get the following interpolating polynomial

$$P(x_n+hu)=f(x_n)+u\nabla f(x_n)+\frac{u(u+1)}{2!}\nabla^2 f(x_n)+\ldots+\frac{u(u+1)\ldots(u+n-1)}{n!}\nabla^n f(x_n)$$

$$=\sum_{i=0}^{n}(-1)^i\binom{-u}{i}\nabla^i f_n \qquad (2.19)$$

The polynomial (2.19) is called the Gregory-Newton backward difference interpolating polynomial.

2.7 ILLUSTRATIVE EXAMPLES

Example 2.7.1: Using the Newton's backward difference interpolation, construct the interpolating polynomial that fits the data

x	0.1	0.3	0.5	0.7	0.9	1.1
$f(x)$	-1.699	-1.073	-0.375	0.443	1.429	2.631

Estimate the value of $f(x)$ at $x=0.6$ and $x=1$.

Solution: The difference table is obtained as

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$
0.1	-1.699					
		0.626				
0.3	-1.073		0.072			
		0.698		0.048		
0.5	-0.375		0.120		0	
		0.818		0.048		0
0.7	0.443		0.168		0	
		0.986		0.048		
0.9	1.429		0.216			
		1.202				
1.1	2.631					

The backward difference polynomial is obtained by

$$P(x) = 2.631 + \frac{(x-1.1)}{0.2} \times 1.202 + \frac{(x-1.1)}{0.2}\left(\frac{x-1.1}{0.2}+1\right)\frac{1}{2!} \times 0.216$$
$$+ \frac{(x-1.1)}{0.2}\left(\frac{x-1.1}{0.2}+1\right)\left(\frac{x-1.1}{0.2}+2\right)\frac{1}{3!} \times 0.048$$
$$= 2.631 + (x-1.1) \times 6.01 + (x-1.1)(x-0.9) \times 2.7 + (x-1.1)(x-0.9)(x-0.7) \times 1$$
$$= x^3 + 3x - 2$$

Hence, $f(0.6) \approx P(0.6) = 0.016$ and $f(1) \approx P(1) = 2$.

2.8 CENTRAL INTERPOLATION

Newton's forward and backward formulas are good when we want to find a value near the beginning or end of a table of values. But if the value we're looking for is somewhere in the middle, these formulas may not give a good result and can have more errors. To fix this problem, other formulas worked better for the middle of the table. These are called central difference formulas, and they help give more accurate answers when the point is in the centre of the data.

Stirling and Bessel Interpolations

From the central difference Table 2.3, it may be seen that there are many ways in which the central differences along the horizontal line can be used to construct the interpolations. Here, we discuss two interpolating polynomials with central differences.

For *n* even we assume that the nodal points are $x_{-p}, x_{-(p-1)}, \ldots, x_{-1}, x_0, x_1, x_2, \ldots, x_{p-1}, x_p$.

The **Stirling interpolating** polynomial is given by

$$P(x)=f(x_0)+\frac{u}{2}\left[\delta f_{1/2}+\delta f_{-1/2}\right]+\frac{u^2}{2!}\delta^2 f_0+\frac{u^2\left(u^2-1^2\right)}{3!}\frac{1}{2}\left[\delta^3 f_{1/2}+\delta^3 f_{-1/2}\right]+\ldots$$
$$+\frac{u^2\left(u^2-1^2\right)\ldots\left(u^2-(p-1)^2\right)}{(2p-1)!}\frac{1}{2}\left[\delta^{2p-1} f_{1/2}+\delta^{2p-1} f_{-1/2}\right]$$
$$+\frac{u^2\left(u^2-1^2\right)\left(u^2-2^2\right)\ldots\left(u^2-(p-1)^2\right)}{(2p)!}\delta^{2p} f_0$$

(2.20)

where $u=\dfrac{x-x_0}{h}$.

For *n* odd we assume that the nodal points are $x_{-p}, x_{-(p-1)},\ldots, x_{-1}, x_0, x_1, x_2,\ldots, x_p, x_{p+1}$. Then the **Bessel interpolating** polynomial is given by

$$P(x)=\frac{1}{2}\left(f_0+f_1\right)+v\delta f_{1/2}+\frac{v^2-\frac{1}{4}}{2!}\frac{1}{2}\left(\delta^2 f_0+\delta^2 f_1\right)+\frac{v\left(v^2-\frac{1}{4}\right)}{3!}\delta^3 f_{1/2}+\ldots$$
$$+\frac{v\left(v^2-\frac{1}{4}\right)\left(v^2-\frac{9}{4}\right)\ldots\left(v^2-\frac{(2p-1)^2}{4}\right)}{(2p)!}\frac{1}{2}\left(\delta^{2p} f_0+\delta^{2p} f_1\right)$$
$$+\frac{v\left(v^2-\frac{1}{4}\right)\left(v^2-\frac{9}{4}\right)\ldots\left(v^2-\frac{(2p-1)^2}{4}\right)}{(2p+1)!}\delta^{2p+1} f_{1/2}$$

(2.21)

where $v=u-1/2$.

The Newton-Gregory interpolations with forward and backward differences are used if the interpolation is desired near the beginning and ending, respectively, of the table values. The Stirling interpolation is used for calculations between $x_0-h/4$ and $x_0+h/4$, and also when the first neglected highest difference is odd order. The Bessel interpolation is used for calculations between $x_0+h/4$ and $x_1-h/4$, and also when the first neglected highest difference is even order.

2.9 HERMITE INTERPOLATION

The Hermit interpolating polynomial interpolates not only the function $f(x)$ but also its derivatives at a given set of tabular points. The simple interpolating conditions are

given in (2.6). We now provide an explicit expression for the interpolating polynomial satisfying (2.6), that is

$$P(x_i) = f(x_i)$$
$$P'(x_i) = f'(x_i),\ i = 0,1,2,...,n$$

Since there are 2*n*+2 conditions to be satisfied, $P(x)$ must be a polynomial of degree $\leq 2n+1$. The required polynomial may be written as

$$P(x) = \sum_{i=0}^{n} A_i(x) f(x_i) + \sum_{i=0}^{n} B_i(x) f'(x_i),$$

(2.22)

where $A_i(x)$ and $B_i(x)$ are polynomial of degree $\leq 2n+1$ and satisfy

(a) $A_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

(b) $A_i'(x_j) = 0$ for all *i* and *j*

(c) $B_i(x_j) = 0$ for all *i* and *j*

(d) $B_i'(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$

(2.23)

Using the Lagrange fundamental polynomials $l_i(x)$, we write

$$A_i(x) = \gamma_i(x) l_i^2(x)$$
$$B_i(x) = \delta_i(x) l_i^2(x).$$

(2.24)

Since $l_i^2(x)$ is a polynomial of degree 2n, $\gamma_i(x)$ and $\delta_i(x)$ must be a linear polynomial.

Let $\gamma_i(x) = a_i x + b_i,\ \delta_i(x) = c_i x + d_i.$

(2.25)

Using (2.23), we get

$$a_i = -2l_i'(x_i),\ b_i = 1 + 2x_i l_i'(x_i),\ c_i = 1,\ \text{and}\ d_i = -x_i$$

(2.26)

Substituting (2.26) in (2.25) using (2.24), the equation (2.22) becomes

$$P(x) = \sum_{i=0}^{n}\left[1 - 2(x - x_i)l_i'(x_i)\right]l_i^2(x_i)f(x_i) + \sum_{i=0}^{n}(x - x_i)l_i^2(x_i)f(x_i).$$

(2.27)

This polynomial is called Hermite interpolating polynomial.

2.10 ILLUSTRATIVE EXAMPLES

Example 2.10.1 Determine the parameters in the formula

$P(x) = a_0(x-a)^3 + a_1(x-a)^2 + a_2(x-a) + a_3$ such that

$$P(a) = f(a), P'(a) = f'(a), P(b) = f(b), P'(b) = f'(b).$$

Using the interpolatory conditions in the given formula we obtain

$$P(a) = f(a) = a_3$$

$$P'(a) = f'(a) = a_2$$

$$P(b) = f(b) = a_0(b-a)^3 + a_1(b-a)^2 + a_2(b-a) + a_3$$

$$P'(b) = f'(b) = 3a_0(b-a)^2 + 2a_1(b-a) + a_2$$

Solving these equations, we get

$$a_3 = f(a),\ a_2 = f'(a), a_1 = \frac{3\{f(b)-f(a)\}}{(b-a)^2} - \frac{2f'(a)+f'(b)}{b-a}, a_0 = \frac{2\{f(a)-f(b)\}}{(b-a)^3} + \frac{f'(a)+f'(b)}{(b-a)^2}.$$

2.11 LET US SUM UP

- The unit focused on the fundamental concept of interpolation and its importance in approximating complex functions.
- Discussed various interpolation techniques, including:
  - Lagrange interpolation
  - Newton interpolation

- Explained the use of divided differences and finite difference operators in interpolation.
- Introduced central interpolation methods, such as:
  - Stirling's formula
  - Bessel's formula
- Highlighted their accuracy near the center of data sets.
- Covered Hermite interpolation, which incorporates both function values and derivatives.
- Concluded with examples to strengthen practical understanding of the concepts.

✓ 2.12 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q. (a) $f(0.05) = 1.0246875$; Error=0.0625 at $x = 1$.

Ans to Q. (b) $8x^2 - 19x + 12;\ \ 1.5$

2.13 FURTHER READING

1. Gupta, R. K. Numerical Methods: Fundamentals and Applications. 1st edition, Cambridge University Press, 2019.
2. Thangaraj, P. Computer Oriented Numerical Methods. PHI Learning Pvt. Ltd, 2013.
3. Jain, M. K., Iyengar, S. R. K. and Jain, R. K. Numerical Methods for Scientific & Engineering Computation. New Age International, 2012.
4. Burden R. L. and Faires J. D. Numerical Analysis. 9th Edition, Cengage Learning, 2011.
5. Chapra, S. C. and Canale, R. P. Numerical Methods for Engineers. McGraw Hill, International Edition, 1998.
6. Mathews, J. H. Numerical Methods for Mathematics, Science and Engineering. Prentice- Hall, International Editions, 1992.

2.14 MODEL QUESTIONS

1. Obtain a second-degree polynomial approximation of $f(x)=e^{2x}\sin x$ over $[0,0.5]$ by means of the Taylor expansion about $x=0$. Use this polynomial to approximate $f(0.1)$ and obtain a bound of the error.

2. If $f(x)=\frac{1}{x^2}$, find the divided difference $f[x_1,x_2,x_3,x_4]$.

3. If $f(x)=e^{ax}$, show that $\Delta^n f(x)=\left(e^{ah}-1\right)^n e^{ax}$.

4. A function $f(x)$ is approximated by the interpolating polynomial

   $P(x)=c_0+c_1(x-1)+c_2(x-1)^2+c_3(x-1)^3, 1\le x\le 2.$

   Determine the parameters c_0,c_1,c_2,c_3 such that

   $P(1)=f(1), P(2)=f(2), P'(1)=f'(1), P'(2)=f'(2).$

5. Given the values of $f(x)$ and $f'(x)$

x	$f(x)$	$f'(x)$
-1	1	-5
0	1	1
1	3	7

estimate the values of $f(-0.5)$ and $f(0.5)$ using the Hermite interpolation. The exact values are $f(-0.5)=\frac{33}{64}$ and $f(0.5)=\frac{97}{64}$.

6. Construct the Hermite interpolation polynomial that fits data

x	$f(x)$	$f'(x)$
2	29	50
3	105	105

Interpolate $f(x)$ at $x=2.5$.

*** *** ***