UNIT 3: SPLINE INTERPOLATION

UNIT STRUCTURE

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3.1 LEARNING OBJECTIVES

After studying this unit, you will be able to:

- Understand the concept and purpose of spline interpolation.
- Apply quadratic and cubic spline interpolation techniques to given data sets.
- Derive and solve the systems of equations necessary for fitting splines.
- Analyze the limitations of different spline methods.
- Estimate function values at intermediate points using spline functions.

3.2 INTRODUCTION

When computers were not available, the draftsman used a device to draw a smooth curve through a given set of points such that the slope and the curvature are also continuous along the curve, that is $f(x), f'(x)$ and $f''(x)$ are continuous on the curve. Such a device was called a spline and the plotting of the curve was called spline fitting.

The given interval $[a,b]$ in subdivided into n subintervals $[x_0,x_1],[x_1,x_2],...,[x_{n-1},x_n]$ where $a=x_0<x_1<x_2<...<x_n=b$. The nodes (knots) $x_1,x_2,...,x_{n-1}$ are called internal nodes.

Definition: A spline function of degree n with knots (nodes) $x_i, i=0,1,...,n$ is a function $F(x)$ satisfying the properties

(i) $F(x_i)=f(x_i), i=0,1,...,n$.

(ii) On each subinterval $[x_{i-1},x_i]$, $1\le i\le n$, $F(x)$ is a polynomial of degree n.

(iii) $F(x)$ and its first $(n-1)$ derivatives are continuous on (a,b).

3.3 QUADRATIC SPLINE INTERPOLATION

A quadratic spline satisfies the following properties:

(i) $F(x_i)=f(x_i), i=0,1,...,n$.

(ii) On each subinterval $[x_{i-1},x_i]$, $1\le i\le n$, $F(x)$ is a second-degree polynomial, except in the first or the last interval.

(iii) $F(x)$ and $F'(x)$ are continuous on (a,b).

We denote $F''(x)=M_i$, if the second derivative exists. On each subinterval $[x_{i-1},x_i]$, we approximate $f(x)$ by a second-degree polynomial as

$$F(x)=P(x_i)=a_ix^2+b_ix+c_i, i=1,2,...,n.$$

There are $3n$ unknowns to be determined which are $a_1,b_1,c_1;\ a_2,b_2,c_2;\ ...;\ a_n,b_n,c_n$.

Since $F(x)$ is continuous at the internal nodes $x_1,x_2,...,x_{n-1}$, we obtain the equations

on $[x_{i-1},x_i]: P_i(x_i)=f_i=a_ix_i^2+b_ix_i+c_i$

(3.1)

on $[x_i,x_{i+1}]: P_{i+1}(x_i)=f_i=a_{i+1}x_i^2+b_{i+1}x_i+c_{i+1}$

(3.2)

$i = 1, 2, ..., n-1$. From this set, we have $2n-2$ equations.

Since $F'(x)$ is also continuous at the internal nodes, we obtain the equations

$$P_i'(x_i) = P_{i+1}'(x_i) \text{ or } 2a_i x_i + b_i = 2a_{i+1}x_i + b_{i+1},\ i = 1, 2, ..., n-1 \tag{3.3}$$

From (3.3) we have *n*-1 equations. At the points x_0, x_n, interpolatory conditions give the equations

$$f_0 = a_1 x_0^2 + b_1 x_0 + c_1 \text{ and } f_n = a_n x_n^2 + b_n x_n + c_n. \tag{3.4}$$

Now, we have total $(2n-2)+(n-1)+2 = 3n-1$ equations to determine 3*n* unknowns.

We need one more equation to determine the polynomial uniquely. To find extra condition the following ways can be applied.

(i) Prescribe $M_0 = f''(x_0) = P_1''(x_0) = p$. This gives $f''(x_0) = 2a_1 = p \Rightarrow a_1 = \frac{p}{2}$.

If $p = 0$, then $a_1 = 0$. Hence, in the first subinterval $[x_0, x_1]$, we are using linear approximation, that is the first two points are joined by a straight line.

(ii) Prescribe $M_n = f''(x_n) = P_n''(x_n) = q$. This gives $f''(x_n) = 2a_n = q \Rightarrow a_n = \frac{q}{2}$.

If $q = 0$, then $a_n = 0$. Hence, in the last subinterval $[x_{n-1}, x_n]$, we are using linear approximation, that is the last two points are joined by a straight line.

Now, the system of $3n \times 3n$ linear algebraic equations are solved for $a_i, b_i, c_i, i = 1, 2, ..., n.$ However, by arranging the equations in proper order, it is possible to solve 3×3 equations for each set of unknowns $a_i, b_i, c_i, i = 1, 2, ..., n.$

Suppose we have three subintervals $[x_0, x_1], [x_1, x_2], [x_2, x_3]$, then from (3.1) -(3.4), we have

$$a_1 x_1^2 + b_1 x_1 + c_1 = f_1, a_2 x_1^2 + b_2 x_1 + c_2 = f_1 \tag{3.5 a, b}$$

$a_2x_2^2+b_2x_2+c_2=f_2, a_3x_2^2+b_3x_2+c_3=f_2$ (3.6 a, b)

$2a_1x_1+b_1=2a_2x_1+b_2, 2a_2x_2+b_2=2a_3x_2+b_3$ (3.7 a, b)

$a_1x_0^2+b_1x_0+c_1=f_0, a_3x_3^2+b_3x_3+c_3=f_3$. (3.8 a, b)

Let us choose $M_0=f''(x_0)=0$. This implies $a_1=0$. Then the above equations become

$$\left.\begin{aligned} b_1x_0+c_1&=f_0\\ b_1x_1+c_1&=f_1\end{aligned}\right\} \quad (3.9)$$

$$\left.\begin{aligned} a_2x_1^2+b_2x_1+c_2&=f_1\\ a_2x_2^2+b_2x_2+c_2&=f_2\\ 2a_2x_2+b_2&=2a_3x_2+b_3\end{aligned}\right\} \quad (3.10)$$

$$\left.\begin{aligned} a_3x_2^2+b_3x_2+c_3&=f_2\\ 2a_2x_2+b_2&=2a_3x_2+b_3\\ a_3x_3^2+b_3x_3+c_3&=f_3\end{aligned}\right\} \quad (3.11)$$

The system of equations (3.9) is solved for $b_1 \& c_1$. After that (3.10) and (3.11) are solved. i.e., the system of equations is solved in forward direction. If $M_3=f''(x_3)=0$ is considered, then we rearrange the equations so that the solution is obtained in the backward direction.

Quadratic splines have two disadvantages. They are

(a) A straight line connects the first two or last two points.

(b) The spline for the last interval or the first interval may swing high in the above cases.

For this reasons, quadratic splines are not often used.

3.4 ILLUSTRATIVE EXAMPLES

Example 3.4.1: Given the data

x	0	1	2	3
$f(x)$	1	2	33	244

fit quadratic splines with $M(0)=f''(0)=0$. Hence, find an estimate of $f(2.5)$.

Solution: Let us suppose

$$P_1(x) = a_1x^2 + b_1x + c_1,\ 0 \le x \le 1$$

$$P_2(x) = a_2x^2 + b_2x + c_2,\ 1 \le x \le 2$$

$$P_3(x) = a_3x^2 + b_3x + c_3,\ 2 \le x \le 3.$$

Since, $M(0) = f''(0) = 0$, we have $a_1 = 0$.

Also, here $x_0 = 0, x_1 = 2, x_2 = 2, x_3 = 3, f_0 = 1, f_1 = 2, f_2 = 33, f_3 = 244$. Putting these values in (3.9) -(3.11), we have

$$\left.\begin{aligned} b_1(0) + c_1 &= 1 \\ b_1 + c_1 &= 2 \end{aligned}\right\}$$

(i)

$$\left.\begin{aligned} a_2 + b_2 + c_2 &= 2 \\ 4a_2 + 2b_2 + c_2 &= 33 \\ 2a_2 + b_2 &= 2a_1 + b_1 \end{aligned}\right\}$$

(ii)

$$\left.\begin{aligned} 4a_3 + 2b_3 + c_3 &= 33 \\ 9a_3 + 3b_3 + c_3 &= 244 \\ 4a_3 + b_3 &= 4a_2 + b_2 \end{aligned}\right\}$$

(iii)

From (i), we get $b_1 = 1,\ c_1 = 1$. Then the equations (ii) become

$$\begin{aligned} a_2 + b_2 + c_2 &= 2 \\ 4a_2 + 2b_2 + c_2 &= 33 \\ 2a_2 + b_2 &= 1 \end{aligned}$$

The solution of this system is $a_2 = 30, b_2 = -59,\ c_2 = 31$.

From (iii), we get

$$\begin{aligned} 4a_3 + 2b_3 + c_3 &= 33 \\ 9a_3 + 3b_3 + c_3 &= 244 \\ 4a_3 + b_3 &= 61 \end{aligned}$$

The solution of this system is $a_3 = 150, b_3 = -539,\ c_3 = 511$.

Therefore, the corresponding intervals can be written as

$P_1(x) = x + 1,\ 0 \le x \le 1$

$P_2(x) = 30x^2 - 59x + 31,\ 1 \le x \le 2$

$P_3(x) = 150x^2 - 539x + 511,\ 2 \le x \le 3.$

Hence, $f(2.5) = 150 \times 2.5^2 - 539 \times 2.5 + 511 = 101.$

CHECK YOUR PROGRESS 1

1. (a) In the following problems, obtain the quadratic splines representing the function defined by the given data

(i)

x	0	1	2	3
$f(x)$	1	3	11	31

Assume $f''(0) = M(0) = 0$. Interpolate at $x = 1.5$ and $x = 2.5$.

(ii)

x	-1	0	1	2
$f(x)$	-4	1	0	5

Assume $f''(2) = M(2) = 0$. Interpolate at $x = -0.5$.

3.5 CUBIC SPLINE INTERPOLATION

A cubic spline satisfies the following properties:

(i) $F(x_i) = f(x_i), i = 0,1,...,n.$

(ii) On each subinterval $[x_{i-1}, x_i],\ 1 \le i \le n,\ F(x)$ is a third-degree polynomial.

(iii) $F(x), F'(x)$ and $F''(x)$ are continuous on (a,b).

We denote $F'(x_i) = m_i$ and $F''(x_i) = M_i$.

On each subinterval $[x_{i-1}, x_i]$, we approximate $f(x)$ by a cubic polynomial as

$F(x) = P_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i, i = 1,2,...,n.$

We have $4n$ unknowns $a_i, b_i, c_i, d_i, i = 1,2,...,n$ to be determined.

Using the continuity of $F(x), F'(x)$ and $F''(x)$ we have the following equations.

(a) Continuity of $F(x)$:

$$\text{On } [x_{i-1}, x_i]: P_i(x_i) = f_i = a_i x_i^3 + b_i x_i^2 + c_i x_i + d_i \tag{3.12a}$$

$$\text{On } [x_i, x_{i+1}]: P_{i+1}(x_i) = f_i = a_{i+1} x_i^3 + b_{i+1} x_i^2 + c_{i+1} x_i + d_{i+1} \tag{3.12b}$$

where $i = 1, 2, ..., n-1$.

(b) Continuity of $F'(x)$:

$$3a_i x_i^2 + 2b_i x_i + c_i = 3a_{i+1} x_i^2 + 2b_{i+1} x_i + c_{i+1}, i = 1, 2, ..., n-1. \tag{3.13}$$

(c) Continuity of $F''(x)$:

$$6a_i x_i + 2b_i = 6a_{i+1} x_i + 2b_{i+1}, i = 1, 2, ..., n-1. \tag{3.14}$$

At the end points x_0 and x_n, we have the interpolatory conditions

$$f_0 = a_1 x_0^3 + b_1 x_0^2 + c_1 x_0 + d_1 \tag{3.15}$$

$$f_n = a_n x_n^3 + b_n x_n^2 + c_n x_n + d_n. \tag{3.16}$$

Now, we have total $(2n-2)+(n-1)+(n-1)+2 = 4n-2$ equations to determine $4n$ unknowns. We need two more equations to determine the polynomial uniquely. To find extra conditions we choose $F''(x_0) = M_0 = p$ and $F''(x_n) = M_n = q$.

The end conditions, $M_0 = 0, M_n = 0$, lead to a natural spline. It is called a natural spline since the drafting spline always behaves in this fashion. However, we can use the conditions as $p \neq 0$ and/ or $q \neq 0$. If the above conditions are imposed, then we have $4n$ equations in $4n$ unknowns. These equations can be written in the matrix form and solution can be obtained.

3.6 ILLUSTRATIVE EXAMPLES

Example 3.6.1: Obtain the cubic spline approximation for the function defined by the data

x	0	1	2	3
$f(x)$	1	2	33	244

with $M(0)=0, M(3)=0$. Hence, find an estimate of $f(2.5)$.

Solution: Suppose $F(x)=P_1(x)=a_1x^3+b_1x^2+c_1x+d_1,\ 0\le x\le 1$

$$F(x)=P_2(x)=a_2x^3+b_2x^2+c_2x+d_2,\ 1\le x\le 2$$

$$F(x)=P_3(x)=a_3x^3+b_3x^2+c_3x+d_3,\ 2\le x\le 3.$$

Continuity of $F(x)$ gives

$$P_1(x_1)=f_1=a_1x_1^3+b_1x_1^2+c_1x_1+d_1=a_1+b_1+c_1+d_1$$

$$P_2(x_1)=f_1=a_2+b_2+c_2+d_2$$

$$P_2(x_2)=f_2=a_2x_2^3+b_2x_2^2+c_2x_2+d_2=8a_2+4b_2+2c_2+d_2$$

$$P_3(x_2)=f_2=8a_3+4b_3+2c_3+d_3$$

Continuity of $F'(x)$ gives the following equations

$$3a_1x_1^2+2b_1x_1+c_1=3a_2x_1^2+2b_2x_1+c_2\Rightarrow 3a_1+2b_1+c_1=3a_2+2b_2+c_2$$

$$3a_2x_2^2+2b_2x_2+c_2=3a_3x_2^2+2b_3x_2+c_3\Rightarrow 12a_2+4b_2+c_2=12a_3+4b_3+c_3$$

Continuity of $F''(x)$ gives the following equations

$$6a_1x_1+2b_1=6a_2x_1+2b_2\Rightarrow 6a_1+2b_1=6a_2+2b_2$$

$$6a_2x_2+2b_2=6a_3x_2+2b_3\Rightarrow 12a_2+2b_2=12a_3+2b_3$$

At the end points, we have the interpolatory conditions

$$f_0=a_1x_0^3+b_1x_0^2+c_1x_0+d_1=d_1$$

$$f_3=a_3x_3^3+b_3x_3^2+c_3x_3+d_3=27a_3+9b_3+3c_3+d_3$$

From the given conditions, $M(0)=0, M(3)=0$, we get $b_1=0, 9a_3+b_3=0$ and $d_1=1$.

Substituting these values, we get the following equations

$a_1 + c_1 = 1$

$a_2 + b_2 + c_2 + d_2 = 2$

$8a_2 + 4b_2 + 2c_2 + d_2 = 33$

$-28a_3 + 2c_3 + d_3 = 33$

$3a_1 + c_1 - 3a_2 - 2b_2 - c_2 = 0$

$12a_2 + 4b_2 + c_2 + 24a_3 - c_3 = 0$

$3a_1 - 3a_2 - b_2 = 0$

$6a_2 + b_2 + 3a_3 = 0$

$-54a_3 + 3c_3 + d_3 = 244$

The solution of this system is given as

$a_1 = -4, b_1 = 0, c_1 = 5, d_1 = 1;$
$a_2 = 50, b_2 = -162, c_2 = 167, d_2 = -53;$
$a_3 = -46, b_3 = 414, c_3 = -985, d_3 = 715.$

Hence, $P_1(x) = -4x^3 + 5x + 1,\ 0 \le x \le 1$

$P_2(x) = 5x^3 - 162x^2 + 167x - 53,\ 1 \le x \le 2$

$P_3(x) = -46x^3 + 414x^2 - 985x + 715,\ 2 \le x \le 3.$

Therefore, $f(2.5) = -46 \times 2.5^3 + 414 \times 2.5^2 - 985 \times 2.5 + 715 = 121.25.$

CHECK YOUR PROGRESS 2

1. (a) The following values of x and y are given:

   x : 1 2 3 4

   y : 1 2 5 11

   Find the cubic splines and evaluate $y(1.5)$ and $y'(3)$.

   (b) Find the cubic spline interpolation for the data:

   x : 1 2 3 4 5

   y : 1 0 1 0 1

3.7 LET US SUM UP

In this unit, we discussed Spline Interpolation, focusing on Quadratic and Cubic Splines. The key takeaways are:

- A spline function is a piecewise polynomial function that maintains continuity of function values and derivatives at the data points (called knots).
- Quadratic splines use second-degree polynomials, ensuring continuity of the first derivative. However, they may suffer from issues at the endpoints.
- Cubic splines use third-degree polynomials and ensure continuity of the first and second derivatives, producing smoother interpolations.
- The natural cubic spline imposes zero second derivatives at the end points.
- Examples were provided to illustrate the construction of both quadratic and cubic spline interpolations.

3.8 ANSWERS TO CHECK YOUR PROGRESS 1

Answer to Q.1: (a)

(i) Let us suppose

$$P_1(x) = a_1x^2 + b_1x + c_1,\ 0 \le x \le 1$$

$$P_2(x) = a_2x^2 + b_2x + c_2,\ 1 \le x \le 2$$

$$P_3(x) = a_3x^2 + b_3x + c_3,\ 2 \le x \le 3.$$

Since, $M(0) = f''(0) = 0$, we have $a_1 = 0$.

Here $x_0 = 0, x_1 = 2, x_2 = 2, x_3 = 3, f_0 = 1, f_1 = 3, f_2 = 11, f_3 = 31$.

Since

$$P_1(x_1) = f_1 = a_1x_1^2 + b_1x_1 + c_1 \Rightarrow a_1 + b_1 + c_1 = 3$$
$$P_2(x_2) = f_2 = a_2x_2^2 + b_2x_2 + c_2 \Rightarrow 4a_2 + 2b_2 + c_2 = 11$$
$$P_2(x_1) = f_1 = a_2x_1^2 + b_2x_1 + c_2 \Rightarrow a_2 + b_2 + c_2 = 3$$
$$P_2(x_2) = f_2 = a_3x_2^2 + b_3x_2 + c_3 \Rightarrow 4a_3 + b_3 + c_3 = 11$$

Also,

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2 \Rightarrow 2a_1 + b_1 = 2a_2 + b_2$$
$$2a_2x_2 + b_2 = 2a_3x_2 + b_3 \Rightarrow 4a_2 + b_2 = 4a_3 + b_3$$

Also,

$f_0 = a_1 x_0^2 + b_1 x_0 + c_1 \Rightarrow c_1 = 1$

$f_3 = a_3 x_3^2 + b_3 x_3 + c_3 \Rightarrow 9a_3 + 3b_3 + c_3 = 31$

After solving the above equations, we get

$a_1 = 0, b_1 = 2, c_1 = 1; a_2 = 6, b_2 = -10, c_2 = 7; a_3 = 6, b_3 = -10, c_3 = 7$.

Therefore,

$$P_1(x) = 2x + 1, \ 0 \le x \le 1$$

$$P_2(x) = 6x^2 - 10x + 7, \ 1 \le x \le 2$$

$$P_3(x) = 6x^2 - 10x + 7, \ 2 \le x \le 3.$$

Hence, $f(1.5) = 5.5$ and $f(2.5) = 19.5$.

Answer to Q.1: (b)

(ii) Try yourself.

✓ 3.9 ANSWERS TO CHECK YOUR PROGRESS 2:

Ans to Q. 1. a) Try yourself.

b) Try yourself.

3.10 FURTHER READING

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3.11 MODEL QUESTIONS

1. Obtain the cubic spline approximation valid $[3,4]$, for the function given in the tabular form

x	1	2	3	4
$f(x)$	3	10	29	65

under the natural spline conditions $f''(1) = M(1) = 0$ and $f''(4) = M(4) = 0$.

2. Obtain the cubic splines fit for the data

x	0	1	2	3
$f(x)$	1	4	10	8

under the conditions $f''(0) = 0 = f''(3)$ and valid in the interval $[1,2]$. Hence, obtain the estimate of $f(1.5)$.

3. Fit the following four points by the cubic splines:

x	1	2	3	4
y	1	5	11	8

Use the end conditions $y''(1) = 0 = y''(4)$. Hence, compute $y(1.5)$ and $y'(2.5)$.

4. Find the value of α and β such that the function

$$f(x) = \begin{cases} x^2 - \alpha x + 1, & 1 \le x \le 2 \\ 3x - \beta, & 2 \le x \le 3 \end{cases}$$

is a quadratic spline.

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