UNIT 4: SOLUTION OF TRANSCENDENTAL AND POLYNOMIAL EQUATIONS

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4.1 LEARNING OBJECTIVES

After studying this unit, you will be able to:

- Understand the various numerical methods used to find roots of transcendental and polynomial equations.
- Learn both direct and iterative methods for root-finding including Bisection, Secant, Regula-Falsi, and Newton-Raphson methods.

4.2 INTRODUCTION

Finding the root of an equation $f(x)=0$ is crucial in applied mathematics and engineering. These equations can be polynomial or transcendental in nature. A root is a value where the function equals zero, i.e., where the graph of $y=f(x)$ intersects the x-axis. Solutions can be simple or multiple roots, and root-finding techniques can be either direct (yielding exact results in finite steps) or iterative (yielding approximate results through repeated approximations).

4.3 SOLUTION OF EQUATIONS

Finding the root of an equation of the form

$$f(x)=0 \tag{4.1}$$

is very important in applied mathematics and engineering. The function $f(x)$ may be given explicitly, for example

$$f(x)=P_n(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n,\ a_0\neq 0$$

a polynomial of degree n in x, or $f(x)$ may be known only implicitly as a transcendental function.

***Definition* 4.1:** A number ξ is a solution of $f(x)=0$ if $f(\xi)\equiv 0$. Such a solution ξ is a root or a zero of $f(x)=0$.

Also, a root of the equation (4.1) is the value of x at which the graph of $y=f(x)$ intersects the x- axis.

***Definition* 4.2:** If we can write (4.1) as

$f(x)=(x-\xi)^m g(x)=0$

where $g(x)$ is bounded and $g(\xi)\neq 0$, then ξ is called a multiple root of multiplicity m.

In this case, $f(\xi)=f'(\xi)=\ldots=f^{(m-1)}(\xi)=0, f^{(m)}(\xi)\neq 0$.

For $m=1$, the number ξ is said to be a simple root.

There are generally two types of methods used to find the roots of the equation (4.1).

4.3.1 Direct Method

These methods give the exact value of the roots in a finite number of steps. For example, a direct method gives the root of a linear or first-degree equation

$$a_0x+a_1=0,\ a_0\neq 0 \tag{4.2}$$

$\Rightarrow x = -\frac{a_1}{a_0}$.

Similarly, the roots of the quadratic equation

$$a_0 x^2 + a_1 x + a_2 = 0,\ a_0 \neq 0 \tag{4.3}$$

are given by

$$x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0}.$$

4.3.2 Iterative methods and related concepts

These methods are based on the idea of successive approximations, i.e., starting with one or more initial approximations to the root, we obtain a sequence approximations or iterates $\{x_k\}$, which in the limit converges to the root. The methods give only one root at a time. For example, to solve the equation (2.3) we can choose any one of the following iteration methods:

$$\text{(i)} \quad x_{k+1} = -\frac{a_2 + a_0 x_k^2}{a_1},\ k = 0,1,2,...$$

$$\text{(ii)} \quad x_{k+1} = -\frac{a_2}{a_0 x_k + a_1},\ k = 0,1,2,...$$

$$\text{(iii)} \quad x_{k+1} = -\frac{a_2 + a_1 x_k}{a_0 x_k},\ k = 0,1,2,... \tag{4.4}$$

The convergence of the sequence $\{x_k\}$ to the number ξ, the root of the equation (4.3) depends on the rearrangement (4.4) and the choice of the starting approximation x_0.

***Definition* 4.3** A sequence of iterates $\{x_k\}$ is said to converge to the root ξ, if

$$\lim_{k\to\infty} |x_k - \xi| = 0 \ \text{ or } \ \lim_{k\to\infty} x_k = \xi.$$

If $x_k, x_{k-1}, ..., x_{k-m+1}$ are m approximations to the root, then a multipoint iteration method is defined as

$$x_{k+1} = \phi(x_k, x_{k-1}, ..., x_{k-m+1}). \tag{4.5}$$

The function ϕ is called multipoint iteration function.

For $m = 1$, we get the one-point iteration method

$$x_{k+1} = \phi(x_k). \tag{4.6}$$

Thus, given one or more initial approximations to the root, we require a suitable iteration function ϕ for a given function $f(x)$, such that the sequence of iterates obtained from (4.5) or (4.6) converges to the root ξ. If it is not possible to find ξ which satisfy the given equation exactly. We therefore, attempt to find an approximate root ξ^* such that either

$$\left|f\left(\xi^*\right)\right| < \varepsilon$$

$$\text{or } \left|x_{k+1} - x_k\right| < \varepsilon \,, \tag{4.7}$$

where x_k and x_{k+1} are two consecutive iterates and ε is the prescribed error tolerance.

Initial Approximation

Initial approximations to the root are often known from the physical considerations of the problem. Otherwise, first approximations to the root are typically obtained using graphical approaches. Since the value of x at which the graph of $y = f(x)$ intersect the x-axis gives the root of $f(x) = 0$, any value in the neighbourhood of this point may be taken as an initial approximation to the root. If the equation $f(x) = 0$ can be conveniently written in the form $f_1(x) = f_2(x)$, then the point of intersection of the graphs of $y = f_1(x)$ and $y = f_2(x)$ gives the root of $f(x) = 0$ and therefore any value in the neighbourhood of this point can be taken as an initial approximation to the root.

Another method for obtaining the initial approximation to the root is based on the Intermediate Value Theorem.

Theorem: If $f(x)$ is a continuous function on some interval $[a,b]$ and $f(a)f(b) < 0$, the equation $f(x) = 0$ has at least one real root or an odd number of real roots in the interval (a,b).

4.4 BISECTION METHOD

The bisection method is based on the repeated application of the Intermediate Value Theorem. If we know that a root of $f(x) = 0$ lies in the interval $I_0 = (a_0, b_0)$, we bisect I_0 at the point $m_1 = (a_0 + b_0)/2$. Suppose $I_1 = (a_0, m_1)$ if $f(a_0) f(m_1) < 0$ or $I_1 = (m_1, b_0)$ if $f(m_1) f(b_0) < 0$. Then the interval I_1 also contains the root. We bisect the interval I_1 and get a subinterval I_2 at whose end points $f(x)$ takes the value of opposite signs and therefore contains the root. Continuing this procedure, we obtain a sequence of nested sets of subintervals $I_0 \supset I_1 \supset I_2 \ldots$ such that each subintervals contains the root. After repeating this process q times, we either find the root or find the interval I_q of length $(b_0 - a_0)/2^q$ which contains the root. We take the midpoint of the

last subinterval as the desired approximation to the root. This root has an error not greater than one-half of the length of the interval of which it is the midpoint.

Thus, we have

$$m_{k+1} = a_k + \frac{1}{2}(b_k - a_k),\ k = 0,1,2,\ \ldots$$

where

$$(a_{k+1}, b_{k+1}) = \begin{cases} (a_k, m_{k+1}), \text{ if } f(a_k) f(m_{k+1}) < 0 \\ (m_{k+1}, b_k), \text{ if } f(m_{k+1}) f(b_k) < 0. \end{cases}$$

This method is used only at the endpoints of the interval $[a_k, b_k]$ where $f(a_k) f(b_k) < 0$ and not the values of $f(x)$ at these end points, to obtain the next approximation to the root. In this method, the sequence of approximation always converges to the root for any $f(x)$ which is continuous in the interval that contains the root. If the permissible error is ε, then the approximate number of iterations required may be determined from the relation

$$\frac{b_0 - a_0}{2^n} \le \varepsilon \text{ or } n \ge \frac{\log(b_0 - a_0) - \log \varepsilon}{\log 2}$$

Since *n* is an integer, we take *n* as the next nearest integer.

The minimum number of iterations required for convergence to a root in the interval $(0,1)$ for a given ε are listed below:

ε	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
n	7	10	14	17	20	24

4.5 ILLUSTRATIVE EXAMPLES

Example 4.5.1: Perform five iterations of the bisection method to obtain the smallest positive root of the equation $f(x) = x^3 - 5x + 1 = 0$.

Solution: Since $f(0) > 0$ and $f(1) < 0$, the smallest positive root lies in the interval $(0,1)$. Taking $a_0 = 0, b_0 = 1$, we get

$$m_1 = \frac{1}{2}(0+1) = 0.5$$

Now, $f(m_1) = f(0.5) = -1.375$ and $f(a_0) f(m_1) < 0$.

Thus, the root lies in the interval $(0, 0.5)$.

Now, taking $a_1 = 0, b_1 = 0.5$, we get

$$m_2 = \frac{1}{2}(0+0.5) = 0.25$$

$f(m_2) = f(0.25) = -0.234375$ and $f(a_1) f(m_2) < 0$.

Thus, the root lies in the interval $(0, 0.25)$. Taking $a_2 = 0, b_2 = 0.25$, we get

$$m_3 = \frac{1}{2}(0 + 0.25) = 0.125$$

$f(m_3) = f(0.125) = 0.37695$ and $f(a_2) f(m_3) > 0$ and $f(m_3) f(b_2) < 0$.

Thus, the root lies in the interval $(0.125, 0.25)$. Taking $a_3 = 0.125, b_3 = 0.25$, we get

$$m_4 = \frac{1}{2}(0.125 + 0.25) = 0.1875$$

$f(m_4) = f(0.1875) = 0.06909$ and $f(m_4) f(b_3) < 0$.

Thus, the root lies in the interval $(0.1875, 0.25)$. Taking $a_4 = 0.1875, b_4 = 0.25$, we get

$$m_5 = \frac{1}{2}(0.1875 + 0.25) = 0.21875$$

$f(m_5) = f(0.21875) = -0.08328$ and $f(m_5) f(a_4) < 0$.

Thus, the root lies in the interval $(0.1875, 0.21875)$.

CHECK YOUR PROGRESS 1

(a) Find a root of the equation $x^3 - 4x - 9 = 0$, using the bisection method correct to three decimal places.

(b) Using bisection method, find the negative root of the equation $x^3 - 4x - 9 = 0$.

4.6 THE METHOD OF ITERATION

Here, we have studied the iteration methods which will produce exact results whenever $f(x) = 0$ is a first-degree equation. Thus, if we approximate $f(x)$ by a first-degree equation in the neighbourhood of the root, then we may write

$$f(x) = a_0 x + a_1 = 0. \tag{4.8}$$

The solution of (4.8) is $x = -\frac{a_1}{a_0}$ (4.9)

where $a_0 \neq 0$ and a_1 are arbitrary parameters to be determined by prescribing two appropriate conditions on $f(x)$ and/or its derivatives.

4.6.1 Secant method

If x_{k-1} and x_k are two approximations to the root, then we determine a_0 and a_1 in (4.8) by using the conditions

$f_{k-1} = a_0 x_{k-1} + a_1$

$f_k = a_0 x_k + a_1$

where $f_{k-1} = f(x_{k-1})$ and $f_k = f(x_k)$.

Solving, we get, $a_0 = \dfrac{f_k - f_{k-1}}{x_k - x_{k-1}}$ and $a_1 = \dfrac{x_k f_{k-1} - x_{k-1} f_k}{x_k - x_{k-1}}$

(4.10)

Therefore $x_{k+1} = \dfrac{x_{k-1} f_k - x_k f_{k-1}}{f_k - f_{k-1}}$.

(4.11)

Which may also be written as

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k, \; k = 1, 2, \; \ldots$$

(4.12)

This is called the secant or the chord method.

4.6.2 Regula-Falsi Method

Geometrically, in the secant method, we replace the function $f(x)$ by a straight line or a chord passing through the points (x_k, f_k) and (x_{k-1}, f_{k-1}) and take the point of intersection of the straight line with the x-axis as the next approximation to the root. If the approximations are such that $f_k f_{k-1} < 0$, then the method (4.11) or (4.12) is known as the Regula-Falsi method .

Since $(x_{k-1}, f_{k-1}), (x_k, f_k)$ are known before the start of the iteration, the secant and Regula-Falsi methods require one function evaluation per iteration.

4.7 ILLUSTRATIVE EXAMPLES:

Example 4.7.1: A real root of the equation $f(x) = x^3 - 5x + 1 = 0$ lies in the interval $(0,1)$. Perform four iterations of the secant method and the Regula-Falsi method to obtain this root.

Solution: We have $x_0 = 0, x_1 = 1, f(x_0) = 1, f(x_1) = -3$.

Secant Method:

$$x_2 = x_1 - \left(\frac{x_1 - x_0}{f_1 - f_0} \right) f_1 = 0.25, f_2 = f(x_2) = -0.234375$$

$$x_3 = x_2 - \left(\frac{x_2 - x_1}{f_2 - f_1}\right) f_2 = 0.186441, f_3 = f(x_3) = 0.074276$$

$$x_4 = x_3 - \left(\frac{x_3 - x_2}{f_3 - f_2}\right) f_3 = 0.201736, f_4 = f(x_4) = -0.000470$$

$$x_5 = x_4 - \left(\frac{x_4 - x_3}{f_4 - f_3}\right) f_4 = 0.201640.$$

Regula-Falsi Method:

$$x_2 = x_1 - \left(\frac{x_1 - x_0}{f_1 - f_0}\right) f_1 = 0.25, f_2 = f(x_2) = -0.234375.$$

Since $f(x_0) f(x_2) < 0, \xi \in (x_0, x_2)$. Therefore,

$$x_3 = x_2 - \left(\frac{x_2 - x_0}{f_2 - f_0}\right) f_2 = 0.202532, f_3 = f(x_3) = -0.004352.$$

Since $f(x_0) f(x_3) < 0, \xi \in (x_0, x_3)$. Therefore,

$$x_4 = x_3 - \left(\frac{x_3 - x_0}{f_3 - f_0}\right) f_3 = 0.201654, f_4 = f(x_4) = -0.000070.$$

Since $f(x_0) f(x_4) < 0, \xi \in (x_0, x_4)$. Therefore,

$$x_5 = x_4 - \left(\frac{x_4 - x_0}{f_4 - f_0}\right) f_4 = 0.201640.$$

Example 4.7.2: Find a real root of the equation $x^3 - 2x - 5 = 0$ by the Regula-Falsi Method correct to three decimal places.

Solution:

Let $f(x) = x^3 - 2x - 5$

So that $f(2) = -1$ and $f(3) = 16$.

i.e., root lies between 2 and 3.

Therefore, taking $x_0 = 2, x_1 = 3, f(x_0) = -1, f(x_1) = 16$, in the Regula-Falsi method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2.0588$$

(i)

Then, $f(x_2) = f(2.0588) = -0.3908$.

Therefore, the root lies between 2.0588 and 3.

Putting $x_0 = 2.0588, x_1 = 3, f(x_0) = -0.3908, f(x_1) = 16$, in (i), we get

$$x_3 = 2.0588 - \frac{3 - 2.0588}{16 - (-0.3908)}(-0.3908) = 2.0813 .$$

Repeating this process the successive approximations are

$$x_4 = 2.0862, x_5 = 2.0915, x_6 = 2.0934, x_7 = 2.0941, x_8 = 2.0943 .$$

Hence, the root correct to three decimal places is 2.094.

4.8 NEWTON-RAPHSON METHOD

We determine a_0 and a_1 in (4.8) using the conditions

$$\begin{aligned} f_k &= a_0 x_k + a_1 \\ f_k' &= a_0 \end{aligned} \qquad (4.13)$$

where a prime denotes differentiation with respect to x.

Substituting a_0 and a_1 from (4.13) in (4.9), and representing the approximate value of x by x_{k+1}, we have

$$x_{k+1} = x_k - \frac{f_k}{f_k'}, k = 0,1, \ \ldots \qquad (4.14)$$

This method is called the Newton-Raphson method. The method (4.14) may also be obtained directly from (4.12) by taking the limit $x_{k-1} \to x_k$. In the limit when $x_{k-1} \to x_k$, the chord passing through the point (x_k, f_k) and (x_{k-1}, f_{k-1}) becomes the tangent at the point (x_k, f_k). Hence, finding the root of the equation (4.1) is equivalent to finding the point of intersection of the tangent to the curve $y = f(x)$ at the point (x_k, f_k) with the x-axis. This method is shown graphically in Figure 4.1.

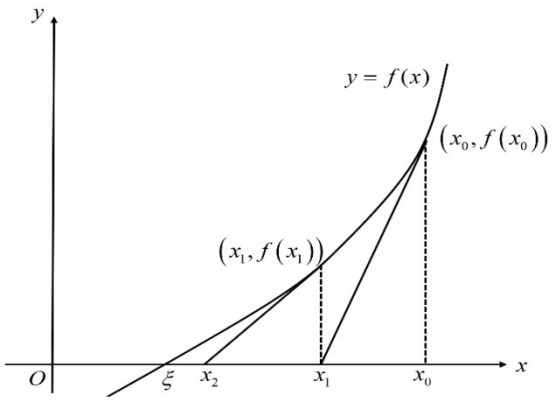


Figure 4.1: Newton-Raphson method

Alternative Method

Let x_k be an approximation to the root of the equation $f(x)=0$. Let Δx be an increment in x such that $x_k+\Delta x_k$ is an exact root. Therefore,

$f\left(x_k+\Delta x_k\right)\equiv 0$.

Expanding in Taylor series about the point x_k, we get

$$f\left(x_k\right)+\Delta x f'\left(x_k\right)+\frac{1}{2!}\left(\Delta x\right)^2 f''\left(x_k\right)+\ \ldots=0.$$

Neglecting the second and higher powers of Δx, we obtain

$$f\left(x_k\right)+\Delta x f'\left(x_k\right)\approx 0 \ \text{ or } \ \Delta x\approx-\frac{f\left(x_k\right)}{f'\left(x_k\right)},\ k=0,1,2,\ \ldots$$

Hence, we obtain

$$x_{k+1}=x_k+\Delta x=x_k-\frac{f\left(x_k\right)}{f'\left(x_k\right)},\ k=0,1,2,\ \ldots$$

4.9 ILLUSTRATIVE EXAMPLES:

Example 4.9.1: Perform four iterations of the Newton-Raphson method to find the smallest positive root of the equation $f(x)=x^3-5x+1$. The smallest positive root lies in the interval $(0,1)$. Take the initial approximation as $x_0=0.5$.

We have $f(x)=x^3-5x+1$ and $f'(x)=3x^2-5$.

Using the Newton-Raphson method

$$x_{k+1}=x_k-\frac{f\left(x_k\right)}{f'\left(x_k\right)}$$

we get $x_{k+1}=x_k-\dfrac{x_k^3-5x_k+1}{3x_k^2-5}=\dfrac{2x_k^3-1}{3x_k^2-5},\ k=0,1,2,\ldots$

Since, $x_0=0.5$, so we get

$x_1=0.176471,$

$x_2=0.201568,$

$x_3=0.201640,$

$x_4=0.201640.$

Therefore, the exact value correct to six decimal places is 0.201640.

Example 4.9.2: Find by Newton's method the real root of the equation $3x=\cos x+1$, correct to four decimal places.

Solution:

Let $f(x)=3x-\cos x-1$

$f(0) = -2 < 0, f(1) = 3 - 0.5403 - 1 > 0$.

So, the root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1. Let us take $x_0 = 0.6$.

Also $f'(x) = 3 + \sin x$

Therefore, Newton's iteration formula gives

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{3x_k - \cos x_k - 1}{3 + \sin x_k} = \frac{x_k \sin x_k + \cos x_k + 1}{3 + \sin x_k}. \quad \text{(i)}$$

Putting $k = 0$, the first approximation x_1 is given by

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{0.6\sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)} = 0.6071.$$

Similarly putting $k = 1$ in (i), the second approximation is $x_2 = 0.6071$.

Here, we get $x_1 = x_2$. Hence, the required root is 0.6071, correct to four decimal places.

CHECK YOUR PROGRESS 2

(a) Find the positive root of $x^4 - x = 10$ correct to three decimal places, using the Newton-Raphson method.

(b) Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to 2 decimal places by Newton's method.

4.10 ROOTS OF A POLYNOMIAL

Let $P_n(x) = a_0 x^n + a_1 x^{n-1} + \ \ldots\ + a_{n-1} x + a_n,\ a_0 \neq 0,$ (4.15)

where $a_0, a_1, \ \ldots\ , a_n$ are real numbers. However, if we are interested in determining all the roots (real or complex) of a polynomial equation, then it is advantageous from a computational viewpoint to use direct methods of solution.

It is useful to know the following information about the roots:

(a) The exact number of real or complex roots, along with their multiplicity.

(b) The interval in which each real root lies.

***Definition**:* When a polynomial (terms being written in order), a positive sign follows a positive sign or a negative sign follows a negative sign, a continuation or a permanence of signs is said to occur. But if a positive sign follows a negative sign or a negative e sign follows a positive sign, then a change of sign is said to occur.

For example, the polynomial

$$P_5(x) = 8x^5 + 12x^4 - 10x^3 + 17x^2 - 18x + 5 = 0 \quad (4.16)$$

has four changes and one continuation of signs.

Descartes' Rule of Signs

The number of positive real roots of $P_n(x)=0$ cannot exceed the number of sign changes in $P_n(x)$ and the number of negative real roots of $P_n(x)=0$ cannot exceed the number of sign changes in $P_n(-x)$.

Thus, the polynomial (4.16) has maximum of four positive roots and one negative root. This rule does not give the exact number of real roots but gives only their upper bounds.

4.11 LET US SUM UP

In this unit, we studied various numerical methods used to find the roots of transcendental and polynomial equations. The key techniques include:

- The unit covered numerical methods for solving both polynomial and transcendental equations.
- Methods discussed include direct methods and iterative methods such as Bisection, Secant, Regula-Falsi, and Newton-Raphson.
- These methods are essential for obtaining both real and complex roots, particularly in engineering and scientific computations.

4.12 ANSWERS TO CHECK YOUR PROGRESS 1

Answer to Q. (a)

(a) Let $f(x)=x^3-4x-9$

Since $f(2)$ is negative and $f(3)$ is positive, a root lies between 2 and 3.

Therefore, the first approximation to the root is

$x_1=\frac{1}{2}(2+3)=2.5$.

Thus $f(x_1)=(2.5)^3-4(2.5)-9=-3.375<0$.

Therefore, root lies between x_1 and 3. The second approximation to the root is

$x_2=\frac{1}{2}(x_1+3)=2.75$.

So, $f(x_2)=(2.75)^3-4(2.75)-9=0.7969>0$

Therefore, root lies between x_1 and x_2. The third approximation to the root is

$x_3=\frac{1}{2}(x_1+x_2)=2.625$.

Then $f(x_3)=(2.625)^3-4(2.625)-9=-1.4121<0$.

Therefore, root lies between x_2 and x_3. The fourth approximation to the root is

$x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875$.

Repeating this process, the successive approximations are

$x_5 = 2.71875, x_6 = 2.70313, x_7 = 2.71094, x_8 = 2.70703,$

$x_9 = 2.70508, x_{10} = 2.70605, x_{11} = 2.70654, x_{12} = 2.70642.$

Hence, the root is 2.7064.

(b) If α, β, γ are the roots of the given equation, then $-\alpha, -\beta, -\gamma$ are the roots of $(-x)^3 - 4(-x) + 9 = 0$.

Therefore, the negative root of the given equation is the positive root of $x^3 - 4x - 9 = 0$ which we have found above to be 2.7064.

✓ 4.13 ANSWERS TO CHECK YOUR PROGRESS 2

Answer to Q.

(a) Let $f(x) = x^4 - x - 10$

So that $f(1) = -10 < 0, f(2) = 4 > 0$.

Therefore, a root of $f(x) = 0$ lies between 1 and 2.

Let us take $x_0 = 2$.

Also, $f'(x) = 4x^3 - 1$.

Newton-Raphson's formula is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \qquad \text{(i)}$$

Putting $k = 0$, the first approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{4}{31} = 1.871.$$

Putting $k = 1$ in (i), the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{f(1.871)}{f'(1.871)} = 1.856.$$

Putting $k = 2$ in (i), the second approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{f(1.856)}{f'(1.856)} = 1.856.$$

Hence, $x_2 = x_3$. Hence, the desired root is 1.856 correct to three decimal places.

Try yourself.

4.14 FURTHER READING

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4.15 MODEL QUESTIONS

1. Find the interval in which the smallest positive root of the following equations lies:

   (i) $\tan x + \tanh x = 0$ (ii) $x^3 - x - 4 = 0$

   Determine the roots correct to two decimals using the bisection method.

2. The negative root of the smallest magnitude of the equation
   $f(x) = 3x^3 + 10x^2 + 10x + 7 = 0$
   is to be obtained.
   (i) Find an interval of unit length which contains this root.
   (ii) Perform two iterations of the bisection method.
   (iii) Taking the end points of the last interval as initial approximations, perform three iterations of the (a) secant method, (b) Regula-Falsi method.
3. The smallest positive root of the equation
   $f(x) = x^4 - 3x^2 + x - 10 = 0$
   is to be obtained.
   (i) Find an interval of unit length which contains this root.
   (ii) Perform two iterations of the bisection method.
   (iii) Taking the midpoint of the last interval as the initial approximation, perform three iterations of the Newton-Raphson method.
4. Perform five iterations of the bisection method to obtain a root of the equation $f(x) = \cos x - xe^x = 0$.

5. Given the following equations:

   (i) $x^4 - x - 10 = 0$

   (ii) $x - e^{-x} = 0$

   Determine the initial approximations to find the smallest positive root. Use these to find the roots correct to three decimal places with the following methods:

   (a) Regula-Falsi method,
   (b) Secant method,
   (c) Newton-Raphson method.

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