Unit 5: INTERPOLATION (WITH EQUAL INTERVALS)

UNIT STRUCTURE

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5.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Understand Newton's forward interpolation.
- Understand Backward interpolation formula.
- Know about Central difference interpolation.

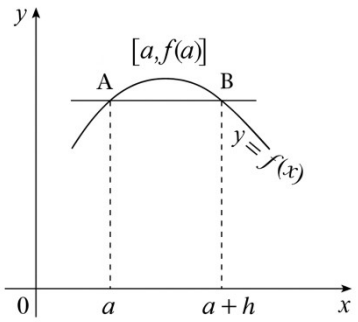
5.2 INTRODUCTION

This unit introduces numerical interpolation techniques used to estimate values between known points. It covers Newton's forward and Backward interpolation formulas for equally spaced data, along with central difference formulas. These methods are essential in numerical analysis for approximating functions and solving real-world scientific and computational applications.

5.3 NEWTON'S FORWARD INTERPOLATION

Let, $y = f(x)$ denote a function which takes the values $f(a), f(a+), f(a+2), \ldots\ldots, f(a+n)$ for $(n+1)$ equidistant values $a, a+, a+2, \ldots, a+n$ of the independent variable x and let $P_n(x)$ be a polynomial in x of degree n.

Then,



$$P_n(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-) + A_3(x-a)(x-a-)(x-a-2) + \cdots \ldots \ldots . + A_n(x-a)(x-a-) \ldots (x-a-\overline{n-1.}) \quad (1)$$

We choose the coefficients $A_0, A_1, \ldots \ldots, A_n$ so that

$$P_n(a) = f(a), \qquad P_n(a+) = f(a+), \ldots \ldots, \qquad P_n(a+n) = f(a+n)$$

Putting successive in (1), $x = a, a+, \ldots, a+n$, and writing the values of $P_n(a), P_n(a+), \ldots, P_n(a+n)$; we obtain,

$$f(a) = A_0$$

$$f(a+) = A_0 + A_1$$

$$\Rightarrow A_1 = \frac{f(a+) - f(a)}{} = \frac{1}{2}\Delta f(a)$$

$$f(a+2) = A_0 + 2A_1 + 2.1.A_2$$

$$\Rightarrow A_2 = \frac{f(a+2) - 2f(a+) + f(a)}{2^{\ 2}}$$

$$= \frac{1}{2!\square^2}\Delta^2 f(a)$$

Similarly,

$$A_3 = \frac{1}{3!^{\ 3}}\Delta^3 f(a)$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$A_n = \frac{1}{n!^{\ n}}\Delta^n f(a)$$

Substituting these values in (1), we get

$$P_n(x) = f(a) + \frac{1}{2}\Delta f(a)(x-a) + \frac{1}{2!^{\ 2}}\Delta^2 f(a)(x-a)(x-a-) + \frac{1}{3!^{\ 3}}\Delta^3 f(a)(x-a)(x-a-)(x-a-2) + \cdots \ldots \ldots . + \frac{1}{n!^{\ n}}\Delta^n f(a)(x-a)(x-a-) \ldots + (x-a-\overline{n-1.})$$

This is Newton's formula for Forward interpolation.

Another Form:

Setting $\frac{x-a}{\square} = u \ \ or \ \ x = a+u$, the formula becomes;

$$P_n(x) = P_n(a+u)$$

$$= f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \cdots \ldots$$

$$+ \frac{u(u-1) \ldots . . (u-n+1)}{n!}\Delta^n f(a) \quad (1)$$

Or,

$$P_n(x) = f(a) + u^{(1)}\Delta f(a) + \frac{1}{2!}u^{(2)}\Delta^2 f(a) + \frac{1}{3!}u^{(3)}\Delta^3 f(a) + \cdots \ldots + \frac{1}{n!}u^{(n)}\Delta^n f(a)$$

Where,

$$u^{(n)} = u(u-1) \ldots . . (u-n+1)$$

$$x = a + \square u$$

This is the new form of Newton's formula.

Example 1: Form a difference table and interpolate the value of $f(x)$ when $x = 4$, given

x	3	5	7	9
$f(x)$	180	150	120	90

Solution: Given that, $x = 4$

Here, $x_0 = 3$, $\square = 2$

$u = \frac{x - x_0}{\square} = \frac{1}{2} = 0.5$

Thus, the difference table is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
3	180			
		-30		
5	150		0	
		-30		0
7	120		0	
		-30		
9	90			

Now from the Newton's forward interpolation formula we have,

$$P_n(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$\Rightarrow P_n(4) = 180 + 0.5 \times (-30) + \frac{0.5(0.5-1)}{2!} \times 0 + \frac{0.5(0.5-1)(0.5-2)}{3!} \times 0$$

$$\Rightarrow P_n(4) = 165$$

Example 2: Find by suitable interpolation formula the value of $f(2.5)$ from the following data.

x	2	3	4	4
$f(x)$	14.5	16.3	17.5	18

Solution: Here, $= 1$, $x = 2.5$, $x_0 = 2$

$\therefore \quad u = \frac{x - x_0}{\square} = \frac{2.5-2}{1} = 0.5$

Thus, the difference table is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	14.5			
		1.8		
3	16.3		-0.6	
		1.2		-0.1
4	17.5		-0.7	
		0.5		
5	18			

Now from the Newton's forward interpolation formula we have,

$$P_n(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$\Rightarrow P_n(2.5) = 14.5 + 0.5 \times 1.8 + \frac{0.5(0.5-1)}{2!} \times (-0.6) + \frac{0.5(0.5-1)(0.5-2)}{3!} \times (-0.1)$$

$$\Rightarrow P_n(2.5) = 14.5 + 0.9 + \frac{0.15}{2} - \frac{0.0375}{6}$$

$$\Rightarrow P_n(2.5) = 15.4678$$

Example 3: Estimate the population in 1895 from the following statistics-

Year	1891	1901	1911	1921	1931
Population	46	66	81	93	101

Solution: Here,

$$x = 1895$$
$$x_0 = 1891$$
$$= 10$$
$$\therefore \quad u = \frac{x - x_0}{} = \frac{1895 - 1891}{10} = 0.4$$

Thus, the difference table is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Now from the Newton's forward interpolation formula we have,

$P_n(x) =$

$f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)$

$= 46 + 0.4 \times 20 + \frac{0.4(0.4-1)}{2} \times (-5) + \frac{0.4(0.4-1)(0.4-2)}{6} \times 2 + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24} \times (-3)$

$\Rightarrow P_n(1895) = 54.8528$

CHECK YOUR PROGRESS

Q1) Using an appropriate formula for interpolation, estimate the number of students who obtained marks between 40 and 45 from the following table:

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

5.4 NEWTON'S FORMULA FOR BACKWARD INTERPOLATION

The formula of the proceeding section cannot be used for interpolating a value of $f(x)$ near the end of the tabular values. To derive a formula in this case, we write,

$$P_n(x) = A_0 + A_1(x-a-n) + A_2(x-a-n)(x-a-n+) + A_3(x-a-n)(x-a-n\leftarrow+)(x-a-n\leftarrow+2) + \cdots \ldots \ldots + A_n(x-a-n)(x-a-n\leftarrow+)\ldots\ldots(x-a-) \quad (1)$$

We determine the coefficients $A_0, A_1, \ldots\ldots, A_n$; so as to make

$$P_n(a+n\leftarrow) = f(a+n\leftarrow), \ldots\ldots\ldots, P_n(a) = f(a)$$

Putting successively in (1), $x = a+n\leftarrow, a+n\leftarrow-, \ldots\ldots$ and also putting $P_n(a+n\leftarrow) = f(a+n), \ldots\ldots$ we get,

$$f(a+n) = A_0 \quad \text{or} \quad A_0 = f(a+n)$$

$$f(a+n\leftarrow-) = A_0 + A_1(-)$$

$$\Rightarrow A_1 = \frac{f(a+n) - f(a+n\leftarrow-)}{\leftarrow}$$

$$\Rightarrow A_1 = \frac{1}{\leftarrow}\nabla f(a+n)$$

Where ∇ is called the backward operator and is defined by the relation

$$\nabla f(x) = f(x) - f(x-\leftarrow)$$

Also, $f(a+n\leftarrow-2) = A_0 + A_1(-2) + A_2(-2)(-)$

$$\Rightarrow A_2 = \frac{f(a+n) - 2f(a+\overline{n-1}) + f(a+\overline{n-2})}{2^{\ 2}}$$

$$\Rightarrow A_2 = \frac{1}{2!\square^2}\nabla^2 f(a+n)$$

Similarly,

$$A_3 = \frac{1}{3!\ \ {}_3}\nabla^3 f(a+n)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$A_n = \frac{1}{n!\ \ {}_n}\nabla^n f(a+n)$$

Substituting the values of $A_0, A_1, \dots\dots, A_n$ in (1), we get-

$$P_n(x) = f(a+n) + \frac{\nabla f(a+n)}{}(x - \overline{a+n}) + \frac{\nabla^2 f(a+n)}{2!\ \ {}_2}(x - \overline{a+n})(x - \overline{a+n-}) + \cdots + \frac{\nabla^n f(a+n)}{n!\ \ {}_n}(x - \overline{a+n})(x - \overline{a+n-}) \dots (x - \overline{a+})$$

This is Newton's formula for backward interpolation.

Setting $u = \frac{x-(a+n\square)}{\square}$ or $x = a + n + u$, we find

$$P_n(x) = P_n(a+n+u)$$

$$\Rightarrow P_n(x) = f(a+n) + u\nabla f(a+n) + \frac{u(u+1)}{2!}\nabla^2 f(a+n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+n) + \cdots + \frac{u(u+1)\dots(u+n-1)}{n!}\nabla^n f(a+n) \quad (2)$$

This is the form in which Newton's formula for backward interpolation is usually written.

This is the backward interpolation formula and is so called because it contains values of the tabulated function from $f(a+n\square)$ backward to the left and none to the right of $f(a+n\square)$. This formula is used mainly for interpolating the value of $f(x)$ near the end of a set of tabular values.

Working table for backward formula:

x	y	∇	∇^2	∇^3	∇^4
$a-4\square$	y_4				
		∇y_3			
$a-3\square$	y_3		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_1$	
$a-2\square$	y_2		$\nabla^2 y_1$		$\nabla^4 y_0$
		∇y_1		$\nabla^3 y_0$	
$a-\square$	y_1		$\nabla^2 y_0$		
		∇y_0			
a	y_0				

Example 4: Find the cubic polynomial interpolation which takes on the values $f_0 = 5$, $f_1 = 1,\ f_2 = 9,\ f_3 = 25,\ f_4 = 55$

Solution: Here we find f_5

x	f	∇	∇^2	∇^3	∇^4
0	5				
		-4			
1	1		12		
		8		-4	
2	9		8		10
		16		6	
3	25		14		
		30			
4	55				

Here, $a = 4,\ \square = 1,\ a - n\square = 5 \Rightarrow n = -1$

Using Newton's backward difference formula, we get,

$f(a - n\square) = (1 - \nabla)^n f(a)$

$$= f(a) - n\nabla f(a) + \frac{n(n-1)}{2!}\nabla^2 f(a) - \frac{n(n-1)(n-2)}{3!}\nabla^3 f(a) + \frac{n(n-1)(n-2)(n-3)}{4!}\nabla^4 f(a)$$

$$\Rightarrow f(5) = f(4) - (-1)\nabla f(4) + \frac{(-1)(-2)}{2!}\nabla^2 f(4) - \frac{(-1)(-2)(-3)}{3!}\nabla^3 f(4) + \frac{(-1)(-2)(-3)(-4)}{4!}\nabla^4 f(4)$$

$$= 55 + 30 + 14 + 6 + 10$$

$$= 115 \text{ approx.}$$

Example 5: Write down the Newton's backward difference formula and use it to obtain the numerical value of the first derivative at $x = 0.4$ of the function $f(x)$ defined as under:

x:	0.1	0.2	0.3	0.4
$f(x)$:	1.10517	1.22140	1.34986	1.49182

Solution: Newton's backward difference formula is

Newton's backward difference formula is,

$f(a - n\square) = (1 - \nabla)^n f(a)$

$$= f(a) - n\nabla f(a) + \frac{n(n-1)}{2!}\nabla^2 f(a) - \frac{n(n-1)(n-2)}{3!}\nabla^3 f(a) + \cdots \ldots \ldots \ldots$$

Differentiating both sides with respect to n, we obtain

$$f'(a-n\square) = \frac{1}{\square}[\nabla f(a) - \frac{2n-1}{2!}\nabla^2 f(a) + \frac{3n^2-6n+2}{3!}\nabla^3 f(a) - \cdots \ldots] \quad (1)$$

Difference table is:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
0.1	1.10517			
		0.11623		
0.2	1.22140		0.01223	
		0.12846		0.00127
0.3	1.34986		0.01350	
		0.14196		
0.4	1.49182			

For the given question, we have

$a = 0.4, \quad \square = 0.1, \quad a - n\square = 0.4 \Rightarrow n = 0$

Substituting these values of a, n and $\square$ in (1), we obtain,

$$f'(0.4) = \frac{1}{0.1}\left[0.14196 + \frac{1}{2}(0.013150) + \frac{1}{3}(0.00127)\right]$$

$$= 1.149133$$

Example 6: Estimate the population in 1925 from the following statistics:

Year:	1891	1901	1911	1921	1931
Population:	46	66	81	93	101

Solution: Difference table is:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Here,

$$a = 1931, \quad \square = 10$$

$$a + \square u = 1925$$

$$\Rightarrow u = -0.6$$

Using Newton's backward interpolation formula, we get,

$$f(a+\square u)$$
$$= f(a) + \frac{u}{1!}\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a)$$

$$\Rightarrow f(1925) = 101 + \frac{-0.6}{1}\times 8 + \frac{(-0.6)(-0.6+1)}{2}\times(-4) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{6}\times(-1) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}\times(-3)$$
$$= 96.8368 \approx 97$$

CHECK YOUR PROGRESS

Q2) The population of a country in the decennial census were as under. Estimate the population for the year 1975.

Year x:	1941	1951	1961	1971	1981
Population (in lakh) y:	46	67	83	95	102

Q3) Find the lowest degree polynomial $y(x)$ that fit the data, find $y(5)$.

x:	0	2	4	6	8
y:	5	9	61	209	501

5.5 CENTRAL-DIFFERENCE INTERPOLATION

Central Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2	y_{-2}				
		Δy_{-2}			
-1	y_{-1}		$\Delta^2 y_{-2}$		
		Δy_{-1}		$\Delta^3 y_{-2}$	
0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
1	y_1		$\Delta^2 y_0$		
		Δy_1			
2	y_2				

Gauss's Forward Interpolation Formula:

$$y(a+u\square) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots \ldots \ldots \ldots$$

Gauss's Backward Interpolation Formula:

$$y(a+u\square) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u-1)u(u+1)}{3!}\Delta^3 y_{-2} + \frac{(u-1)u(u+1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots \ldots \ldots \ldots$$

Example 7: Use Gauss forward formula to find y_9 from the following table:

x	0	4	8	12	16
y	14	24	32	35	40

Solution: Central Difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	14				
		10			
4	24		-2		
		8		-3	
8	32		-5		10
		3		7	
12	35		2		
		5			
16	40				

Here,

$$a = 8, \quad = 4$$

$$a + u = 9 \Rightarrow u = \frac{1}{4} = 0.25$$

Now, using Gauss's forward interpolation formula-

$$y(a+u\square) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2}$$

$$= 32 + 0.25 \times 3 + \frac{0.25(0.25-1)}{2} \times (-5) + \frac{(0.25+1)0.25(0.25-1)}{6} \times 7 + \frac{(0.25+1)0.25(0.25-1)(0.25-2)}{24} \times 10$$

$$= 33.161$$

Example 8: Interpolation by mean of Gauss's backward formula, the population for the year 1962 from the following table:

Year	1931	1941	1951	1961	1971	1981
Population (in lakh)	12	15	20	27	39	52

Solution: Central Difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1931	12					
		3				
1941	15		2			
		5		0		
1951	20		2		3	
		7		3		-10
1961	27		5		-7	
		12		-4		
1971	39		1			
		13				
1981	52					

Here,

$$a = 1961, \quad = 10$$
$$a + u = 1962 \Rightarrow u = 0.1$$

Now, using Gauss's backward interpolation formula-

$$y(a+u) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u-1)u(u+1)}{3!}\Delta^3 y_{-2} + \frac{(u-1)u(u+1)(u+2)}{4!}\Delta^4 y_{-2} + \frac{(u-2)(u-1)u(u+1)(u+2)}{5!}\Delta^5 y_{-3}$$

$$\Rightarrow f(1962) = 27 + 0.1 \times 7 + \frac{0.1(0.1+1)}{2} \times 5 + \frac{(0.1-1)(0.1)(0.1+1)}{6} \times 3 + \frac{(0.1-1)(0.1)(0.1+1)(0.1+2)}{24} \times (-7) + \frac{(0.1-2)(0.1-1)(0.1)(0.1+1)(0.1+2)}{120} \times (-10)$$

$$= 27.9532$$

CHECK YOUR PROGRESS

Q4) Use Gauss's forward formula to find the value of y when $x = 3.75$ from the following table:

x:	2.5	3.0	3.5	4.0	4.5	5.0
y:	24.145	28.043	20.225	18.644	17.262	46.047

5.6 LET US SUM UP

- For a function $y = f(x)$, the Newton's formula for forward interpolation is given by,

$$P_n(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \cdots \ldots + \frac{u(u-1)\ldots\ldots(u-n+1)}{n!}\Delta^n f(a)$$

  Where the symbols have their usual meaning.

- The Newton's formula for backward interpolation is usually written as:

$$P_n(x) = f(a+n) + u\nabla f(a+n) + \frac{u(u+1)}{2!}\nabla^2 f(a+n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+n) + \cdots + \frac{u(u+1)\ldots(u+n-1)}{n!}\nabla^n f(a+n)$$

- Another form of Newton's backward difference formula is-

$$f(a-n) = f(a) - n\nabla f(a) + \frac{n(n-1)}{2!}\nabla^2 f(a) - \frac{n(n-1)(n-2)}{3!}\nabla^3 f(a) + \cdots \ldots \ldots$$

  Where the symbols have their usual meaning.

- For Central-difference interpolation; Gauss's forward and backward interpolation formulas are respectively written as:

$$y(a+u) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots \ldots \ldots \ldots$$

  And,

$$y(a+u\square) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u-1)u(u+1)}{3!}\Delta^3 y_{-2} + \frac{(u-1)u(u+1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots \ldots \ldots \ldots$$

  Where the symbols have their usual meaning.

✓ 5.7 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: The difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	31				
		42			
Below 50	73		9		
		51		-25	
Below 60	124		-16		37
		35		12	
Below 70	159		-4		
		31			
Below 80	190				

Here, $x = 45, \quad x_0 = 40, \quad = 10$

$$u = \frac{x - x_0}{} = \frac{5}{10} = 0.5$$

Using Newton's forward interpolation formula we have,

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$= 31 + 0.5 \times 42 + \frac{0.5(0.5-1)}{2} \times 9 + \frac{0.5(0.5-1)(0.5-2)}{6} \times (-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24} \times 37$$

$$= 47.8672 \approx 48$$

Therefore, the number of students who scored below 45 = 48.

Number of who secured marks between 40 to 45 is =48-31=17

Answer to Q2: The difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1941	46				
		21			
1951	67		-5		
		16		1	
1961	83		-4		-2
		12		-1	
1971	95		-5		
		7			
1981	102				

Here, $x = 1975, \quad \square = 10, \quad x_n = 1981$

$$u = \frac{x - x_n}{\square} = \frac{1975 - 1981}{10} = -0.6$$

Hence, using Newton's backward interpolation formula we have,

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n$$

$$= 102 + (0.6) \times 7 + \frac{(-0.6)(-0.6+1)}{2} \times (-5) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{6} \times (-1)$$
$$+ \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24} \times (-2)$$

$= 98.3104$ approx

Answer to Q3: The difference table is:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0	5				
		4			
2	9		48		
		52		48	
4	61		96		0
		148		48	
6	209		144		
		292			
8	501				

Here, $a = 0,$ $= 2$

Let, $a + u = x \Rightarrow u = \frac{x}{2}$

Now, using Newton's forward interpolation formula we have,

$$f(a + u\square)$$
$$= f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a)$$
$$+ \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)$$

$$\Rightarrow f(x) = 5 + \frac{x}{2} \times 4 + \frac{\frac{x}{2}\left(\frac{x}{2}-1\right)}{2} \times 48 + \frac{\frac{x}{2}\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)}{6} \times 48 + \frac{\frac{x}{2}\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)\left(\frac{x}{2}-3\right)}{24} \times 0$$

$$\Rightarrow f(x) = x^3 - 2x + 5$$

Hence,

$$f(5) = 5^3 - 2 \times 5 + 5 = 120$$

Answer to Q4: The difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2.5	24.145					
		-2.102				
3.0	22.043		0.284			
		-1.818		-0.047		
3.5	20.225		0.237		0.009	
		-1.581		-0.038		-0.003
4.0	18.644		0.199		0.006	
		-1.382		-0.032		
4.5	17.262		0.167			
		-1.215				
5.0	16.047					

Here, $x = 3.75, \quad a = 3.5, \quad \square = 0.5$

$$u = \frac{x-a}{\square} = 0.5$$

Now, using Gauss's forward formula-

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{4!}\Delta^5 y_{-2}$$

$$= 20.225 + 0.5 \times (-1.581) + \frac{0.5(0.5-1)}{2} \times 0.237 + \frac{(0.5+1)0.5(0.5-1)}{6} \times (-0.038) + \frac{(0.5+1)0.5(0.5-1)(0.5-2)}{24} \times 0.009 + \frac{(0.5+2)(0.5+1)0.5(0.5-1)(0.5-2)}{120} \times (-0.003)$$

$= 19.407$ approx

5.8 FURTHER READING

1. Gupta, R.K. *Numerical Methods: Fundamentals and Applications*. 1st **edition**, Cambridge University Press, 2019.
2. Jain, M.K., **Iyengar**, S.R.K., and Jain, R.K. *Numerical Methods for Scientific & Engineering Computation*. New Age International, 2012.
3. Burden, R.L. and **Faires**, J.D. *Numerical Analysis*, 9th Edition, Cengage Learning, 2011.

5.9 MODEL QUESTIONS

Q1. A third-degree polynomial passes through the points $(0,-1), (1,1), (2,1)$ and $(3,-2)$. Find the polynomial.

Q2. Find the lowest degree polynomial u_x which takes the following values:

(i) $u_1 = 1,\ u_2 = 9,\ u_3 = 25,\ u_4 = 55,\ u_5 = 105$

(ii)

x =	0	1	2	3	4	5
$f(x)$ =	0	3	8	15	24	35

Q3. Show that Newton's formula can be written as

$$P_n(x) = f(0) + x\Delta f(0) - xa_1\Delta^2 f(0) + xa_1a_2\Delta^3 f(0) - \cdots \ldots \ldots \ldots$$

Where, $a_1 = 1 - \frac{1}{2}(x+1)$, $a_2 = 1 - \frac{1}{3}(x+1)$, ... etc

Q4. The following data gives the melting points (m.p.) of an alloy of lead and zinc.

Percentage in alloy (P):	50	60	70	80
Temperature (Q□):	125	225	248	274

Find the m.p. of the alloy containing *54%* of lead, using appropriate interpolation formula.

Q5. From the following data, estimate the number of persons earning wages between 60 and 70 rupees.

Wages (in Rs.)	No. of persons (in thousands)
Below 40	250
40-60	120
60-80	100
80-100	70
100-120	50

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