Unit 6: INTERPOLATION (WITH UNEQUAL INTERVALS)

UNIT STRUCTURE

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6.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- learn about Lagrange's interpolation formula.
- understand the concept of divided difference.
- learn about Hermite's interpolation formula.

6.2 INTRODUCTION

This unit explores polynomial interpolation techniques, focusing on Lagrange's interpolation formula, divided-differences and Hermite's interpolation formula. These methods are essential for constructing approximating polynomials through known data points, enabling accurate function estimation, and numerical analysis in engineering, science and mathematics.

6.3 LAGRANGE'S INTERPOLATION FORMULA

Let $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ be the values of the function $y = f(x)$ correspond to the $(n+1)$ arguments $x_0, x_1, x_2, \ldots, x_n$ not necessarily equally spaced. Suppose that $P_n(x)$is a polynomial in x of degree n. Then

$$P_n(x) = A_0(x-x_1)(x-x_2)\ldots.(x-x_n) + A_1(x-x_0)(x-x_2)\ldots.(x-x_n) + \cdots + A_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \quad (1)$$

where A's are constants. We determine the $(n+1)$ constants $A_0, A_1, \ldots., A_n$ so as to make

$$P_n(x_0) = f(x_0),\ \ P_n(x_1) = f(x_1),\qquad \ldots,\ \ P_n(x_n) = f(x_n).$$

To determine A_0, put $x = x_0$ and $P_n(x_0) = f(x_0)$, so that

$$f(x_0) = A_0(x_0-x_1)(x_0-x_2)\ldots.(x_0-x_n)$$

$$\therefore \quad A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\ldots.(x_0-x_n)}$$

Similarly,

$$A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \ldots . (x_1 - x_n)}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \ldots . (x_n - x_{n-1})}$$

Substituting the values of A's in (1), we get

$$P(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x)} f(x_0) + \frac{(x - x_0)(x - x_2)\ldots(x - x)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x)} f(x_1) + \cdots \ldots \ldots . . + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} f(x_n) \quad (2)$$

Which is Lagrange's interpolation formula.

$$P_n(x) = \sum_{r=0}^{n'} \frac{(x - x_0)(x - x_1) \ldots (x - x)}{(x_r - x_0)(x_r - x_1) \ldots (x_r - x)} f(x_r)$$

$$= \sum_{r=0}^{n'} \left[\frac{\prod_n(x) f(x_r)}{x - x_r}\right]\left[\frac{1}{(x_r - x_0)(x_r - x_1) \ldots (x_r - x)}\right]$$

Where

$$\prod\nolimits_n(x) = \prod_{r=0}^{n} (x - x_r)$$

And therefore,

$$\prod\nolimits_n{}'(x_r) = \left[-\frac{d}{dx}\prod\nolimits_n(x)\right]_{x=x_r}$$
$$= (x_r - x_0)(x_r - x_1) \ldots \ldots \ldots (x_r - x_n)$$

Hence,

$$P_n(x) = \sum_{r=0}^{n'} \frac{\prod_n(x) f(x_r)}{(x - x_r)\prod_n{}'(x_r)} \quad (3)$$

Example 1: Given
$\log_{10} 654 = 2.8156, \log_{10} 658 = 2.8182, \log_{10} 659 = 2.8189, \log_{10} 661 = 2.8202$; find $\log_{10} 656$.

Solution: We have,

x	654	658	659	661
$\log_{10} x$	2.8156	2.8182	2.8189	2.8202

Let, $y = f(x) = \log_{10} x$
Here, $x_0 = 654, x_1 = 658, x_2 = 659, x_3 = 661, x = 656$
Substituting these values in Lagrange's formula we get,

$$\log_{10} 656 = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times 2.8156 + \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \times 2.8182 + \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \times 2.8189 + \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \times 2.8202$$

$$\Rightarrow \log_{10} 656 = 0.6033 + 7.0455 - 5.6378 + 0.8059 = 2.8169$$

Thus, the estimated value of $\log_{10} 656$ is 2.8169.

Example 2: The mode of a certain frequency curve $y = f(x)$ is very near $x = 9$ and the value of the frequency density $f(x)$ for $x = 8.9, 9.0 \; and \; 9.3$ are respectively equal to $0.30, 0.35 \; and \; 0.25$. Calculate the approximate value of the mode.

Solution: We have,

x	8.9	9.0	9.3
$f(x)$	0.30	0.35	0.25

We shall first find the form of the frequency density $f(x)$. By Lagrange's formula, we get

$$f(x) = \frac{(x-9)(x-9.3)}{(8.9-9)(8.9-9.3)} \times 0.30 + \frac{(x-8.9)(x-9.3)}{(9-8.9)(9-9.3)} \times 0.35 + \frac{(x-8.9)(x-9)}{(9.3-8.9)(9.3-9)} \times 0.25$$

$$\Rightarrow f(x) = -\frac{25}{12}x^2 + \frac{453.5}{12}x - \frac{2052.3}{12}$$

This is the form of the frequency density. It will be maximum for the value of x lying in the range $(8.9, 9.3)$ for which $f'(x) = 0$ and $f''(x)$is negative.

Now, $f'(x) = 0$

$$\Rightarrow -\frac{5}{12}(10x - 90.7) = 0$$

$$\Rightarrow x = 9.07$$

and $f"(x) = -\frac{25}{6}$, which is a negative quantity.

Hence, the mode of the frequency curve $y = f(x)$ is at $x = 9.07$.

Example 3: Using Lagrange's interpolation formula, prove that $y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2\,(y_{-3} - y_{-5})$ approximately.

Solution: If we shift the origin to 5, then we are given y_0, y_1, y_3 and y_{10} . We are to find y_6.

Applying Lagrange's formula, we have

$$y_6 = \frac{(6-2)(6-8)(6-10)}{(0-2)(0-8)(0-10)}y_0 + \frac{(6-0)(6-8)(6-10)}{(2-0)(2-8)(2-10)}y_2 + \frac{(6-0)(6-2)(6-10)}{(8-0)(8-2)(8-10)}y_8 + \frac{(6-0)(6-2)(6-8)}{(10-0)(10-2)(10-8)}y_{10}$$

$$= -0.2y_0 + 0.5y_2 + y_8 - 0.3y_{10}$$

Shifting back the origin, we get

$$y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$$

CHECK YOUR PROGRESS

Q1) Four equidistant values u_{-1}, u_0, u_1 and u_2 being given a value is interpolated by Lagrange's interpolation formula. Show that it may be written in the form-

$$u_x = yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0$$

Where $x + y = 1$.

Q2) The following table gives the normal weight of a baby during the six months of life.

Age in Months	0	2	3	5	6
Weights in lbs	5	7	8	10	12

Estimate the weight of the baby at the age of 4 months.

6.4 DIVIDED DIFFERENCE

Let $f(x_0), f(x_1), \ldots\ldots, f(x_n)$ be the entries corresponding to the arguments $x_0, x_1, x_2, \ldots., x_n$ where the intervals $x_1 - x_0, x_2 - x_1, \ldots\ldots., x_n - x_{n-1}$ may not be equal, i.e., not necessarily equally spaced. Then, the first divided differences of $f(x)$ for the two arguments $x_0, x_1; x_1, x_2$ etc are defined as,

$$\Delta'_{x_1} f(x_0) = f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$\Delta'_{x_2} f(x_1) = f(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

……..etc (1)

The second divided differences of $f(x)$ for the three arguments $x_0, x_1 \; and \; x_2$ etc are defined as,

$$\Delta'^2_{x_0,x_1} f(x_0) = \Delta'^2_{x_1,x_2} f(x_0) = f(x_0, x_1, x_2) = \frac{f(x_0,x_1) - f(x_1,x_2)}{x_0 - x_2} \quad (2)$$

And the n^{th} divided differences of $f(x)$ for the $n+1$ arguments $x_0, x_1, \ldots\ldots, x_n$ are defined as,

$$\Delta'^n_{x_1,x_2,\ldots,x_{n-1}} f(x_0) = f(x_0, x_1, \ldots, x_n) = \frac{f(x_0,x_1,\ldots,x_{n-1}) - f(x_1,x_2,\ldots,x_n)}{x_0 - x_n} \quad (3)$$

A divided difference table can be constructed for any set of tabulated values of $f(x)$; it has the appearance of table-1.

Table 1: Divided Difference table

Argument x	Entry $f(x)$	First Divided difference	Second Divided difference	Third Divided difference

		$\Delta' f(x)$	$\Delta'^2 f(x)$	$\Delta'^3 f(x)$
x_0	$f(x_0)$			
		$f(x_0, x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		
x_3	$f(x_3)$			

Newton's Divided Difference Interpolation Formula for Unequal Interval:

Let $f(x_0), f(x_1), \ldots\ldots, f(x_n)$ be the values of $f(x)$ corresponding to the arguments $x_0, x_1, \ldots\ldots, x_n$; not necessarily equal spaced. From the definition of divided difference we have,

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow f(x) = f(x_0) + (x - x_0) f(x, x_0) \quad (1)$$

Also,

$$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$$\Rightarrow f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1) \quad (2)$$

Similarly,

$$f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2) \quad (3)$$

……

$$f(x, x_0, x_1, \ldots, x_{n-1}) = f(x_0, x_1, \ldots., x_n) + (x - x_n) f(x, x_0, x_1, \ldots., x_n) \quad (4)$$

Multiplying (2) by $(x - x_0)$, (3) BY $(x - x_0)(x - x_1)$; and similarly, the others, at last (4) by $(x - x_0)(x - x_1) \ldots\ldots (x - x_{n-1})$ and adding (1), we get

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \cdots \ldots\ldots\ldots + (x - x_0)(x - x_1) \ldots. (x - x_{n-1}) f(x_0, x_1, \ldots, x_n) + R_n$$

Where the remainder R_n is given by,

$$R_n = (x - x_0)(x - x_1) \ldots\ldots\ldots (x - x_n) f(x, x_0, x_1, \ldots., x_n)$$

Assuming that $f(x)$ is a polynomial of degree n, so that $f(x, x_0, x_1, \ldots., x_n)$vanishes. Thus,

$f(x) =$

$f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \ldots\ldots\ldots\ldots + (x - x_0)(x - x_1) \ldots. (x - x_{n-1}) f(x_0, x_1, \ldots, x_n)$ (5)

This formula is called Newton's divided difference interpolation formula.

Example 4: (a) Find the third divided difference with arguments 2,4,9,10 of the function $f(x) = x^3 - 2x$

(b) Find a polynomial satisfied by $(-4,1245), (-1,33), (0,5), (2,9)$ and $(5,1335)$

Solution: (a) Given, $f(x) = x^3 - 2x$

x	$f(x)$	$\Delta' f(x)$	$\Delta'^2 f(x)$	$\Delta'^3 f(x)$
2	4			
		$\frac{56-4}{4-2} = 26$		
4	56		$\frac{131-26}{9-2} = 15$	
		$\frac{711-56}{9-4} = 131$		$\frac{23-15}{10-2} = 1$
9	711		$\frac{269-131}{10-4} = 23$	
		$\frac{980-711}{10-9} = 269$		
10	980			

Hence the third divided difference is 1.

(b) The divided difference table is:

x	$f(x)$	$\Delta' f(x)$	$\Delta'^2 f(x)$	$\Delta'^3 f(x)$	$\Delta'^4 f(x)$
-4	1245				
		-404			
-1	33		94		
		-28		-14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

$$\therefore \quad f(x) = 1245 - 404(x+4) + 94(x+4)(x+1) - 14(x+4)(x+1)x + 3(x+4)(x+1)x(x-2)$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

Example 5: By means of Newton's divided difference interpolation formula, find the value of $f(2), f(8)$ and $f(15)$ from the following table:

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution: The divided difference table is:

x	$f(x)$	$\Delta' f(x)$	$\Delta'^2 f(x)$	$\Delta'^3 f(x)$	$\Delta'^4 f(x)$
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

Using Newton's divided difference interpolation formula, we have

$$f(x) = 48 + 52(x-4) + 15(x-4)(x-5) + 1(x-4)(x-5)(x-7)$$
$$= x^2(x-1)$$

$\therefore$ $f(2) = 2^2(2-1) = 4$

$f(8) = 8^2(8-1) = 448$

$f(15) = 15^2(15-1) = 3150$

CHECK YOUR PROGRESS

Q3) Show that the n^{th} divided difference of x^n is 1.

Q4) Find a polynomial satisfied by $(-4,1245), (-1,33), (0,5), (2,9)$ by the use of Newton's divided difference formula.

6.5 HERMITE'S INTERPOLATION FORMULA

So far, we have considered the interpolation formula which make use only if a certain number of function values. We now derive an interpolation formula in which both the function and its derivatives are to be assigned to each point of interpolation.

Theorem: If f' is continuous on $[a,b]$ and $x_0, x_1, \ldots\ldots, x_n \in [a,b]$ are distinct than the unique polynomial of least degree agreeing with f and f' at $x_0, x_1, \ldots\ldots, x_n$ is a polynomial of degree at most $2n+1$ given by

$$p(x) = \sum_{i=0}^{n} [1 - 2(x - x_i)] l_i(x)^2 f(x_i) + \sum_{i=0}^{n} (x - x_i) l_i(x)^2 f'(x_i)$$

Where $l_i(x)^2 = [l_i(x)]^2$

The Hermite interpolation polynomial interpolation are not only the function $f(x)$ but also its derivatives at a given set of tabular points i.e.,

$$p(x_i) = f(x_i), \qquad p'^{(x_i)} = f'(x_i), \qquad i = 0,1, \dots \dots, n$$

Since there are $(2n+2)$ conditions to be satisfied, $p(x)$ must be a polynomial of $\leq 2n+1$. The required polynomial may be written as

$$p(x) = \sum_{i=0}^{n} A_i(x) f(x_i) + \sum_{i=0}^{n} B_i(x) f'(x_i) \qquad (1)$$

Where $A_i(x)$ and $B_i(x)$ are polynomials of degree $\leq 2n+1$ and satisfy

(i) $A_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

(ii) $A'_i(x_j) = 0$ for all i and j

$B_i(x_j) = 0$ for all i and j

(iii) $B'_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

(2)

Using the Lagrange's theorem, $l_i(x)$ where

$$l_i(x) = \frac{(x-x_0)(x-x_1)\dots\dots(x-x_{i-1})(x-x_{i+1})\dots\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots\dots(x_i-x_n)}$$

And

$$\begin{cases} A_i(x) = \gamma_i(x) l_i^2(x), & l_i^2(x) = \{l_i(x)\}^2 \\ B_i(x) = \delta_i(x) l_i^2(x) \end{cases} \qquad (3)$$

Since $l_i^2(x)$ is a polynomial of degree $2n$; $\gamma_i(x)$ and $\delta_i(x)$ are both the linear polynomials.

Thus let $\gamma_i(x) = a_i x + b_i$ and $\delta_i(x) = c_i x + d_i$ (4)

Where $a_i, b_i, c_i \; and \; d_i$ are constants to be determined.

Using the conditions (2) we obtain

$$\begin{aligned} a_i &= -2l.\, l_i(x_i) \\ b_i &= l + 2x_i l_i(x_i) \\ c_i &= 1 \\ d_i &= -x_i \end{aligned} \qquad (5)$$

Substituting (5) in (4) we obtain $\gamma_i(x)$. Now, using (3), the equation (1) and $\delta_i(x) = x - x_i$, then (1) becomes

$$p(x) = \sum_{i=0}^{n} [1 - 2(x-x_i) l_i(x_i) l_i^2(x_i) f(x_i)] + \sum_{i=0}^{n} [(x-x_i) l_i^2(x) f'(x_i)]$$

Which is the required Hermite Interpolation formula.

Example 6: For the following tabular values of $f(x)$ and $f'(x)$, estimate the values of $f(x)$ at

(i) $x = 0.5$ (ii) $x = -0.5$

Using Hermite interpolation polynomial:

x	-1	0	1
$f(x)$	1	1	3

$f'(x)$	-5	1	7

Solution: By Lagrange's formula for arguments -1,0,1; the function $f(x)$ is given by

$$f(x) = \frac{(x-0)(x-1)}{(-1-0)(-1-1)}.1 + \frac{(x+1)(x-1)}{(0+1)(-1)}.1 + \frac{(x-0)(x+1)}{(1+1)(0+1)}.3$$
$$= \frac{1}{2}(x^2-x) - (x^2-1) + \frac{3}{2}(x^2+x)$$
$$l_0 = \frac{1}{2}(x^2-x), \quad l_1 = -(x^2-1), \quad l_2 = \frac{3}{2}(x^2+x)$$
$$x_0 = -1, \quad x_1 = 0, \quad x_2 = 1$$

The Hermite's interpolation is:

$$p(x) = \sum_{i=0}^{2} A_i(x) f(x_i) + \sum_{i=0}^{2} B_i(x) f'(x_i) \qquad (1)$$

Where,

$$A_0(x) = [1 - 2(x-x_0)l_0'(x_0)]l_0^2(x)$$
$$A_1(x) = [1 - 2(x-x_1)l_1'(x_1)]l_1^2(x)$$
$$A_2(x) = [1 - 2(x-x_2)l_2'(x_2)]l_2^2(x)$$
$$B_0(x) = (x-x_0)l_0^2(x)$$
$$B_1(x) = (x-x_1)l_1^2(x)$$
$$B_2(x) = (x-x_2)l_2^2(x)$$
$$l_0(x) = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{1}{2}(x^2-x); \qquad l_0'(-1) = -\frac{3}{2}$$
$$l_1(x) = \frac{(x+1)(x-1)}{(0+1)(-1)} = -(x^2-1); \qquad l_1'(0) = 0$$
$$l_2(x) = \frac{(x-0)(x+1)}{(1+1)(0+1)} = \frac{3}{2}(x^2+x); \qquad l_2'(1) = \frac{3}{2}$$
$$A_0(x) = [1 + 3(x+1)]\left[\frac{x^2(x-1)^2}{4}\right] = \frac{1}{4}[3x^5 - 2x^4 - 5x^3 + 4x^2]$$
$$A_1(x) = [1 - 2(x-0).0].(x^2-1)^2 = x^4 - 2x^2 + 1$$
$$A_2(x) = [1 - 3(x-1)]\left[\frac{x^2(x+1)^2}{4}\right] = \frac{1}{4}[-3x^5 - 2x^4 + 5x^3 + 4x^2]$$
$$B_0(x) = \frac{(x-1)x^2(x+1)^2}{4} = \frac{1}{4}(x^5 - x^4 - x^3 + x^2)$$
$$B_1(x) = x(x^2-1)^2 = x^5 - 2x^3 + x$$
$$B_2(x) = \frac{1}{4}(x^5 + x^4 - x^3 - x^2)$$

Substituting these values in (1), we get

$$p(x) = \frac{1}{4}[3x^5 - 2x^4 - 5x^3 + 4x^2] \times 1 + (x^4 - 2x^2 + 1) \times 1$$
$$+ \frac{1}{4}[-3x^5 - 2x^4 + 5x^3 + 4x^2] \times 3 + \frac{1}{4}(x^5 - x^4 - x^3 + x^2) \times (-5)$$
$$+ (x^5 - 2x^3 + x) \times 1 + \frac{1}{4}(x^5 + x^4 - x^3 - x^2) \times 7$$
$$= 2x^4 - x^2 + x + 1$$

Putting $x = -0.5$ and $x = 0.5$, we get

$$f(-0.5) \cong \frac{3}{8}, \text{ exact value is } \frac{33}{64}$$
$$f(0.5) \cong \frac{11}{8}, \text{ exact value is } \frac{97}{64}$$

CHECK YOUR PROGRESS

Q5) Using Hermite's interpolation formula, find an approximate value of $\log(3.2)$, from the following table:

x	3	3.5	4.0
$y = lnx$	1.09861	1.25276	1.38629
$y' = \frac{1}{x}$	0.3333	0.28571	0.2500

6.6 LET US SUM UP

- For the function $y = f(x)$, Lagrange's interpolation formula is given by,
  $P(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x)} f(x_0) + \frac{(x-x_0)(x-x_2)\ldots(x-x)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x)} f(x_1) + \cdots\ldots\ldots\ldots +$
  $\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} f(x_n)$
  Where the symbols have their usual meanings.
- For the function $y = f(x)$, Newton's divided difference interpolation formula is given by,

  $f(x) =$
  $f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) +$
  $\ldots\ldots\ldots\ldots + (x-x_0)(x-x_1)\ldots.(x-x_{n-1})f(x_0,x_1,\ldots,x_n)$
  Where the symbols have their usual meanings.
- In f' is continuous on $[a,b]$ and $x_0, x_1, \ldots\ldots, x_n \in [a,b]$ are distinct, then the Hermite's interpolating polynomial is given by;
  $$p(x) = \sum_{i=0}^{n}[1-2(x-x_i)]l_i(x)^2 f(x_i) + \sum_{i=0}^{n}(x-x_i)l_i(x)^2 f'(x_i)$$
  Where the symbols have their usual meanings.

6.7 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: We have,

$$\Delta^2 u_{-1} = \Delta u_0 - \Delta u_{-1}$$
$$= (u_1 - u_0) - (u_0 - u_{-1})$$

$$= u_1 - 2u_0 + u_{-1}$$

Similarly, $\Delta^2 u_0 = u_2 - 2u_1 + u_0$

Therefore, Right hand side

$= (1-x)u_0 + xu_1 + \frac{(1-x)x(x-2)}{3}(u_1 - 2u_0 + u_{-1}) + \frac{(x-1)x(x+1)}{3!}(u_2 - 2u_1 + u_0)$

$= -\frac{x(x-1)(x-2)}{6}u_{-1} + \frac{(x-2)(x-1)(x+1)}{6}u_0 - \frac{(x-2)x(x+1)}{2}u_1 + \frac{(x-1)x(x+1)}{6}u_2$

$= (x+1)x(x-1)(x-2)\left[-\frac{u_{-1}}{6(x+1)} + \frac{u_0}{6x} - \frac{u_1}{2(x-1)} + \frac{u_2}{6(x-2)}\right] = (x+1)x(x-1)(x-2)[u_{-1}.\frac{1}{(-1-0)(-1-1)(-1-2)(x+1)} + u_0\frac{1}{(0+1)(0-1)(0-2)x} + u_1.\frac{1}{(1+1)(1-0)(1-2)(x+1)} + u_2.\frac{1}{(2+1)(2-0)(2-1)(x-2)}]$

$= u_x,$ by Lagrange's formula.

Answer to Q2: . Let,

$$x_0 = 0, \quad x_1 = 2, \quad x_2 = 3, \quad x_3 = 5, \quad x_4 = 6$$
$$y_0 = 5, \quad y_1 = 7, \quad y_2 = 8, \quad y_3 = 10, \quad y_4 = 12$$

Given, $x = 4$.

Now, the Lagrange's interpolation formula is

$= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_2)(x_0-x_4)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_2)(x_1-x_4)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}y_3 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}y_4$

$= \frac{(2)(1)(-1)(-6)}{(-2)(-3)(-5)(-6)} \times (5) + \frac{(4)(1)(-1)(-2)}{(2)(-1)(-3)(-4)} \times (7) + \frac{(4)(2)(-1)(-2)}{(3)(1)(-2)(-3)} \times (8) + \frac{(4)(2)(1)(-2)}{(5)(3)(2)(-1)} \times (10) + \frac{(4)(2)(1)(-1)}{(6)(4)(3)(1)} \times (12)$

$= 0.111 - 2.333 + 7.111 + 5.333 - 1.333$

$= 8.889 \ lbs$

Hence, the weight of the baby at the age of 4 months is 8.889 lbs.

Answer to Q3: We have,

$$\frac{x^{n+1}}{(x-x_0)(x-x_1)\dots\dots(x-x_n)} = 1 + \sum \frac{x_0^{n+1}}{(x_0-x_1)\dots\dots(x_0-x_n)}.\frac{1}{x-x_0}$$

Put $x = 0$ on both sides we have,

$$0 = 1 - f(x_0, x_1, \dots\dots, x_n)$$
$$\Rightarrow f(x_0, x_1, \dots\dots, x_n) = 1$$

Answer to Q4: The divided difference table is-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-4	1245			
		-404		
-1	33		94	
		-28		-14
0	5		10	
		2		

2	9			

Now, using Newton's divided difference formula, we have,

$$f(x) = 1245 + (-404)(x+4) + 94(x+4)(x+1) + (-14)(x+4)(x+1)(x-0)$$
$$= -14x^3 + 24x^2 + 10x + 5$$

Answer to Q5: Hermite's interpolation formula is;

$$H_{2n+1}(x) = \sum_{i=0}^{n}[1-2(x-x_i)][l_i(x)]^2 y_i + \sum_{i=0}^{n}(x-x_i)[l_i(x)]^2 y_i' \quad (1)$$

First, we shall find $l_i(x)$; $i = 0,1,2.$

$$\therefore \; l_0(x) = \frac{1}{0.5}(x^2 - 7.5x + 14) = 2(x^2 - 7.5x + 14)$$
$$l_1(x) = -4(x^2 - 7x + 12)$$
$$l_2(x) = 2(x^2 - 6.5x + 10.5)$$

Again,

$$l_0'(x) = 2(2x - 7.5)$$
$$l_1'(x) = -4(2x - 7)$$
$$l_2'(x) = 2(2x - 6.5)$$

Next,

$$l_0'(x_0) = l_0'(3) = -3$$
$$l_1'(x_1) = l_1'(3.5) = 0$$
$$l_2'(x_2) = l_2'(4) = 3$$

From (1), we have

$$H_5(x) = [1-2(x-x_0)l_0'(x_0)][l_0(x)]^2 y_0 + [1-2(x-x_1)l_1'(x)][l_1(x)]^2 y_1$$
$$+ [1-2(x-x_2)l_2'(x_2)][l_2(x)]^2 y_2 + (x-x_0)[l_0(x)]^2 y_0'$$
$$+ (x-x_1)[l_1(x)]^2 y_1' + (x-x_2)[l_2(x)]^2 y_2'$$
$$= [1-2(x-3)(-3)].4.(x^2 - 7.5x + 14)^2(1.09861)$$
$$+[1-2(x-3.5)(0)].16.(x^2 - 7x + 12)^2(1.25276)$$
$$+[1-2(x-4)(3)].4.(x^2 - 6.5x + 10.5)^2(1.38629)$$
$$+(x-3).4.(x^2 - 7.5x + 14)^2(0.3333)$$
$$+(x-3.5).16.(x^2 - 7x + 12)^2(0.28571)$$
$$+(x-4).4.(x^2 - 6.5x + 10.5)^2(0.25)$$

Putting $x = 3.2$ and calculating and simplifying the expressions, we get,

$$\log(3.2) = 0.556863 + 0.513130 + 0.115782 + 0.0153584 - 0.03510804 - 0.00288$$
$$= 1.16314542$$

6.8 FURTHER READING

1. Gupta, R.K. *Numerical Methods: Fundamentals and Applications*. 1st **edition**, Cambridge University Press, 2019.

2. Jain, M.K., **Iyengar**, S.R.K., and Jain, R.K. *Numerical Methods for Scientific & Engineering Computation*. New Age International, 2012.
3. Burden, R.L. and **Faires**, J.D. *Numerical Analysis*, 9th Edition, Cengage Learning, 2011.

6.9 MODEL QUESTIONS

1. Given $y_{35.0} = 1175,\ y_{35.5} = 1280,\ y_{39.5} = 2180,\ y_{40.5} = 2420$. Obtain a value for y_{40}
2. Given the following data, find $f(x)$ as a polynomial in powers of $(x-5)$

	0	2	3	4	7	9
$f(x)$:	4	26	58	112	466	922

3. By means of Lagrange's interpolation formula, prove that approximately-

$$f(x) = f(3) - 0.3[f(-3)] + 0.2[f(-2) - f(-5)]$$

4. Find out the divided difference of y_n, given that

x:	1	2	3	4	5
$f(x)$:	22	30	82	106	216

5. Show that the divided difference of x^n is unity, no matter which of the arguments $x_0, x_1, \ldots\ldots, x_n$ are used.
6. Construct the polynomial $p_3(x)$ for which $p(0) = 1, p(1) = 2, p(-1) = 5, p"(0) = 0$.
7. Using Hermite interpolation formula, find an approximate value of $\log(2.7)$, from the following table:

x :	2.0	2.5	3.0
$y = lnx$:	0.69315	0.91629	1.09861
$y' = \frac{1}{x}$:	0.5	0.4000	0.3333

*** *** ***