UNIT 7: THE ACCURACY OF INTERPOLATION FORMULAS

UNIT STRUCTURE

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7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Understand the concept of error in polynomial interpolation.
- Derive and interpret the remainder term in Newton's Forward Interpolation Formula and Newton's Backward Interpolation Formula
- Learn about the remainder term in Lagrange's Interpolation Formula and Stirling's Interpolation Formula
- Estimate interpolation errors using remainder terms.

7.2 INTRODUCTION

In numerical analysis, interpolation is a fundamental technique used to estimate unknown values of a function based on known data points. While interpolation provides useful approximations, it is essential to assess how accurate these approximations are. This leads to the concept of error analysis in interpolation. Understanding the remainder term or error term in interpolation formulas allows us to estimate the deviation between the true value of the function and the interpolated value.

This unit focuses on the accuracy of various interpolation formulas—Newton's forward and backward methods, Lagrange's formula, and Stirling's central difference formula. It presents theoretical expressions for their respective remainder terms and

demonstrates how these expressions can be used to analyze and control interpolation errors in practical computations.

7.3 ERROR IN POLYNOMIAL INTERPOLATION

Let the function $y(x)$, defined by the $(n + 1)$ points

$$(x_i, y_i), \qquad i = 0,1,2, \dots, n$$

be continuous and differentiable $(n + 1)$ times, and let $y(x)$ be approximated by a polynomial $p_n(x)$ of degree not exceeding n such that

$$p_n(x_i) = y_i, \qquad i = 0,1,2, \dots, n$$

Using the polynomial $p_n(x)$ of degree n, we can obtain approximate value of $y(x)$ at some points other x_i, $0 \le i \le n$. Since the expression $y(x) - p_n(x)$ vanishes for $x = x_0, x_1, \dots, x_n$, we put

$$y(x) - p_n(x) = L\Pi_{n+1}(x) \qquad (1)$$

Where,

$$\Pi_{n+1}(x) = (x - x_0)(x - x_1) \dots (x - x_n)$$

and L is to be determined such that the equation (1) holds for any intermediate value of x, say $x = x' \in (x_0, x_n)$. Clearly,

$$L = \frac{y(x') - p_n(x')}{\Pi_{n+1}(x')} \qquad (2)$$

We construct a function $F(x)$ such that

$$F(x) = y(x) - p_n(x) - L\Pi_{n+1}(x) \qquad (3)$$

where L is given by the equation (2) above.

It is clear that

$$F(x_0) = F(x_1) = \cdots = F(x_n) = F(x') = 0$$

that is, $F(x)$ vanishes $(n + 2)$ times in the interval $x_0 \le x \le x_n$

Consequently, by the repeated application of Rolle's theorem, $F'(x)$ must vanish $(n + 1)$ times, $F''(x)$ must vanish n times, etc., in the interval $x_0 \le x \le x_n$. In particular, $F^{(n+1)}(x)$ must vanish once in the interval.

Let this point be given by $x = \xi$, $x_0 < \xi < x_n$. On differentiating the equation (3) $(n + 1)$ times with respect to x and putting $x = \xi$, we obtain

$$F^{(n+1)}(\xi) = 0 = y^{(n+1)}(\xi) - L(n + 1)!$$

so that

$$L = \frac{y^{(n+1)}(\xi)}{(n+1)!} \qquad (4)$$

Comparison of (2) and (4) yields the results

$$y(x') - p_n(x') = \frac{y^{(n+1)}(\xi)}{(n+1)!}\,\Pi_{n+1}(x')$$

Dropping the prime on x', we obtain, for some $x_0 < \xi < x_n$

$$y(x) - p_n(x) = \frac{\Pi_{n+1}(x)}{(n+1)!} y^{(n+1)}(\xi) \qquad (5)$$

which is the required expression for the error. Since $y(x)$ is, generally, unknown and hence we do not have any information concerning $y^{(n+1)}(x)$, formula (5) is almost useless in practical computations. On the other hand, it is extremely useful in theoretical work in different branches of numerical analysis. We can also simply denoted error term by $E_n(x)$ and represented as

$$E_n(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!} y^{(n+1)}(\xi)$$

Example 1: Given $f(x) = e^x$, and you interpolate it using a first-degree polynomial at $x_0 = 0$ and $x_1 = 1$, estimate the maximum error at $x = 0.5$.

Solution:

We know the error formula of first-degree polynomial

$$E_n(x) = \frac{(x-x_0)(x-x_1)}{(n+1)!} f^{(n+1)}(\xi)$$

$$\therefore E_1(0.5) = \frac{(0.5-0)(0.5-1)}{2!} f^2(\xi)$$

$$= \frac{(0.5).(-0.5)}{2} e^{\xi} = -\frac{e^{\xi}}{8}$$

Since $\xi \in (0,1)$, the maximum error occurs when $\xi = 1$:

$$|E_1(0.5)| \leq \left|-\frac{e^1}{8}\right| \approx \frac{2.718}{8} \approx 0.34$$

7.4 REMAINDER TERM IN NEWTON'S FORWARD INTERPOLATION FORMULA

To find the remainder term(error) committed in replacing the function $y(x)$ by means of the polynomial $p_n(x)$, we obtain

$$y(x) - p_n(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(n+1)!} y^{(n+1)}(\xi)$$

for some $x_0 < \xi < x_n$

As remarked earlier we do not have any information concerning $y^{(n+1)}(x)$, and therefore the above formula is useless in practice. Nevertheless, if $y^{(n+1)}(x)$, does not vary too rapidly in the interval, a useful estimate of the derivative can be obtained in the following way. Expanding $y(x + h)$ by Taylor's series, we obtain

$$y(x + h) = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \cdots$$

Neglecting the terms containing h^2 and higher powers of h, this gives

$$y'(x) = \frac{[y(x + h) - y(x)]}{h} = \frac{\Delta y(x)}{h}$$

Writing $y'(x)$ as $Dy(x)$ where $D \equiv \frac{d}{dx}$, the differentiation operator, the above equation gives the operator relations

$$D \equiv \frac{1}{h}\Delta \qquad \text{and so} \quad D^{n+1} \equiv \frac{1}{h^{n+1}}\Delta^{n+1}$$

We thus obtain

$$y^{(n+1)}(x) \approx \frac{1}{h^{n+1}}\Delta^{n+1} y(x)$$

Hence the equation can be written as (equally spaced nodes, $x = x_0 + ph$)

$$y(x) - p_n(x) = \frac{p(p-1)(p-2)\dots(p-n)}{(n+1)!} \Delta^{(n+1)} y(\xi)$$

for some $\xi \in (x_0, x_n)$, which is suitable for computation.

This is the error term in the Newton's forward difference interpolation formula.

Additionally, the remainder term $R_n(x)$ can alternatively be expressed as follows:

$$R_n(x) = \frac{p(p-1)(p-2)\dots(p-n)}{(n+1)!} h^{(n+1)} y^{(n+1)}(\xi)$$

Example 2: Use Newton's forward interpolation with 3 nodes: $x_0 = 1,\ x_1 = 1.1,\ x_2 = 1.2,\ x_3 = 1.3$ to estimate the remainder term at $x = 1.15$ for $f(x) = \ln x$.

Solution:

We will use the remainder term formula:

$$R_n(x) = \frac{p(p-1)(p-2)\dots(p-n)}{(n+1)!} h^{(n+1)} f^{(n+1)}(\xi)$$

Here, $h = 0.1,\ x = 1.15, x_0 = 1$

$$p = \frac{1.15 - 1}{0.1} = 1.5$$

$n = 3$ (Since 4 points are used)

We need $f^4(\xi)$ for $f(x) = \ln x$

$$f^4(x) = \frac{6}{x^4}$$

On interval [1,1.3], maximum of $|f^4(x)|$ occurs at $x = 1$, $f^4(1) = 6$

Now,

$$R_3(1.15) \leq \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{4!}.(0.1)^4.6$$

$$= \frac{1.5(0.5).(-0.5).(-1.5)}{24} 0.0006 = 0.00001406$$

Hence, $R_3(1.15) \leq 1.406 \times 10^{-5}$

7.5 REMAINDER TERM IN NEWTON'S BACKWARD INTERPOLATION FORMULA

The remainder term in the Newton's backward difference interpolation formula is given by

$$y(x) - p_n(x) = \frac{p(p+1)(p+2)\dots(p+n)}{(n+1)!} \nabla^{(n+1)} y(\xi)$$

Where $\xi \in (x_0, x_n)$ and $x = x_n + ph$

Additionally, the remainder term $R_n(x)$ can alternatively be expressed as follows:

$$R_n(x) = \frac{p(p+1)(p+2)\dots(p+n)}{(n+1)!} h^{(n+1)} y^{(n+1)}(\xi)$$

Example 3: Use Newton's backward interpolation with 3 nodes: $x_0 = 1,\ x_1 = 1.1,\ x_2 = 1.2,\ x_3 = 1.3$ to estimate the remainder term at $x = 1.25$ for $f(x) = \ln x$.

Solution:

We will use the remainder term formula:

$$R_n(x) = \frac{p(p-1)(p-2)\dots(p-n)}{(n+1)!} h^{(n+1)} f^{(n+1)}(\xi)$$

Here, $h = 0.1,\ x = 1.25, x_n = 1.3$

$$p = \frac{x - x_n}{h} = \frac{1.25 - 1.3}{0.1} = -0.5$$

$n = 3$ (Since 4 points are used)

We need $f^4(\xi)$ for $f(x) = \ln x$

$$f^4(x) = \frac{6}{x^4}$$

On interval $[1,1.3]$, maximum of $|f^4(x)|$ occurs at $x = 1$, $f^4(1) = 6$

Now,

$$R_3(1.25) \leq \frac{-0.5(-0.5+1)(-0.5+2)(-0.5+3)}{4!} \times (0.1)^4 \times 6$$

$$= \frac{-0.5(0.5).(1.5).(2.5)}{24} 0.0006 = -2.34375 \times 10^{-5}$$

Hence, $|R_3(1.25)| \leq 2.34375 \times 10^{-5}$

CHECK YOUR PROGRESS 1

Q1. Given $f(x) = \cos x$ and interpolated using a quadratic polynomial through $x_0 = 0$, $x_1 = \frac{\pi}{4}$, and $x_2 = \frac{\pi}{2}$. Estimate the error at $x = \frac{\pi}{6}$.

Q2. Use Newton's forward interpolation with 3 points: $x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2$. Estimate the remainder term at $x = 0.2$ for $f(x) = e^x$.

Q3. Given Data points: $x_0 = 0.6,\ x_1 = 0.7,\ x_2 = 0.8,\ x_3 = 0.9$. Use Newton's backward formula to estimate the error term for $f(x) = sin(x)$ at $x = 0.85$.

7.6 REMAINDER TERM IN LAGRANGE'S INTERPOLATION FORMULA

The error (remainder term) formula of polynomial interpolation is

$$y(x) - p_n(x) = \frac{\Pi_{n+1}(x)}{(n+1)!} y^{(n+1)}(\xi)$$

for some $x_0 < \xi < x_n$

This equation can be used to estimate the error of the Lagrange interpolation formula for the class of functions which have continuous derivatives of order up to $(n+1)$ on $[a,b]$. We therefore, have

$$y(x) - L_n(x) = R_n(x) = \frac{\Pi_{n+1}(x)}{(n+1)!} y^{(n+1)}(\xi), \qquad a < \xi < b$$

and the quantity E_L, where

$$E_L = \max_{[a,b]} |R_n(x)|$$

May be taken as an estimate of error. Further, if we assume that

$$\left|y^{(n+1)}(\xi)\right| \leq M_{n+1}, \qquad a \leq \xi \leq b$$

Then

$$E_L \leq \frac{M_{n+1}}{(n+1)!} \max_{[a,b]} |\Pi_{n+1}(x)|$$

Example 4: Given $y = \ln x$ and interpolated using a Lagrange interpolating polynomial at $x = 2, 2.5, 3$. Estimate the maximum error at $x = 2.7$.

Solution:

Since $y = \ln x$, we get

$$y' = \frac{1}{x}, \qquad y'' = -\frac{1}{x^2}, \qquad y''' = \frac{2}{x^3}$$

It follows that $y'''(\xi) = \frac{2}{\xi^3}$

Thus, the continuity conditions on $y(x)$ and its derivatives are satisfied in $[2,3]$. Hence,

$$R_n(x) = \frac{(x-2)(x-2.5)(x-3)}{6} \frac{2}{\xi^3}, \qquad 2 < \xi < 3$$

But

$$\left|\frac{2}{\xi^3}\right| < \frac{1}{2^3} = \frac{1}{8}$$

When $x = 2.7$, we therefore obtain

$$|R_n(x)| \leq \left|\frac{(2.7-2)(2.7-2.5)(2.7-3)}{6} \frac{2}{8}\right| = \frac{0.7 \times 0.2 \times 0.3}{3 \times 8} = 0.00175$$

Hence, $|R_2(2.7)| \leq 0.00175$

7.7 REMAINDER TERM IN STIRLING'S FORMULA

Stirling's interpolation formula is a central difference interpolation method used when the value of the argument is near the center of the given tabulated values. It is particularly suitable for equally spaced data. Let's define the general form of Stirling's formula for interpolation:

$$y_p = y_0 + p\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{p^2}{2}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

where:

$p = \frac{x - x_0}{h}$,

x_0 is the central value,

h is the constant interval between tabulated values,

Δ represents the forward difference operator.

The remainder (or error) term in Stirling's interpolation formula provides an estimate of the accuracy of the interpolated value. The general form of the remainder term $R_n(x)$ for interpolation is given by:

$$R_n(x) = \frac{(x - x_0)(x - x_1) \ldots (x - x_n)}{(n+1)!} y^{(n+1)}(\xi)$$

For Stirling's formula, since it is derived from Newton-Gregory interpolation using central differences, the error term depends on the $(2n+1)$th derivative of the function.

Thus, the remainder term in Stirling's interpolation formula is:

$$R_n(x) = \frac{p(p^2-1)(p^2-4) \ldots (p^2-n^2)}{(2n+1)!} h^{2n+1} y^{(2n+1)}(\xi)$$

Where, $\xi \in [x_0 - nh,\ x_0 + nh]$, $p = \frac{x - x_0}{h}$ and $y^{(2n+1)}(\xi)$ is the $(2n+1)$th derivative of the function evaluated at some point ξ in the interpolation interval.

Let, x_0 be the central value of the dataset, h be the uniform spacing and $x = x_0 + ph$, where $p = \frac{x - x_0}{h}$. The function $y(x)$ is sufficiently differentiable over the interval. The Stirling formula is built from the average of Newton's forward and backward interpolation formulas.

The interpolation polynomial of degree $2n$ with the remainder is:

$$y(x) = P_{2n}(x) + R_{2n}(x)$$

Where, $P_{2n}(x)$ is the interpolation polynomial of degree $2n$ and $R_{2n}(x)$ is the remainder term (error).

By Taylor's theorem:

$$R_{2n}(x) = \frac{y^{(2n+1)}(\xi)}{(2n+1)!}(x - x_{-n})(x - x_{-n+1}) \dots (x - x_n)$$

For some $\xi \in (x_{-n}, x_n)$

We now convert the product $(x - x_{-n})(x - x_{-n+1}) \dots (x - x_n)$ into a function of $p = \frac{x - x_0}{h}$

Let us denote the product:

$$\prod_{k=-n}^{n}(x - x_k) = h^{2n+1}\prod_{k=1}^{n}(p^2 - k^2)$$

Since,

$$x_k = x_0 + kh \qquad \Rightarrow x - x_k = x_0 + ph - (x_0 + kh) = h(p - k)$$

Hence,

$$\prod_{k=-n}^{n}(x - x_k) = h^{2n+1}p\prod_{k=1}^{n}(p^2 - k^2)$$

Thus, the remainder term becomes:

$$R_{2n}(x) = \frac{y^{(2n+1)}(\xi)}{(2n+1)!}h^{2n+1}p\prod_{k=1}^{n}(p^2 - k^2)$$

Therefore, final Expression of the Remainder Term is as follows

$$R_{2n}(x) = \frac{p(p^2 - 1)(p^2 - 4) \dots (p^2 - n^2)}{(2n+1)!}h^{2n+1}y^{(2n+1)}(\xi)$$

Example 5: Given the function $y = \ln x$, interpolate using Stirling's interpolation formula around $x_0 = 2$ and node $n = 3$ with step size $h = 1$. Find the remainder term at $x = 2.2$.

Solution:

Here,

$$p = \frac{x - x_0}{h} = \frac{2.2 - 2}{1} = 0.2$$

Also,

$$p(p^2 - 1)(p^2 - 4)(p^2 - 9) = 0.2(0.04 - 1)(0.04 - 4)(0.04 - 9)$$
$$= 0.2(-0.96)(-3.96)(-8.96) = 0.2(-33.949) \approx -6.79$$

Since, $y^7(x) = (-1)^6 \frac{6!}{x^7} = \frac{720}{x^7}$, so $y^7(2) = \frac{720}{128} = 5.625$

$$R_{2n}(x) = \frac{p(p^2 - 1)(p^2 - 4) \dots (p^2 - n^2)}{(2n + 1)!} h^{2n+1} y^{(2n+1)}(\xi)$$
$$= \frac{-6.79}{7!} \times 1 \times 5.625 \approx -0.00758$$

The remainder term is $R_6(2.2) \approx -0.0076$

CHECK YOUR PROGRESS 2

Q1. Given $y = \sin x$ and interpolated using a Lagrange interpolating polynomial at $x = 0, \frac{\pi}{4}, \frac{\pi}{2}$. Find the remainder term $R_2\left(\frac{\pi}{6}\right)$.

Q2. Given the function $y = e^x$, interpolate Stirling's interpolation formula around $x_0 = 0$ and node $n = 2$ (i.e., 5 points: -2 to 2) with step size $h = 1$. Find the remainder term at $x = 0.3$.

7.8 LET US SUM UP

A summary of the lessons we learnt from this unit is provided below.

- **Error in Polynomial Interpolation**
  - When a function $y(x)$ is approximated by a polynomial $p_n(x)$ of degree n, the error term is given by:

$$E_n(x) = \frac{y^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i), \qquad \xi \in [x_o, x_n]$$

  - This is a general formula applicable to all polynomial interpolations with distinct nodes.
- **Remainder Term in Newton's Forward Interpolation Formula**

- For equally spaced nodes with spacing h, and $p = \frac{x-x_0}{h}$, the remainder term is:

$$R_n(x) = \frac{p(p-1)(p-2)\dots(p-n)}{(n+1)!} h^{(n+1)} y^{(n+1)}(\xi)$$

- This error applies when interpolation is done near the beginning of the data table.

- **Remainder Term in Newton's Backward Interpolation Formula**

- For equally spaced nodes with $p = \frac{x-x_n}{h}$, the remainder term is:

$$R_n(x) = \frac{p(p+1)(p+2)\dots(p+n)}{(n+1)!} h^{(n+1)} y^{(n+1)}(\xi)$$

- Used when interpolating near the end of the dataset.

- **Remainder Term in Lagrange's Interpolation Formula**

- For $n+1$ distinct nodes $x_0, x_1, \dots, x_n$ the error is:

$$R_n(x) = \frac{y^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i), \qquad \xi \in [x_o, x_n]$$

- This provides an exact form of the error under the assumption that y is $(n+1)$ times continuously differentiable.

- **Remainder Term in Stirling's Interpolation Formula**

- Stirling's formula is used for central interpolation with equally spaced nodes.

- The error (remainder) term is given by:

$$R_n(x) = \frac{p(p^2-1)(p^2-4)\dots(p^2-n^2)}{(2n+1)!} h^{2n+1} y^{(2n+1)}(\xi)$$

Here, $p = \frac{x-x_0}{h}$, and ξ is within the interpolation interval.

7.9 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1

Answer to Q.1.

We know the error formula of quadratic polynomial

$$E_n(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(n+1)!} f^{(n+1)}(\xi)$$

We need $f^3(\xi)$ for $f(x) = \sin x$

$$f^3(x) = \cos x$$

So, the maximum in $\left[0, \frac{\pi}{2}\right]$ is

$$|f^3(x)| \leq 1$$

$$\therefore E_2\left(\frac{\pi}{6}\right) = \frac{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{4}\right)\left(\frac{\pi}{6}-\frac{\pi}{2}\right)}{3!} f^3(\xi)$$

$$\leq \frac{\frac{\pi}{6}\left(-\frac{\pi}{12}\right)\cdot\left(-\frac{\pi}{3}\right)}{6} = \frac{\pi^3}{1296} \approx \frac{31}{1296} \approx 0.0239$$

Hence,

$$\left|E_2\left(\frac{\pi}{6}\right)\right| \leq 2.39 \times 10^{-3}$$

Answer to Q 2.

We will use the remainder term formula:

$$R_n(x) = \frac{p(p-1)(p-2)\ldots(p-n)}{(n+1)!} h^{(n+1)} f^{(n+1)}(\xi)$$

Here, $h = 0.1,\ x = 0.2, x_0 = 0$

$$p = \frac{0.2-0}{0.1} = 2$$

$n = 2$ (Since 3 points are used)

We need $f^3(\xi)$ for $f(x) = e^x$

$$f^3(x) = e^x$$

On interval $[0,0.2]$, maximum of $|f^3(x)|$ occurs at $x = 0.2$, $f^3(0.2) = e^{0.2} \approx 1.2214$

Now,

$$R_2(2) \leq \frac{2(2-1)(2-2)}{3!} \times (0.1)^3 \times 1.2214 = 0$$

Hence, Interpolation at the data point $x = x_2 \Rightarrow$ error is zero.

Answer to Q 3.

We will use the remainder term formula:

$$R_n(x) = \frac{p(p-1)(p-2)\ldots(p-n)}{(n+1)!} h^{(n+1)} f^{(n+1)}(\xi)$$

Here, $h = 0.1,\ x = 0.85, x_n = 0.9$

$$p = \frac{x - x_n}{h} = \frac{0.85 - 0.9}{0.1} = -0.5$$

$n = 3$ (Since 4 points are used)

We need $f^4(\xi)$ for $f(x) = \sin x$

$$f^4(x) = \sin x$$
$$\therefore |f^4(x)| \leq 1$$

Now,

$$R_3(0.85) \leq \frac{-0.5(-0.5+1)(-0.5+2)(-0.5+3)}{4!} \times (0.1)^4 \times 1$$
$$= \frac{-0.5(0.5).(1.5).(2.5)}{24} 0.0001 = -0.0000039$$

Hence, $|R_3(0.85)| \leq 3.9 \times 10^{-6}$

Check your progress 2

Answer to Q 1.

Since $y = \sin x$, we get

$$y' = \cos x\,, \qquad y'' = -\sin x\,, \qquad y''' = -\cos x$$

Hence, $|y'''(\xi)| < 1$

Thus, the continuity conditions on $y(x)$ and its derivatives are satisfied in $\left[0, \frac{\pi}{2}\right]$.

Hence,

$$R_n(x) = \frac{(x-0)\left(x-\frac{\pi}{4}\right)\left(x-\frac{\pi}{2}\right)}{6}, \qquad 0 < \xi < \frac{\pi}{2}$$

When $x = \frac{\pi}{6}$, we therefore obtain

$$|R_n(x)| \leq \left|\frac{\frac{\pi}{6}\left(\frac{\pi}{6}-\frac{\pi}{4}\right)\left(\frac{\pi}{6}-\frac{\pi}{2}\right)}{6}\right| = \frac{\pi \times \pi \times \pi}{6 \times 6 \times 12 \times 3} = 0.02392$$

The remainder term is $R_2\left(\frac{\pi}{6}\right) \approx 0.02392$

Answer 2.

Here,

$$p = \frac{x - x_0}{h} = \frac{0.3 - 0}{1} = 0.3$$

Also,

$$p(p^2-1)(p^2-4) = 0.3(0.09-1)(0.09-4) = 0.3(-0.91)(-3.91) \approx 1.0674$$

Since, $y^5(x) = e^x$, so $y^5(0.3) = e^{0.3} \approx 1.3499$

$$R_{2n}(x) = \frac{p(p^2-1)(p^2-4)\ldots(p^2-n^2)}{(2n+1)!}h^{2n+1}y^{(2n+1)}(\xi)$$

$$= \frac{1.0674}{5!} \times 1 \times 1.3499 \approx 0.01199$$

The remainder term is $R_4(0.3) \approx 0.01199$

7.10 FURTHER READING

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⑦ 7.11 MODEL QUESTIONS

1. Show that

$$\sum_{i=0}^{n} \frac{\Pi_{n+1}(x)}{(x-x_i)\Pi'_{n+1}(x_i)} = 1$$

2. Derive expressions for the remainder term in Newton's formulae of forward differences.

3. Derive expressions for the remainder term in Newton's formulae of backward differences.

4. Show that the error term in interpolating polynomial is

$$E_n(x) = \frac{\Pi_{n+1}(x)}{(n+1)!} y^{(n+1)}(\xi)$$

for some $x_0 < \xi < x_n$.

5. $f(x) = \ln x$ is interpolated using a first-degree polynomial (linear interpolation) at points $x_0 = 1,\ x_1 = 2$. Estimate the error at $x = 1.5$.

6. Approximate $sin(x)$ using Newton's forward interpolation from points:

$x_0 = 0,\ x_1 = \frac{\pi}{6}, x_2 = \frac{\pi}{3}$, and $x_3 = \frac{\pi}{2}$. Estimate the maximum remainder at $x = \frac{\pi}{4}$

7. Use Newton's backward interpolation with 3 points: $x_0 = 1.6,\ x_1 = 1.7,\ x_2 = 1.8,\ x_3 = 1.9$. Estimate the remainder term at $x = 1.85$ for $f(x) = e^x$.

8. Given $f(x) = \ln x$ and interpolated using a Lagrange interpolating polynomial at $x = 1, 2, 4$. Estimate the maximum error at $x = 3$.

9. Derive the expression for the remainder term in Stirling's interpolation formula. Clearly state the assumptions and the conditions under which the formula is valid.

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