UNIT 8: NUMERICAL DIFFERENTIATION

UNIT STRUCTURE

8.1 Learning Objective

8.2 Introduction

8.3 Numerical Differentiation

8.4 Formulae for Derivatives

8.5 Let Us Sum Up

8.6 Answers to check your progress

8.7 Further Reading

8.8 Model Questions

8.1 LEARNING OBJECTIVES

- To understand how to approximate derivatives using numerical methods.
- To apply interpolation techniques—like Newton's, Stirling's, and Bessel's formulas—for computing derivatives from tabulated values.
- To handle cases with both **equi-spaced** and **non-equi-spaced** data in numerical differentiation.

8.2 INTRODUCTION

Numerical differentiation is used when you have values of a function at specific points but don't know the exact function itself. Instead of computing derivatives analytically, you replace the unknown function with an **interpolating polynomial** and differentiate it. The method used depends on where in the dataset the derivative is needed and whether the data points are evenly spaced.

8.3. NUMERICAL DIFFERENTIATION

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of values (x, y). To compute dy/dx, we first replace the exact relation $y = f(x)$ by the best interpolating polynomial $y = \varphi(x)$ and then differentiate the latter as many times as we desire. The choice of the interpolating polynomial to be used, will depend on the assigned value of x at which dy/dx is desired. If the values of x are equi-spaced and dy/dx is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula. For values near the middle of the table, dy/dx is calculated by means of Stirling's or Bessel's formulae. If the values of x are not equi-spaced, we use Lagrange's formula or Newton's divided difference

formula to represent the function. Hence corresponding to each of the interpolation formulae, we can derive a formula for finding the derivative.

8.4. FORMULAE FOR DERIVATIVES

Consider the function $y = f(x)$ which is tabulated for the values $x = x_0 + ih$, $i = 0, 1, 2, \ldots, n$.

(1) **Derivatives using Newton's forward difference formula**

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Differentiating both sides w.r.t. p, we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots$$

Since $p = \frac{(x-x_0)}{h}$, therefore $\frac{dp}{dx} = \frac{1}{h}$

Now $\frac{dy}{dx} = \frac{dy}{dp}.\frac{dp}{dx} = \frac{1}{h}[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \ldots]$

(1)

At $x = x_0, p = 0$. Hence putting $p = 0$,

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right]$$

(2)

Again differentiating (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dp}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12\ p^2-36p+22}{4!}\Delta^4 y_0 + \ldots\right]\frac{1}{h}$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \ldots\right]$$

(3)

Similarly $\left(\frac{d^3y}{dx^3}\right)_{x_0} = \frac{1}{h^3}[\Delta^3 y_0 - (3/2)\Delta^4 y_0 + \ldots]$

(4)

Otherwise: We know that $1 + \Delta = E = e^{hD}$

$$hD = log(1 + \Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots$$

or $$D = \frac{1}{h}[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots]$$

and $$D^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + \ldots\right]$$

and $$D^3 = \frac{1}{h^3}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \ldots\right]$$

Now applying the above identities to y_0, we get

$$Dy_0 \; i.e. \; \left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \ldots\right]$$

and $$\left(\frac{d^3y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right]$$

which are the same as (2), (3) and (4) respectively.

(2) **Derivatives using Newton's backward difference formula**

Newton's backward interpolation formula (p. 187) is

$$y = y + p\nabla y + \frac{p(p+1)}{2!}\nabla^2 y + \frac{p(p+1)(p+2)}{3!}\nabla^3 y + \ldots$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dp} = \nabla y + \frac{2p+1}{2!}\nabla^2 y + \frac{3p^2+6p+2}{3!}\nabla^3 y + \ldots$$

Since $p = \frac{x-x}{h}$, therefore $\frac{dp}{dx} = \frac{1}{h}$

Now $$\frac{dy}{dx} = \frac{dy}{dp}.\frac{dp}{dx} = \frac{1}{h}\left[\nabla y + \frac{2p+1}{2!}\nabla^2 y + \frac{3p^2+6p+2}{3!}\nabla^3 y + \ldots\right] \tag{5}$$

At $x = x, p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_x = \frac{1}{h}\left[\nabla y + \frac{1}{2}\nabla^2 y + \frac{1}{3}\nabla^3 y + \frac{1}{4}\nabla^4 y + \frac{1}{5}\nabla^5 y + \frac{1}{6}\nabla^6 y + \ldots\right] \tag{6}$$

Again differentiating (5) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \left(\frac{d}{dp}\right)\left(\frac{dy}{dx}\right)\left(\frac{dp}{dx}\right) = \frac{1}{h^2}\left[\frac{2}{2!}\nabla^2 y + \frac{6p+6}{3!}\nabla^3 y + \frac{6p^2+18p+11}{12}\nabla^4 y + \ldots\right]$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_x = \frac{1}{h^2}\left[\nabla^2 y + \nabla^3 y + \frac{11}{12}\nabla^4 y + \frac{5}{6}\nabla^5 y + \frac{137}{180}\nabla^6 y + \ldots\right] \tag{7}$$

Similarly, $$\left(\frac{d^3y}{dx^3}\right)_x = \frac{1}{h^3}\left[\nabla^3 y + \frac{3}{2}\nabla^4 y + \ldots\right]$$

(8)

Otherwise: We know that $1 - \nabla = E^{-1} = e^{-hD}$

$\therefore$ $$-hD = log(1 - \nabla) = -\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\right]$$

or $$D = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\right]$$

$\therefore \quad D^2 = \frac{1}{h^2}[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \ldots] = \frac{1}{h^2}[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \ldots]$

Similarly, $D^3 = \frac{1}{h^2}[\nabla^3 + \frac{3}{2}\nabla^4 + \ldots]$

Applying these identities to y, we get

Dy *i.e.* $\left(\frac{dy}{dx}\right)_x = \frac{1}{h}\left[\nabla y + \frac{1}{2}\nabla^2 y + \frac{1}{3}\nabla^3 y + \frac{1}{4}\nabla^4 y + \frac{1}{5}\nabla^5 y + \frac{1}{6}\nabla^6 y + \ldots\right]$

$$\left(\frac{d^2y}{dx^2}\right)_x = \frac{1}{h^2}\left[\nabla^2 y + \nabla^3 y + \frac{11}{12}\nabla^4 y + \frac{5}{6}\nabla^5 y + \frac{137}{180}\nabla^6 y + \ldots\right]$$

And $\left(\frac{d^3y}{dx^3}\right)_x = \frac{1}{h^3}[\nabla^3 y + \frac{3}{2}\nabla^4 y + \ldots]$

which are the same as (6), (7) and (8).

(3) **Derivatives using Stirling's central difference formula**

Stirling's formula is

$$y = y^0 + \frac{p}{1!}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \ldots$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dp} = \left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{2p}{2!}\Delta^2 y^{-1} + \frac{3p^2-1}{3!}.\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{4p^3-2p}{4!}\Delta^4 y_{-2} + \ldots$$

Since $p = \frac{x-x_0}{h}$, $\therefore \frac{dp}{dx} = \frac{1}{h}$

Now $\frac{dy}{dx} = \left(\frac{dy}{dp}\right)\left(\frac{dp}{dx}\right) = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} + p\Delta^2 y_{-1} + \left(\frac{3p^2-1}{6}\right).\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{2p^3-p}{12}\Delta^4 y_{-2} + \ldots\right]$

At $x = x_0, p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \left(\frac{1}{6}\right).\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\Delta^5 y_{-3} + \ldots\right]$$

Similarly $\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + \ldots]$

(4) **Derivatives using Bessel's central difference formula**

Bessel's formula is

$$y = y^0 + p\Delta y^0 + \left[\frac{p(p-1)}{2!}\right].\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \left(p - \frac{1}{2}\right)\frac{p(p-1)}{3!}\Delta^3 y^{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}.\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dp} = \Delta y^0 + \frac{2p-1}{2!}.\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(3p^2-3p+½)}{3!}\Delta^3 y_{-1} + \frac{4p^3-6p^2-2p+2}{4!}.\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots$$

Since $p = \frac{x-x_0}{h}, \therefore \frac{dp}{dx} = \frac{1}{h}$

Now $\frac{dy}{dx} = \left(\frac{dy}{dp}\right)\left(\frac{dp}{dx}\right) = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}.\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{3p^2-2p+½}{3!}\Delta^3 y_{-1} + \frac{4p^3-6p^2-2p+2}{4!}.\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots\right]$

At $x = x_0, p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{12}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots\right]$$

Similarly $\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} - \frac{1}{12}\Delta^4 y^{-2} + \ldots\right]$

(5) **Derivatives using unequally spaced values of argument**

(i) **Lagrange's interpolation formula** is

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x)} f(x_0) + \frac{(x-x_0)(x-x_2)\ldots(x-x)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x)} f(x_1) + \ldots$$

Differentiating both sides w.r.t. x, we get $f'(x)$.

(ii) **Newton's divided difference formula** is

$f(x) = f(x_0) + (x - x_0)\Delta f(x_0) + (x - x_0)(x - x_1)\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\Delta^3 f(x_0) + \ldots$

Differentiating both sides w.r.t. x, we obtain

$$f'(x) = \Delta f(x_0) + [2x - (x_0 + x_1)]\Delta^2 f(x_0) + [3x^2 - 2x(x_0 + x_1 + x_2) + (x_0x_1 + x_1x_2 + x_2x_0)]\Delta^3 f(x_0) + \ldots$$

Example 8.1. Given that

x:	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y:	7.989	8.403	8.781	9.129	9.451	9.750	10.031,

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(a)\, x = 1.1$ $(b)\, x = 1.6$

Sol. (a) The difference table is:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989						
		0.414					
1.1	8.403		-0.036				
		0.378		0.006			
1.2	8.781		-0.030		-0.002		
		0.348		0.004		0.001	
1.3	9.129		-0.026		-0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		-0.023		0.002		
		0.299		0.005			
1.5	9.750		-0.018				
		0.281					
1.6	10.031						

We have

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right]$$

(i)

and $$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \ldots\right]$$

(ii)

Here $h = 0.1, x_0 = 1.1, \Delta y_0 = 0.378, \Delta^2 y_0 = -0.030$, etc.

Substituting these values in (i) and (ii), we get

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(-0.001) + \frac{1}{5}(0.003)\right] = 3.952$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.1} = \frac{1}{(0.1)^2}\left[(-0.03) - (0.004) + \frac{11}{12}(-0.001) - \frac{5}{6}(0.003)\right] = -3.74$$

(b) We use the above difference table and the backward difference operator ∇ instead of Δ.

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y + \frac{1}{2}\nabla^2 y + \frac{1}{3}\nabla^3 y + \frac{1}{4}\nabla^4 y + \frac{1}{5}\nabla^5 y + \frac{1}{6}\nabla^6 y + \ldots\right] \quad \text{(i)}$$

And $$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y + \nabla^3 y + \frac{11}{12}\nabla^4 y + \frac{5}{6}\nabla^5 y + \frac{137}{180}\nabla^6 y + \ldots\right] \quad \text{(ii)}$$

Here $h = 0.1, x = 1.6, \nabla y = 0.281, \nabla^2 y = -0.018$ etc.

Putting these values in (i) and (ii), we get

$$\left(\frac{dy}{dx}\right)_{1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(0.002) + \frac{1}{5}(0.003) + \frac{1}{6}(0.002)\right] = 2.75$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[(-0.018) + (0.005) + \frac{11}{12}(0.002) + \frac{5}{6}(0.003) + \frac{137}{180}(0.002)\right] = -0.715$$

Example 8.2. The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds. Find the initial acceleration using the data:

$Time\ t\ (sec)$:	0	5	10	15	20
$Velocity\ v\ (m/sec)$:	0	3	14	69	228

Sol. The difference table is:

x	y	Δ	Δ²	Δ³	Δ□
0	0				
		3			
5	3		8		
		11		36	
10	14		44		24
		55		60	

15	69		104		
		159			
20	228				

An initial acceleration (i.e. $\frac{dv}{dt}$) at $t = 0$ is required, we use Newton's forward formula:

$$\left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \ldots\right]$$

$$\therefore \quad \left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{5}\left[3 - \frac{1}{2}(8) + \frac{1}{3}(36) - \frac{1}{4}(24)\right]$$

$$= \frac{1}{5}[3 - 4 + 12 - 6] = 1$$

Hence the initial acceleration is $1\ m/sec^2$.

Example 8.3. Find the value of $cos(1.74)$ from the following table:

x:	1.7	1.74	1.78	1.82	1.86
$\sin x$:	0.9916	0.9857	0.9781	0.9691	0.9584

Sol. Let $y = f(x) = \sin x$, so that $f'(x) = \cos x$.

The difference table is

x	y	Δ	Δ²	Δ³	Δ□
1.7	0.9916				
		-0.0059			
1.74	0.9857		-0.0017		
		-0.0076		0.0003	
1.78	0.9781		-0.0014		-0.0006
		-0.0090		-0.0003	
1.82	0.9691		-0.0017		
		-0.0107			
1.86	0.9584				

Since we require $f'(1.74)$, we use Newton's forward formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right] \qquad \text{(i)}$$

Here $h = 0.04, x_0 = 1.7, \Delta y_0 = -0.0059, \Delta^2 y_0 = -0.0017$ etc.

Substituting the values in (i), we get

$\left(\frac{dy}{dx}\right)_{1.74} = \frac{1}{0.04}\left[(-0.0059) - \frac{1}{2}(-0.0017) + \frac{1}{3}(0.0003) - \frac{1}{4}(-0.0006)\right] = \frac{1}{0.04}(-0.007) = 0.175$

Hence $\cos(1.74) = 0.175$

Example 8.4. A slider in a machine moves along a fixed straight rod. Its distance x cm along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when $t = 0.3$ second.

$t =$	0	0.1	0.2	0.3	0.4	0.5	0.6
$x =$	30.13	31.62	32.87	33.64	33.95	33.81	33.24

Sol. The difference table is:

x	y	Δ	Δ^2	Δ^3	Δ□	Δ□	Δ□
0	30.13						
		1.49					
0.1	31.62		-0.24				
		1.25		-0.24			
0.2	32.87		-0.48		0.26		
		0.77		0.02		-0.27	
0.3	33.64		-0.46		-0.01		0.29
		0.31		0.01		0.02	
0.4	33.95		-045		0.01		
		-0.14		0.02			
0.5	33.81		-0.43				
		-0.57					
0.6	33.24						

As the derivatives are required near the middle of the table, we use Stirling's formulae:

$$\left(\frac{dx}{dt}\right)_{t_0} = \frac{1}{h}\left(\frac{\Delta x_0 + \Delta x_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2}\right) + \ldots] \quad \text{(i)}$$

$$\left(\frac{d^2x}{dt^2}\right)_{t_0} = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} + \ldots\right] \quad \text{(ii)}$$

Here $h = 0.1, t_0 = 0.3, \Delta x_0 = 0.31, \Delta x_{-1} = 0.77, \Delta^3 x_{-1} = -0.46$ etc.

Putting these values in (i) and (ii), we get

$$\left(\frac{dx}{dt}\right)_{0.3} = \frac{1}{0.1}\left[\frac{0.31 + 0.77}{2} - \frac{1}{6}\left(\frac{0.01 + 0.02}{2}\right) + \frac{1}{30}\left(\frac{0.02 - 0.27}{2}\right) + \ldots\right] = 5.33$$

$$\left(\frac{d^2x}{dt^2}\right)_{0.3} = \frac{1}{(0.1)^2}\left[(-0.46) - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29) + \ldots\right] = -45.6$$

Hence the required velocity is $5.33\ cm/sec$ and acceleration is $-45.6\ cm/sec^2$.

Example 8.5 Find $f'(10)$ from the following data:

x:	3	5	11	27	34

$$f(x): -13 \quad 23 \quad 899 \quad 17315 \quad 35606.$$

Sol. As the values of x are not equi-spaced, we shall use Newton's divided difference formula. The divided difference table is

x	$f(x)$	1st div. diff.	2nd div. diff.	3rd div. diff.	4th div. diff.
3	-13				
		18			
5	23		16		
		146		0.998	
11	899		39.96		0.0002
		1025		1.003	
27	17315		69.04		
		2613			
34	35606				

Fifth differences being zero, Newton's divided difference formula for the first derivative, we get

$$f'(x) = f(x_0, x_1) + (2x - x_0 - x_1) f(x_0, x_1, x_2)$$
$$+ [3x^2 - 2x(x_0 + x_1 + x_2) + (x_0x_1 + x_1x_2 + x_2x_0)] f(x_0, x_1, x_2, x_3)$$
$$+ [4x^3 - 3x^2(x_0 + x_1 + x_2 + x_3) + 2x(x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3) - (x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3)] f(x_0, x_1, x_2, x_3, x_4)$$

Putting $x_0 = 3, x_1 = 5, x_2 = 11, x_3 = 27, x_4 = 34, x = 10$, we obtain

$$f'(10) = 18 + 12 \times 16 + 23 \times 0.998 - 420 \times 0.002 = 232.869.$$

CHECK YOUR PROGRESS

Q1. Find $y'(0)$ and $y''(0)$ from the following table

x:	0	1	2	3	4	5
y:	4	8	15	7	6	2

Q2. Find the first, second and third derivatives of $f(x)$ at $x = 1.5$ if

x:	1.5	2.0	2.5	3.0	3.5	4.0
y:	3.375	7.000	13.625	24.000	38.875	59.000

Q3. Find $f'(6)$ from the following data:

x:	2	4	6	10	13
$f(x)$:	−5	3	25	255	624

8.5 LET US SUM UP

Numerical differentiation is a method used to estimate the derivative of a function when its analytical form is unknown, but its values at specific points are given. By constructing an

interpolating polynomial that approximates the function, its derivative can be calculated at a desired point. The choice of method—such as Newton's forward or backward formulas, Stirling's or Bessel's central difference formulas, or Lagrange's and Newton's divided differences—depends on whether the data points are evenly spaced and where the required value lies within the dataset. These techniques provide a powerful alternative to analytical differentiation, especially useful when dealing with empirical or experimental data.

✓ 8.6 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q 1 : We use Newton's forward difference formula since $x = 0$ is at the beginning and values are equally spaced with $h = 1$.

Forward Difference Table:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	4					
		4				
1	8		3			
		7		-18		
2	15		-15		40	
		-8		22		-72
3	7		7		-32	
		-1		-10		
4	6		-3			
		-4				
5	2					

First Derivative at x = 0

$$y'(0) = (1/h) \times [\Delta y_0 - (1/2)\Delta^2 y_0 + (1/3)\Delta^3 y_0 - (1/4)\Delta^4 y_0]$$

$$\therefore y'(0) = 1 \times [4 - (1/2)(3) + (1/3)(-18) - (1/4)(40)]$$
$$= 4 - 1.5 - 6 - 10$$
$$= -13.5$$

Second Derivative at $x = 0$

$$y''(0) = (1/h^2) \times [\Delta^2 y_0 - \Delta^3 y_0 + (11/12)\Delta^4 y_0]$$

$y''(0) = 1 \times \left[3 - (-18) + \left(\frac{11}{12}\right)(40)\right]$

$= 3 + 18 + 36.67$

$= 57.67$

Hence the results are $'(0) = -13.5$, $y''(0) = 57.67$

Answer to Q 2. We use Newton's Forward Difference method as the values are equally spaced with $h = 0.5$, and the point $x = 1.5$ lies at the start of the table.

x	$f(x)$	Δ	Δ^2	Δ^3	Δ□	Δ□
1.5	3.375	3.625	3.000	1.250	0.250	0.125
2.0	7.000	6.625	4.250	1.500	0.375	
2.5	13.625	10.375	5.750	1.875		
3.0	24.000	14.875	7.625			
3.5	38.875	20.125				
4.0	59.000					

$f'(x_0) = (1 / h) \times [\Delta y_0 - (1/2)\Delta^2 y_0 + (1/3)\Delta^3 y_0 - (1/4)\Delta^4 y_0 + (1/5)\Delta^5 y_0]$

$= (1 / 0.5) \times [3.625 - 1.500 + 0.417 - 0.063 + 0.025]$

$= 2 \times 2.504 = 5.008$

$f''(x_0) = (1 / h^2) \times [\Delta^2 y_0 - \Delta^3 y_0 + (11/12)\Delta^4 y_0 - (5/6)\Delta^5 y_0]$

$= (1 / 0.25) \times [3.000 - 1.250 + 0.229 - 0.104]$

$= 4 \times 1.875 = 7.500$

$f'''(x_0) = (1 / h^3) \times [\Delta^3 y_0 - (3/2)\Delta^4 y_0]$

$= (1 / 0.125) \times [1.250 - 0.375]$

$= 8 \times 0.875 = 7.000$

Hence $'(1.5) = 5.008$, $f''(1.5) = 7.500$, $f'''(1.5) = 7.000$

Answer to Q 3.

x	$f(x)$	1st div. diff.	2nd div. diff.	3rd div. diff.	4th div. diff.
2	-5				
		8			
4	3		0.75		
		11		0.875	
6	25		7.75		-0.0633
		57.5		0.1786	
10	255		9.357		
		123			
13	624				

$f'(x) = f[x_0, x_1] + (2x - x_0 - x_1)f[x_0, x_1, x_2] + [3x^2 - 2x(x_0 + x_1 + x_2) + (x_0x_1 + x_1x_2 + x_2x_0)]f[x_0, x_1, x_2, x_3] + higher\ order\ terms\ldots$

Let's take: $x_0 = 2, x_1 = 4, x_2 = 6, x_3 = 10$

Then $f[x_0, x_1] = 8.0, f[x_0, x_1, x_2] = 0.75, f[x_0, x_1, x_2, x_3] = 0.875$

$$\therefore f'(6) = 8.0 + (2 \times 6 - 2 - 4)(0.75) + [3 \times 6^2 - 2 \times 6 \times (2 + 4 + 6) + (2 \times 4 + 4 \times 6 + 6 \times 2)](0.875)$$

Thus $f'(6) = 8.0 + 4.5 + 7.0 = 19.5$

Hence $f'(6) = 19.5$

8.7 FURTHER READING

1. Gupta, R. K. *Numerical Methods: Fundamentals and Applications.* 1st edition, Cambridge University Press, 2019.
2. Thangaraj, P. *Computer Oriented Numerical Methods*. PHI Learning Pvt. Ltd, 2013.
3. Jain, M. K., Iyengar, S. R. K. and Jain, R. K. *Numerical Methods for Scientific & Engineering Computation*. New Age International, 2012.

8.8 MODEL QUESTIONS

1. Find the first and second derivatives of the function tabulated below at the point $x = 1.1$

x: 1.0 1.2 1.4 1.6 1.8 2.0

y: 0 0.128 0.544 1.296 2.432 4.000

2. Given x: 0 1 2 3 4 5

y: 4 8 15 7 6 2

Find: $y'(0)$ and $y''(0)$

3. Given x: 1.0 1.2 1.4 1.6 1.8 2.0

y: 0 0.128 0.544 1.296 2.432 4.000

Find $y'(1.1)$ and $y''(1.1)$.

4. Given x: 3 5 11 27 34

$f(x)$: −13 23 899 17315 35606

Find $f'(10)$.

*** *** ***