UNIT 9: NUMERICAL INTEGRATION

UNIT STRUCTURE

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9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- Understand the concept of approximating definite integrals using numerical methods.
- Learn the formulation of quadrature techniques based on equally spaced intervals.
- Explore the use of linear approximations to estimate definite integrals.
- Apply parabolic fitting to approximate integrals over even intervals and extend Simpson's method using cubic polynomials for better accuracy.
- Generalize numerical integration using polynomial interpolation formulas.

9.2 INTRODUCTION

In calculus, we learn how to find the exact value of a definite integral using formulas and techniques such as substitution or integration by parts. However, in many real-world situations, finding an exact value is either very difficult or impossible. This happens, for example, when the function is too complicated, when it doesn't have a simple antiderivative, or when we only know the values of the function at certain points, such as from experiments or measurements.

In such cases, we need to use numerical methods to approximate the value of the integral. This process is called numerical integration or numerical quadrature.

The basic idea is to replace the function with a simpler one-usually a polynomial that is easy to integrate. Example, we might connect a few points on the graph of the function with straight lines or curves, and then calculate the area under these approximations instead of the exact curve. The result is an estimate of the definite integral.

In this chapter, we will learn about several common methods of numerical quadrature, including:

- The Trapezoidal Rule, which approximates the function using straight lines.
- Simpson's Rule, which uses parabolas to better fit the function.
- More general formulas, like the Newton-Cotes formulas, which use higher degree polynomials.

We will also see how to derive these rules, how to apply them, and how accurate they are. These techniques are useful not only in mathematics but also in physics, engineering, and other sciences where exact answers are hard to find and good approximations are needed.

9.3 NUMERICAL QUADRATURE

In many practical situations, we encounter definite integrals that cannot be easily or exactly evaluated. Moreover, if the values of the integrand are known only at a few discrete points, analytical integration is not possible.

In such cases, numerical integration (or numerical quadrature) provides a practical alternative. Quadrature is the process of numerically evaluating a definite integral, particularly when the integrand is a function of a single variable. The general approach is as follows:

1. Approximate the integrand $f(x)$ using an interpolation polynomial (based on tabulated values).
2. Integrate this polynomial over subintervals of the full integration range.
3. Sum the contributions from all subintervals to approximate the value of the integral.

9.4 QUADRATURE FORMULA FOR EQUALLY SPACED ARGUMENTS

Let

$$I = \int_a^b f(x)\,dx,$$

where $f(x)$ is known at equally spaced points

$x_0=a, \quad x_1=x_0+h=a+h \quad x_2=x_0+2h=a+2h...$

$$x_n = a + nh = b.$$

Thus, the interval $[a, b]$ is divided into n equal parts of width h, and the function values (ordinates) are

$$y_0=f(x_0),\ y_1=f(x_1),\ldots.\ y_n=f(x_n)..$$

We approximate $f(x)$ using Newton's forward difference interpolation polynomial

$$f(x) \approx y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots,$$

where

$$u = \frac{x - x_0}{h}, \qquad dx = h\,du.$$

Substituting into the integral

$$I = \int_a^b f(x)\,dx = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\right] du.$$

Integrating term-by-term, we obtain

$$I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{n^3 - n^2}{6}\Delta^2 y_0 + \frac{n^4 - 4n^3 + 6n^2}{24}\Delta^3 y_0 + \cdots\right] \tag{9.1}$$

This is the general quadrature formula, and it gives the approximate value of the integral using $n + 1$ equally spaced ordinates.

By choosing specific values of *n*, we can derive standard numerical integration rules

- *n* = 1: Trapezoidal Rule
- *n* = 2: Simpson's $\frac{1}{3}$ Rule
- *n* = 3: Simpson's $\frac{3}{8}$ Rule

9.5 THE TRAPEZOIDAL RULE

Putting *n* = 1 in the general quadrature formula (equation 9.1), and neglecting the second and higher-order differences, we get

$$\int_{x_0}^{x_0+h} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{y_1 - y_0}{2}\right] = \frac{h}{2}(y_0 + y_1)$$

Similarly, for other intervals

$$\int_{x_1}^{x_1+h} y\,dx = \frac{h}{2}(y_1 + y_2), \quad \ldots, \quad \int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding all these n integrals together, we obtain

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

Or,

$$\int_{x_0}^{x_0+nh} y\,dx = h\left[\frac{1}{2}(y_0 + y_n) + (y_1 + y_2 + \cdots + y_{n-1})\right]$$

This is known as the Trapezoidal Rule

= Distance between two consecutive ordinates × [mean of the first and last ordinate + sum of all intermediate ordinates]

Note: Here, we approximate the curve between two points (xo, yo) and $(xo + h, y1)$ by a straight line. That is, we assume y is a linear function of x, say

$$y = a + bx$$

Example 1:

Approximate the value of the integral

$$\int_0^4 f(x)\,dx$$

using the trapezoidal rule with $n = 4$ subintervals, where

$$f(x) = x_2$$

Solution:

For n subintervals of width $h = (b-a)/n$, the trapezoidal rule is

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Here, $a = 0, b = 4, n = 4, h = (4-0)/4 = 1$

So, the x-values are: $x_0=0,\ x_1=1,\ x_2=2,\ x_3=3,\ x_4=4$

Now evaluate $f(x) = x\text{^}2$ at each point, we get

$$f(0) = 0, f(1) = 1, f(2) = 4, f(3) = 9, f(4) = 16$$

Now, apply the Trapezoidal rule,

$$\begin{aligned}\int_0^4 x^2\,dx &= \frac{1}{2}\left[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right]\\ &= \frac{1}{2}\left[0 + 2(1) + 2(4) + 2(9) + 16\right]\\ &= \frac{1}{2}\left[0 + 2 + 8 + 18 + 16\right]\\ &= \frac{1}{2}\cdot 44 = 22\end{aligned}$$

Therefore,

$$\int_0^4 x^2\,dx = 22$$

The exact value is $$\int_0^4 x^2\,dx = \frac{x^3}{3}\Big|_0^4 = \frac{64}{3} \approx 21.33$$

So, the trapezoidal rule gives a good approximation.

9.6 SIMPSON'S 1/3 RD RULE

Put *n* = 2 in 9.1 and neglect third and higher differences

$$\int_{x_0}^{x_0+2h} y\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly

$$\int_{x_2}^{x_4} y\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

In general, for even *n*

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[y_0 + y_n + 4\sum_{\text{odd } i} y_i + 2\sum_{\substack{\text{even } i \\ 1\le i\le n-1}} y_i\right]$$

Or,

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\right]$$

This is known as **Simpson's One-Third Rule**.

Note: Since we neglect all differences above the second, this rule assumes that *y* is a quadratic polynomial

$$y = ax2 + bx + c$$

Example 2:

Show that Simpson's one-third rule applied thrice gives

$$\int_0^6 u_x\,dx = \frac{1}{3}\left[(u_0 + u_6) + 4(u_1 + u_3 + u_5) + 2(u_2 + u_4)\right]$$

Solution:

For three equally spaced points $x_0\,x_1\,x_2$ with spacing *h*, Simpson's rule is

$$\int_{x_0}^{x_2} u_x\,dx = \frac{h}{3}\left[u_0 + 4u_1 + u_2\right]$$

Let the points be

$x_0=0,\ x_1=1,\ x_3=3,\ x_4=4,\ x_5=5,\ x_6=6$

We divide the interval [0,6] into three subintervals

- Interval 1: [0,2] with points $x_0\,x_1\,x_2$
- Interval 2: [2,4] with points $x_2\,x_3\,x_4$
- Interval 3: [4,6] with points $x_4\,x_5\,x_6$

Here, *h* = 1 (since the spacing between points is 1).

Now, apply Simpson's rule on each interval

$$\int_0^2 u_x\,dx = \frac{1}{3}(u_0 + 4u_1 + u_2)$$
$$\int_2^4 u_x\,dx = \frac{1}{3}(u_2 + 4u_3 + u_4)$$
$$\int_4^6 u_x\,dx = \frac{1}{3}(u_4 + 4u_5 + u_6)$$

Adding these three approximations, we get

$$\int u(x)dx\ u0$$

$$\int_0^6 u\,(x)dx = 1/3\ [(u_0 + 4u_{0,} + u_2) + (u_2 + 4u_3 + u_4) + (u_4 + 4u_5 + u_6)$$

$$= 1/3[(u_0 + u_6) + (u_1 + u_3 + u_5) + 2(u_2 + u_4)$$

9.7 SIMPSON'S 3/8 TH RULE

Put *n* = 3 in equation (9.1) and neglect all differences above the third, then

$$\int_{x_0}^{x_0+3h} y\,dx = h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{5}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

This simplifies to

$$\int_{x_0}^{x_0+3h} y\,dx = \frac{3h}{8}\,[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly,

$$\int_{x_3}^{x_6} y\,dx = \frac{3h}{8}\,[y_3 + 3y_4 + 3y_5 + y_6]$$

Adding such expressions from $x_0\ to\ x_n$ (where *n* is a multiple of 3), we obtain the Simpson's Three-Eighth Rule

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\,[y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots + y_{n-3})]$$

On applying Simpson's Three-Eighth Rule twice, we obtain

$$\int_0^6 u(x)\,dx = \frac{3}{8}\,[u_0 + u_6 + 3(u_1 + u_2 + u_4 + u_5) + 2u_3]$$

Since we have neglected all differences above the third, the function y is assumed to be a cubic polynomial

$$y\ =\ ax3\ +\ bx2\ +\ cx\ +\ d$$

Change of Scale:

Let $y\ =\ hx$, so that the values of y are at equidistant intervals of *h* and $dy\ =\ hdx$. Then

$$\int_0^{2h} u(y)\,dy = h\int_0^{2} u(hx)\,dx = \frac{h}{3}\left[u_0 + 4u_h + u_{2h}\right]$$

Thus, in order to change the scale by a factor of *h*, we must multiply the right-hand side by *h* and each suffix in *u* by *h*.

Example 3: Approximate

$$\int_0^3 (1+x^2)\,dx$$

using Simpson's 3/8 Rule.

Solution:

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

Here, $a = 0, b = 3$ so $h = (3-0)/3 = 1$

Points:

$$x_0 = 0, \qquad x_1 = 1, \qquad x_2 = 2, \qquad x_3 = 3$$

Function values:

$$f(0) = 1,\ f(1) = 2, f(2) = 5, f(3) = 10$$

Now, applying the rule, we get

$$\begin{aligned}\int_0^3 (1+x^2)\,dx &\approx \frac{3}{8}\left[1 + 3(2) + 3(5) + 10\right]\\ &= \frac{3}{8}\cdot(1+6+15+10)\\ &= \frac{3}{8}\cdot 32 = 12\end{aligned}$$

Therefore,

$$\int_0^3 (1+x^2)\,dx = 12$$

The exact value is

$$\int_0^3 (1+x^2)\,dx = \left[x + \frac{x^3}{3}\right]_0^3 = 3 + 9 = 12$$

So, the Simpson's 3/8 Rule gives the exact value in this case.

CHECK YOUR PROGRESS

Q1) What is the step size *h* in numerical integration? Explain the significance of step size *h* in numerical integration.

Q2) Calculate an approximate value of

$$\int_{-3}^{3} x^4\,dx$$

using Simpson's Rule with seven equidistant ordinates. Compare the result with the exact value and the value obtained using the Trapezoidal Rule.

Q3) What is the order of error in the Trapezoidal Rule and Simpson's Rule?

Q4) What kind of functions benefit most from Simpson's Rule?

9.8 NEWTON-COTE'S FORMULA

Consider Lagrange's formula

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}f(x_0) + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}f(x_1) + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}f(x_n)$$

Integrating between the limits x_0 and $x_0 + nh$, we have

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = H_0 f(x_0) + H_1 f(x_1) + \cdots + H_i f(x_i) + \cdots + H_n f(x_n) \tag{9.2}$$

where

$$H_i = \int_{x_0}^{x_0+nh} \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_n)}\frac{1}{x-x_i}\,dx$$

Let $x_i = x_0 + ih$ for all i so that $x_i = x_0 + ih$. Further, let

$$x = x_0 + th \quad \text{so that} \quad dx = hdt,$$

where t is the new real variable. Then

$$H_i = \frac{1}{(i!)(-1)^{n-i}(n-i)!h^n}\int_0^n \frac{t(t-1)\cdots(t-n)h^{n+1}}{t-i}dt,$$

which simplifies to

$$H_i = \frac{(-1)^{n-i}h}{i!(n-i)!}\int_0^n \frac{t(t-1)\cdots(t-n)}{t-i}dt \tag{9.3}$$

Equation (9.2) is known as Cotes' formula with H_i given by (9.3). Replacing H_i by nhC_i^n, we have

$$I = nh\sum_{i=0}^{n} f(x_i)C_i^n = (x_n - x_0)\sum_{i=0}^{n} f(x_i)C_i^n = (b-a)\sum_{i=0}^{n} f(x_i)C_i^n,$$

where the numbers C_i^n, $0 \le i \le n$, are called Cotes' numbers.

Properties of Cotes' numbers

(i) $C_k^n = \binom{n}{k} = \binom{n}{n-k}$ or $H_i = H_{n-i}$

(ii) $\sum_{k=0}^{n} C_k^n = 1$

We shall prove (i) and leave (ii) for the reader to prove.

Proof (i): In (9.3), put $n - i$ for i and $t = n - s$. Then we get

$$\begin{aligned} H_{n-i} &= \frac{(-1)^i h}{i!(n-i)!} \int_n^0 \frac{(-1)^{n-2}(s-n)(s-n+1)\cdots s}{(n-s)-(n-i)}(-ds) \\ &= \frac{(-1)^i(-1)^n h}{i!(n-i)!} \int_0^n \frac{s(s-1)\cdots(s-n)}{(s-i)} ds \\ &= \frac{(-1)^{n-i} h}{i!(n-i)!} \int_0^n \frac{s(s-1)\cdots(s-n)}{(s-i)} ds = H_i \end{aligned}$$

(Note: $(-1)^i = (-1)^{-i}$)

Note: Thus, the coefficients of Newton-Cotes formula are symmetric from both ends. Also, it is sufficient to calculate only half the coefficients, and the remaining half are obtained by the property of symmetry.

Proof (ii): Let $f(x) = 1$. Then $\int_a^b$

$$\int_a^b 1.dx = b - a$$

Using the Newton-Cotes formula

$$\int_a^b f(x)\,dx = \sum_{k=0}^{n} w_k f(x_k) = \sum_{k=0}^{n} w_k$$

Since $f(x_k) = 1$, we get

$$\sum_{k=0}^{n} w_k = b - a$$

Define the normalized Cotes' weights

$$C_k^n = \frac{w_k}{b-a}$$

Then:

$$\sum_{k=0}^{n} C_k^n = \frac{1}{b-a} \sum_{k=0}^{n} w_k = \frac{b-a}{b-a} = 1$$

Corollary: Derivation of Trapezoidal and Simpson's rule Put $n = 1$ in(9.2), we get

$$I = \int_{x_0}^{x_0+h} f(x)\,dx = H_0 f(x_0) + H_1 f(x_1)$$

where

$$H_0 = h\int_0^1 \frac{t-1}{0-1}dt = \frac{h}{2}$$

$$H_1 = \int_0^1 \frac{t+0}{1+0}dt = \frac{t^2}{2}\Big|_0^1 = \frac{h}{2}$$

and

Thus,

$$\int_{x_0}^{x_0+h} f(x)\,dx = \frac{h}{2}[f(x_0)+f(x_1)],$$

which is the Trapezoidal rule.

Put $n = 2$ in (9.2), so that

$$\int_{x_0}^{x_0+2h} f(x)dx = H_0 f(x_0) + H_1 f(x_1) + H_2 f(x_2)$$

$$H_0 = \frac{(-1)^2 h}{0!2!}\int_0^2 (t-1)(t-2)dt = \frac{1}{3}h,$$

$$H_1 = \frac{(-1)^1 h}{1!1!}\int_0^2 (t-2)dt = \frac{4}{3}h,$$

$$H_2 = H_0 = \frac{1}{3}h.$$

where and

Hence,

$$\int_{x_0}^{x_0+2h} f(x)\,dx = \frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)],$$

which is Simpson's one-third rule. Put $n = 3$ in (9.2), so that

$$\int_{x_0}^{x_0+3h} f(x)dx = H_0 f(x_0) + H_1 f(x_1) + H_2 f(x_2) + H_3 f(x_3)$$

where

$$H_0 = \frac{(-1)^3 h}{0!3!}\int_0^3 (t-1)(t-2)(t-3)dt = \frac{3}{8}h,$$

similarly,

$$H_1 = \frac{(-1)^2 h}{0!2!}\int_0^3 t(t-2)(t-3)dt = \frac{9}{8}h,$$

$$H_2 = \frac{9}{8}h,$$

$$H_3 = \frac{3}{8}h$$

Hence,

$$\int_{x_0}^{x_0+3h} f(x)\,dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)],$$

which is Simpson's three-eighth rule.

Cotes' numbers for some values of *n*

n	C_0^n	C_1^n	C_2^n	C_3^n	C_4^n	C_5^n	C_6^n
1	$\frac{1}{2}$	$\frac{1}{2}$					
2	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$				
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$			
4	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$		
5	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$	
6	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	...		

CHECK YOUR PROGRESS

Q5) Write the general form of an *n*-point Newton-Cotes quadrature formula.
Q6) How does increasing the number of subintervals affect the accuracy of quadrature?
Q7) If y_x is a polynomial in *x* of the third degree, find an expression for

$$\int y_x dx$$

in terms of $y_0, y_1, y_2, and\ y_3$. Use this result to show that

$$\int_1^2 y_x\,dx = \frac{1}{24}(-y_0 + 13y_1 + 13y_2 - y_3)$$

9.9 LET US SUM UP

- Numerical integration, also known as numerical quadrature, is used to approximate the definite integral of a function when an analytical evaluation is difficult or impossible.

- The basic quadrature formula for equally spaced arguments divides the interval $[a, b]$ into *n* subintervals of equal width $h = \frac{b-a}{h}$, and approximates the integral by summing weighted function values at these points.
- **Trapezoidal Rule:** Approximates the area under the curve as a series of trapezoids. The formula is:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right],$$

where $x_i = a + ih$, and *n* is the number of subintervals.

- **Simpson's $\frac{1}{3}$ Rule**: Applies a quadratic polynomial through every three points. The formula (for even *n*) is:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4\sum_{\text{odd } i=1}^{n-1} f(x_i) + 2\sum_{\text{even } i=2}^{n-2} f(x_i) + f(x_n)\right].$$

- **Simpson's $\frac{3}{8}$ Rule**: Uses cubic polynomials over three intervals. The formula (for *n* divisible by 3) is:

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f(x_0) + 3\sum_{i\not\equiv 0 \pmod 3}^{n-1} f(x_i) + 2\sum_{i\equiv 0 \pmod 3}^{n-3} f(x_i) + f(x_n)\right].$$

- **Newton-Cotes formulas** generalize the trapezoidal and Simpson's rules by fitting higher degree polynomials to equally spaced nodes. The general formula is

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i)$$

where w_i are weights determined by polynomial interpolation.

- These numerical techniques provide useful approximations when only discrete values of the function are known or when the integrand is difficult to integrate analytically.

✓ 9.10 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q1: Here $h = \frac{b-a}{h}$, where *n* is the number of subintervals.

The step size *h* determines the width of each subinterval. Smaller *h* generally increases accuracy but requires more computations.

Answer to Q2: Here, *h* = 1, and we compute the values of $y = x^4$ at the seven points

$$y_{-3} = (-3)^4 = 81,$$

$$y_{-2} = (-2)^4 = 16,$$

$$y_{-1} = (-1)^4 = 1,$$

$$y_0 = 0^4 = 0,$$

$$y_1 = 1^4 = 1,$$

$$y_2 = 2^4 = 16,$$

$$y_3 = 3^4 = 81.$$

Using Simpson's 1⁄3 Rule, with h = 1, we have

$$\int_{-3}^{3} x^4\,dx = \frac{1}{3}\left[(y_{-3} + y_3) + 2(y_{-1} + y_1) + 4(y_{-2} + y_0 + y_2)\right]$$

$$= \frac{1}{3}\left[(-3)^4 + 3^4 + 2\left((-1)^4 + 1^4\right) + 4\left((-2)^4 + 0 + 2^4\right)\right]$$

$$= \frac{1}{3}\left[81 + 81 + 2(1 + 1) + 4(16 + 0 + 16)\right]$$

$$= 98$$

Now, compute the exact value of the integral

$$\int_{-3}^{3} x^4\,dx = \left[\frac{1}{5}x^5\right]_{-3}^{3} = \frac{1}{5}\left(3^5 - (-3)^5\right)$$

$$= \frac{1}{5}(243 - (-243)) = \frac{1}{5} \cdot 486 = 97.2$$

Now, apply the Trapezoidal rule

$$\int_{-3}^{3} x^4\,dx \approx \frac{1}{2}h\left[y_{-3} + y_3 + 2(y_{-2} + y_{-1} + y_0 + y_1 + y_2)\right]$$

$$= \frac{1}{2}\left[81 + 81 + 2(16 + 1 + 0 + 1 + 16)\right]$$

$$= 115$$

This shows that the Trapezoidal Rule gives a less accurate result in this case. In general, Simpson's Rule provides a better approximation than the Trapezoidal Rule.

Answer to Q3: The error in the Trapezoidal Rule is of order O(h^2), and in Simpson's Rule it is O(h^4), where h is the subinterval width.

Answer to Q4: Functions that are smooth and can be well approximated by quadratics benefit most from Simpson's Rule.

Answer to Q5: Here,

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i)$$

where x_i are equally spaced points and w_i are weights depending on *n*.

Answer to Q6: Increasing subintervals usually improves accuracy by better capturing the function's behavior, but also increases computational cost.

Answer to Q7: We have

$$y_x = \frac{(x-1)(x-2)(x-3)}{(-1)(-2)(-3)}y_0 + \frac{x(x-2)(x-3)}{1(-1)(-2)}y_1 + \frac{x(x-1)(x-3)}{2(1)(-1)}y_2 + \frac{x(x-1)(x-2)}{3\cdot 2\cdot 1}y_3$$

which simplifies to

$$y_x = -\frac{1}{6}(x^3 - 6x^2 + 11x - 6)y_0 + \frac{1}{2}(x^3 - 5x^2 + 6x)y_1 - \frac{1}{2}(x^3 - 4x^2 + 3x)y_2 + \frac{1}{6}(x^3 - 3x^2 + 2x)y_3$$

Integrating term-wise from 0 to 1, we get

$$\int_0^1 y_x\,dx = -\frac{1}{6}\left(\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x\right)y_0 + \frac{1}{2}\left(\frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2\right)y_1 - \frac{1}{2}\left(\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2\right)y_2 + \frac{1}{6}\left(\frac{x^4}{4} - x^3 + x^2\right)y_3$$

9.11 FURTHER READING

1. Gupta, R. K. Numerical Methods: Fundamentals and Applications. 1st edition, Cambridge University Press, 2019.
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3. Jain, M. K., Iyengar, S. R. K. and Jain, R. K. Numerical Methods for Scientific & Engineering Computation. New Age International, 2012.
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5. Chapra, S. C. and Canale, R. P. Numerical Methods for Engineers. McGraw Hill, International Edition, 1998.
6. Mathews, J. H. Numerical Methods for Mathematics, Science and Engineering. Prentice-Hall, International Editions, 1992.

9.12 MODEL QUESTIONS

1. *Show that if $f(x)$ is a cubic, then*

$$\int_0^{2h} f(x)\,dx = \frac{h}{3}[f(0) + 4f(h) + f(2h)],$$

Hence, deduce Simpson's rule for an approximate value of the area between two ordinates, the x-axis and the curve.

2: Apply Simpson's rule to estimate the value of the integral

$$\int_1^2 \frac{dx}{x}$$

by dividing the interval (1,2) into 4 equal parts.

3. Show that 1.62 is an approximate value of

$$\int_1^5 \frac{dx}{x}$$

If $h = 1$ in Simpson's rule.

4. Calculate (upto 3 decimal places)

$$\int_2^{10} \frac{dx}{1+x}$$

By dividing the range into eight equal parts.

5. Use Simpson's $\frac{3}{8}$th rule to obtain the value of

(i) $\int_0^{0.3}(1-8x^3)^{1/2}dx$

(ii) $\int_{1.6}^{2.2} f(x)dx$; given

x:	1.6	1.8	2.0	2.2
$f(x)$:	4.953	6.050	7.389	9.025

6. Calculate the approximate value of

$$\int_2^6 \frac{dx}{1+x^2}$$

Using the Newton-Cote formula and compare the results with the true value of $\tan^{-1} 6 = 1.406$

7. Evaluate

$$\int_0^1 e^x dx$$

Approximately, in steps of 0.05 using

i. Trapezoidal rule

ii. Simpson's one-third rule

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