GBA S1 03

BUSINESS MATHEMATICS

SEMESTER – 1

BACHELOR IN BUSINESS ADMINISTRATION

BLOCK-2

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Reprint, 2018

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Printed and published by Registrar on behalf of the Krishna Kanta Handiqui State Open University.

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The University acknowledges with thanks the financial support provided by the **Distance Education Bureau**, **UGC** for the preparation of this study material.

BLOCK INTRODUCTION

This is the second block of the course '*Business Mathematics*'. The block consists of four units.

The sixth unit will focus on Logarithms.

The seven unit is an elementary introduction to Binomial Theory and its uses.

The eighth unit is about Matrix theory. In this unit,you will able to explores some of application of matrix .

In nineth unit ,we discuss limits and continuity of a functions.

While going through a unit, you will notice some along-side boxes, which have been included to help you know some of the difficult, unseen terms. Some 'ACTIVITY' (s) has been included to help you apply your own thoughts. Again, we have included some relevant concepts in 'LET US KNOW' along with the text. And, at the end of each section, you will get 'CHECK YOUR PROGRESS' questions. These have been designed to self-check your progress of study. It will be better if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with 'ANSWERS TO CHECK YOUR PROGRESS' given at the end of each unit.

BACHELOR IN BUSINESS ADMINISTRATION
BUSINESS MATHEMATICS
Block 2

DETAILED SYLLABUS

UNIT 6: Logarithms **Pages: 5-15**

Definition, Properties of Logarithms, Application of Logarithms.

UNIT 7: Binomial Theorm **Page : 16-31**

Binomial Theorem for any positive Integer, General and Middle Terms

UNIT 8: Matrices **Page : 32-100**

Definition of Matrix and Examples, Types of matrices,Transpose of a matrix,symmetric and skew-symmetric matrix,algebra of matrices:Addition of matrices,scalar multiplication,subtraction of matrices,multiplication of matrices: Adjoint and inverse of a matrix and its existence, Rank of a matrix, Solution of a system of linear equations by matrix method, Solution of a system of linear equations by Crammer's rule.

UNIT 9: Limits and continuity **Page : 101-114**

Limit of a function, Continuity of a function.

UNIT 6: LOGARITHM

UNIT STRUCTURE

6.1 Learning Objectives

6.2 Introduction

6.3 Logarithms

- 6.3.1 Properties of Logarithm
- 6.3.2 Common Logarithm
- 6.3.3 Antilogarithm
- 6.3.4 Application of Logarithm

6.4 Let Us Sum Up

6.5 Further Readings

6.6 Answer To check Your Progress

6.7 Model Questions

6.1 LEARNING OBJECTIVES

After going through this unit you will be able to:

- define logarithm
- explain the properties of logarithms
- explain common logarithm

6.2 INTRODUCTION

Logarithms play an important role in numerical calculations. In this unit, we are going to discuss about logarithm and its properties. Here we shall discuss about the laws of logarithms and their applications. In statistical work, they are particularly useful while constructing ratio graphs, in computing geometric mean, in finding powers and extracting roots and fitting trend lines. Thus through this unit you will able to understand the importance of logarithms and its application in different field.

6.3 LOGARITHMS

The logarithm of a given number to a certain base is the power to

which the base must be raised in order to obtain that number. Thus, the logarithms or simply 'log' of 15625 to base 5 is 6. It means that if 5 is the base, raised to the power 6, we shall get the required number i.e. 5x5x5x5x5x5=15625.

Let *a* be a positive real number other than 1. Then for any real number *x*, a^x is a positive real number. Let

$$a^x = N$$

Here we say that *a* is the base and *x* is the index of the power of *a*. The index *x* is called the logarithm of *N* with respect to the base *a*, and we write $x = \log_a N$.

Thus, if $a^x = N$ ($a > 0, a \neq 1$), then $\log_a N = x$,

or if $\log_a N = x$, then $a^x = N$.

Examples:

(i) $2^3 = 8$, hence $\log_2 8 = 3$. (ii) $3^{-2} = \frac{1}{9}$, hence $\log_3(\frac{1}{9}) = -2$.

(iii) $(125)^{\frac{1}{3}} = 5$, hence $\log_{125} 5 = \frac{1}{3}$ etc.

Remark: Logarithm of any negative number is not defined. Because, for any $a > 0, a \neq 1$, we can not find any rational number *x* such that a^x is negative.

Special Cases: Let $a > 0$ and $a \neq 1$.

1. We have $a^0 = 1$, hence $\log_a 1 = 0$.

2. Since $a^1 = a$, we have $\log_a a = 1$.

Exercise 9: Find $\log_a N$, where $a = .01$ and $N = .0001$.

Solution: Let $\log_a N = x$. Then $a^x = N$.

Thus, $\log_{.01}(.0001) = 2$.

6.3.1 Properties of Logarithms

1) $\log_a(mn) = \log_a m + \log_a n, \quad (a > 0, a \neq 1, m, n > 0)$

Proof: Let $\log_a m = x$ and $\log_a n = y$. Hence, $a^x = m, \quad a^y = n$.

$$\Rightarrow mn = a^x a^y = a^{x+y} \Rightarrow \log_a(mn) = x + y$$

$$\Rightarrow \log_a(mn) = \log_a m + \log_a n$$

Hence, the logarithm of the product of two numbers is equal to the sum of their logarithms.

Examples: (i) $\log_3 35 = \log_3(5\times 7) = \log_3 5 + \log_3 7$

(ii) $\log_{\sqrt{2}} 57 = \log_{\sqrt{2}}(3\times 19) = \log_{\sqrt{2}} 3 + \log_{\sqrt{2}} 19$

Remark: We can extend this property to any product of finite numbers. For example,

$$\log_2 105 = \log_2(3\times(7\times 5)) = \log_2 3 + \log_2(7\times 5$$
$$= \log_2 3 + \log_2 7 + \log_2 5$$

2) $\log_a(\frac{m}{n}) = \log_a m - \log_a n, \quad (a>0, a\neq 1, m, n>0)$

Proof: Let $\log_a m = x$ and $\log_a n = y$. Hence, $a^x = m,\ a^y = n$.

Now, $\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$. Hence, $\log_a(\frac{m}{n}) = x - y = \log_a m - \log_a n$.

Examples:

(i) $\log_2(\frac{7}{5}) = \log_2 7 - \log_2 5$

(ii) $\log_2(\frac{14}{5}) = \log_2 14 - \log_2 5 = \log_2 2 + \log_2 7 - \log_2 5$

(iii) $\log_a(\frac{p.q}{r}) = \log_a p + \log_a q - \log_a r$

3) $\log_a(m^p) = p\log_a m, \quad (a>0, a\neq 1, m>0)$

Proof: Let $\log_a m = x$. Hence, $a^x = m$.

Now, $m^p = (a^x)^p = a^{px} \Rightarrow \log_a(m^p) = px = p\log_a m$.

Examples: (i) $\log_5 2^3 = 3\log_5 2$ (ii) $\log_2 5^{\frac{1}{2}} = \frac{1}{2}\log_2 5$, etc.

4) (Change of base):

$$\log_a m = \log_b m \times \log_a b, \quad (a, b>0, a\neq 1, b\neq 1, m>0)$$

Proof: Let , $\log_a m = x$, $\log_b m = y$ and $\log_a b = z$.

Then, $a^x = m, \quad b^y = m,$ and $a^z = b$. This implies $a^x = b^y$.

Now, $a^x = b^y \Rightarrow a^x = (a^z)^y = a^{zy} \Rightarrow x = zy, \quad (\mathbf{Q}\, b = a^z)$.

This implies that $\log_a m = \log_b m \times \log_a b$.

Remark: If we put *a* for *m* in both sides of

$\log_a m = \log_b m \times \log_a b$, we get $\log_a a = \log_b a \times \log_a b$.

$\Rightarrow \log_b a \times \log_a b = 1 \Rightarrow \log_b a = \dfrac{1}{\log_a b}$

Examples: (i) $\log_3 5 = \log_2 5 \times \log_3 2$ (ii) $\log_7 25 = \log_3 25 \times \log_7 3$.

5) $a^{\log_a m} = m, \quad (a > 0, a \neq 1, m > 0)$.

Proof: Let $a^{\log_a m} = x$. Then,

$\log_a x = \log_a m \Rightarrow x = m \Rightarrow a^{\log_a m} = m$.

Corollary: $a^{x \log_a m} = a^{\log_a m^x} = m^x$.

If $x = -1$, $a^{-\log_a m} = a^{\log_a m^{-1}} = m^{-1} = \dfrac{1}{m}$.

Examples: (i) $2^{\log_2 3} = 3$ (ii) $5^{\frac{1}{2}\log_5 3} = \sqrt{3}$.

Exercise 10: Prove that $\log_3(\log_2(\log_2 256)) = 1$.

Solution: Since $256 = 16 \times 16 = 2^8$, we have

$$\log_3(\log_2(\log_2 256)) = \log_3(\log_2(8\log_2 2)) = \log_3(\log_2 8) = \log_3(\log_2 2^3)$$
$$= \log_3(3\log_2 2) = \log_3 3 = 1$$

Hence the result.

Exercise 11: Simplify $\dfrac{\log\sqrt{27} + \log 8 - \log\sqrt{1000}}{\log 1.2}$, where the base of all the logarithms are same.

Solution: $\dfrac{\log\sqrt{27} + \log 8 - \log\sqrt{1000}}{\log 1.2} = \dfrac{\log 3^{\frac{3}{2}} + \log 2^3 - \log 10^{\frac{3}{2}}}{\log 1.2}$

$$= \frac{\frac{3}{2}\log 3 + 3\log 2 - \frac{3}{2}\log 10}{\log 1.2} = \frac{\frac{3}{2}(\log 3 + 2\log 2 - \log 10)}{\log 1.2}$$

$$= \frac{3}{2}\frac{\log\frac{3\times 4}{10}}{\log 1.2} = \frac{3}{2}\frac{\log 1.2}{\log 1.2} = \frac{3}{2}$$

Hence the required answer is $\dfrac{3}{2}$.

Exercise 12: If $\dfrac{\log a}{b-c} = \dfrac{\log b}{c-a} = \dfrac{\log c}{a-b}$, thus $a^a b^b c^c = 1$

Solution: Let each fraction = K

log a = K (b-a)

log b= K (c-a)

log c= K (a- b)

Let, $A = a^a b^b c^c$

Log A = log $(a^a b^b c^c)$ = log a^a +log b^b+ log c^c= a log a + b log b +c log c

= a K (b-c) +b K (c-a) + c K (a-b) = K {ab- ac+ bc- ab+ ac- bc}

= 0 = log 1

Hence, A = 1 = $a^a b^b c^c$, Proved.

Exercise 13: Prove that

Solution: $\frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_c(abc)} = 1$

L.H.S = $\log_{abc} a + \log_{abc} b + \log_{abc} C = \log_{abc}$ (abc) = 1, proved



CHECK YOUR PROGRESS

Q 1: State whether the following statements are True (T) or False (F):

(i) $\log_3(\frac{1}{3}) = -1$ (T / F)

(ii) $\log_{.5}(\frac{1}{2}) = 1$ (T / F)

(iii) $\log_a a^y = y$ (T / F)

(iv) $\log_a(\frac{1}{a^m}) = m$ (T / F)

(v) $\log_a x < 1$ if $0 < x < 1$. (T / F)

Q 2: If $\log_a x > 1$ and $a > 1$, then which one of the following is true?

(i) $a > x$ (ii) $a < x$ (iii) $a = x$ (iv) $x = \frac{1}{a}$

Q 3: Fill up the blanks:

(i) $\log_6(\log_{\sqrt{3}} 27) =$ .....

(ii) $\log_2 \log_{\sqrt{2}} \log_3 81 =$ .....

(iii) If $\log_{10} 8.72 = 0.9404$, then $\log_{10} 8720 =$ .....

(iv) If $\log_a(mn) = \log_a(m+n)$, then

(v) If $x^2 + y^2 = 7xy$, and $\log_a(\frac{x+y}{3}) = m(\log_a x + \log_a y)$, then m = ...

6.3.2 Common Logarithms

We have learnt that any positive number other than 1 can be used as base in logarithms. In particular, the logarithm in which 10 is used as base is known as common logarithm. Generally, the base is not written in **common logarithm**. Of course, there is another particular logarithm known as **Natural or Napierian logarithm** in which the base is the irrational number. The value of e lies between 2 and 3. It should be noted that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \mathbf{L}$ where for any positive integer n, $n! = 1 \times 2 \times \mathbf{L} \times (n-1) \times n$. The approximate value of e is 2.7182. Also in Unit-3, we shall discuss this in more detail.

(a). COMPUTATION WITH COMMON LOGARITHMS:

From the definition, we have

$$10^0 = 1 \quad \Rightarrow \log_{10} 1 = 0$$
$$10^1 = 10 \quad \Rightarrow \log_{10} 10 = 1$$
$$10^2 = 100 \quad \Rightarrow \log_{10} 100 = 2$$
$$10^3 = 1000 \quad \Rightarrow \log_{10} 1000 = 3, \quad \mathbf{L}$$

As we have seen, the logarithm of a number lying between 1 and 10 lies between 0 and 1. Hence it is of the form:

0 + some positive proper decimal number.

For example, consider the number 7. Since $1 < 7 < 10$, $0 < \log 7 < 1$, and hence it is of the form

0 + some proper decimal number.

Similarly, the logarithm of a number lying between 10 and 100 will be of the form: 1 + some positive proper decimal number, and so on. Thus we have seen that common logarithm of a number greater than or equal to 1 has two parts: the first part is the integral part and the second part is the decimal part. The integral part is called the

Characteristic and the decimal part is called the Mantissa. If the number is an integral power of 10, then the **Mantissa** is always zero. Also, the characteristic of a number greater than or equal to 1 is the number of digits in the integral part of the number reduced by 1.

Again,

$$10^{0} = 1 \qquad \Rightarrow \log_{10} 1 = 0$$
$$10^{-1} = 0.1 \qquad \Rightarrow \log_{10} 0.1 = -1$$
$$10^{-2} = 0.01 \qquad \Rightarrow \log_{10} 0.01 = -2$$
$$10^{-3} = 0.0001 \qquad \Rightarrow \log_{10} 0.0001 = -3, \quad \mathbf{L}$$

As we have seen, the logarithm of a number lying between 0.1 and 1 lies between -1 and 0. Hence it is of the form:

-1 + some positive proper decimal number.

For example, consider the number 0.7. Since $0 < 0.7 < 1$, $-1 < \log 0.7 < 0$, and hence it is of the form -1 + some proper decimal number.

Similarly, the logarithm of a number lying between 0.01 and 0.1 will be of the form: -2 + some positive proper decimal number, and so on. Thus, the characteristic of the logarithm of a positive number less than 1 is a negative integer and it is one more than the number of zeros just after the decimal point. That is, if a positive number less than one 1 has *n* zeros just after the decimal point, then the characteristic of the logarithm of the number is *–(n* +1).

The mantissa of the logarithm of a number is obtained by using a standard **"Log -table".**

(b) Important steps to find characteristic of a positive number *N* :

(1) If $N > 1$, count the number of digits in the integral part of *N*, say it is *m*. Then the characteristic of $\log_{10} N$ is $(m-1)$.

(2) If $0 < N < 1$, count the number of zeros just after the decimal point, say it is . Then the characteristic of the logarithm of the number is $-(n+1)$.

Thus, characteristic of 234, 23.4, 2.34, 0.000234 are 2, 1, 0, - 4, respectively.

(c) Important Properties of Mantissa: The mantissa does not depend on the position of the decimal point in the number. That is mantissa of the numbers 234, 23.4, 2.34, 0.000234 are same. Let *N* be any positive number. If we multiply *N* by 10 or powers of 10, then the significant digits shall remain same in the same order, only the position of the decimal point will change. Similarly, when we divide *N* by 10 or by powers of 10, the position of the decimal point in will change. Let, $\log N = m + 0.abcd$, where $0.abcd$ represents a decimal number. Now,

$$\log(N \times 10^p) = \log N + \log 10^p = m + 0.abcd + p$$
$$= (m + p) + 0.abcd \qquad \mathbf{L}\,(1)$$

Again,

$$\log\left(\frac{N}{10^p}\right) = \log N - \log 10^p = m + 0.abcd - p$$
$$= (m - p) + 0.abcd \qquad \mathbf{L}\,(2)$$

Thus in both the cases, mantissa remains same, only the characteristic changes.

Exercise 12: If the mantissa of log 2 is 0.3010 and mantissa of log23 is 0.3617, find the following logarithms. (You should find these mantissa values using the standard Log-table.)

(i) log 2 (ii) log 200 (iii) log 0.0002.

Solution: (i) log 2 = 0 + 0.3010 = 0.3010.

(ii) log 200 = 2 + 0.3010 = 2.3010.

(iii) log 0.0002 = - 4 + 0.3010 and we denote it by $\bar{4}.3010$.



CHECK YOUR PROGRESS

Q 4: Fill up the blanks.

(i) Characteristic of the logarithm of 0.00023 is .....

(ii) If log 32 = 1.5051, then log 32000 = ...

(iii) If log 0.000672 = $\bar{4}.8274$, that is, -4 + 0.8274, then log 6720 = ...

Q 5: If the mantissa of log 2 is 0.3010 and mantissa of log23 is 0.3617, find the following logarithms: (i) log 230 (ii) log 46

6.3.3 Antilogarithm

In many cases we require to find the number of which logarithm is given. If $\log x = m$, then x is the antilogarithm of m and we write $anti\log m = x$. In the antilog-table, the number corresponding to the mantissa of the logarithm of a number is given. The process of finding the number from the antilog-table corresponding to the given mantissa is similar to that of finding the mantissa from the log-table.

6.3.4 Application of Logarithms

Exercise 13: Find the number of digits of 5^{25}, if log 2 = 0.3010.

Solution: First we find the characteristic of log 5^{25}. Now,

$$\log 5^{25} = 25\log 5 = 25\log\frac{10}{2} = 25\times(\log 10 - \log 2)$$
$$= 25\times(1-0.3010) = 17.475.$$

Hence the characteristic is 17 and this implies that the number of digits of 5^{25} is 17 + 1 = 18.

Exercise 14: Find the compound interest on Rs.5000.00 for 2 years and 6 months at 6% compound interest per annum, interest being compounded half-yearly. Given $\log 2 = 0.3010$, $\log 103 = 2.0128$ and $\log 5.794 = 0.7630$.

Solution: We know that the formula for finding amount (A) on a given principal (P) at r% compound interest per annum for n years is given by $A = P\times\left(1+\frac{r}{100}\right)^n$ and for half-yearly calculation of interest the formula is: $A = P\times\left(1+\frac{r/2}{100}\right)^{2n}$.

Here, P = Rs.5000.00, r = 6, n = 2.5 years and hence

$$A = 5000 \times \left(1 + \frac{6/2}{100}\right)^{2\times 2.5}$$

$$\Rightarrow \log A = \log 5000 + 5 \times \log(1.03) = \log 5 + \log 1000 + 5 \times 0.0128$$

$$= (1 - 0.3010) + 3 + 0.0640 = 3.7630.$$

$\Rightarrow A = anti \log 3.7630 = 5794$ (Find this value from antilog-table).

Thus the compound interest = Rs.5794.00- Rs.5000.00 = Rs. 794.00.



6.4 LET US SUM UP

We defined the real number *x* as the logarithm of a^x, where *a* was a positive real number other than 1. We then derived several properties of logarithms. We have learnt about logarithms with base 10 known as common logarithms. Common logarithm of a number greater than or equal to 1 has two parts: the first part is the integral part and the second part is the decimal part. The integral part is called the **Characteristic** and the decimal part is called the **Mantissa**. Characteristic of a number can be found easily, but the one requires log-table to find Mantissa.



6.5 FURTHER READING

1. Khanna V. K., Zameeruddin Qazi & Bhambri S.K.(1995). Business Mathematics, New Delhi, Vikas Publishing House Pvt Ltd. .
2. Hazarika P.L. Business Mathematics, New Delhi. S.Chand & Co.



6.6 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1:

(i) T, (ii) F, (iii) T, (iv) F, (v) T.

Ans to Q No 2: (ii) $a < x$.

Ans to Q No 3: (i) 1, (ii) 2, (iii) 3.9404, (iv) $\frac{n}{n-1}$, (v) $\frac{1}{2}$.

Ans to Q No 4: (i) -4, (ii) 4.5051, (iii) 3.8274.

Ans to Q No 5: (i) =2.3617, (ii) 1.6627.



6.7 MODEL QUESTIONS

1. If $a = b^x = c^y = d^z$, prove that $\log_a(abcd) = 1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.
2. Solve $\frac{\log(1+x)}{\log x} = 2$.
3. Prove that $\log_5 2$ is irrational.
4. Find the values of $\log_{\sqrt{2}}(1/64)$ and $\log_3 \log_2 \log_{\sqrt{3}} 81$.
5. If $\log_{10} 8.72 = 0.9405$, find the value of $\log_{10} 87200$.
6. Solve
   (i) $\log_2(2x+1) - \log_2(3x-1) = 2$
   (ii) $\log_{10}(3x+2) - \log_{10}(x-1) = 1$.
   (iii) $(\log a)^2 - (\log b)^2 = (\log ab).\log$
   (iv) $\log\frac{81}{8} - 2\log 3 \log\frac{2}{3} + \log\frac{3}{4} = 0$
   (v) $\log 2 + 16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80} = 1$
7. If $a^{2x-3}.b^{2x} = a^{6-x}.b^{5x}$, prove that $3\log a = x\log\frac{a}{b}$
8. Show that, $x^{\log y - \log z}.y^{\log z - \log x}.z^{\log x - \log y} = 1$.
9. Find x if $\log_{10} x + \log_{\sqrt{10}} x + \log_{\sqrt[3]{10}} x = 27$
10. Simplify $\log_a b.\log_b c.\log_c d.\log_d a$

*** ***** ***

UNIT 7: BINOMIAL THEOREM

UNIT STRUCTURE

7.1 Learning Objectives

7.2 Introduction

7.3 Binomial Theorem

- 7.3.1 Pre-Requisites
- 7.3.2 Statement of Binomial Theorem
- 7.3.3 Some Particular Forms of the Binomial Theorem
- 7.3.4 Middle Term (Middle Terms)

7.4 Properties of Binomial Coefficients

- 7.4.1 Greatest Coefficients
- 7.4.2 Greatest Term

7.5 Let Us Sum Up

7.6 Further Readings

7.7 Answers To Check Your Progress

7.8 Model Questions

7.1 LEARNING OBJECTIVES

After going through this unit you will be able to:

- discuss Binomial Theorem and Binomial coefficients.
- discuss the properties of Binomial coefficients.
- explain the middle terms and equidistant terms and how to find them in a given binomial expansion.
- determine greatest coefficient and greatest term in a given binomial expansion.

7.2 INTRODUCTION

We are already familiar with algebraic expressions which are very common in mathematics. An algebraic expression having two terms is called a binomial expression. One can easily find the square or cube of binomial expressions like $a+b$ or $a-b$. But it will be difficult to expand $(a+b)^7$ by

repeated multiplication. Binomial theorem gives an easier way to expand $(a+b)^n$, where n is an integer or a rational number. In this unit, *we* shall consider n to be always a positive integer. In mathematics, the binomial coefficient ${}^nC_k$ is the coefficient of the x^k term in the polynomial expansion of the binomial power $(1 + x)^n$. In combinatorics, ${}^nC_k$ is interpreted as the number of k-element subsets of an n-element set, that is the number of ways that k things can be 'chosen' from a set of n things.

7.3 BINOMIAL THEOREM

We know that for two real numbers *a* and *b* ,

$$(a+b)^0 = 1$$
$$(a+b)^1 = a^1 + b^1$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = (a+b)(a^3 + 3a^2b + 3ab^2 + b^3)$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4, \mathbf{L}$$

We observe that the powers of a go on decreasing by 1 and the powers of b go on increasing by 1 in the successive terms. We arrange the co-efficients in the above expressions as follows, and the diagram is called the Pascal's triangle.

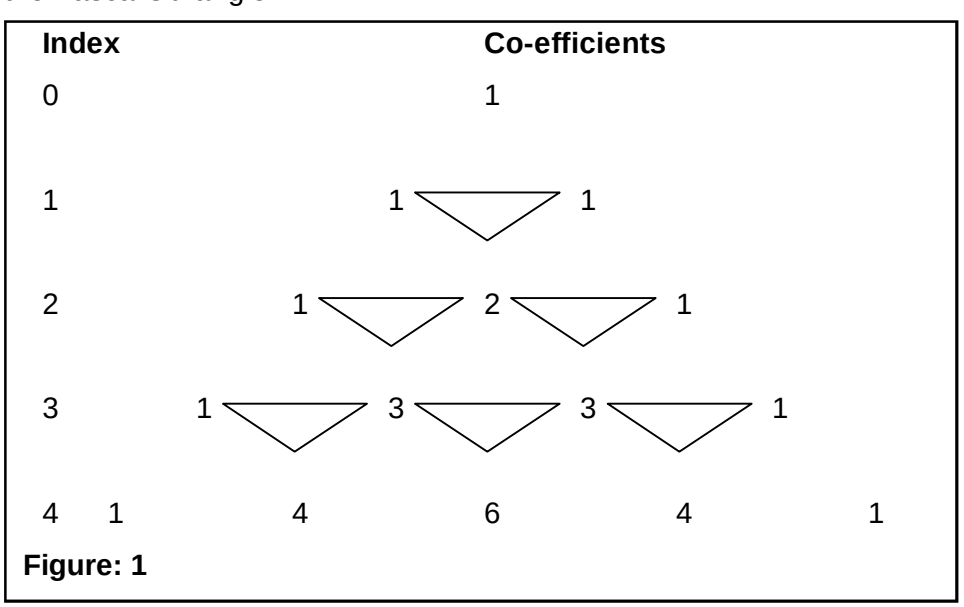
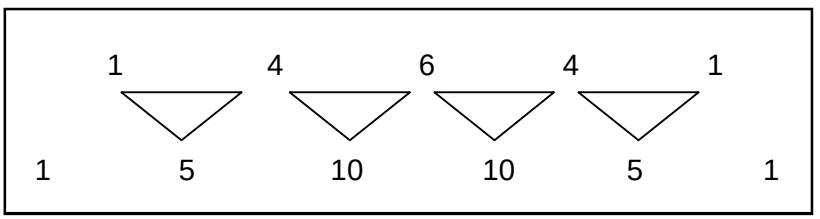


Figure: 1

We have seen that for $n = 0, 1, 2, 3, 4$ there are $n+1$ terms in the expansion of $(a+b)^n$. Also, the co-efficients of a^n and b^n are equal to 1 for all n. The intermediate number 2 in 3rd row in Fig-1 is the sum of the two numbers 1 and 1 of the preceding row which has been shown by the triangle. Thus using Fig-1, we can find the co-efficients in the expansion of $(a+b)^5$ as follows:



Hence, $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

The expansion with the help of Pascal's triangle becomes lengthy. Now, we discuss about Binomial theorem which makes the expansion easier.

7.3.1 Pre-requisites

(i) $^nC_r = \dfrac{n!}{r!(n-r)!} = \dfrac{1\times 2\times \mathbf{L}\times (n-1)\times n}{(1\times 2\times \mathbf{L}\times r)(1\times 2\times \mathbf{L}\times (n-r))}.$

Hence, $^nC_0 = 1, \quad ^nC_1 = n, \quad ^nC_2 = \dfrac{n(n-1)}{2}, \mathbf{L}, \ ^nC_n = 1.$

(ii) $^nC_r = {}^nC_{n-r}$, (iii) $^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$.

7.3.2 Statement of Binomial Theorem

If n is a positive integer, then for all values of a and x,

$$(a+x)^n = {}^nC_0a^n + {}^nC_1a^{n-1}x + \mathbf{L} + {}^nC_ra^{n-r}x^r + \mathbf{L} + {}^nC_{n-1}ax^{n-1} + {}^nC_nx^n$$

Proof: For $n=1$, L.H.S. = $(a+x)^1 = a+x$ and R.H.S. = $a+x$. Hence the theorem is true for $n=1$.

For $n=2$, L.H.S. = $(a+x)^2 = a^2 + 2ax + x^2$ = R.H.S. Hence the theorem is also true for $n=2$. From this we can assume that the theorem may be true for $n=m$. Then,

$$(a+x)^m = {}^mC_0 a^m + {}^mC_1 a^{m-1}x + \mathbf{L} + {}^mC_r a^{m-r}x^r + \mathbf{L} + {}^mC_{m-1}ax^{m-1} + {}^mC_m x^m.$$

Multiplying both sides by *(a+x)*, we get

$$\begin{aligned}(a+x)^{m+1} = (a+x)(a+x)^m &= (a+x)[{}^mC_0 a^m + {}^mC_1 a^{m-1}x + \mathbf{L} \\ &\quad + {}^mC_r a^{m-r}x^r + \mathbf{L} + {}^mC_{m-1}ax^{m-1} + {}^mC_m x^m] \\ &= [{}^mC_0 a^{m+1} + {}^mC_1 a^m x + \mathbf{L} + {}^mC_r a^{m+1-r}x^r + \mathbf{L} + {}^mC_m ax^m] \\ &\quad + [{}^mC_0 a^m x + {}^mC_1 a^{m-1}x^2 + \mathbf{L} + {}^mC_r a^{m-r}x^{r+1} + \mathbf{L} + {}^mC_m x^{m+1}] \\ &= {}^{m+1}C_0 a^{m+1} + ({}^mC_1 + {}^mC_0)a^m x + \mathbf{L} + ({}^mC_r + {}^mC_{r-1})a^{m-r}x^r \\ &\quad + \mathbf{L} + {}^{m+1}C_{m+1}x^{m+1}, \quad [\mathbf{Q}\ {}^mC_0 = {}^{m+1}C_0 = 1,\ {}^mC_m = {}^{m+1}C_{m+1} = 1] \\ &= {}^{m+1}C_0 a^{m+1} + {}^{m+1}C_1 a^{(m+1)-1}x + \mathbf{L} + {}^{m+1}C_r a^{(m+1)-r}x^r + \mathbf{L} \\ &\quad + {}^{m+1}C_{m+1}x^{m+1}, [\mathbf{Q}\ {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r].\end{aligned}$$

This shows that we can get the expansion of $(a+x)^{m+1}$ by writing $m+1$ for n in the Binomial theorem. Therefore the theorem is true for $n = m+1$ if we assume that the theorem is true for $n = m$. But the theorem is true for $n = m = 1$ and $n = m = 2$. Hence the theorem is true for $n = 2+1 = 3$. Since it is true for 3, it is also true for 4, and so on. From this we conclude that the theorem is true for any positive integer n. This completes the proof of the theorem.

Remark: In the expansion of $(a+x)^n$, we see that

1. total number of terms is $n+1$,
2. degree of each term is n, and
3. the coefficients ${}^nC_0, {}^nC_1, \mathbf{L}, {}^nC_n$ are called **Binomial coefficients** and they are written in short as $C_0, C_1, \mathbf{L}, C_n$

General Term: In the expansion of $(a+x)^n$, ${}^nC_r a^{n-r}x^r$ occurs as $(r+1)$ th term. This term is known as general term and it is denoted by T_{r+1}. Hence,

$$T_{r+1} = {}^nC_r a^{n-r}x^r$$

Putting $r = 0, 1, 2, \mathbf{L}$ we get T_1, T_2, T_3, ... etc.

Equidistant Terms: In the expansion of $(a+x)^n$, the $(r+1)$ th term from the beginning is ${}^nC_r a^{n-r}x^r$. As the $(r+1)$ th term from the

end is the $(n-r+1)$ th term from the beginning and this term is ${}^{n}C_{n-r}\, a^{n-(n-r)}x^{n-r}$ which is nothing but ${}^{n}C_{n-r}\, a^{r}x^{n-r}$.

But we know that ${}^{n}C_{r}={}^{n}C_{n-r}$. Hence the binomial coefficients of the terms equidistant from the beginning and from the end are equal.

7.3.3 Some Particular Forms of the Binomial theorem

(1) Writing $-x$ for x in $(a+x)^n$ and its expansion, we find

$$\begin{aligned}(a-x)^n &= {}^{n}C_0\, a^n + {}^{n}C_1\, a^{n-1}(-x) + \mathbf{L} + {}^{n}C_r\, a^{n-r}(-x)^r + \mathbf{L} \\ &\quad + {}^{n}C_{n-1}\, a(-x)^{n-1} + {}^{n}C_n(-x)^n \\ &= a^n - {}^{n}C_1\, a^{n-1}x + {}^{n}C_2 a^{n-2}x^2 - \mathbf{L} + (-1)^r\, {}^{n}C_r a^{n-r}x^r \\ &\quad + \mathbf{L} + (-1)^n x^n.\end{aligned}$$

Hence the general term: $T_{r+1} = (-1)^r\ {}^{n}C_r\, a^{n-r}x^r$.

(2) $(1+x)^n = {}^{n}C_0 + {}^{n}C_1 x + \mathbf{L} + {}^{n}C_r x^r + \mathbf{L} + {}^{n}C_{n-1}x^{n-1} + {}^{n}C_n x^n$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \mathbf{L} + x^n$$

Hence the general term: $T_{r+1} = {}^{n}C_r\, x^r$.

(3) $(1-x)^n = {}^{n}C_0 - {}^{n}C_1 x + \mathbf{L} + (-1)^r\ {}^{n}C_r x^r + \mathbf{L} + (-1)^n\ {}^{n}C_n x^n$

$$= 1 - nx + \frac{n(n-1)}{2!}x^2 + \mathbf{L} + (-1)^n x^n$$

Hence the general term: $T_{r+1} = (-1)^r\ {}^{n}C_r\, x^r$.

Exercise 3.1: Expand $\left(\frac{3}{x}+\frac{x}{3}\right)^6$.

Solution:

$$\begin{aligned}\left(\frac{3}{x}+\frac{x}{3}\right)^6 &= \left(\frac{3}{x}\right)^6 + {}^{6}C_1\left(\frac{3}{x}\right)^5\left(\frac{x}{3}\right) + {}^{6}C_2\left(\frac{3}{x}\right)^4\left(\frac{x}{3}\right)^2 + {}^{6}C_3\left(\frac{3}{x}\right)^3\left(\frac{x}{3}\right)^3 \\ &\quad + {}^{6}C_4\left(\frac{3}{x}\right)^2\left(\frac{x}{3}\right)^4 + {}^{6}C_5\left(\frac{3}{x}\right)\left(\frac{x}{3}\right)^5 + {}^{6}C_6\left(\frac{x}{3}\right)^6 \\ &= \frac{729}{x^6} + \frac{486}{x^4} + \frac{135}{x^2} + 20 + \frac{5x^2}{3} + \frac{2x^4}{27} + \frac{x^6}{729}.\end{aligned}$$

Exercise 3.2: Find the 9th term in the expansion of $(2x-3y)^{12}$.

Solution: Here *n =12* and the 9th term is:

$$T_9 = T_{8+1} = {}^{12}C_8 (2x)^{12-8}(-3y)^8 = {}^{12}C_4 2^4 x^4 3^8 y^8 = 51963120x^4y^8$$

Exercise 3.3: Find the co-efficient of x^5 in the expansion of $(2-3x)^8$.

Solution: Here $n=8$ and the $(r+1)$ th term is:

$$T_{r+1} = {}^8C_r\, 2^{8-r}(-3x)^r = (-3)^r\ {}^8C_r 2^{8-r} x^r$$

If this term has x^5 as a factor, then r must be 5. Thus, the co-efficient of x^5 in the expansion of $(2-3x)^8$ is

$${}^8C_5\, 2^{8-5}(-3)^5 = -\frac{8\times7\times6}{1\times2\times3}\times2^3\times3^5 = -108864 \quad [\mathbf{Q}\ {}^8C_5 = {}^8C_3]$$

Exercise 3.4: If n is a natural number, using binomial theorem prove that $4^n - 3n - 1$ is divisible by 9.

Solution: If $n=1$, then $4^n - 3n - 1 = 0$ and hence divisible by 9. For $n \geq 2$, using binomial theorem, we get

$$\begin{aligned} 4^n &= (1+3)^n = 1 + {}^nC_1\times3 + {}^nC_2\times3^2 + \mathbf{L} + {}^nC_{n-1}\times3^{n-1} + 3^n \\ &= 1 + 3n + 3^2\times\left[{}^nC_2 + {}^nC_3\times3 + \mathbf{L} + 3^{n-2}\right] \\ &= 1 + 3n + 9k \end{aligned}$$

where *k* is an integer. This implies that $4^n - 3n - 1$ is divisible by 9.

7.3.4 Middle term (Middle terms)

If the number of terms in the expansion of $(a+x)^n$ is odd then there is exactly one middle term, but the expansion will have two middle terms if the number of terms is even.

Case 1: When n is even, say $n=2m$, then the number of terms is equal to $2m+1$ which is odd. Hence the middle term is the $(m+1)$ th term, that is, $(\frac{n}{2}+1)$ th term.

Case 2: If *n* is odd, say $n=2m+1$, then the number of terms is equal to *2m +2* which is even. Hence there are two middle terms and these are $(m+1)$ th term and *(m+2)*th term. Putting $m=\frac{n-1}{2}$,

we get the middle terms as $\frac{n+1}{2}$ th term and $\frac{n+3}{2}$ th term.

Exercise 3.5: Find the middle term in the expansion of $\left(x-\frac{1}{2x}\right)^8$.

Solution: Here $n = 8$ and hence there is one middle term in the expansion which is T_5. Hence $T_5 = {}^8C_4x^4(-\frac{1}{2x})^4 = \frac{35}{8}$.

Exercise 3.6: Find the middle terms in the expansion of $\left(2x^2+\frac{1}{x}\right)^7$.

Solution: Here $n = 7$ and hence there are two middle terms which are T_4 and T_5. Now,

$$T_4 = {}^7C_3(2x^2)^4\left(\frac{1}{x}\right)^3 = 560x^5, \quad T_5 = {}^7C_4(2x^2)^3\left(\frac{1}{x}\right)^4 = 280x^2.$$



CHECK YOUR PROGRESS

Q 1: Fill up the blanks.

i. The number of terms in the expansion of $(a+x)^n$ = ...

ii. The 6th term in the expansion of $(x-1)^7$ = ...

iii. The coefficient of x^6 in $(x+2)^9$ = ...

iv. The coefficient of middle term in the expansion of $(1+x)^{10}$ is .....

Q 2: Write True (T) or False (F):

i. If n is even, then there is one middle term in $(a+x)^n$. (T / F)

ii. If n is odd, then there are two middle terms in $(a+x)^n$. (T / F)

iii. If there is a term in $\left(x-\frac{1}{x}\right)^n$ that is independent of x, then n is even. (T / F)

7.4 PROPERTIES OF BINOMIAL COEFFICIENTS

Following are some basic properties of binomial coefficients.

(1) Sum of the binomial coefficients is equal to 2^n, that is,

$$C_0 + C_1 + \mathbf{L} + C_r + \mathbf{L} + C_{n-1} + C_n = 2^n.$$

Proof: Let us consider the expansion:

$$(1+x)^n = C_0 + C_1 x + \mathbf{L} + C_r x^r + \mathbf{L} + C_{n-1} x^{n-1} + C_n x^n .$$

Putting 1 for *x* in both sides, we get

$$2^n = C_0 + C_1 + \mathbf{L} + C_r + \mathbf{L} + C_{n-1} + C_n$$

Example: The sum of the binomial coefficients in the expansion of $(1+x)^6$ is $2^6 = 64.$

(2) The sum of the binomial coefficients of the odd terms is equal to the sum of the binomial coefficients of the even terms and each is equal to $2^{n-1}.$

Proof: We have,

$$(1+x)^n = C_0 + C_1 x + \mathbf{L} + C_r x^r + \mathbf{L} + C_{n-1} x^{n-1} + C_n x^n$$

Putting –1 for *x* in both sides, we get

$$\begin{aligned} 0 &= {}^nC_0 - {}^nC_1 + {}^nC_2 - \mathbf{L} + (-1)^n \, {}^nC_n x^n \\ &= (C_0 + C_2 + C_4 + \mathbf{L}) - (C_1 + C_3 + C_5 + \mathbf{L}) \\ \Rightarrow\ & C_0 + C_2 + C_4 + \mathbf{L} = C_1 + C_3 + C_5 + \mathbf{L} \end{aligned}$$

Suppose that $C_0 + C_2 + C_4 + \mathbf{L} = C_1 + C_3 + C_5 + \mathbf{L} = k$. Then,

$2k = (C_0 + C_2 + C_4 + \mathbf{L}) + (C_1 + C_3 + C_5 + \mathbf{L})$

$\Rightarrow 2k = C_0 + C_1 + C_2 + \mathbf{L} + C_n = 2^n \Rightarrow k = 2^{n-1}$, [Using property (1)].

Hence, $C_0 + C_2 + C_4 + \mathbf{L} = C_1 + C_3 + C_5 + \mathbf{L} = 2^{n-1}.$

Exercise 3.7: Find the value of ${}^5C_0 + {}^5C_2 + {}^5C_4$.

Solution: ${}^5C_0 + {}^5C_2 + {}^5C_4$ is the sum of the binomial coefficients of the even terms of the expansion of $(1+x)^5$ and hence equal to $2^4 = 16$.

Exercise 3.8: Prove that

(1) $C_1 + 2 \times C_2 + 3 \times C_3 + \mathbf{L} + n \times C_n = n \times 2^{n-1}.$

(2) $\dfrac{C_0}{1} + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \mathbf{L} + \dfrac{C_n}{n+1} = \dfrac{1}{n+1}(2^{n+1} - 1).$

Solution: (1) L.H.S. $= n + 2\times\frac{n(n-1)}{1\times 2} + 3\times\frac{n(n-1)(n-2)}{1\times 2\times 3} + \mathbf{L} + n\times 1$

$$= n + n\times(n-1) + \frac{n(n-1)(n-2)}{1\times 2} + \mathbf{L} + n\times 1$$

$$= n\times\left\{1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \mathbf{L} + 1\right\}$$

$$= n\times\left\{{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \mathbf{L} + {}^{n-1}C_{n-1}\right\} = n\times 2^{n-1}$$

= R.H.S.

(2) L.H.S. $= \frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} + \mathbf{L} + \frac{C_n}{n+1}$

$$= 1 + \frac{n}{2} + \frac{n\times(n-1)}{2\times 3} + \mathbf{L} + \frac{1}{n+1}$$

$$= \frac{1}{n+1}\times\left\{(n+1) + \frac{(n+1)n}{2!} + \frac{(n+1)n(n-1)}{3!} + \mathbf{L} + 1\right\}$$

$$= \frac{1}{n+1}\times\left\{{}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \mathbf{L} + {}^{n+1}C_{n+1}\right\}$$

$$= \frac{1}{n+1}\times\left\{{}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \mathbf{L} + {}^{n+1}C_{n+1} - {}^{n+1}C_0\right\}$$

$$= \frac{1}{n+1}\left(2^{n+1} - 1\right), \quad [\mathbf{Q}\ {}^{n+1}C_0 = 1].$$

= R.H.S. Hence proved.

7.4.1 Greatest Coefficients

The binomial coefficients in the expansion of $(a+x)^n$ are ${}^nC_r$ where $r = 0, 1, \mathbf{L}, n$.

Case 1: If *n* is even, then ${}^nC_r$ has the greatest value for $r = \frac{n}{2}$.

Hence, if *n* is even, then the greatest coefficient is ${}^nC_{n/2}$.

Case 2: If *n* is odd, then ${}^nC_r$ has the greatest value for $r = \frac{n-1}{2}$ or for $r = \frac{n+1}{2}$. Hence, if *n* is odd, then the greatest coefficient is

${}^nC_{n-1/2}$ or ${}^nC_{n+1/2}$, both being equal.

Exercise 3.9: Find the greatest coefficient in the expansion of $(3x^2 + 4y)^{11}$.

Solution: Here $n = 11$ and this is odd. Hence the greatest coefficient is ${}^{11}C_{11-1/2} = {}^{11}C_5$ or ${}^{11}C_{11+1/2} = {}^{11}C_6$ and both are equal.

Now, ${}^{11}C_5 = \frac{11!}{5! \times 6!} = \frac{11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2} = 11 \times 7 \times 3 \times 2 = 462$, which is the required greatest coefficient.

7.4.2 Greatest Term

In the expansion of $(a + x)^n$ where $a, \quad x > 0$, we have

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r a^{n-r} x^r}{{}^nC_{r-1} a^{n-r+1} x^{r-1}} = \frac{n-r+1}{r} \times \frac{x}{a} \quad \text{L L } (3.1)$$

From equation (3.1), $T_{r+1} >=< T_r$ if $\frac{n-r+1}{r} \times \frac{x}{a} >=< 1$,

that is, if $\frac{(n+1)x}{ra} >=< 1 + \frac{x}{a}$,

that is, if $(n+1)x >=< r(a+x)$,

that is, if $r <=> \frac{(n+1)x}{a+x}$.

Thus,

$T_{r+1} > T_r$ when $r < \frac{(n+1)x}{a+x}$,

$T_{r+1} = T_r$ when $r = \frac{(n+1)x}{a+x}$

and $T_{r+1} < T_r$ when $r > \frac{(n+1)x}{a+x}$.

This shows that the terms go on increasing till $r < \frac{(n+1)x}{a+x}$ and start decreasing when *r* exceeds $\frac{(n+1)x}{a+x}$.

Case 1: If $\frac{(n+1)x}{a+x} = m$, an integer, then

$T_{r+1} > T_r$ for $r = 1, 2, \mathbf{L}, m-1$;

$T_{r+1} = T_r$ for $r = m$;

$T_{r+1} < T_r$ for $r > m$.

Hence the greatest terms are T_m and T_{m+1} both being the same.

Case 2: If $\frac{(n+1)x}{a+x} = k + f$, where *k* is an integer and *f* is a positive proper fraction, then

$T_{r+1} > T_r$ for $r = 1, 2, \mathbf{L}, k$;

$T_{r+1} < T_r$ for $r = k+1, k+2, \mathbf{L}, n$.

Hence the greatest term is T_{k+1}.

Remark: If $k = 0$, the T_1, that is, the 1st term is the greatest term.

(2) If $\frac{(n+1)x}{a+x} > n$, that is, $x > na$, then T_{n+1} is the greatest term which is nothing but the last term.

(3) Since the corresponding terms in the expansions of $(a+x)^n$ and $(a-x)^n$ are numerically equal, so the numerically greatest term in the expansion of $(a+x)^n$ is the same as that of $(a-x)^n$.

Exercise 3.10: Which term is the greatest in the expansion of $(2+3x)^{12}$ when $x = \frac{5}{6}$?

Solution: We have, $(2+3x)^{12} = 2^{12} \times \left(1 + \frac{3x}{2}\right)^{12}$.

In the expansion of $\left(1 + \frac{3x}{2}\right)^{12}$, *r* th term $T_r = {}^{12}C_{r-1}\left(\frac{3x}{2}\right)^{r-1}$ and the *r +1)*th term $T_{r+1} = {}^{12}C_r\left(\frac{3x}{2}\right)^r$

$\therefore \quad \frac{T_{r+1}}{T_r} = \frac{{}^{12}C_r}{{}^{12}C_{r-1}} \times \frac{3x}{2} = \frac{12! \times (r-1)! \times (12-r+1)!}{12! \times r! \times (12-r)!} \times \frac{3x}{2} = \frac{(13-r) \times 3x}{2r}$

When $x = \frac{5}{6}$, we get $\frac{T_{r+1}}{T_r} = \frac{5\times(13-r)}{4r}$. This implies that

$T_{r+1} \geq T_r$ if $\frac{5\times(13-r)}{4r} \geq 1$, that is, if $r \leq \frac{65}{9} = 7 + \frac{2}{9}$.

Thus, in this problem, *k* =7 and $f = \frac{2}{9}$, which is a positive proper fraction.

Hence the greatest term in the expansion of $\left(1+\frac{3x}{2}\right)^{12}$ is

$T_8 = {}^{12}C_7 \times \left(\frac{3x}{2}\right)^7 = {}^{12}C_7 \times \left(\frac{5}{4}\right)^7$ when $x = \frac{5}{6}$. This implies that the greatest term in the expansion of $(2+3x)^{12}$ when $x = \frac{5}{6}$ is equal to $2^{12} \times {}^{12}C_7 \times \left(\frac{5}{4}\right)^7$.



CHECK YOUR PROGRESS

Q 3: Write True (T) or False (F):

(i) If $m \in N$, then sum of the binomial coefficients in the expansion of $(1+x)^m$ is 2^m. (T / F)

(ii) ${}^7C_1 + {}^7C_3 + {}^7C_5 + {}^7C_7 = 64$. (T / F)

(iii) ${}^8C_3 + {}^8C_5 + {}^8C_7 = 120$. (T / F)

(iv) If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \mathbf{L} + a_{2n}x^{2n}$, then

$a_0 + a_1 + a_2 + \mathbf{L} + a_{2n} = 3^{2n}$. (T / F)

(v) If $(2-3x+x^2)^n = a_0 + a_1x + a_2x^2 + \mathbf{L} + a_{2n}x^{2n}$, then

$a_0 - a_1 + a_2 - \mathbf{L} + a_{2n} = 0$ (T / F)

(vi) If *n* is even, the greatest coefficient is ${}^nC_{\frac{n}{2}+1}$. (T / F)

(vii) If *n* is odd, the greatest coefficient is ${}^nC_{\frac{n+1}{2}}$. (T / F)

Q 4: Fill up the blanks:

(i) In the expansion of $(1+x)^{50}$, the sum of the coefficients of

odd powers of x is .....

(ii) In the expansion of $(1+x)^{100}$, the sum of the coefficients of even powers of x is .....

(iii) The greatest coefficient in the expansion of $(1+x)^{16}$ is equal to .....

(iv) In the expansion of $(a+x)^n$ where $a, \quad x>0$, the terms T_r go on increasing till .....

(v) In the expansion of $(a+x)^n$ where $a, \quad x>0$, the terms T_r go on decreasing when r exceeds .....



7.5 LET US SUM UP

1. We have learnt about the Binomial theorem on the expansion of $(a+x)^n$. Without writing the complete expansion of $(a+x)^n$, we know how to get any term of the expansion using the general term: $T_{r+1} = {}^nC_r\, a^{n-r} x^r$.
2. We have also learnt about middle terms. If n is even, then there is one middle term and if n is odd, then there are two middle terms in the expansion of $(a+x)^n$. If n is even, then $\left(\frac{n}{2}+1\right)$th term is the middle term. If n is odd, then $\left(\frac{n-1}{2}+1\right)$th and $\left(\frac{n-1}{2}+2\right)$th terms are the two middle terms.
3. We have discussed about binomial coefficients and also we derived certain nice properties of binomial coefficients.
4. If we consider the terms equidistant from the beginning and from the end, then we have noticed that the corresponding binomial coefficients are same.
5. If n is even, then ${}^nC_{n/2}$ is the greatest binomial coefficient in the

expansion of $(a+x)^n$. Similarly, if n is odd, then the greatest coefficient in the expansion of $(a+x)^n$ is ${}^nC_{n-1/2}$ or ${}^nC_{n+1/2}$, both being equal.

6. Finally we have learnt about the greatest term in the expansion of $(a+x)^n$ where $a,\quad x>0$. If $\frac{(n+1)x}{a+x}=m$ is an integer, then the greatest terms are T_m and T_{m+1} both being the same. Again if $\frac{(n+1)x}{a+x}=k+f$, where k is an integer and f is a positive proper fraction, then the greatest term is T_{k+1}.



7.6 FURTHER READINGS

1. Business Mathematics, Qazi Zameeruddin, V K Khanna, and S K Bhambri, Vikas Publishing House Pvt Ltd, Second Reprint, 1995.
2. Business Mathematics, Dr. P.L. Hazarika, S.Chand.



7.7 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (i) $n+1$ (ii) $-21x^2$ (iii) 672 (iv) 252

Ans to Q No 2: T, T, T.

Ans to Q No 3: T, T, T, F, F, F, T.

Ans to Q No 4: (i) 2^{49} (ii) 2^{99} (iii) ${}^{16}C_8$

(iv) $r<\frac{(n+1)x}{a+x}$ (v) $\frac{(n+1)x}{a+x}$



7.8 MODEL QUESTIONS

1. Find the value of the term which is independent of x in the expansion of $\left(2x+\frac{1}{3x^2}\right)^9$

2. Using Binomial theorem find the value of $(1.01)^5$.

3. Using Binomial theorem prove that $2^{3n}-7n-1$ is divisible by 49. [Hint: Expand $2^{3n}=8^n=(1+7)^n$]

4. Expand $\left(x^2+\frac{3}{x}\right)^6$ by using binomial theorem and write down the general term.

5. Show that the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is equal to the sum of the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$.

6. Find the coefficient of x^9 in the expansion of $\left(2x^2+\frac{1}{x}\right)^{20}$.

7. Find the coefficient of x^8 in the expansion of $(2+3x-4x^2)(1-x)^8$.

8. Find the coefficient of x^{-17} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$.

9. Find the middle term(s) in the expansion of $\left(2x-\frac{1}{y}\right)^8$ and $\left(x+\frac{1}{x}\right)^{21}$.

10. If three successive coefficients in the expansion of $(1+x)^n$ are 220, 495 and 792 respectively, find the value of n.

11. Using binomial theorem find the value of ${}^9C_0+{}^9C_2+{}^9C_4+{}^9C_6+{}^9C_8$.

12. If $(1+x)^n=C_0+C_1x+C_2x^2+\mathbf{L}+C_nx^n$, prove that

    (i) $C_0+2C_1+3C_2+\mathbf{L}+(n+1)C_n=(n+2)\times 2^{n-1}$.

    (ii) $C_0-\frac{C_1}{2}+\frac{C_2}{3}-\mathbf{L}+(-1)^n\frac{C_n}{n+1}=\frac{1}{n+1}$.

13. Prove that $(C_0 + C_1 + C_2 + \mathbf{L} + C_n)^2 = {}^{2n}C_0 + {}^{2n}C_1 + \mathbf{L} + {}^{2n}C_{2n}$.

14. In the expansion of $\left(\sqrt{x} - \frac{a}{x^2}\right)^{10}$, the term free from *x* is 405. Find the value of *a*.

15. Find the greatest term in the expansion of $\left(\frac{2a}{x} + \frac{3x}{a}\right)^6$ when $a = \frac{1}{2}$ and $x = \frac{1}{3}$.

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UNIT 8 : MATRICES

UNIT STRUCTURE

8.1 Learning objectives

8.2 Introduction

8.3 Definition of Matrix

8.4 Types of Matrix

8.5 Equality of Matrices

8.6 Addition and Subtraction of Matrices

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8.8 Multiplication of Matrices

8.9 Transpose of a Matrix

8.10 Symmetric and Skew-Symmetric Matrix

8.11 Adjoint and Inverse of Matrices and its existence

8.12 Rank of a Matrix

8.13 Solution of system of linear equations by matrix method

8.14 Solution of system of linear equations by Crammers rule.

8.15 Answers to check your progress

8.16 Further Readings

8.17 Model Questions

8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- understand the definition of a Matrix
- describe different types of Matrix
- learn matrix operations e.g. addition, subtraction and multiplication
- learn about adjoint, inverse and rank of Matrices
- solve system of linear equations by matrix method and crammer's rule

8.2 INTRODUCTION

Matrix is one of the most powerful tools of modern mathematics. Matrix notations and operations are used in Computer graphics, programming and implementation of electronic spreadsheet programmes. Matrix were introduced by the English Mathematician Arthur Cayley in 1858.Matrices are initially connected with linear transformation. Matrix theory now occupies important position in Physics, Economics, Statistics, Engineering etc.

8.3 DEFINITION OF MATRIX

A set of mn numbers (real or complex) arranged in the form of a rectangular array having m rows and n columns is called an mxn matrix (to be read as 'm' by 'n' matrix).

An m x n matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The compact representation of the above matrix is

$A = [a_{ij}]$, $i = 1, 2, \dots m$, $j = 1, 2, \dots n$.

or, simply, $A = [a_{ij}]_{mxn}$. Sometimes, the braket [ ] can be replaced by ( ) .

The general element of the matrix A is a_{ij} which belongs to the i-th row and j-th column. a_{ij} is sometimes denoted by (i,j)th element of the matrix. Here i, the first suffix denotes the number of row and j, the second suffix denotes the number of column in which the element a_{ij} occurs.

Note :

1. A matrix having m-rows and n-columns is called a matrix of order mn (read as 'm' by 'n') or simply m x n matrix.

2. In a matrix, the number of rows need not be equal to the number of columns.

3. A matrix is denoted by the capital letters A, B, C… etc. whereas any elements of a matrix is denoted by small letters such as $a_{ij,} b_{ij}, c_{ij,} \cdots\cdots$ etc.

Illustrative example :

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 5 \end{bmatrix}_{2\times3} \text{ or } \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 5 \end{pmatrix}_{2\times3}$$

is a 2 x3 matrix i.e. it has 2 rows and 3 columns. For A,

a_{12} = (1,2)th element = 2

a_{23} = (2,3)th element = 5

a_{13} = (1,3)th element = 0

$$B = \begin{bmatrix} 1+i & 2 & \frac{1}{3} \\ 4 & 0 & 1 \\ \sqrt{3} & 2-i & \sqrt{2} \end{bmatrix}_{3\times3}$$

is a 3x3 matrix. For B,

b_{11} = 1+ i, $b_{31} = \sqrt{3}$, b_{22} =0 etc.

8.4 TYPES OF MATRIX

There are different types of matrices. We will discuss one by one.

(i) Row Matrix : Any 1x n matrix which has only one row and n column is called a row matrix.

For example,

A = [2, 5,- 3, 0, $\sqrt{3}$] is a row matrix order 1×5.

In general, B = $[a_{ij}]_{1\times n}$ is a row matrix of order 1x n

(ii)**Column Matrix :** Any m x1 matrix which has m rows and only one column is called a column matrix.

For example,

$$B = \begin{bmatrix} 6 \\ 0 \\ -3 \\ \sqrt{2} \end{bmatrix}$$ is a column matrix of order 4×1

In general, $B = [b_{ij}]_{m\times 1}$ is a column matrix of order m×1.

(iii) **Square Matrix :** A matrix whose numbers of rows is equal to the number of columns is called a square matrix.

For the matrix A= $[\ a_{ij}\ ]_{m\times n}$, if m = n, then the matrix A is said to be square matrix of order m.

Example:

(a)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$ is a square matrix of order 2.

(b)

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 5 & 0 & 1 \end{bmatrix}$$ is a square matrix of order 3

In a square matrix A = $[\ a_{ij}\ ]_{m\times m}$, the elements a_{11}, a_{22}, a_{33} ..... are called the **Diagonal Elements** and the line along which they lie is called the **Principal Diagonal** of the matrix.

(iv)**Diagonal Matrix :** A square matrix A is said to be a diagonal matrix if all its non-diagonal elements be zero.

Examples:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

are diagonal matrices of order 2 and 3 respectively.

(v)**Scalar Matrix :** A diagonal matrix (i.e. all non-diagonal elements being zero) where all the diagonal elements are equal is called a Scalar Matrix.

Examples :

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

are two scalar matrices of order 2 and 3 respectively.

Thus, the square matrix $A = [a_{ij}]_{m\times m}$ is a scalar matrix if

a_{ij} = 0, when i $\neq$ j

a_{ij} = k(say) when i = j

(vi) **Unit (or Identity) Matrix :** A square matrix in which the diagonal elements are unity and non-diagonal elements are all zero is called a unit (or Identity)matrix.

Examples :

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Unit matrices are denoted by I.

I_2 and I_3 are Unit matrices of order 2 and 3 respectively.

Thus, the square matrix $A = [a_{ij}]_{m\times m}$ is a unit matrix if

a_{ij} = 0 when i $\neq$ j

a_{ij} = 1 when i = j

(vii) **Null Matrix or Zero Matrix :** A matrix of order m x n in which all the elements are zero is called a null matrix (a zero matrix). It is denoted by O.

Thus,

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are all zero matrices of order 2 x2 and 4x3

respectively.

(viii) **Upper Triangular Matrix and Lower Triangular Matrix:** Let A be a square matrix of order n.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots a_{2n} \\ a_{31} & a_{32} & a_{33} \ldots a_{3n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1} & a_{n2} & a_{n3} \ldots a_{nn} \end{bmatrix}$$

The diagonal elements of A are $a_{11}, a_{12}, \ldots, a_{nn}$ i.e. a_{ij} for i= j.

All these elements of A above the principal diagonal are a_{12}

$a_{13} \ldots\ldots a_{1n}, a_{23} \ldots\ldots a_{2n} \ldots\ldots$ etc. i.e. a_{ij} for $i < j$

All these elements of A below the principal diagonal are a_{21}, a_{31},

$a_{32}, \ldots\ldots a_{n1}, a_{n2}, \ldots\ldots$ i.e. a_{ij} for $i > j$.

If all a_{ij} for $i > j$ be zeros in a square matrix A, then A is called **Upper Triangular matrix.**

Example :

$$A = \begin{bmatrix} 3 & 1 & 2 & 6 \\ 0 & 2 & -4 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$ is Upper Triangular Matrix

If all a_{ij} for $i < j$ be zeros is a square matrix A, the A is called **Lower Triangular Matrix.**

Example :

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & -1 & 4 & 0 \\ -2 & 1 & 3 & 5 \end{bmatrix}$$ is Lower Triangular Matrix

Remarks : A diagonal matrix is both upper and lower triangular.

8.5 EQUALITY OF MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if

(i) they are of the same size or order

(ii) each element of one matrix is equal to the corresponding element of the other matrix i.e. $a_{ij} = b_{ij}$ for all i and j.

If A and B are equal, then we write A = B.If A and B are not equal, then we write $A \neq B$.

Example :

1. If $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}_{2\times 2}, B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}_{2\times 2}$ then A = B .

2. If $A=\begin{pmatrix}1^2 & 3^2\\ 2^2 & 4^2\end{pmatrix}_{2\times2}$, $B=\begin{pmatrix}1 & 9\\ 4 & 16\end{pmatrix}_{2\times2}$ then A = B.

3. If $A=\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}_{2\times2}$, $B=\begin{pmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{pmatrix}_{2\times3}$ then $A \neq B$.

4. If $\begin{pmatrix}x & a\\ y & b\\ z & c\end{pmatrix}_{3\times2} = \begin{pmatrix}-3 & 0\\ 1 & 2\\ 4 & \sqrt{3}\end{pmatrix}_{3\times2}$

then, x = - 3, y = 1, z = 4, a = 0, b = 2, c = $\sqrt{3}$

Remarks: The inequality of two matrices arised due to the following reasons :

(i) This orders may not be equal.

(ii) Elements in the corresponding places may not be equal.



CHECK YOUR PROGRESS 1

Q 1: If a matrix has 10 elements what are the possible orders it can have?

Q 2: Find the values of a, b, c, d when

$$\begin{pmatrix}2a+b & 5c-d\\ a-2b & 4c+d\end{pmatrix}=\begin{pmatrix}4 & 11\\ -3 & 25\end{pmatrix}$$

Q 3: Are the following matrices equal where

$$A=\begin{pmatrix}2 & 4 & 3\\ 1 & 0 & 5\\ 0 & 0 & 6\end{pmatrix}, B=\begin{pmatrix}2 & 4 & 3\\ 1 & 0 & 5\\ 0 & 0 & 0\end{pmatrix}$$

8.6 ADDITION AND SUBTRACTION OF MATRICES

Let $A = [a_{ij}]_{mxn}$ and $B = [b_{ij}]_{mxn}$ be two matrices of same order mxn. Then their sum(or difference) denoted by A+B (or A-B) is defined as another matrix $C= [c_{ij}]$ of the order m xn such that any element of C is the sum (or difference) of the corresponding element of A and B.

i.e. $c_{ij} = a_{ij}+b_{ij}$ for all i, j

Thus, $C =[c_{ij}] = [a_{ij}+b_{ij}]_{mxn}$

Example : If $A = \begin{pmatrix} 5 & 1 \\ 3 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & -5 \\ -2 & 1 \end{pmatrix}$

Then, $C = A+B = \begin{pmatrix} 5+2 & 1-5 \\ 3+(-2) & 0+1 \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ 1 & 1 \end{pmatrix}$

Similarly, $D=A-B = \begin{pmatrix} 5-2 & 1-(-5) \\ 3-(-2) & 0-1 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 5 & -1 \end{pmatrix}$

Remark : Two matrices A and B are said to be conformable for addition or subtraction if they are of the same order.

Negative of a Matrix : If A be a given matrix, then the negative of A denoted by – A is the matrix whose elements are the negative of the corresponding elements of A.

Example:

$$\text{If } A = \begin{pmatrix} 2 & -1 \\ 0 & 4 \\ -5 & 6 \end{pmatrix} \text{ then } -A = \begin{pmatrix} -2 & 1 \\ 0 & -4 \\ 5 & -6 \end{pmatrix}$$

Properties of Matrix Addition :

The addition of matrices satisfies the following properties :

(i) **Commutative Law :** If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same order, say m xn, then

$$A + B = B + A$$

Proof :

$$\begin{aligned} A+B &= [a_{ij}] + [b_{ij}] \\ &= [a_{ij} + b_{ij}] \\ &= [b_{ij} + a_{ij}] \quad (\because \text{addition of numbers is commutative}) \\ &= [b_{ij}] + [a_{ij}] \\ &= B + A \end{aligned}$$

(ii)**Associative Law :** If $A = [a_{ij}]$, $B=[b_{ij}]$ and $C = [c_{ij}]$ are matrices of the same order mxn, then $(A+B) + C = A + (B+C)$

Proof: $(A+B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$$\begin{aligned} &= [a_{ij} + b_{ij}] + [c_{ij}] \\ &= [(a_{ij} + b_{ij}) + c_{ij}] \end{aligned}$$

$= [a_{ij}] + [(b_{ij} + c_{ij})]$

$= [a_{ij}] + ([b_{ij}] + [c_{ij}])$

$= A + (B+C)$

i.e. $(i,j)^{th}$ element of $(A+B)+C$ = $(i,j)^{th}$ element of $A+(B+C)$

(iii)**Existence of Additive Identity :** If O be the zero matrix of order $m \times n$ and $A=[a_{ij}]$ be an $m \times n$ matrix, then $A+ O = O + A = A$

Proof :

$A + O = [a_{ij}] + [0]$

$= \lfloor a_{ij} + 0 \rfloor = [a_{ij}] = A$

and $O + A = [0] + [a_{ij}]$

$= [0+a_{ij}]$

$= [a_{ij}]$

$= A$

$\therefore A + O = O + A = A$

Hence, the zero matrix of the type $m \times n$ plays the role of additive identity.

(iv)**Existence of Additive inverse :** If $A = [a_{ij}]$ be a matrix of order $m \times n$, then we have another matrix $- A = [- a_{ij}]$ such that

$A+(–A) = (–A) + A = 0$

Proof :

$A + (-A) = [a_{ij}] + [-a_{ij}]$

$= [a_{ij}- a_{ij}]$

$= [0] = 0$

$(–A) + A = [-a_{ij}] + [a_{ij}]$

$= [- a_{ij} + a_{ij}]$

$= [0] = 0$

Hence, $A + (–A) = (–A) + A = 0$

Here, –A acts as the additive inverse of A.

(v)**Subtraction of two Matrices :** If A and B be two matrices of order $m \times n$, then we define $A – B = A + (–B)$.

In other words, the difference A – B is obtained by subtracting from each element of A the corresponding element of B.

(vi)**Cancellation Laws for addition of Matrices :** If A, B, C are three

matrices of the same order say m×n, then

$A + B = A + C \Rightarrow B = C$ [Left Cancellation law]

$B + A = C + A \Rightarrow B = C$ (Right Cancellation Law)

Proof : We have

$A + B = A + C$

$\Rightarrow -A + (A+B) = -A + (A+C)$ [Adding – A to both sides]

$\Rightarrow (-A+A) + B = (-A+A) + C$ [Q matrix addition is associative]

$\Rightarrow O + B = O + C$

$\Rightarrow B = C$

Similarly :

$B + A = C + A$

$\Rightarrow (B+A) + (-A) = (C+A) + (-A)$

$\Rightarrow B + (A - A) = C + (A - A)$

$\Rightarrow B + O = C + O$

$\Rightarrow B = C$

Example:(1) If $A = \begin{pmatrix} 5 & 0 \\ -3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$, find A + B and A – B

Solution :

$$A + B = \begin{pmatrix} 5 & 0 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 1 \\ 9 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 1 \\ 9 & 3 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 5 & 0 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5-3 & 0-1 \\ 3-6 & 1-2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -3 & -1 \end{pmatrix}$$

(2) If $A = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 3 \\ 5 & -2 \end{pmatrix}$, $C = \begin{pmatrix} 3 & 5 \\ 2 & 0 \end{pmatrix}$

Show that (A+B) + C = A + (B+C)

Solution :

$$A+B = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+3 \\ 4+5 & 3-2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 \\ 9 & 1 \end{pmatrix}$$

$$(A+B)+C = \begin{pmatrix} 1 & 3 \\ 9 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 5 \\ 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1+3 & 3+5 \\ 9+2 & 1+0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 8 \\ 11 & 1 \end{pmatrix}$$

Again, $B+C = \begin{pmatrix} 0 & 3 \\ 5 & -2 \end{pmatrix} + \begin{pmatrix} 3 & 5 \\ 2 & +0 \end{pmatrix}$

$$= \begin{pmatrix} 0+3 & 3+5 \\ 5+2 & -2+0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 8 \\ 7 & -2 \end{pmatrix}$$

$$\therefore A+(B+C) = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 8 \\ 7 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+3 & 0+8 \\ 4+7 & 3-2 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 11 & 1 \end{pmatrix}$$

$$\therefore (A+B)+C = A+(B+C)$$

(3) **Find the additive inverse of**

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix}_{2\times 3}$$

Solution:

A is a matrix of order 2×3.

– A will also be a matrix of order 2×3.

$$-A=\begin{pmatrix}-2 & -3 & -1\\ 0 & -(-1) & -5\end{pmatrix}$$

$$=\begin{pmatrix}-2 & -3 & -1\\ 0 & 1 & -5\end{pmatrix}_{2\times 3}$$



CHECK YOUR PROGRESS 2

Q 1: Is it possible to define A+B when–

(i) A has 3 rows ,2 columns and B has 2 rows, 2 columns.

(ii) A has 3 rows ,3 columns and B has 3 rows, 3 columns.

(iii) A and B are square matrices of same order.

Q 2: What is the additive identity of 2x3 matrices ?

8.7 MULTIPLICATION OF A MATRIX BY A SCALAR

Multiplication of a matrix by a scalar is defined as follows:
If $A = [a_{ij}]_{m\times n}$ is a matrix of order m×n and k is a scalar, then

$kA = k[a_{ij}]_{m\times n} = [ka_{ij}]_{m\times n}$

i.e., (i, j)th element of kA is k times the (i,j) th element of A .

Example:

If $k = 3$ and $A=\begin{pmatrix}2 & -1\\ -2 & 3\\ 0 & 1\end{pmatrix}_{2\times 3}$,then

$$3A=3\begin{pmatrix}2 & -1\\ -2 & 3\\ 0 & 1\end{pmatrix}=\begin{pmatrix}3\times 2 & 3\times(-1)\\ 3\times(-2) & 3\times 3\\ 3\times 0 & 3\times 1\end{pmatrix}$$

$$=\begin{pmatrix}6 & -3\\ -6 & 9\\ 0 & 3\end{pmatrix}_{3\times 2}$$

and,

$$\frac{1}{2}A=\frac{1}{2}\begin{pmatrix}2 & -1\\ -2 & 3\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\times 2 & \frac{1}{2}\times(-1)\\ \frac{1}{2}\times(-2) & \frac{1}{2}\times 3\\ \frac{1}{2}\times 0 & \frac{1}{2}\times 1\end{pmatrix}=\begin{pmatrix}1 & -\frac{1}{2}\\ -1 & \frac{3}{2}\\ 0 & \frac{1}{2}\end{pmatrix}_{3\times 2}$$

PROPERTIES OF MULTIPLICATION OF A MATRIX BY A SCALAR :

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order m x n ; k and l are two scalars, then

(i) $k(A+B) = kA + kB$

(ii) $(k+l)A = kA + lA$

(iii) $k(lA) = (kl)A$

(iv) $(-k)A = -(kA) = k(-A)$

Proof of (i) :

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$

Then $k(A+B)=k([a_{ij}]+[b_{ij}])$

$= k([a_{ij}+b_{ij}])$

$=([ka_{ij}+kb_{ij}])$

$=[ka_{ij}+kb_{ij}]$

$= [ka_{ij}]+[kb_{ij}]$

$= k[a_{ij}]+k[b_{ij}]$

$= kA + kB.$

Example :

$$\text{If } A = \begin{pmatrix}3 & 9\\ 1 & 8\end{pmatrix}_{2\times 2}, B=\begin{pmatrix}4 & 0\\ 7 & 2\end{pmatrix}_{2\times 2}$$

Then verify that $2(A+B) = 2A + 2B$

Solution :

We have

$$A + B = \begin{bmatrix}3 & 9\\ 1 & 8\end{bmatrix}+\begin{bmatrix}4 & 0\\ 7 & 2\end{bmatrix}$$

$$=\begin{bmatrix}3+4 & 9+0\\ 1+7 & 8+2\end{bmatrix}$$

$$= \begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix}$$

$$\therefore 2\,(A+B) = 2\begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix} = \begin{bmatrix} 2\times 7 & 2\times 9 \\ 2\times 8 & 2\times 10 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 18 \\ 16 & 20 \end{bmatrix}$$

Again, $2A+2B = 2\begin{bmatrix} 3 & 9 \\ 1 & 8 \end{bmatrix} + 2\begin{bmatrix} 4 & 0 \\ 7 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2\times 3 & 2\times 9 \\ 2\times 1 & 2\times 8 \end{bmatrix} + \begin{bmatrix} 2\times 4 & 2\times 0 \\ 2\times 7 & 2\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 18 \\ 2 & 16 \end{bmatrix} + \begin{bmatrix} 8 & 0 \\ 14 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+8 & 18+0 \\ 2+14 & 16+4 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 18 \\ 16 & 20 \end{bmatrix}$$

$\therefore 2(A+B) = 2A + 2B$ verified.

Proof of (ii)

Let $A = [a_{ij}]$, then

$(k+l)\,A = (k+l)\,[a_{ij}]$

$$= \left[ka_{ij} + la_{ij}\right]$$

$$= \left[ka_{ij}\right] + \left[la_{ij}\right]$$

$$= k[a_{ij}] + l[a_{ij}]$$

$$= kA + lA$$

Example :

If $k = 3$, $l = -2$ and $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

Then verify that $(k+l)\,A = kA + lA$

$$(k+l)A = (3-2)\begin{bmatrix}3 & 2\\ 1 & 0\\ 2 & 1\end{bmatrix} = (1)\begin{bmatrix}3 & 2\\ 1 & 0\\ 2 & 1\end{bmatrix}$$

Again,

$$kA + lA = 3A - 2A = 3\begin{bmatrix}3 & 2\\ 1 & 0\\ 2 & 1\end{bmatrix} - 2\begin{bmatrix}3 & 2\\ 1 & 0\\ 2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}9 & 6\\ 3 & 0\\ 6 & 3\end{bmatrix} - \begin{bmatrix}6 & 4\\ 2 & 0\\ 4 & 2\end{bmatrix}$$

$$= \begin{bmatrix}9-6 & 6-4\\ 3-2 & 0-0\\ 6-4 & 3-2\end{bmatrix}$$

$$= \begin{bmatrix}3 & 2\\ 1 & 0\\ 2 & 1\end{bmatrix}$$

$\therefore (k+l)A = kA + lA$. Verified

Proof of (iii) :

Let $A = [a_{ij}]$, then

$k(lA) = k(l[a_{ij}])$

$= k[l\, a_{ij}]$

$= [k(l\, a_{ij})]$

$= [(kl)\, a_{ij}]$

$= (kl)\,[a_{ij}] = (kl)\,A.$

Example : If k=4, l=2 and $A = \begin{bmatrix}3 & 2\\ 1 & 0\\ 2 & 1\end{bmatrix}$, then

Verify that k(lA) = (kl)A.

Solution :

$$k(lA) = 4(2A) = 4\begin{bmatrix} 2\times3 & 2\times2 \\ 2\times1 & 2\times0 \\ 2\times2 & 2\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times2\times3 & 4\times2\times2 \\ 4\times2\times1 & 4\times2\times0 \\ 4\times2\times2 & 4\times2\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 8\times3 & 8\times2 \\ 8\times1 & 8\times0 \\ 8\times2 & 8\times1 \end{bmatrix}$$

$$= 8\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$= 8A$

$= (4\times2)\,A$

$= (kl)A.$ verified

Proof of (iv) :

If $A = [a_{ij}]$ then

$(-k)\,A = [(-k)\,a_{ij}]$

$= [-(k\ a_{ij})]$

$= -[k\ a_{ij}] = -(kA)$

Also $k(-A) = k[-a_{ij}]$

$= [k.-(a_{ij})]$

$= -[k\ a_{ij}]$

$= -(kA)$

$\therefore (-k)\,A = -(kA) = k(-A)$

Example :

$$\text{If } k = 5 \text{ and } A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Verify that $(-k)A = -(kA) = k(-A)$

Solution :

$$(-k)\,A = (-5)\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-5)\times 3 & -(5\times 2) \\ (-5)\times 1) & -(5\times 0) \\ -(5\times 2) & -(5\times 1) \end{bmatrix}$$

$$= -\begin{bmatrix} (5)\times 3 & (5)\times 2 \\ (5)\times 1 & (5\times 0) \\ (5\times 2) & (5\times 1) \end{bmatrix} = -(kA)$$

Also, $k(-A) = 5\begin{bmatrix} -3 & -2 \\ -1 & 0 \\ -2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 5\times(-3) & 5\times(-2) \\ 5\times(-1) & 5\times 0 \\ 5\times(-2) & 5\times(-1) \end{bmatrix}$$

$$= -\begin{bmatrix} 5\times 3 & 5\times 2 \\ 5\times 1 & 5\times 0 \\ 5\times 2 & 5\times 1 \end{bmatrix}$$

$$= -5\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} = -(kA)$$ Verified.

Example : If $A = \begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$, find $3A - 2B$.

Solution :

$$3A - 2B = 3\begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 2 \end{bmatrix} - 2\begin{bmatrix} 4 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 9 & 6 \\ 3 & 12 & 6 \end{bmatrix} - \begin{bmatrix} 8 & 0 & -6 \\ 0 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0-8 & 9-0 & 6-(-6) \\ 3-0 & 12-2 & 6-4 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 9 & 12 \\ 3 & 10 & 2 \end{bmatrix}$$

8.8 MULTIPLICATION OF TWO MATRICES

Two matrices A and B are conformable for the product AB if and only if the number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$,

Then the product A,B,C is the matrix of order m×p

i.e. $AB = C = [c_{jk}]_{m\times p}$ where

$$C_{jk} = \sum_{j=1}^{n} a_{ij}\, b_{jk} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + \ldots\ldots\ldots\ldots + a_{in}\, b_{nk}$$

i.e. the (j, k) th element of the matrix C= AB is found by multiplying the corresponding element of the j-th row of A and the k-th column of B and then, adding the product.

The rule of multiplication is *row by column multiplication*.

Example:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2\times 2}$$

$$\text{Then } AB = \begin{bmatrix} a_{11}b_{11} + a_{12}\ b_{21} & a_{11}b_{12} + a_{12}\ b_{22} \\ a_{21}b_{11} + a_{22}\ b_{21} & a_{21}b_{12} + a_{22}\ b_{22} \\ a_{31}b_{11} + a_{32}\ b_{21} & a_{31}b_{12} + a_{32}\ b_{22} \end{bmatrix}_{3\times 2}$$

Note : 1) If the product AB exists, then it is not necessary that the product BA will also exist.

Example : $A = [2\ \ 3]_{1\times 2}$ and $B = \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix}_{2\times 3}$

Here,

$$AB = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 8+3 & 0+6 & 10+9 \end{bmatrix} = \begin{bmatrix} 11 & 6 & 19 \end{bmatrix}$$

But $BA = \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \end{bmatrix}$ which does not exist

$\therefore AB \neq BA$

Matrix multiplication is not commutative

2) In the case when both A and B are square matrices of the same order then also both AB and BA are defined but still $AB \neq BA$.

Example : If $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$

$$\text{Then } AB = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2x3 + 1x4 & 2x5 + 1x1 \\ 0x3 + 2x4 & 0x5 + 2x1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 11 \\ 8 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3x3 + 5x0 & 3x1 + 5x2 \\ 4x2 + 1x0 & 4x1 + 1x2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 13 \\ 8 & 6 \end{bmatrix}$$

It is seen that $AB \neq BA$

Example :

$$\text{If } A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 0 \\ -2 & 1 & 3 \end{bmatrix}$$

Can you find AB and BA ?

Solution :

Here, A is of order 2×2

B is of order 2×3

$\therefore$ AB will be order 2×3

Again, B is of order 2×3

A is of order 2×2

$\therefore$ BA does not exist because the number of column of B is 3 and

the number of rows of A is 2.

So, they are not conformable for the product BA.

PROPERTIES OF MATRIX MULTIPLICATION :

The multiplication of matrices possesses the following properties:

(1) The multiplication of matrices is not always commutative.

(a) Whenever AB exists, it is not necessary that BA should exist.

(b) Whenever AB and BA both exist, it is not always necessary that they should be matrices of same order.

(c) Whenever AB and BA exists and are of the same order, it is not necessary that AB = BA.

(d) In some cases, AB = BA.

(e) Matrix multiplication is associative i.e. A(BC) = (AB)C where A, B, C are matrices of order m×n, n×p, p×q respectively.

(f) Matrix multiplication is distribution w. r. t. addition of matrix .i.e, A(B+C) = AB+AC

Where, A, B, C are matrices of order m×n, n×p and n×p respectively.

(g) For every square matrix A, there exists the identity matrix I of same order of A such that AI = IA = A.

MATRIX MULTIPLICATION IS ASSOCIATIVE

If A, B, C are matrices of order m×n, n×p, p×q, then
A(BC) = (AB) C

Proof :

Let A= $[a_{ij}]_{m\times n}$, B = $[b_{jk}]_{n\times p}$, C = $[c_{kl}]_{p\times q}$

Then AB = U = $[u_{ik}]_{m\times p}$ where

$$u_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$$

Also, $BC=V=[v_{jl}]_{n\times q}$ where $v_{jl} = \sum_{k=1}^{n} b_{jk} c_{kl}$

$\therefore$A(BC) = W = $[w_{il}]_{m\times q}$ when (i, l)th element of A(BC)

$$= W_{il} = \sum_{j=1}^{n} a_{ij}\ v_{jl}$$

$$= \sum_{j=1}^{n}\left[a_{ij}\left\{\sum_{K=1}^{n} b_{jk} c_{kl}\right\}\right]$$

$$= \sum_{K=1}^{n}\left[\left\{\sum_{j=1}^{n} a_{ij}\ b_{kl}\right\} C_{kl}\right]$$

$$= \sum_{k=1}^{n} u_{ik} c_{kl}$$

= (i, j)th element of (AB) C

$\therefore$ A(BC) = (AB) C

MULTIPLICATION OF MATRICES IS DISTRIBUTIVE WITH RESPECT TO ADDITION OF MATRICES :

If A, B and C are matrices of order m×n, n×p, n×p respectively, then A(B+C) = AB + AC

Proof :

Let $A=[a_{ij}]_{m\times n}$, $B=[b_{jk}]_{n\times p}$, $C=[c_{jk}]_{n\times p}$

We have B+C = $[b_{jk}+c_{jk}]_{n\times p}$

$\therefore$(i, k)th element of A(B+C) = $\sum_{j=1}^{n} a_{ij}(b_{jk}+c_{jk})$

$$= \sum_{j=1}^{n} a_{ij}\ b_{jk} + \sum_{j=1}^{n} a_{ij}\ c_{jk}$$

= (i, k)th element of AB + (i, k)th element of AC

= (i, k)th element of AB +AC

Here, A(B+C) = AB + AC

Similarly, we can show that,

(B+C) D = BD + CD

where B, C and D are matrices of suitable types.

Example :

If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}_{2\times 2}$, $B=\begin{bmatrix}3 & 1 & 2\\ 0 & 2 & 1\end{bmatrix}_{2\times 3}$

$C = \begin{bmatrix} 5 & 0 & 1 \\ 4 & 6 & 0 \end{bmatrix}_{2\times 3}$ then show that

$A(B+C) = AB + AC$

Solution :

$$B+C = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 1 \\ 4 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 1 & 3 \\ 4 & 8 & 1 \end{bmatrix}_{2\times 3}$$

$A(B+C)$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \begin{bmatrix} 8 & 1 & 3 \\ 4 & 8 & 1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1\times 8+2\times 4 & 1\times 1+2\times 8 & 1\times 3+2\times 1 \\ 3\times 8+4\times 4 & 3\times 1+4\times 8 & 3\times 3+4\times 1 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 16 & 17 & 5 \\ 40 & 35 & 13 \end{bmatrix}$$

Again, $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\times 3+2\times 0 & 1\times 1+2\times 2 & 1\times 2+2\times 1 \\ 3\times 3+4\times 0 & 3\times 1+4\times 2 & 3\times 2+4\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & 4 \\ 9 & 11 & 10 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 4 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 5+2\times 4 & 1\times 0+2\times 6 & 1\times 1+2\times 0 \\ 3\times 5+4\times 4 & 3\times 0+4\times 6 & 3\times 1+4\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 12 & 1 \\ 31 & 24 & 3 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 3 & 5 & 4 \\ 9 & 11 & 10 \end{bmatrix} + \begin{bmatrix} 13 & 12 & 1 \\ 31 & 24 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 17 & 5 \\ 40 & 35 & 13 \end{bmatrix}$$

$\therefore A(B+C) = AB + AC$

Example :

$$\text{If } A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}_{2\times 2}, B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}_{3\times 2}$$

$$C = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}_{2\times 2}$$

Find A(BC) and (AB)C and show that A(BC) ≠ (AB) C.

Solution :

$$BC = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 1\times 1+3\times 2 & 1\times 2+3\times 0 \\ 0\times 1+2\times 2 & 0\times 2+2\times 0 \\ -1\times 1+4\times 2 & -1\times 2+4\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ 4 & 0 \\ 7 & 2 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 4 & 0 \\ 7 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 7+1\times 4+(-1)\times 7 & 1\times 2+1\times 0+(-1)\times(-2) \\ 2\times 7+0\times 4+3\times 7 & 2\times 2+0\times 0+3\times(-2) \\ 3\times 7+(-1)\times 4+2\times 7 & 3\times 2+(-1)\times 0+2\times(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 \\ 35 & -2 \\ 31 & 2 \end{bmatrix}$$

$$\text{and } AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 1+1\times 0+(-1)(-1) & 1\times 3+1\times 2+(-1)4 \\ 2\times 1+0\times 0+3(-1) & 2\times 3+0\times 2+3\times 4 \\ 3\times 1+(-1)\times 0+2\times(-1) & 3\times 3+(-1)\times 2+2\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times1+1\times2 & 2\times2+1\times0 \\ (-1)\times1+18\times2 & (-1)\times2+18\times0 \\ 1\times1+15\times2 & 1\times2+15\times0 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 35 & -2 \\ 31 & 2 \end{bmatrix}$$



CHECK YOUR PROGRESS 3

Q 1: Given

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ C = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix},\ D = \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

B(C+D) = BC + BD

(B+C) D = BD + CD

Q 2: For the matrices A,B,C put the numerical values of their orders, so that (A+B) C = AC + BC

Q 3: Find A and B if

$$A + B = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix},\ A - B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Q 4: Given $A = \begin{bmatrix} 0 & 1 \\ 21 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, Show that

$A^2 = B^2 = I$ (unit matrix)

Q 5: If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$

Q 6: If $A_a = \begin{bmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{bmatrix}$, then prove that

$$(A_a)^3 = \begin{bmatrix} \cos 3a & \sin 3a \\ -\sin 3a & \cos 3a \end{bmatrix}$$

8.9 TRANSPOSE OF A MATRIX

Let $A = [a_{ij}]_{m\times n}$. Then the n×m matrix obtained from A by changing the rows of A into columns and columns into rows is called the Transpose of matrix A

and is denoted by A^{l} or A^{T}.

Therefore, A^{l} or A^{T} will have n-rows and m-columns.

Example :

If $A = \begin{bmatrix} 2 & 0 & 5 \\ 1 & 2 & 4 \end{bmatrix}_{2\times 3}$ Then,

$$A^{/} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 5 & 4 \end{bmatrix}_{3\times 2}$$

SOME PROPERTIES OF TRANSPOSE OF MATRICES :

(i) The transpose of transpose of a matrix is the matrix itself i.e. if A is a matrix, then $(A^{/})^{/} = A$

Note :

i) $(kA)^{/} = kA^{/}$

ii) $(A+B)^{/} = A^{'/} + B^{/}$

iii) $(AB)^{/} = AB$ if $AB = BA$

(ii) If A is a matrix f order mxn and k is a scalar, then $(kA)^{/} = kA^{/}$

(iii) If A and B are matrices of same order, then $(A+B)^{/} = A^{/} + B^{/}$

(iv) If A and B are two matrices conformable to multiplication, the

$(AB)^{/} = B^{/}A^{/}$.

This law is known as "reversal Law for transpose".

Example:

If $A = \begin{bmatrix} 3 & 4 \\ 4 & 2 \\ 5 & 3 \end{bmatrix}$ then show that

$\left(A^{/}\right)^{/} = A$

Solution :

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & 3 \end{bmatrix}, \text{ then } A^{/} = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

$$(A^{/})^{/} = \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & 3 \end{bmatrix}$$

Hence, $(A^{/})^{/} = A$

Note : The reversal law for transpose can be extended for more than two matrices.

$(ABC)^{/} = C^{/}B^{/}A^{/}$

$(ABCD)^{/} = D^{/}C^{/}B^{/}A^{/}$ etc.

Example :

If $A=\begin{bmatrix}2 & 3\\ 4 & 1\end{bmatrix}, B=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$

Show that (i) $(2A)^{/}=2A^{/}$

(ii) $(A+B)^{/}=A^{/}+B^{/}$

(iii) $(AB)^{/}=B^{/}A^{/}$

Solution :

(i) $A=\begin{bmatrix}2 & 3\\ 4 & 1\end{bmatrix}$

$2A=\begin{bmatrix}4 & 6\\ 8 & 2\end{bmatrix}, A^{/}=\begin{bmatrix}2 & 4\\ 3 & 1\end{bmatrix}$

$\therefore (2A)^{/}=\begin{bmatrix}4 & 8\\ 6 & 2\end{bmatrix}, 2A^{/}=2\begin{bmatrix}2 & 4\\ 3 & 1\end{bmatrix}=\begin{bmatrix}4 & 8\\ 6 & 2\end{bmatrix}$

$\therefore \quad (2A)^{/}=2A^{/}$

(ii) $A+B=\begin{bmatrix}2 & 3\\ 4 & 1\end{bmatrix}+\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$

$=\begin{bmatrix}2+1 & 3+2\\ 4+3 & 1+4\end{bmatrix}=\begin{bmatrix}3 & 5\\ 7 & 5\end{bmatrix}$

$(A+B)^{/}=\begin{bmatrix}3 & 7\\ 5 & 5\end{bmatrix}$

Again, $A^{/}+B^{/}=\begin{bmatrix}2 & 4\\ 3 & 1\end{bmatrix}+\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$

$=\begin{bmatrix}2+1 & 4+3\\ 3+2 & 1+4\end{bmatrix}=\begin{bmatrix}3 & 7\\ 5 & 5\end{bmatrix}$

Hence, $(A+B)^{/}=A^{/}+B^{/}$

(ii) $AB=\begin{bmatrix}2 & 3\\ 4 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$

$=\begin{bmatrix}2\times1+3\times3 & 2\times2+3\times4\\ 4\times1+1\times3 & 4\times2+1\times4\end{bmatrix}$

$=\begin{bmatrix}11 & 16\\ 7 & 12\end{bmatrix}$ and $(AB)^{/}=\begin{bmatrix}11 & 7\\ 16 & 12\end{bmatrix}$

Note :

i) If A is a symmetric matrix, then $A^{/}=A$

ii) A is a skew-symmetric matrix then $A^{/}=-A$

Again, $B'A' = \begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}\begin{bmatrix}2 & 4\\ 3 & 1\end{bmatrix}$

$$= \begin{bmatrix}1\times2+3\times3 & 1\times4+3\times1\\ 2\times2+4\times3 & 2\times4+4\times1\end{bmatrix}$$

$$= \begin{bmatrix}11 & 7\\ 16 & 12\end{bmatrix}$$

Hence, $(AB)' = B'A'$



CHECK YOUR PROGRESS 4

Q 1: If $A = \begin{bmatrix}-4 & 3\\ 1 & 1\\ 0 & 2\end{bmatrix}_{3\times2}$ and $B = \begin{bmatrix}1 & 3 & 4\\ 2 & 1 & 2\end{bmatrix}_{2\times3}$

Show that $(AB)' = B'A'$

Q 2: With suitable example show that $(A')' = A$

Q 3: With suitable example, show that $A' = -A$

8.10 SYMMETRIC AND SKEW-SYMMETRIC MATRIX

SYMMETRIC MATRIX :

A square matrix $A=[a_{ij}]$ is said to be symmetric if for all values of i and j, i.e, $a_{ij} = a_{ji}$

i.e. $(i,j)^{th}$ element is same as the $(j,i)^{th}$ element of A

For example

$$\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}, \begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}$$

are symmetric matrix.

Theorem : The necessary and sufficient condition for a matrix A to be symmetric is that $A = A'$

Proof : Let $A=[a_{ij}]_{nxn}$ is a n-rowed square matrix.

Necessary : $a_{ij} = a_{ji}$

Now, A/ which is the transpose of A is also a square matrix.

$\therefore$ $(i,j)^{th}$ element of A/ =$(j, i)^{th}$ element of A

$= a_{ji}$ =(i j)th element of A

Hence A is symmetric.

Sufficient : Let $A^{/} = A$

$\therefore$ (i, j)th elements of A = (j, i) th element of A$^{/}$

= (j, i)th element of A

Hence A is symmetric.

Skew symmetric Metrix : A square matrix A=[a_{ij}] is said to be skew-synmetric if for all value of i and j, $a_{ij} = -a_{ji}$

i.e. (i,j)th element is the negative of (j,i)th element of A

For example

$$\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$$

are skew symmetric matrix.

If A is skew-symmetric then –

$a_{ij} = -a_{ji} \quad \forall$ i, j

$\therefore \; a_{ii} = -a_{ii} \quad \forall$ i

$\Rightarrow 2\, a_{ii} = 0$

$\Rightarrow a_{ii} = 0$

Thus the diagonal elements a_{11}, a_{22}, a_{33} ..... of a skew-symmetric matrix are all zero.

Theorem : Necessary and sufficient condition for a matrix A to be skew-symmetric is that. A$^{/}$= – A.

Prof : Necessary :

Let A= [a_{ij}]$_{nxn}$ be an n rowed skew symmetric matrix. Then $a_{ij} = -a_{ji}$

$\therefore$ (i,j) th element of $A^{/}$ =(j, i) th element of A

$= a_{ji}$

$= -a_{ij}$

= – (i,j)th element of A

$\therefore \quad A^{/} = -A$

Sufficiency part :

Let $A^{/} = -A$

$\therefore$ $(i, j)^{th}$ element of $A^{/}$ = –$(i, j)^{th}$ element of A

$\Rightarrow$ (j, i)th element of A = –(i, j)th element of A

$\Rightarrow a_{ji} = -a_{ij}$

$\Rightarrow a_{ij} = -a_{ji}$

Hence A is skew-symmetric.

Example 1:

If A is symmetric (skew symmetric) matrix then kA is also symmetrix (skew symmetric).

Solution :

i) Let A be symmetric. Then $A^{/} = A$

Now $(kA)^{/} = kA^{/} = kA$

$\Rightarrow$ kA is also symmetric.

ii) Let A be skew symmetric. Then $A^{/} = -A$

Now, $(kA)^{/} = kA^{/} = k(-A) = -(kA)$

$\Rightarrow$ kA is also symmetric.

Example 2: Show the matrix A+B is symmetric or skew-symmetric according as A and B are symmetric or skew-symmetric matrix.

Solution :

i) A and B be two symmetric matrics

$\therefore A^{/}=A, B^{/}=B$

Now, $(A+B)^{/} = A^{/} + B^{/} = A+B$

$\therefore$ A+B is symmetric

ii) A and B be to skew-symmetric matrices

$\therefore A^{/}=-A, B^{/}=-B$

Now, $A^{/}+B^{/}=A^{/}+B^{/} = -A-B = -(A+B)$

$\therefore$ A+B is skew-symmetric.

Example 3:

If A and B are symmetric matices, then AB is symmetic if and only if A and B commute i.e. AB=BA

Solution : A and B are symmetric. To prove that $(AB)^{/} = AB$

$\therefore A^{/}=A, B^{/}=B$

Also, AB = BA

Now, $(AB)^{/} = B^{/}A^{/}$

$= BA$

$= AB$

$\therefore$ AB is symmetric.

Conversely, Let AB be symmetric. To prove that AB=BA

$\therefore (AB)' = AB$ – (1)

Also $(AB)' = B'A'$

$= BA$ – (2)

From (1) and (2), is equel

AB = BA Proved

Example 4:

If A is a n-squared matrix, then AA' and $A'A$ are both symmetric matrices.

Solution :

we have

$$(AA')' = (A')'A'$$

$$= AA' \qquad \mathbf{Q}\,(A')' = A$$

$\therefore AA'$ is symmetric

Also, $\left(A'A\right)' = A'\left(A'\right)' = A'A$

$A'A$ is symmetric .

Example 5:

Show that every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric metrix.

Solution : Let A be any square matrix.

We can write

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

$$= P + Q$$

Where $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

Now,

$$P' = \left\{\frac{1}{2}(A + A')\right\}' = \frac{1}{2}(A + A')' \qquad \mathbf{Q}\,(kA)' = kA'$$

$$= \frac{1}{2}\left\{A' + (A')'\right\}$$

$$= \frac{1}{2}(A' + A)$$

$$= \frac{1}{2}(A + A') = P$$

$\therefore$ P is symmetric matrics.

and

$$Q' = \left\{\frac{1}{2}(A - A')\right\}'$$

$$= \frac{1}{2}(A - A')'$$

$$= \frac{1}{2}\{A' - (A')'\}$$

$$= \frac{1}{2}(A' - A)$$

$$= -\frac{1}{2}(A - A')$$

$$= -Q$$

$\therefore$ Q is skew-symmetric matrix.

Uniqueness : To prove that the representation is unique,let us take another representation. A=R+S. where R is symmetric and S is skew-symmetric. To prove that R=P, S=Q.

$$A' = (R+S)' = R' + S'$$

$$= R - S$$

$$\therefore A + A' = R + S + R - S = 2R$$

$$\Rightarrow R = \frac{1}{2}(A + A')$$

and

$$A - A' = R + S - R + S = 2S$$

$$\Rightarrow S = \frac{1}{2}(A - A')$$

Thus R=P and S=Q.

$\therefore$ The representation is unique.



CHECK YOUR PROGRESS – 5

Q 1: $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, find AA' and $A'A$

Q 2: Let $A=\begin{bmatrix}1 & 6 & 3\\ 2 & 8 & 5\\ 4 & 1 & 7\end{bmatrix}$, then find $\frac{1}{2}(A+A')$ and

$\frac{1}{2}(A-A')$. show that $\frac{1}{2}(A+A')$ is symmetric

$\frac{1}{2}(A-A')$ is skew−symmetric.

Q 3: Show that $\begin{bmatrix}0 & -3i & -4\\ 3i & 0 & 8\\ 4 & -8 & 0\end{bmatrix}$ is skew-symmetric matrix.

8.10 ADJOINT OF A SQUARE MATRIX

$$\text{Let } A=\begin{bmatrix}a_{11} & a_{12}\ldots & a_{1m}\\ a_{21} & a_{22}\ldots & a_{2m}\\ \ldots\ldots\ldots\ldots\ldots & & \\ a_{m1} & a_{m2}\ldots & a_{mm}\end{bmatrix}=[a_{ij}]_{mxn}$$

The determinant of the matrix A,

$$|A|=\begin{bmatrix}a_{11} & a_{12}\ldots & a_{1m}\\ a_{21} & a_{22}\ldots & a_{2m}\\ \ldots\ldots\ldots\ldots\ldots & & \\ a_{m1} & a_{m2}\ldots & a_{mm}\end{bmatrix}$$

Let the co-factors of the element a_{ij} be A_{ij} considering the matrix whose elements are the co-factors of the corresponding elements of A, we get

$$\Delta=\begin{bmatrix}A_{11} & A_{12} & \ldots & A_{1m}\\ A_{21} & A_{22} & \ldots & A_{2m}\\ . & \ldots\ldots\ldots\ldots\ldots & & \\ A_{m1} & A_{m2}\ldots & & A_{mm}\end{bmatrix}$$

$$\Delta'=\begin{bmatrix}A_{11} & A_{21} & \ldots & A_{m1}\\ A_{12} & A_{22} & \ldots & A_{m2}\\ . & \ldots\ldots\ldots\ldots\ldots & & \\ A_{1m} & A_{2m} & \ldots & A_{mm}\end{bmatrix}=B(\text{say})$$

$B=\Delta'$ is called the adjoint of the matrix A and is denoted by adj A.

i.e. B = adj. A

Theorem:

If A be any n-rowed square matrix, then

(adj A) A= A(adj A) = |A| I_n where I_n is the n-rowed unit matrix.

Proof :

Let A= $[a_{ij}]_{n\times n}$ be any n-rowed square matrix and

let adj A = B = $[b_{ij}]_{n\times n}$

Then b_{ij} = co-factor of a_{ij} in |A|

= A_{ji}

Since A and adj A are both n-rowed square matrices, therefore they are conformable for multiplication

i.e. Both A (adj A) and (adj A) A exist and are of the type n×n.

Now, (i, j)th element of A (adj A)

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$= \sum_{K=1}^{n} a_{ik}\ A_{jk}$$

$$= \begin{cases} 0 & \text{if } i \neq j \\ |A| & \text{if } i = j \end{cases}$$

In other words, all the elements of A(adj A) along the principal diagonal are equal to |A| and the non-diagonal elements are all zero.

$$\therefore A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 \ldots\ldots\ldots\ldots 0 \\ 0 & |A| & 0 \quad\quad 0 \\ 0 & 0 & |A| \ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & 0 & 0 \ldots\ldots\ldots |A| \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 0 & 0 \ldots\ldots\ldots 0 \\ 0 & 1 & 0 \ldots\ldots\ldots 0 \\ 0 & 0 & 1 \ldots\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & 0 & 0 \ldots\ldots\ldots\ldots 1 \end{bmatrix}$$

$$= |A|\ I_n$$

Similarly,

(i, j)th element of (adj A) A

$$= \sum_{K=1}^{n} b_{ik} a_{kj}$$

$$= \sum_{k=1}^{n} A_{ki} a_{kj}$$

$$= \begin{cases} 0 & \text{if } i \neq j; \\ |A| & \text{if } i = j \end{cases}$$

Therefore, $(\text{adj } A)\, A = |A| I_n$

Hence,

$$A(\text{adj } A) = (\text{adj } A)\, A = |A| I_n \text{ and } A^{-1} = \frac{1}{|A|}(\text{adj.A.}) .$$

Example :

Find the adjoint of the matrix A

$$\text{where } A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix}$$

Solution :

We have

$$|A| = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix}$$

Now, A_{11} = cofactor of $(1,1)^{th}$ element of $|A|$

$$= \begin{vmatrix} -1 & 4 \\ 0 & 1 \end{vmatrix} = -1 - 0 = -1$$

A_{12} = cofactor of (1,2)th element of |A|

$$= \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -(3 - 8) = 5$$

Similarly,

$$A_{13} = \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = -0 - (-1)\, 2 = 2$$

$$A_{21} = - \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2$$

$$A_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, A_{23} = - \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = 4$$

$$A_{31} = \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} = 8, A_{32} = - \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4, \quad A_{33} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -1 - 6 = -7$$

$$\therefore \text{adj } A = \begin{bmatrix} -1 & -2 & 8 \\ 5 & 1 & -4 \\ 2 & 4 & -7 \end{bmatrix}$$

Note :

(1) To find the Adj A, write down the determinant |A| and find the co-factor of various element of |A|. Replace each element in A by its co-factor and then take the transpose to get adj A.

(2) If A is a n-rowed square matrix, then A is said to be singular matrix if its determinant is zero i.e |A| = 0.

$\therefore$ A(adj A) = (Adj A)A= $|A| I_n = 0$ (null matrix)

Hence, in this case all the elements of A(adj A) and (adj A)A will be zero's.

(3) If $|A| \neq 0$, then

$$|A \text{Adj } A| = \left\| |A| I_n \right| = \begin{vmatrix} |A| & 0 & 0 \ldots & 0 \\ 0 & |A| & 0 \ldots & 0 \\ 0 & 0 & |A| \ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots & \\ 0 & 0 & 0 \ldots & |A| \end{vmatrix}$$

$$= |A|^n .$$

$$\Rightarrow |\text{Adj } A| = |A|^{n-1}$$

8.11 INVERSE OF A MATRIX

Let A and B any n-squared matrices such that

$AB = BA = I_n$

Then B is called inverse of A and is denoted by A^{-1} i.e. $B=A^{-1}$

Note : It is to be noted that every square matrix does not possess its inverse.

PROPERTIES OF THE INVERSE :

(a) The inverse of a matrix is unique.

Let A be n-squared matrix whose inverse exists.

Let B and C be two inverses of A.

Then $AB = BA = I_n$ —— (1)

$AC = CA = I_n$ —— (2)

For (1) $C(AB) = C\,I_n = C$ —— (3)

For (2) $(CA)\,B = I_n B = B$ —— (4)

$\therefore$ $C(AB) = (CA)\,B$

$\therefore$ $C = B$

which shows that an invertible matrix possesses a unique inverse.

(b) The necessary and sufficient condition for a square matrix A to possess an inverse is that A is non-singular i.e. $|A| \neq 0$

Proof: Necessary :

Let A be a square matrix of order n and B be its inverse

Then $AB = I_n$

$\therefore |AB| = |I_n|$

$\Rightarrow |A||B| = 1$

$\Rightarrow |A| \neq 0$

Sufficient :

Let $|A| \neq 0$, we define a matrix B so that $B = \frac{1}{|A|}(\text{adj.A.})$

Then $AB = A\,\frac{1}{|A|}\text{adj}\,A$

$= \frac{1}{|A|}A(\text{adj}\,A)$

$= \frac{1}{|A|}|A|\,I_n = I_n$

Similarly, $BA = \left(\frac{1}{|A|}\text{adj}\,A\right)A$

$= \frac{1}{|A|}(\text{adj}\,A)\,A = \frac{1}{|A|}.|A|\,I_n = I_n$

Thus, $AB = BA = I_n$.

Hence, A is invertible and B, is the inverse of A.

$\therefore B = A^{-1} = \frac{1}{|A|}\text{adj}\,A$

(c) **Reversal law for the inverse of a product.**

If A, B be two n-rowed non-singular, then AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.i.e. the inverse of the product is the product of the inverses in the reverse order.

Proof : Let A and B be two non-singular matrices i.e. $|A| \neq 0$, $|B| \neq 0$,

So their inverses exist.

We have $|AB| = |A||B| \neq 0$

Hence, the matrix AB is also invertible .

Let C be the inverse of AB.

Let A^{-1} and B^{-1} be the inverses of A and B respectively and

$AA^{-1} = A^{-1}A = I_n$ and $B\ B^{-1} = B^{-1}B = I_n$

$\therefore C(AB) = (AB)\ C = I_n$

Choosing $C = B^{-1}A^{-1}$ we get

$C(AB) = (B^{-1}A^{-1})(AB)$

$= B^{-1}(A^{-1}A)B$

$= B^{-1}\ I_n\ B$

$= B^{-1}\ B$

$= I_n$ ... (1)

Again, $(AB)\ C = (AB)\ (B^{-1}A^{-1})$

$= A(BB^{-1})\ A^{-1}$

$= AI_nA^{-1}$

$= AA^{-1}$

$= I_n$ ... (2)

By (1) and (2)

$C(AB) = (AB)\ C = I_n$

Hence, C is the inverse of AB

or $B^{-1}A^{-1}$ is the inverse of AB

Thus, $(AB)^{-1} = B^{-1}A^{-1}$

(d) The operation of transposing and inverting are commutative

i.e. $(A')^{-1} = (A^{-1})'$

Proof :

Let A be a non-singular sequare matrix i.e. $|A| \neq 0$

$\therefore A^{-1}$ exists

We get $AA^{-1} = A^{-1}A = I_n$

$\therefore (AA^{-1})' = (A^{-1}A)' = I'_n$

$\Rightarrow (A^{-1})'A' = A'(A^{-1})' = I_n$

which shows that $(A^{-1})'$ is the inverse of A^{-1}.

Hence, $(A')^{-1} = (A^{-1})'$

Note : If A, B are two n-rowed non-singular matrices. Then

$(AB)^{-1} = B^{-1}A^{-1}$.

Example :

Find the adjoint and inverse of the matrix $\begin{bmatrix} 0 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

Solution :

$$\text{Let } A = \begin{bmatrix} 0 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 0 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= 0 + (-1)\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= (-1)(-1) + 3(6-1)$$

$$= 1 + 15 = 16 \neq 0$$

$\therefore A^{-1}$ exists

Now, Co-factors of the elements of 1st row are –

$$\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}, -\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \text{ i.e. } -2,\ 1,\ 5$$

Co-factors of the elements of 2nd row are –

$$-\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \text{ i.e. } 6, -3, 1$$

Co-factors of the elements of 3rd row are

$$\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} \text{ i.e. } -2, 9, -3$$

Matrix formed by the corresponding co-factors of the elements of |A| is

$$C = \begin{bmatrix} -2 & 1 & 5 \\ 6 & -3 & 1 \\ -2 & 9 & -3 \end{bmatrix}$$

$$\therefore \text{adj } A = \text{transpose of } C = \begin{bmatrix} -2 & 6 & -2 \\ 1 & -3 & 9 \\ 5 & 1 & -3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$= \frac{1}{16}\begin{bmatrix} -2 & 6 & -2 \\ 1 & -3 & 9 \\ 5 & 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{8} & \frac{3}{8} & \frac{-1}{8} \\ \frac{1}{16} & \frac{-3}{16} & \frac{9}{16} \\ \frac{5}{16} & \frac{1}{16} & \frac{-3}{16} \end{bmatrix}$$

Example : Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Solution :

$$|A| = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= (16-9) - 3(4-3) + 3(3-4)$$

$$= 7 - 3 - 3 = 7 - 6 = 1 \neq 0$$

Now, Co-factors of the elements of the 1st row are –

$$\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \text{ i.e. } 7, \ -1, \ -1$$

Co-factors of the elements of the 2nd row are –

$$-\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \text{ i.e. } 3, \ 1, 0$$

Co-factors of the elements of the 3rd row are –

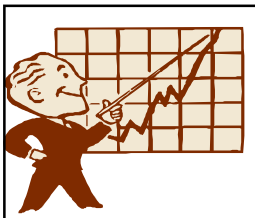
$$\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \text{ i.e. } -3, \ 0, 1$$

$$\therefore \text{adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$



CHECK YOUR PROGRESS 5

Q 1: Find adj A if $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 10 & 1 \end{bmatrix}$

Q 2: Find A^{-1} if $A = \begin{bmatrix} \cos a & -\sin a \\ \sin a & -\cos a \end{bmatrix}$

Q 3: Show that if $|A| \neq 0$, then $|A^{-1}| = |A|^{-1}$

Q 4: If AB = BA, then prove that $A^{-1} B^{-1} = B^{-1}A^{-1}$

Q 5: If $|A| \neq 0$ and A is symmetric, show that A^{-1} is also symmetric.

Q 6: If A be a square matrix, then prove that $|\text{adj } A| = |A|^{n-1}$.

Q 7: If $|A| \neq 0$, then show that adj adj $A = |A|^{n-2}A$.

8.12 RANK OF A MATRIX

Sub-matrix of a matrix : Let A be any matrix of order m×n. Then a marix obtained by leaving some rows and columns from A is called a sub-matrix of A.

Note : A matrix A itself is a sub-matrix of A.

Minors of a matrix : Let A be any matrix of order m×n. by leaving (m–r) rows and(n–r) columns of A, we get a square sub-matrix of order r. The determinant of the square sub–matrix of order r is called a minor of A of order r.

Example :

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 1 & 3 & 2 \end{bmatrix}_{2\times 3}$$

is of order 2×3.

So we get minors of order 1 and 2

Minor of order 2 are $\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} -1 & -2 \\ 3 & 2 \end{vmatrix}$

Now,

$$\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = 9+1=10, \begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} = 6+2=8, \begin{vmatrix} -1 & -2 \\ 3 & 2 \end{vmatrix} = -2+6=4$$

RANK OF A MATRIX :

The rank of a matrix A is r if

(a) Every minor of A of order r+1 is zero.

(b) There is atleast one minor of A of order r which does not vanish.

The rank of a matrix A is denoted by r (A)

Note : (i) The rank of a null matrix is zero.

(ii) Rank of a sub-matrix $\leq$ rank of the matrix.

WORKING RULE :

(1) Calculate the minors of highest possible order of the matrix.

(2) If it is not zero, then the order of the minor is the rank of the matrix.

(3) If it is zero and all other minors of same order are also zero, then calculate the minor of next lower order.

(4) If it is not zero, then follow (2)

(5) If it is zero, then follow (3) and so on.

Example : Find the rank of the matrices

(i) $A = \begin{bmatrix} 2 & 6 & -3 \\ 1 & 4 & -2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 4 & 5 & 6 \\ 4 & 2 & 3 \\ 3 & 4 & 4 \end{bmatrix}$

Solution :

(i) As the matrix A is of order 2×3, the highest possible order of minors will be 2

Minors of order 2 : $\begin{vmatrix} 6 & -6 \\ 4 & -3 \end{vmatrix}$ = 6(-2) - 4 (-3)

$= -12 + 12 = 0$

Another minor of order 2 is $\begin{vmatrix} 2 & 6 \\ 1 & 4 \end{vmatrix} = 2 \times 4 - 1 \times 6$

$= 8 - 6$

$= 2 \neq 0$

Hence, there is atleast one minor of order 2 is non–zero.

$\therefore r$ (A) = 2.

(ii) As the matrix A is of order 3×3, the highest possible order of minor will be 3.

Minor of order 3 is

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\therefore r(\mathrm{A}) = 3$

(iii) As the matrix A is of order 2×2, the highest possible order of minor will be 2.

Minor of order 2 is $\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$

Also minor of order 1 is zero

$\therefore \rho(\mathrm{A}) = 0$

(iv) As the matrix A is of order 3×3, the highest possible order of minor will be 3.

Matrix of order 3 is

$$\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 3 & 4 & 4 \end{vmatrix} = 4(2 \times 4 - 4 \times 3) - 5(4 \times 4 - 3 \times 3 + 4(1 \times 4 - 3 \times 2)$$

$= 4(8–12) – 5(4–9) + 6(4–6)$

$= 4(–4) – 5 (–5) + 6(–2)$

$= -16 + 25 –12$

$= 25 – 28$

$= –3 \neq 0$

$\therefore \rho(\mathrm{A}) = 3$



CHECK YOUR PROGRESS 6

Q 1: Find the rank of the matrices

(i) $A = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 1 & -4 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -4 & -5 \\ 0 & -2 & 1 \\ 0 & 16 & 3 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 13 \\ 4 & -3 & -4 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

8.13 LINEAR EQUATIONS

Linear equation are divided into two types

(i) Homogeneous linear equations

(ii) Non-homogeneous linear equations.

The nature of solutions of a system of linear equations can be studied with the help of matrices. We shall first consider a system of homogeneous linear equations. There are two types of equations

(i) Consistent equation

(ii) Inconsistent equation

(i) **Consistent equation :** A system of equations is said to be consistent if they have one or more solutions.

Example :

(i) $x + 2y = 0$ (ii) $x+2y = 4$

$3x+ y = 2$ $4x + 8y = 16$

(ii)**Inconsistent equations :** A system of equation is said to be inconsistent if they have no solution.

Example :

$x + 2y = 4$

$3x + 6y = 6$

Homogeneous linear equations :

A system of n-linear equations in n-unknown say, $x_1, x_{2,} \cdots, x_n$ is given as follows :

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n &= 0 \\ \text{- -} & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n &= 0 \end{aligned}\right\} \quad (1)$$

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12}\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots a_{2n} \\ a_{m1} & a_{m2}\ldots\ldots a_{mn} \end{bmatrix}_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1}, 0 = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}_{m\times 1}$$

Then the above system of equations can be written in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12}\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots\ldots a_{2n} \\ a_{m1} & a_{m2}\ldots\ldots\ldots a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}, = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$$

i.e. $AX = 0$ (2)

The matrix A is called the co-efficient matrix of the system of equation (1)

Obviously, $x_1 = x_2 = \ldots\ldots\ldots\ldots = x_n = 0$

i.e. X = 0 is a solution of (1) called a trivial solution.

Again, let X_1 and X_2 be two solutions of (2). Then their linear combination $K_1X_1 + K_2X_2$ where K_1 and K_2 are any arbitrary numbers is also a solution of (2).

Therefore, $AX_1 = 0$ and $AX_2 = 0$ and hence

$$\begin{aligned} A(K_1X_1 + K_2X_2) &= K_1(AX_1) + K_2(AX_2) \\ &= K_1\,0 + K_2 0 \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Hence, $K_1X_1 + K_2X_2$ is also a solution of (2)

8.14 SOLUTION OF NON-HOMOGENEOUS SIMULTANEOUS EQUATIONS

Let a system of n simultaneous equation in n-unknown $x_1, x_2, \ldots\ldots x_n$ be

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots\ldots + a_{2n} x_n = b_2$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$a_{n1} x_1 + a_{n2} x_2 + \ldots\ldots\ldots\ldots + a_{nn} x_n = b_n$

This system can be written in the matrix form as

$AX = B \qquad \ldots (1)$

When $A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots\ldots a_{2n} \\ a_{n1} & a_{n2} \ldots\ldots\ldots\ldots a_{nn} \end{bmatrix}_{n \times n}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}_{n x 1}$

If $|A| \neq 0$, then A^{-1} exists.

Pre-multiplying both sides of (1) by A^{-1}

$A^{-1}(AX) = A^{-1} B$

$\Rightarrow (A^{-1}A) X = A^{-1}B$

$\Rightarrow IX = A^{-1} B$

$\Rightarrow X = A^{-1}B$ which is the solution of the given system of equations.

Example : Solve :

$x+y+z = 1, \qquad x+2y+z = 2, \qquad x+y+2z = 0$

Solution :

Given system of equation is –

$x+y+z = 1$

$x+2y + z = 2$

$x + y + 2z = 0$

It can be written as –

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

i.e. $A X = B$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Now,

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1(4-1) - 1(2-1) + 1(1-2)$$

$$= 3 - 1 - 1 = 1 \neq 0$$

Co-factors of the elements of 1st row are –

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \quad \text{i.e.} 3, -1, -1$$

Co-factors of the elements of 2nd row are –

$$-\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \text{ i.e. } -1, 1, 0$$

Co-factors of the elements of 3rd row are –

$$\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \text{ i.e. } -1, 0, 1$$

$$\text{adj } A = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$\therefore A^{-1} = \frac{\text{adj } A}{|A|} = \text{adj } A \qquad \mathbf{Q} |A| = 1$$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ -1+2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

i.e. x = 1, y = 1, z = –1



CHECK YOUR PROGRESS 7

Q 1: Solve x+y+z = 6, x–y+z = 2, 2x+y+z = 1

Q 2: Solve x–3y+z = -1, 2x+y – 4z = –1, 6x–7y+8z = 7

8.15 SOLUTION OF A SYSTEM OF N NON-HOMOGENOUS LINEAR EQUATIONS IN N-UNKNOWNS BY CRAMMAR'S RULE

Let the system of n non-homogenous linear equations in n-unknowns linear

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1 \rightarrow (1)$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2 \rightarrow (2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \rightarrow (n)$$

The system can be written as –

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}_{n\times 1}$$

i.e. $AX = B$

Determinant of the co-efficient matrix A

$=|A| = D$ (say)

Multiplying the equations (1), (2), ……., (n) respectively by the co-factors of $a_{11}, a_{21}, \cdots$ i.e. $A_{11}, A_{21}, \cdots, A_{n1}$ and adding we get

$$A_{11}(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) + A_{21}(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n) + \cdots$$

$$+ A_{n1}(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n) = b_1A_{11} + b_2A_{21} + \cdots + b_nA_{n1}$$

$$\Rightarrow (a_{11}A_1 + a_{21}A_{21} + \cdots + a_{n1}A_{n1})x_1 = b_1A_{11} + b_2A_{21} + \cdots + b_nA_{n1}$$

$$\Rightarrow Dx_1 = D_1$$

Where

$$D = \begin{vmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ \ldots\ldots\ldots\ldots & & \\ a_{n1} & a_{n2} \ldots\ldots & a_{nn} \end{vmatrix} \text{ and } D_1 = \begin{vmatrix} b_1 & a_{12} \ldots\ldots & a_{1n} \\ b_2 & a_{22} \ldots\ldots & a_{2n} \\ \ldots\ldots\ldots\ldots & & \\ b_n & a_{n2} \ldots\ldots & a_{nn} \end{vmatrix}$$

D_1 is the determinant obtained from D by replacing the elements of 1st column by corresponding b's

$$\Rightarrow \frac{x_1}{D_1} = \frac{1}{D} \text{ Provided } D \neq 0$$

Similarly multiplying the equations (1), (2), …… (n) by co-factors of the

elements of 2nd column of |A| and adding, we get –

$$Dx_2 = D_2 \quad \text{where} \quad D_2 = \begin{vmatrix} a_{11} & b_1 ..... a_{1n} \\ a_{21} & b_2 .....a_{2n} \\ a_{n1} & b_n .....a_{nn} \end{vmatrix}$$

$$\Rightarrow \frac{x_2}{D_2} = \frac{1}{D}, D \neq 0$$

As above we will get

$$\frac{x_1}{D_1} = \frac{x_2}{D_2} = ... = \frac{x_n}{D_n} = \frac{1}{D}, D \neq 0$$

The unique solution of the given system of equation provided $D \neq 0$ in the coefficient matrix is non-singular.

i.e. the rank of the co-efficient matrix is n = number of variables.

Note : For a system of n non-homogeneous linear equations with n-unknowns

$$\frac{x_1}{D_1} = \frac{x_2}{D_2} ..... \frac{x_n}{D_n} = \frac{1}{D}, D \neq 0$$

Example : Solve the equations

$3x + y + 2z = 3$
$2x - 3y – z = –3$
$x + 2y + z = 4$
Using Cramer's rule.

Solution : We have

$$D = |A| = \begin{vmatrix} 3 & 1 & -2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$= 3(–3 + 2) – 1(2+1) + 2(4+3)$
$= –3 –3 + 14$
$= 8 \neq 0$

$$D_1 = \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix}$$

$= 3(–3+2) –1 (–3+4) + 2(–6+12)$
$= –3 – 1 + 12$

$= 8$

$$D_2 = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix}$$

$$= 3(-3+4) - 3(2+1) + 2(8+3)$$
$$= 3 - 9 + 22 = 16$$

$$D_3 = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix}$$

$= 3\,(-12+6) - 1(8+3) + 3\,(4+3)$
$= -18 - 11 + 21$
$= -8$

$$\therefore x = \frac{D_1}{D} = \frac{8}{8} = 1$$

$$y = \frac{D_2}{D} = \frac{16}{8} = 2$$

$$z = \frac{D_3}{D} = \frac{-8}{8} = -1$$

Example : Solve completely the equations by Cramer's rule –
$x+2y+3z = 6$, $3x-2y+z=2$, $4x+2y+z=7$

Solution :

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 1(-2-2) - 2(3-4) + 3(6+8)$$
$$= -4 + 2 + 42 = 40 \neq 0$$

$$D_1 = \begin{vmatrix} 6 & 2 & 3 \\ 2 & -2 & 1 \\ 7 & 2 & 1 \end{vmatrix} = 6(-2-2) - 2(2-7) + 3(4+14)$$
$$= -24 + 10 + 54 = 40$$

$$D_2 = \begin{vmatrix} 1 & 6 & 3 \\ 3 & 2 & 1 \\ 4 & 7 & 1 \end{vmatrix}$$

$= 1(2-7) - 6(3-4) + 3(21-8)$
$= -5 + 6 + 39$
$= 40$

$$D_3 = \begin{vmatrix} 1 & 2 & 6 \\ 3 & -2 & 2 \\ 4 & 2 & 7 \end{vmatrix}$$

$= 1(-14-4)-2(21-8)+6(6+8)$
$= -18-26+84$
$= 84 - 44$
$= 40$

$$\therefore x = \frac{D_1}{D} = \frac{40}{40} = 1$$

$$y = \frac{D_2}{D} = \frac{40}{40} = 1$$

$$z = \frac{D_3}{D} = \frac{40}{40} = 1$$



CHECK YOUR PROGRESS 8

Solve completely the system of equations by using Cramer's rule :

Q 1: $3x + 5y = 8$, $-x+2y-z = 0$, $3x-6y+4z=1$

Q 2: $x_1+x_2+x_3=7$, $x_1- x_2+ x_3 = 2$, $2x_1 - x_2+ 3x_3 = 9$



8.16 LET US SUM UP

- A rectangular array of mn numbers arranged in m-rows and n-columns and enclosed in square [ ] or a round bracket ( ), is called a matrix of order m×n (m by n).
- The numbers which constitute the matrix are called the elements of a matrix.
- The element which occurs in the i-th row and j-th column is called the (i,j) the element of the matrix and is denoted by a_{ij} .
- A matrix with exactly one row is called a row matrix. The order of a row matrix is of the type 1×n.
- A matrix with exactly one column is called a column matrix. The order of a column matrix is of the type m×1.

- A matrix $A=[a_{ij}]_{m\times n}$ is a square matrix if m=n.
- A square matrix $a=[a_{ij}]$ is a scalar matrix if $a_{ij} = \begin{cases} k & \text{where } i = j \\ 0 & \text{where } i \neq j \end{cases}$
- A square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$ where $i \neq j$.
- A square matrix $A=[a_{ij}]$ is a unit matrix if $a_{ij} = 1$ where $i = j$
  $= 0$ where $i \neq j$;
  A unit matrix of order n is denoted by I or I_n.
- A matrix in which all elements are zero is called a null matrix or zero matrix. A null matrix is denoted by O.
- A matrix is said to be upper triangular if $a_{ij} = 0$ when $i>j$.
- A matrix is said to be lower triangular if $a_{ij} = 0$ when $i < j$.
- Two matrix are said to be equal if and only of they are of same order and the corresponding elements are equal.
- If A and B are two matrix of same order. Then their sum or difference denoted by A + B ,A–B or B–A can be obtained.
- Matrix addition in commutative i.e. A+B = B+A.
- Matrix addition in associative i.e. (A+B) + C = A+(B+C)
- Existence of addition identity i.e. A+O = A = O +A
- Existence of additive inverse i.e. A+(–A) = (–A) + A
- Scalar multiplication of matrix i..e if (a) $A= [a_{ij}]$ then $kA= [k a_{ij}]$
  (b) k(A+B) = kA+ kB (c) (k+l) A = kA + lA
  (d) k(lA) = (kl) A (e) (–k)A= k(–A)
- Matrix multiplication is not commutative i.e. $AB \neq BA$ (in general)
- Matrix multiplication is associative i.e. (AB)C = A(BC)
- Matrix multiplication is distributive over addition i.e.
  (i) A(B+C) = AB + AC
  (ii) (A+B) C = AC + BC
- If $A = [a_{ij}]_{m\times n}$ then $A^T=[b_{ij}]_{n\times m}$ when $b_{ij} = a_{ij}$
- For matrices A and B of the same order $(A+B)^{/} = A^{/} +B^{/}$
- For matrix A and B conformable to multiplication
  $(AB)^{/} = B^{/}A^{/}$
- The matrix A is called symmetric if $A^{/}=A$

- The matrix A is called skew symmetric if $A' = -A$
- Every square matrix A can uniquely be expressed as A = P+Q where P is a symmetric matrix and Q is a skew-symmetric matrix.
- For a square matrix A (adj A) A = A (adj A) = |A|I
- For a square non-singular matrix A, $A^{-1} = \frac{\text{adj } A}{|A|}$
- For two non-singular matrices A and B , $(AB)^{-1} = B^{-1}A^{-1}$.
- Rank of a sub matrix ≤ Rank of the matrix.
- If x_1 and x_2 be two solutions of AX= 0, the $k_1x_1 + k_2x_2$ is also a solution of AX = 0
- For system of n simultaneous non-homogenous equations with n-unknown written as AX=B, the solution matrix is $X = A^{-1}B$ provided $|A| \neq 0$.
- For a system of n-simultaneous linear non-homogenous equation with n-unknowns

$$\frac{x_1}{D_2} = \frac{x_2}{D_2} = \ldots\ldots\ldots\ldots = \frac{x_n}{D_n} = \frac{1}{D}., D \neq 0$$

8.17 ANSWER TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS 1

Answer Q No 1:

$10 = 1\times10$ $10 = 5\times2$

$10 = 2\times5$ $10 = 10\times1$

$\therefore$ Possible orders of matrices are 1×10, 2×5, 5×2 and 10×1.

Answer Q No 2: By equality of two matrices $\begin{pmatrix} 2a+b & 5c+d \\ a-2b & 4c+d \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ 3 & 25 \end{pmatrix}$

2a+b=4 5c–d=11

a–2b=–3 4c+d=25

Solving these we get

a = 1, b = 2, c = 4, d = 9

Answer Q No 3: For equality of two matrices, their orders and the co-

rresponding elements must be same. Here, the order of the matrices are same. But (3,3)th element of $A \neq (3,3)^{th}$ element of B $\therefore A \neq B$

CHECK YOUR PROGRESS 2

Answer Q No 1:

(i) It is not possible for define A + B, since A is of order 3×2 and B is of order 2×2.

(ii) It is possible to define A+B, since A is of order 3×3 and B is of order 3×3.

(iii) It is possible to define A+B, since A and B are square matrices of the same order.

Answer Q No 2:

$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the additive identity of 2 x3 matrices.

CHECK YOUR PROGRESS 3

Answer Q No 1 :

$$B(C+D) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left(\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 8 & 2 \\ 11 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1x8 + 2x11 & 1x2 + 2x6 \\ 3x8 + 4x11 & 3x2 + 4x6 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 14 \\ 68 & 30 \end{bmatrix}$$

$$BC+BD = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1x3+2x4 & 1x1+2x2 \\ 3x3+4x4 & 3x1+4x2 \end{bmatrix} + \begin{bmatrix} 1x5+2x7 & 1x1+2x4 \\ 3x5+4x7 & 3x1+4x4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 5 \\ 25 & 11 \end{bmatrix} + \begin{bmatrix} 19 & 9 \\ 43 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} 11+19 & 5+9 \\ 25+43 & 11+19 \end{bmatrix} = \begin{bmatrix} 30 & 14 \\ 68 & 30 \end{bmatrix}$$

$$\therefore B(C+D) = BC + CD$$

$$\text{Again } (B+C)D = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}\right)\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3 & 2+1 \\ 3+4 & 4+2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4x5+3x7 & 4x1+3x4 \\ 7x5+6x7 & 7x1+6x4 \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 16 \\ 77 & 31 \end{bmatrix}$$

$$BD+CD = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1x5+2x7 & 1x1+2x4 \\ 3x5+4x7 & 3x1+4x4 \end{bmatrix} + \begin{bmatrix} 3x5+1x7 & 3x1+1x4 \\ 4x5+2x7 & 4x1+2x4 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 9 \\ 43 & 19 \end{bmatrix} + \begin{bmatrix} 22 & 7 \\ 34 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 19+22 & 9+7 \\ 43+34 & 19+12 \end{bmatrix} = \begin{bmatrix} 41 & 16 \\ 77 & 31 \end{bmatrix}$$

$$\therefore (B+C)D = BD + CD$$

Answer Q No 2:

If, A is of order 4×3
Then, B is of order 4×3
and C is of order 3×2

Answer Q No 3:

$$\text{Given, } A+B = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix},\ A-B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\therefore A+B+A-B = 2A = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

Again,

$$A+B-(A-B) = A+B-A+B=2B$$

$$= \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3 & 0-0 \\ 2-0 & 5-3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\therefore B = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Answer Q No 4:

$$A^2 = AA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0x0+1x1 & 0x1+1x0 \\ 1x0+0x1 & 1x1+0x0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$B^2 = BB = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0x0+(-i)i & 0x(-i)+(-i)x0 \\ ix0+0xi & ix(-i)+0x0 \end{bmatrix}$$

$$= \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Answer Q No 5:

$A^2-5A+7I=A.A-5A+7I$

$$= \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3x3+1(-1) & 3x1+1X2 \\ (-1)x3+2(-1) & (-1)x1+2x2 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Answer Q No 6:

$$A_a = \begin{bmatrix} \cos a & \text{Sin}a \\ -\sin a & \cos a \end{bmatrix}$$

$$A_a{}^2 = \begin{bmatrix} \cos a & \text{Sin}a \\ -\sin a & \cos a \end{bmatrix}\begin{bmatrix} \cos a & \text{Sin}a \\ -\sin a & \cos a \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 a & -\text{Sin}^2 a & \cos a \sin a + \sin a \cos a \\ -\sin a \cos a - \cos a \sin a & -\text{Sin}^2 a + \cos^2 a \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2a & \text{Sin}2a \\ -\sin 2a & \cos 2a \end{bmatrix}$$

$$A_a{}^3 = A_a{}^2 A_a = \begin{bmatrix} \cos 2a & \text{Sin}2a \\ -\sin 2a & \cos 2a \end{bmatrix}\begin{bmatrix} \cos a & \text{Sin}a \\ -\sin a & \cos a \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2a.\cos a - \text{Sin}2a.\text{Sin}a & \cos 2a.\text{Sin}a + \text{Sin}2a.\cos a \\ -\sin 2a.\cos a - \cos 2a \text{Sin}a & -\text{Sin}2a.\text{Sin}a + \cos 2a \cos a \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2a + a) & \text{Sin}(a + 2a) \\ -\sin(2a + a) & \cos(2a + a) \end{bmatrix}$$

$$= \begin{bmatrix} \cos 3a & \text{Sin}3a \\ -\sin 3a & \cos 2a \end{bmatrix}$$

CHECK YOUR PROGRESS 4

Answer Q No 6:

$$A' = \begin{bmatrix} -4 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix}_{2x3}, \quad B' = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}_{3x2}$$

$$AB = \begin{bmatrix} -4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4+6 & -12+3 & -16+6 \\ 1+2 & 3+1 & 4+2 \\ 0+4 & 0+2 & 0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -9 & -10 \\ 3 & 4 & 6 \\ 4 & 2 & 4 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} 2 & 3 & 4 \\ -9 & 4 & 2 \\ -10 & 6 & 4 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -4 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4+6 & 1+2 & 0+4 \\ -12+3 & 3+1 & 0+2 \\ -16+6 & 4+2 & 0+4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ -9 & 4 & 2 \\ -10 & 6 & 4 \end{bmatrix}$$

$\therefore (AB)' = B'A'$.

Answer Q No 2:

$$\text{Let } A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$$A' = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$$(A')' = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = A$$

Answer Q No. 3 :

$$A = \begin{bmatrix} o & h & g \\ -h & o & f \\ -g & -f & o \end{bmatrix} \text{ is a skew symmetric matrix}$$

$$A' = \begin{bmatrix} o & -h & -g \\ h & o & -f \\ g & f & o \end{bmatrix} = -\begin{bmatrix} o & h & g \\ -h & o & f \\ -g & -f & o \end{bmatrix} = -A$$

CHECK YOUR PROGRESS 5

Answer Q No. 1 :

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix} \quad \therefore \quad A' = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$AA' = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1x1+3x3+2x2 & 1x3+3x2+2x1 \\ 3x1+2x3+1x2 & 3x3+2x2+1x1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 11 \\ 11 & 14 \end{bmatrix}$$

$$A'A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1x1+3x3 & 1x3+3x2 & 1x2+3x1 \\ 3x1+2x3 & 3x3+2x2 & 3x2+2x1 \\ 2x1+1x3 & 2x3+1x2 & 2x2+1x1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 9 & 5 \\ 9 & 13 & 8 \\ 5 & 8 & 5 \end{bmatrix} \quad \therefore AA' = A'A$$

Answer Q No 2:

$$A = \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix} \quad \therefore A' = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

$$\frac{1}{2}(A + A') = \frac{1}{2}\left(\begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix} \right)$$

$$= \frac{1}{2} \begin{bmatrix} 1+1 & 6+2 & 3+4 \\ 2+6 & 8+8 & 5+1 \\ 4+3 & 1+5 & 7+7 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 8 & 7 \\ 8 & 16 & 6 \\ 7 & 6 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 4 & \frac{7}{2} \\ 4 & 8 & 3 \\ \frac{7}{2} & 3 & 7 \end{bmatrix}$$

$$\frac{1}{2}(A-A') = \frac{1}{2}\left(\begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 1-1 & 6-2 & 3-4 \\ 2-6 & 8-8 & 5-1 \\ 4-3 & 1-5 & 7-7 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -\frac{1}{2} \\ -2 & 0 & 2 \\ \frac{1}{2} & -2 & 0 \end{bmatrix}$$

Answer Q No 3: Let

$$A = \begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 3i & 4 \\ -3i & 0 & -8 \\ -4 & 8 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix} = -A$$

$\therefore$ A is Skew Symmetric

CHECK YOUR PROGRESS 6

Answer Q No 2:

$$|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$$

Co-factors of the elements of 1st row are –

$$\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}, \text{ i.e.} -1, 8, -5$$

Co-factors of the elements of 2nd row are –

$$-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}, \text{ i.e.} 1, -6, 3$$

Co-factors of the elements of 3rd row are –

$$\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix}, -\begin{vmatrix}0 & 2\\ 1 & 3\end{vmatrix}, \begin{vmatrix}0 & 1\\ 1 & 2\end{vmatrix}, \text{ i.e.} -1,\ 2,\ -1$$

$$\therefore \text{adj A} = \begin{vmatrix}-1 & 1 & -1\\ 8 & -6 & 2\\ -5 & 3 & -1\end{vmatrix}$$

Answer Q No 2:

$$|A| = \begin{vmatrix}\cos a & -\sin a\\ \sin a & \cos a\end{vmatrix} \quad \therefore |A| = \cos^2 a + \sin^2 a = 1 \neq 0$$

$\therefore$ Co-factor of the elements of 1st row are

$\cos a, -\sin a$

Co-factor of the element of 2nd row are –

$-(-\sin a),\ \cos a$ i.e. $\sin a, \cos a$

$$\therefore \text{adj A} = \begin{vmatrix}\cos a & \sin a\\ -\sin a & \cos a\end{vmatrix}$$

$$\therefore A^{-1} = \frac{\text{adjA}}{|A|} = \begin{vmatrix}\cos a & \sin a\\ -\sin a & \cos a\end{vmatrix}$$

Answer Q No 3:

$|A| \neq 0 \quad \therefore A^{-1}$ exists

Now,

$$AA^{-1} = I$$

$$\Rightarrow |AA^{-1}| = |I|$$

$$\Rightarrow |A||A^{-1}| = 1$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|} = |A|^{-1}$$

Answer Q No 4:

Given that AB=BA

$$\therefore (AB)^{-1} = (BA)^{-1}$$

$$\Rightarrow B^{-1}A^{-1} = A^{-1}B^{-1}$$

Answer Q No 5:

$|A| \neq 0 \quad \therefore A^{-1}$ exists

also $A^{/}=A$

Now, $(A^{-1})^{/} = (A^{/})^{-1} = A^{-1}$

$\therefore A^{-1}$ is symmetric.

Answer Q No 6:

$$A\,adjA = |A|I_n$$

$$\therefore |A\,adjA| = \|A\|I_n\|$$

$$\Rightarrow |A\|adjA| = |A|^n\ |I_n| \quad \mathbf{Q}\,|kA| = k^n|A|$$

$$= |A|^n \qquad \mathbf{Q}\,|I_n| = 1$$

$$\Rightarrow |adj\,A| = |A|^{n-1} \quad \text{if} \quad |A| \neq 0$$

If $|A| = 0$, then $|adj\ A| = 0 = |A|^{n-1}$

Hence the result.

Answer Q No 7:

$$|A| \neq 0$$

We have $A(adj\ A) = |A|I_n$

Replacing A by adj A ,we get

$$(adj\,A)(adj\,adj\ A) = |adj\,A|I_n$$

$$= |A|^{n-1}.I_n \qquad -(i)$$

Premultiplying both side of (i) by A

$$A\ (adj\,A)(adj\,adj\ A) = A\{|A|^{n-1}I_n\}$$

$$= |A|^{n-1}(AI_n)$$

$$= |A|^{n-1}A$$

$$\Rightarrow \{A(adj\ A)\}(adj\,adj\ A) = |A|^{n-1}A$$

$$\Rightarrow |A|I_n\ (adj\ adj\ A) = |A|^{n-1}A$$

$$\Rightarrow |A|(adj\ adj\ A) = |A|^{n-1}A$$

$$\Rightarrow adj\ adj\ A = |A|^{n-2}A \qquad [\text{Dividy both sides by}|A| \quad \mathbf{Q}\,|A| \neq 0]$$

CHECK YOUR PROGRESS 7

(i) As the matix A is of order 2 x 3, the highest possible order of minor is 2.

$$\text{minor of order } 2 = \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = 0x1 - 2x1 = -2 \neq 0$$

$$\therefore r(A) = 2$$

(ii) As the matix A is of order 4x3, the highest possible order of minor is 3.

$$\text{minor of order } 3 = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & -8 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -3 & -5 \\ -8 & -1 \end{vmatrix} = 3 - 40 = -37 \neq 0$$

$$\therefore r(A) = 3$$

(ii) As the matix A is of order 3x3, the highest possible order of minor is 3.

$$\text{minor of order } 3 = \begin{vmatrix} 2 & 1 & 3 \\ 4 & 3 & 13 \\ 4 & -3 & -1 \end{vmatrix}$$

$$= 2x(-3+39) - (-4-52) + 3(-12-12)$$
$$= 2x36 + 56 + 3(-24)$$
$$= 72 + 56 - 72$$
$$= 56 \neq 0$$
$$\therefore r(A) = 3$$

(iv) As the matix A is of order 2x2, the highest possible order of minor is 2.

minor of order 2

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$$\therefore r(A) = 2$$

CHECK YOUR PROGRESS 8

Answer Q No 1:

x + y + z = 6
x – y + z = 2
2x + y – z =1

This system of equations can be written in matix form as –

Ax=B

$$\text{where } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 1 - 1 - (-1-2) + (1+2)$$
$$= 0 + 3 + 3 = 6 \neq 0$$

Now, Co-factors of the elements of 1st row are –

$$\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}, \text{i.e. } 0,\ 3,\ 3$$

Co-factors of the elements of 2nd row are –

$$-\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}, \begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}, -\begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix}, \text{i.e. } 2, -3, 1$$

Co-factors of the elements of 3rd row are –

$$\begin{vmatrix}1 & 1\\ -1 & 1\end{vmatrix}, -\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}, \begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}, \text{i.e. } 2, 0, -2$$

$$\therefore \text{adj } A = \begin{bmatrix}0 & 2 & 2\\ 3 & -3 & 0\\ 3 & 1 & -2\end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}.\text{adj}.A = \frac{1}{6}\begin{bmatrix}0 & 2 & 2\\ 3 & -3 & 0\\ 3 & 1 & -2\end{bmatrix} = \begin{bmatrix}0 & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{2} & -\frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{3}\end{bmatrix}$$

$$\therefore X = A^{-1}B = \begin{bmatrix}0 & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{2} & -\frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{3}\end{bmatrix}\begin{bmatrix}6\\ 2\\ 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}\frac{2}{3}+\frac{1}{3}\\ 3-1\\ 3+\frac{1}{3}-\frac{1}{3}\end{bmatrix} = \begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

$$\therefore x = 1, \quad y = 2, \quad z = 3.$$

Answer Q No 2:

$x - 3y + z = -1$
$2x + y - 4z = -1$
$6x - 7y + 8z = 7$

The above system of equations can be written in matrix from as $Ax=B$.

When $A = \begin{bmatrix}1 & -3 & 1\\ 2 & 1 & -4\\ 6 & -7 & 8\end{bmatrix}$, $X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$, $= \begin{bmatrix}-1\\ -1\\ 7\end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & -3 & 1 \\ 2 & 1 & -4 \\ 6 & -7 & 8 \end{vmatrix}$

$= \begin{vmatrix} 1 & -4 \\ -7 & 8 \end{vmatrix} + 3\begin{vmatrix} 2 & -4 \\ 6 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 6 & -7 \end{vmatrix}$

$= 8-28+3\,(16+24)+(-14-6)$

$= 8-28+120-20$

$= 80 \neq 0$

Now,

Co-factor of the elements of 1st row are –

$\begin{vmatrix} 1 & -4 \\ -7 & 8 \end{vmatrix}, -\begin{vmatrix} 2 & -4 \\ 6 & 8 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 6 & -7 \end{vmatrix}$, i.e. $-20,\ -40,\ -20$

Co-factor of the elements of 2nd row are –

$-\begin{vmatrix} -3 & 1 \\ -7 & 8 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 6 & 8 \end{vmatrix}, -\begin{vmatrix} 1 & -3 \\ 6 & -7 \end{vmatrix}$, i.e. $17,\ 2,\ -11$

Co-factor of the elements of 3rd row are –

$-\begin{vmatrix} -3 & 1 \\ 1 & -4 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix}, \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix}$, i.e. $11,\ 6,\ 7$

$\therefore \text{adj } A = \begin{bmatrix} -20 & 17 & 11 \\ -40 & 2 & 6 \\ -20 & -11 & 7 \end{bmatrix} = -20\begin{bmatrix} 1 & 17 & 11 \\ 2 & 2 & 6 \\ 1 & -11 & 7 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{80}\begin{bmatrix} 20 & 17 & 11 \\ 40 & 2 & 6 \\ 20 & 11 & 7 \end{bmatrix}$

$\therefore X = A^{-1}\ B = \frac{1}{80}\begin{bmatrix} -20 & 17 & 11 \\ -40 & 2 & 6 \\ -20 & -11 & 7 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 7 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{80}\begin{bmatrix} 20-17+77 \\ 40-2+42 \\ 20+11+49 \end{bmatrix} = \frac{1}{80}\begin{bmatrix} 80 \\ 80 \\ 80 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore x = 1,\quad y = 1,\quad z = 1$

CHECK YOUR PROGRESS 9

Answer Q No 1:

Solve : $3x + 5y + 0z = 8$

$-x + 2y - z = 0$

$3x - 6y + 4z = 1$

$$D=\begin{vmatrix}3 & 5 & 0\\ -1 & 2 & -1\\ 3 & -6 & 4\end{vmatrix}$$

$$=3\begin{vmatrix}2 & -1\\ -6 & 4\end{vmatrix}-5\begin{vmatrix}-1 & -1\\ 3 & 4\end{vmatrix}$$

$= 3\,(8\text{-}6) - 5(-4+3)$

$= 6+5 = 11 \neq 0$

$$D_1=\begin{vmatrix}8 & 5 & 0\\ 0 & 2 & -1\\ 1 & -6 & 4\end{vmatrix}=8\begin{vmatrix}2 & -1\\ -6 & 4\end{vmatrix}-5\begin{vmatrix}0 & -1\\ 1 & 4\end{vmatrix}$$

$= 8(8-6) - 5x1 = 16-5 = 11$

$$D_2=\begin{vmatrix}3 & 8 & 0\\ -1 & 0 & -1\\ 3 & 1 & 4\end{vmatrix}$$

$$=3\begin{vmatrix}0 & -1\\ 1 & 4\end{vmatrix}-8\begin{vmatrix}-1 & -1\\ 3 & 4\end{vmatrix}$$

$= 3(0+1) - 8(-4+3)$

$= 3+8 = 11$

$$D_3=\begin{vmatrix}3 & 5 & 8\\ -1 & 2 & 0\\ 3 & -6 & 1\end{vmatrix}$$

$$=3\begin{vmatrix}2 & 0\\ -6 & 1\end{vmatrix}-5\begin{vmatrix}-1 & 0\\ 3 & 1\end{vmatrix}+8\begin{vmatrix}-1 & 2\\ 3 & -6\end{vmatrix}$$

$= 3(2-0) - 5(-1-0) + 8(6-6)$

$= 6+5 = 11$

$$\therefore x=\frac{D_1}{D}=\frac{11}{11}=1$$

$$y=\frac{D_2}{D}=\frac{11}{11}=1$$

$$y=\frac{D_3}{D}=\frac{11}{11}=1$$

i.e. x =1, y =1, z =1.

Answer Q No 2:

$x_1 + x_2 + x_3 = 6$

$x_1 - x_2 + x_3 = 2$

$2x_1 - x_2 + 3x_3 = 9$

$$\therefore D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = 1x\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} - 1x\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + a\,x\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}$$

$= (-3+1)-(3-2)+(-1+2)$

$= -2-1+1 = -2 \neq 0$

$$\therefore D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 9 & -1 & 3 \end{vmatrix} = 6\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} - 1x\begin{vmatrix} 2 & 1 \\ 9 & 3 \end{vmatrix} + 1x\begin{vmatrix} 2 & -1 \\ 9 & -1 \end{vmatrix}$$

$= 6(-3+1)-(6-9)+(-2+9)$

$= -12+3+7 = -2$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 9 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 9 & 3 \end{vmatrix} - 6x\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix}$$

$= (6-9) -6(3-2) +(9-4)$

$= -3 -6 +5$

$= -9 + 5$

$= -4$

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & -1 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ -1 & 9 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix} + 6\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}$$

$= (-9+2) -(9-4) + 6\ x(-1+2)$

$= -7-5+6$

$= -12+6$

$= -6$

$$x_1 = \frac{D_1}{D} = \frac{-2}{-2} = 1$$

$$x_2 = \frac{D_2}{D} = \frac{-4}{-2} = 2$$

$$x_3 = \frac{D_3}{D} = \frac{-6}{-2} = 3$$



8.19 FURTHER READINGS

- G . Cullen, Charles. *Matrices and Linear Transformation*
- Eves, Howard. *Elementary Matrix Theory*
- N. Franklin, Joel. *Matrix theory*
- W. Lewis, David. *Matrix Theory*
- Bodeing, Ewald, *Matrix Calculus*
- Vatssa, B.S, *Theory of Matrices,* New Age International Publishers
- Kala, V.N. & Rana, Rajeshri. *Matrices*
- Narayan, Shanti & Mittal, A Text Book of Matrices



8.20 MODEL QUESTIONS

Q 1: In a Matrix $\begin{bmatrix} 4 & -3 & 19 & \sqrt{2} \\ 11 & 24 & -7 & -5 \\ 1 & \sqrt{3} & 0 & 1 \end{bmatrix}$

Find (i) Order of the matrix
(ii) The number of elements
(iii) Write the elements a_{23}, a_{12}, a_{32}, $a_{34,}$,$a_{44.}$

Q 2: If $A = \begin{bmatrix} 4 & 5 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 3 & -4 \end{bmatrix}$, find $A + B$ and $A - B$

Q 3: If $A = \begin{bmatrix} 3 & 2 \\ 0 & -2 \\ 1 & 0 \end{bmatrix}$, find $3A$ and $-5A$

Q 4: Find AB and BA if $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 5 & 3 \end{bmatrix}$

Q 5: What is the additive identity of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

Q 6: If $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ -1 & 3 & 5 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 \\ 1 & 3 \\ 0 & 2 \end{pmatrix},$

$C = \begin{pmatrix} 3 & 1 & 2 \\ -2 & 2 & 0 \end{pmatrix}$, find A(BC) and (AB)C

and show that A(BC) = (AB) C

Q 7: Show that A(B+C) = AB + AC

where $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & 5 \\ -1 & 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & -3 \\ 6 & 0 & 4 \end{bmatrix}$

Q 8: If $A = \begin{pmatrix} 4 & 3 & 7 \\ 2 & 5 & 6 \end{pmatrix}$ show that $(A')' = A$

Q 9: If $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$, then show that $(AB)' = B'A'$

Q 10: If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then show that $A^4 = \begin{bmatrix} 1+2\times4 & -4\times4 \\ 4 & 1-2\times4 \end{bmatrix}$

Q 11: Find A^{-1} where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$

Q 12: Find the inverse of the matrix $\begin{pmatrix} 4 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

Q 13: If $A = \begin{bmatrix} 2 & -4 & -2 \\ 4 & 6 & 2 \\ 0 & 10 & -4 \end{bmatrix}$, find adj A

Q 14: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$, find A^{-1}

Q 15: Solve by matrix method :

(a) $3x + y + 2z = 3$, $2x - 3y - z = -3$, $x + 2y + z = 4$

(b) $x + y + z = 0$, $2x + 3y - 5z = 7$, $3x-4y - 2z = -1$

(c) $x + y + z = 6$, $x + 2y - 3z = -4$, $-x - 4y + 9z = 18$

Q 16: Solve completely the system of equations by Cramer's rule.

(a) $2x + 3y + z = 9$, $x+2y + 3z = 6$, $3x + y + 2x = 8$

(b) $4x + y + z = 4$, $x+ 4y - 2z = 4$, $-x + 2y - 4z = 2$

(c) $x + y + z + t = 4$, $x + y + z - t = 2$, $x - y + z - t = 0$

*** ***** ***

UNIT 9: LIMIT AND CONTINUITY

UNIT STRUCTURE

9.1 Learning Objectives

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9.3 Limits

- 9.3.1 Finding Limits Analytically
- 9.3.2 Examples of Evaluations of Limits Using Various Rules9
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9.4 Continuity

- 9.4.1 Basic Definitions and Example
- 9.4.2 Algebra of Continuous Functions

9.5 Let Us Sum Up

9.6 Further Readings

9.7 Answer To Check Your Progress

9.8 Model Questions

9.1 LEARNING OBJECTIVES

After going through this unit you will be able to:

- define the concept of limit.
- evaluate limits by algebraic rules.
- explain the concept of continuity of a function.
- show that some well known functions are continuous.

9.2 INTRODUCTION

In the earlier unit, we have learnt about Set theory and functions.

In this unit we are going to discuss about limit and continuity, which

are again very important and fundamental in Mathematics.

The concept of limit is an abstract one. We will learn various algebraic rules to find limits at various points of well-known functions. We will also learn about left and right hand limits. We will also discuss about the continuity of functions.

9.3 LIMITS

We are already familiar with the concept of a real function and a real variable. The concept of limit is an abstract one. Consider the function $f(x) = \frac{x^2 - 9}{x - 3}$, which is defined at all points except $x = 3$. In the first column of following table, some values of x close to 3 are written and in the second column the corresponding values of $f(x)$ are written.

The value of the variable	The corresponding value of $f(x) = \frac{x^2 - 9}{x - 3}$
2.97	5.97
2.98	5.98
2.99	5.99
3.01	6.01
3.02	6.02
3.03	6.03

From the above table, we observe that if x is very close to 3, then $\frac{x^2 - 9}{x - 3}$ is very close to 6. It is also true that for all values of x close to 3, $\frac{x^2 - 9}{x - 3}$ is close to 6. We then say:

As x tends to 3, the function $\frac{x^2 - 9}{x - 3}$ tends to 6 or equivalently, we say: The function $\frac{x^2 - 9}{x - 3}$ has the limit 6 at the point 3.

In symbols, we write

$\lim_{x\to 3}\frac{x^2-9}{x-3}=6.$

In general, by $\lim_{x\to a} f(x)=l$ we mean that as gets close to *a* then $f(x)$ gets close to a limit of *l*. This can be put in a more precise way as:$|\ f(x)-l\ |$can be made arbitrarily small, by taking $|\ x-a\ |$ to be sufficiently small.

> The value of the function *f* at *a* is immaterial for finding the limit of *f* at *a*. Only the values that are close to *a* but not at *a* matter.

The following two results are simple consequences of the definition.

1. **Constant Rule:** $\lim_{x\to a} c=c$, that is, the limit of the constant function $f(x)=c$ as *x* tends to *a* is the value *c*.

2. **Identity Rule:** $\lim_{x\to a} x=a$, that is, the limit of the identity function $f(x)=x$ as *x* tends to *a* is the value *a*.

Uniqueness of Limit

If $\lim_{x\to a} f(x)$ exists, it is unique. There cannot be two distinct numbers l_1 and l_2 such that when *x* tends to *a* the function *f (x)* tends to both l_1 and l_2.

9.3.1 Finding Limits Analytically

It is possible to find limits by using algebraic techniques. The limit behaves well with respect to the operations of addition, subtraction, multiplication by a constant, multiplication, division, etc. The most basic theorem governing analysis of limits is the Principal Limit Theorem.

Principal Limit Theorem:

Let *a* and *k* be real numbers, $n\geq 0,$ and let *f* and *g* be functions with limits at *a* such that $\lim_{x\to a} f(x)=l$ and $\lim_{x\to a} g(x)=m$. Then

1. **Sum Rule:** $\lim_{x\to a}[\ f(x)+g(x)]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)=l+m.$

2. **Difference Rule:** $\lim_{x\to a}[\ f(x)-g(x)]=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)=l-m.$

3. **Product Rule:** $\lim_{x\to a}[f(x)\bullet g(x)]=\lim_{x\to a}f(x)\bullet\lim_{x\to a}g(x)=l\bullet m.$

4. **Quotient Rule:** $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{l}{m}$ provided $\lim_{x\to a}g(x)=m\neq 0.$

5. **Coefficient Rule:** $\lim_{x\to a}k\,f(x)=k\lim_{x\to a}f(x)=kl.$

6. **Power Rule:** $\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a}f(x)\right]^n=l^n.$

We can state the above rules in language. For example, Sum Rule could be expressed as **"The limit of a sum is the sum of the limits."**

9.3.2 Examples of Evaluation of Limits Using Various Rules

Example 1. Find $\lim_{x\to 2}x^2.$

Solution: $\lim_{x\to 2}x^2=\left(\lim_{x\to 2}x\right)^2$ (by Power Rule)

$= 2^2$ (by Identity Rule)

$= 4$

Example 2. Find $\lim_{x\to a}2x^3+3x^2+4x+5.$

Solution:

$$\lim_{x\to a}2x^3+3x^2+4x+5$$

$= 2\lim_{x\to a}x^3+3\lim_{x\to a}x^2+4\lim_{x\to a}x+\lim_{x\to a}5$ (by Sum and Coefficient Rules)

$= 2\left(\lim_{x\to a}x\right)^3+3\left(\lim_{x\to a}x\right)^2+4\lim_{x\to a}x+\lim_{x\to a}5$ (by Power Rule)

$= 2a^3+3a^2+4a+5.$ (by Identity and Constant Rules)

Remark: It is clear from the above example that for all polynomial functions, the limit at a point can be obtained by substituting that value for the variable *x.* But, for non-polynomial functions, this may not be true always. The next two examples show this fact.

Example 3. Find $\lim_{x\to 1}\frac{2x-1}{x^3-2}$.

Solution:

$$\lim_{x\to 1}\frac{2x-1}{x^3-2} = \frac{\lim_{x\to 1}(2x-1)}{\lim_{x\to 1}(x^3-2} \quad \text{(by Quotient Rule)}$$

$$= \frac{\lim_{x\to 1}2x-\lim_{x\to 1}1}{\lim_{x\to 1}x^3-\lim_{x\to 1}2} \quad \text{(by Difference Rule)}$$

$$= \frac{2\lim_{x\to 1}x-\lim_{x\to 1}1}{(\lim_{x\to 1}x)^3-\lim_{x\to 1}2} \quad \text{(by Coefficient and Power Rules)}$$

$$= \frac{2(1)-1}{(1)^3-2} \quad \text{(by Constant and Identity Rules)}$$

$$= -1.$$

Example 4. Find $\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2}$.

Solution: Since the denominator function x^2-3x+2 has the limit 0 at the point 1, that is, $\lim_{x\to 2}x^2-3x+2=0,$ so Quotient Rule cannot be applied here. Therefore, we have to resort to another method.

Factorizing both the numerator and the denominator, we find that

$$\frac{x^2-4}{x^2-3x+2}=\frac{(x-2)(x+2)}{(x-1)(x-2)}.$$

When $x\neq 2,$ we can cancel the common factor *x – 2.*

Thus,

$$\frac{x^2-4}{x^2-3x+2}=\frac{x+2}{x-1}.$$

Therefore,

$$\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2} = \lim_{x\to 2}\frac{x+2}{x-1} = \frac{\lim_{x\to 2}x+\lim_{x\to 2}2}{\lim_{x\to 2}x-\lim_{x\to 2}1} = \frac{2+2}{2-1} = 3$$



CHECK YOUR PROGRESS

Q 1: Write true (T) or false (F).

(a) $\lim_{x\to 1}(x-1)^2+5=5.$ (T / F)

(b) $\lim_{x\to 2}(x-2)(x+4)=0.$ (T/F)

(c) $\lim_{x\to 3}\frac{x^2-9}{x-3}=0.$ (T / F)

(d) $\lim_{x\to -1}\frac{x^2-1}{x+1}=2.$ (T/F)

9.3.3 Limitis by the Method of Substitution

Consider the example $\lim_{x\to 1}\frac{x^2-1}{x-1}$. We find that

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}=\lim_{x\to 1}(x+1)=2$$

The same limit can be evaluated with the help of substitution. If we substitute $y=x-1,$ i.e. $x=y+1,$ in the expression $\frac{x^2-1}{x-1},$ then $\frac{x^2-1}{x-1}=\frac{(y+1)^2-1}{y}=y+2.$ Also, note that when x tends 1, then y tends 0. Therefore,

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{y\to 0}(y+2)=2.$$

In general,

$$\lim_{x\to a}f(x)=\lim_{h\to 0}f(a+h).$$

Equivalently,

$$\lim_{x\to a}f(x)=\lim_{y\to a+h}f(y),\quad where\ y=x+h.$$

Example 5. Find $\lim_{x\to 1}\frac{(x-1)^4}{x^4-1}$.

Solution: We have

$$\lim_{x\to 1}\frac{(x-1)^4}{x^4-1}=\lim_{y\to 0}\frac{y^4}{(y+1)^4-1}=\lim_{y\to 0}\frac{y^4}{y^4+4y^3+6y^2+4y}.$$

$$=\lim_{y\to 0}\frac{1}{y^3+4y^2+6y+4}=\frac{1}{4}.$$

9.3.4 A Special Limit

Let us evaluate $\lim_{x\to a}\frac{x^n-a^n}{x-a}$, where n is a positive integer.

By the method of substitution, we find that

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{h\to 0}\frac{(a+h)^n-a^n}{a+h-a}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[a^n+\binom{n}{1}a^{n-1}h+\binom{n}{2}a^{n-2}h^2+\ldots+h^n-a^n\right]$$

[By Binomial Theorem]

$$=\lim_{h\to 0}\left[\binom{n}{1}a^{n-1}+\binom{n}{2}a^{n-2}h+\ldots+h^{n-1}\right]=\binom{n}{1}a^{n-1}=na^{n-1}.$$

Thus,

$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$, where n is a positive integer.

The above formula remains true, when n is a rational number, provided a is positive.

Example 6. Find $\lim_{x\to 3}\frac{x^3-27}{x^2-9}$.

Solution:

$$\lim_{x\to 3}\frac{x^3-27}{x^2-9}=\lim_{x\to 3}\left[\frac{x^3-27}{x-3}\div\frac{x^2-9}{x-3}\right]=\lim_{x\to 3}\left[\frac{x^3-27}{x-3}\right]\div\lim_{x\to 3}\left[\frac{x^2-9}{x-3}\right].$$

$$=\lim_{x\to 3}\left[\frac{x^3-3^3}{x-3}\right]\div\lim_{x\to 3}\left[\frac{x^2-3^2}{x-3}\right]=3(3)^2\div 2(3)=27\div 6=\frac{9}{2}.$$

Example 7. If $\lim_{x\to 2}\frac{x^n-2^n}{x-2}=12$ and if n is a positive integer, find n.

Solution: We have $\lim_{x\to 2}\frac{x^n-2^n}{x-2}=n2^{n-1}$. According to the given question, $n2^{n-1}=12=3.2^2$. Therefore, $n = 3$.

9.3.5 Some Other Special Limits

The following limit rules of trigonometric and exponential functions are very useful.

1. **Sine Rule:** $\lim_{x\to 0}\frac{\sin x}{x}=1$, where x is measured in radian.
2. **Cosine Rule:** $\lim_{x\to 0}\cos x=1$, where x is measured in radian.
3. **Exponential Rule:** $\lim_{x\to 0}\frac{e^x-1}{x}=1$.

Example 8. Find $\lim_{q\to 0}\frac{\sin 3q}{\sin 4q}$.

Solution: We have

$$\lim_{q\to 0}\frac{\sin 3q}{\sin 4q}=\lim_{q\to 0}\left(\frac{\sin 3q}{3q}\cdot\frac{3q}{4q}\cdot\frac{4q}{\sin 4q}\right)=\lim_{q\to 0}\frac{\sin 3q}{3q}\cdot\lim_{q\to 0}\frac{3q}{4q}\cdot\lim_{q\to 0}\frac{4q}{\sin 4q}$$

$$=\lim_{3q\to 0}\frac{\sin 3q}{3q}\cdot\frac{3}{4}\cdot\frac{1}{\lim_{4q\to 0}\frac{\sin 4q}{4q}}=1\left(\frac{3}{4}\right)\div 1=\frac{3}{4}.$$

Example 9. Find $\lim_{q\to 0}\frac{\tan q}{q}$.

Solution: We have

$$\lim_{q\to 0}\frac{\sin q}{q\cos q}=\lim_{q\to 0}\frac{\sin q}{q}\div\lim_{q\to 0}\frac{1}{\cos q}=\lim_{q\to 0}\frac{\sin q}{q}\div\frac{1}{\lim_{q\to 0}\cos q}=1.$$

Example 10. Find $\lim_{x\to 0}\frac{e^{4x}-1}{x}$.

Solution: We have

$$\lim_{x\to 0}\frac{e^{4x}-1}{x}=\lim_{x\to 0}\frac{e^{4x}-1}{4x}\cdot 4=4\lim_{y\to 0}\frac{e^y-1}{y}=4.$$

9.3.6 One Sided Limits

In this section we will discuss one sided limits.

The notation $\lim_{x\to a+} f(x) = l$ means that when is close x to a and greater than a then $f(x)$ is close to l.. Similarly, $\lim_{x\to a-} f(x) = l$ means that when is close x to a and less than a, then $f(x)$ is close to l.

$\lim_{x\to a+} f(x)$ is called the **right limit** of f at a and $\lim_{x\to a-} f(x)$ is called the **left limit** of f at a. Both these are called **one-sided limits** of f at a

Example11. Find $\lim_{x\to 0+} \frac{x}{|x|}$ and $\lim_{x\to 0-} \frac{x}{|x|}$.

Solutions: When x takes positive values, then $\frac{x}{|x|} = \frac{x}{x} = 1.$

Therefore, we have $\lim_{x\to 0+} \frac{x}{|x|} = 1.$

On the other hand, when takes negative values, then $\frac{x}{|x|} = \frac{x}{-x} = -1.$ So, $\lim_{x\to 0-} \frac{x}{|x|} = -1.$

If $\lim_{x\to a+} f(x) \neq \lim_{x\to a-} f(x)$ then $\lim_{x\to a} f(x)$ does not exist.

If $\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = l$ then $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = l.$

Example 12. Decide whether $\lim_{x\to 0} \frac{x}{|x|}$ exists or not.

Solution: From the solutions of Example 11, we note that $\lim_{x\to 0+} \frac{x}{|x|} = 1$ and $\lim_{x\to 0-} \frac{x}{|x|} = -1.$ Since the right limit and left limit are not equal, therefore $\lim_{x\to 0} \frac{x}{|x|}$ does not exist.



CHECK YOUR PROGRESS

Q 2: Write true (T) or false (F).

(a) If f is a real function such that $f(x) = f(-x)$ for all x, then $\lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x)$. (T /F)

(b) $\lim_{x\to 1-}[x] = 1$. (T / F)

(c) $\lim_{x\to 1+}[x] = 1$. (T / F)

Q 3: Find the following limits:

(i) $\lim_{x\to 2} \frac{x-2}{x+2}$ (ii) $\lim_{x\to 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$ (iii) $\lim_{x\to 3} \frac{e^x - e^3}{x-3}$

9.4 CONTINUITY

In this section, we will learn about the concept of continuity of a function. We will show that a large class of well known functions is continuous.

9.4.1 Basic Definitions and Examples

Let f be a real function and let a be in the domain of f If the limit of f at a is same as the value of f at a, then f is said to be continuous at a.

Thus, if f is to be continuous at a the following three conditions must be satisfied.

a) $f(a)$ is defined,

b) $\lim_{x\to a} f(x)$ exists,

c) $\lim_{x\to a} f(x) = f(a)$.

Example 13. Show that every constant function is continuous at all points.

Solution: Consider a constant function $f(x) = c$ and an arbitrary point a.

Then $f(a) = c$ and $\lim_{x\to a} f(x) = \lim_{x\to a} c = c$, and so, $\lim_{x\to a} f(x) = f(a)$.

Therefore, f is continuous at a

Example 14. Show that the identity function is continuous at all points.

Solution: Consider the identity function $f(x) = x$ and an arbitrary point *a.*

Then, we have $f(a) = a$ and $\lim_{x \to a} f(x) = \lim_{x \to a} x = a.$

Thus, $\lim_{x \to a} f(x) = f(a),$ which shows that f is continuous at '*a*'.

Example 15. Show that the greatest integer function [*x*] is not continuous at 0.

Solution: Let $f(x) = [x].$

Now, $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} [x] = 0$ and $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} [x] = -1$.

Thus, the right hand and left hand limits of *f* at 0 are not equal. Which means that $\lim_{x \to 0} f(x) = \lim_{x \to 0} [x]$ does not exist. Therefore, though $f(0) = [0] = 0,$ *f* is not continuous at 0.

Some more results are listed in the following table.

(c) The function x^2 is continuous at 3.

(d) The function $\frac{x-4}{x-1}$ is continuous at 2.

(e) The function $\sin x$ is continuous at .

(f) The function $\sin x$ is continuous at 0.

(g) The function $\cos x$ is continuous at 0.

(h) The function e^x is continuous at 0.

(i) The function $\frac{x}{|x|}$ is not continuous at 0.

Definition:

A real function is said to be continuous in an open or closed interval if it is continuous at every point of the interval.

Remark: When a function *f* is considered on a closed interval $[a, b]$, then *f* is said to be continuous at the end point *a* if $\lim_{x \to a+} f(x) = f(a).$ Similarly, *f* is said to be continuous at the end point *b* if $\lim_{x \to b-} f(x) = f(b).$

9.4.2 Algebra of Continuous Functions

Theorem: Let *f* and *g* be two functions, continuous at *a* and let *c* be a real number. Then

a) $f + g$ is continuous at *a*

b) $f - g$ is continuous at *a*

c) *cf* is continuous at *a*

d) *fg* is continuous at *a*

e) $\frac{f}{g}$ is continuous at *a* provided $g(a) \neq 0$.

Proofs. Follow from the Principal Limit Theorem and the definition of continuity.

If *f* and *g* are continuous functions, that is, they are continuous at every point of their domains, then the above theorem can be generalized in the following way.

Theorem: If *f* and *g* are continuous functions, then

(a) $f + g$ is continuous,

(b) $f - g$ is continuous,

(c) cf is continuous at,

(d) *fg* is continuous at,

(e) $\frac{f}{g}$ is continuous, at those points where *g* does not take the value zero.

The statement (a) above means that the sum of two (or even more) continuous functions is continuous. Similarly, statement (b) means that the difference of two continuous functions is continuous, and so on.

Example 16. Show that the function $2x^2 + 3x + 4$ is continuous at all points.

Solution: We have already proved that the identity function *x* is continuous at all points. Thus, x^2 is continuous at all points being the product of *x* and *x*. Again, the constant functions 2 and 4 are continuous at all points. Therefore, $2x^2$, $3x$ and 4 are continuous at all points. Finally, $2x^2 + 3x + 4$ is continuous at all points being the

sum of three continuous functions.

Example 17. At What points is the function $\frac{x-2}{(x-3)(x-5)}$ continuous?

Solution: The given function is not defined at the points 3 and 5. The functions in the numerator, i.e., $x - 2$ and in the denominator, i.e., $(x-3)(x-5)$ are continuous at all points. Therefore, the function $\frac{x-2}{(x-3)(x-5)}$ is continuous at all those points where the denominator does not take the value zero. Thus, it is continuous at all points except at 3 and 5.



9.8 LET US SUM UP

- In this unit, we have learnt about the concept of limits and various algebraic rules to find limits at various points of well-known functions. We have also learnt about the left and right hand limits, and found that they need not be equal always.
- We have learnt about the continuity of functions. In summary, we can say that if the limit exists at a point and it is equal to the value of the function at that point then the function is continuous at that point.



9.9 FURTHER READINGS

1. Khanna V. K., Zameeruddin Qazi & Bhambri S.K.(1995). Business Mathematics, New Delhi, Vikas Publishing House Pvt Ltd. .
2. Hazarika P.L. Business Mathematics, New Delhi. S.Chand & Co.

9.10 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (a) T, (b) T, (c) F, (d) F

Ans to Q No 2: (a) T, (b) F, (c) T.

Ans to Q No 3: (i) 0 (ii) 3, (iii) e^3.

9.11 MODEL QUESTIONS

1. Evaluate the following limits:

   (a) $\lim_{x\to 3} \dfrac{x^2-7x+12}{x^2-4x+3}$ (b) $\lim_{x\to 0} \dfrac{\sqrt{1+x}-1}{x}$

   (c) $\lim_{x\to 0} \dfrac{\sin 10x}{\sin 11x}$ (d) $\lim_{x\to k} \dfrac{x^{10}-k^{10}}{x^5-k^5}$

2. Suppose a function *f* is defined as follows.

$$f(x)=\begin{cases} x, & x\geq 0 \\ -x, & x>0. \end{cases}$$

By evaluating the left and right hand limits, show that $\lim_{x\to 0} f(x)=0.$

3. If *f* is an odd function (i.e., $f(-x)=-f(x), \quad \forall x.$) and $\lim_{x\to 0} f(x)$ exists, then show that the value of this limit is zero.

4. Decide whether the function given in Question 2 is continuous at $x=0$ or not.

5. A real function *f* is defined as follows.

$$f(x)=\begin{cases} \dfrac{\tan x}{x}, & x\neq 0 \\ \dfrac{3}{4}, & x=0. \end{cases}$$

Show that *f* is continuous at 0.

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