GCM(S1) 04-02

KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

Patgaon, Rani Gate, Guwahati - 781 017

FIRST SEMESTER

BACHELOR OF COMMERCE

COURSE: 04

Business Mathematics

Block: 2

CONTENTS

UNIT 7	: Matrices
UNIT 8	: Limit and Continuity
UNIT 9	: Derivative of Functions
UNIT 10	: Application of Derivatives
UNIT 11	: Integration and Methods of Integration
UNIT 12	: Linear Programming Problem
REFERENCES	: For All Units

Subject Experts

Professor Nayan Barua, Gauhati University
Professor H. C. Gautam, Gauhati University
Dr. S. K. Mahapatra, Gauhati University

Course Co-ordinator : Harekrishna Deka, KKHSOU

SLM Preparation Team

UNITS	CONTRIBUTORS
7	Dr. Dipu Nath Sarma, Handique Girls' College
8, 9, & 10	S. K. Jha, Lumding College, Lumding
11	Dr. Dibyajyoti Mahanta, KKHSOU
12	Mrs. Ajanta Majumdar, Gauhati Commerce College

Editorial Team

Content : Prof. N. R. Das, Gauhati University
Dr. Sanjay Dutta, PETC Guwahati

Structure, Format & Graphics : Harekrishna, KKHSOU & Dipankar Malakar, KKHSOU

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Headquarters : Patgaon, Rani Gate, Guwahati-781 017

City Office : Housefed Complex, Dispur, Guwahati-781 006; Web: www.kkhsou.in

BLOCK INTRODUCTION

This is the second block of the course ‘Business Mathematics’. The block consists of six units.

The seven unit is an elementary introduction to Matrix theory. In this unit,you will able to explores some of application of matrix theory in this unit.

In eight unit, we discuss limits and continuity of a functions.

The nineth unit deals with concept of derivatives and the derivatives of various function.

The tenth unit deals with application of derivatives such as finding maxima and minima of functions.

In eleven unit, the basics of the Integral calculus are introduced.In this unit,We will discuss different methods of Integration.

The twelve unit introduces us to the concept of Linear programming problem.In this unit,we will discuss the formation and graphical solution of Linear programming problem.

While going through a unit, you will notice some along-sode boxes, which have been included to help you know some of the difficult, unseen terms. Some ‘ACTIVITY’ (s) has been included to help you apply your own thoughts. Again, we have included some relevant concepts in ‘LET US KNOW’ along with the text. And, at the end of each section, you will get ‘CHECK YOUR PROGRESS’ questions. These have been designed to self-check your progress of study. It will be better if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with ‘ANSWERS TO CHECK YOUR PROGRESS’ given at the end of each unit.

BACHELOR OF COMMERCE

Business Mathematics

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UNIT 8: Limits and Continuity **Pages: 185-199**

Limit of a Function Continuity of a Function

UNIT 9: Derivatives of Functions **Pages: 200-231**

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UNIT 7 : MATRICES

UNIT STRUCTURE

7.1 Learning objectives

7.2 Introduction

7.3 Definition of Matrix

7.4 Types of Matrix

7.5 Equality of Matrices

7.6 Addition and Subtraction of Matrices

7.7 Scalar multiplication of Matrix

7.8 Multiplication of Matrices

7.9 Transpose of a Matrix

7.10 Symmetric and Skew-Symmetric Matrix

7.11 Adjoint and Inverse of Matrices and its existence

7.12 Let Us Sum Up

7.13 Answers to check your progress

7.14 Further Reading

7.15 Model Questions

7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- understand the definition of a Matrix
- describe different types of Matrix
- learn matrix operations e.g. addition, subtraction and multiplication
- learn about adjoint and inverse of Matrices.

7.2 INTRODUCTION

Matrix is one of the most powerful tools of modern mathematics. Matrix notations and operations are used in Computer graphics, programming and implementation of electronic spreadsheet programmes. Matrix were introduced by the English Mathematician Arthur Cayley in

1858.Matrices are initially connected with linear transformation. Matrix theory now occupies important position in Physics, Economics, Statistics, Engineering etc.

7.3 DEFINITION OF MATRIX

A set of mn numbers (real or complex) arranged in the form of a rectangular array having m rows and n columns is called an mxn matrix (to be read as 'm' by 'n' matrix).

An m x n matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

The compact representation of the above matrix is

$A = [a_{ij}]$, i = 1, 2, ... m, j = 1, 2, ... n.

or, simply, $A = [a_{ij}]_{mxn}$. Sometimes, the braket [ ] can be replaced by ( ) .

The general element of the matrix A is a_{ij} which belongs to the i-th row and j-th column. a_{ij} is sometimes denoted by (i,j)th element of the matrix. Here i, the first suffix denotes the number of row and j, the second suffix denotes the number of column in which the element a_{ij} occurs.

Note :

1. A matrix having m-rows and n-columns is called a matrix of order mn (read as 'm' by 'n') or simply m x n matrix.

2. In a matrix, the number of rows need not be equal to the number of columns.

Illustrative example :

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 5 \end{bmatrix}_{2\times3} \text{ or } \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 5 \end{pmatrix}_{2\times3}$$

is a 2 x3 matrix i.e. it has 2 rows and 3 columns. For A,

a_{12} = (1,2)th element = 2

a_{23} = (2,3)th element = 5

a_{13} = (1,3)th element = 0

3. A matrix is denoted by the capital letters A, B, C… etc. whereas any elements of a matrix is denoted by small letters such as $a_{ij}, b_{ij}, c_{ij}, \cdots\cdots$ etc.

$$B = \begin{bmatrix} 1+i & 2 & \frac{1}{3} \\ 4 & 0 & 1 \\ \sqrt{3} & 2-i & \sqrt{2} \end{bmatrix}_{3\times 3}$$

is a 3x3 matrix. For B,

b_{11} = 1+ i, $b_{31} = \sqrt{3}$, b_{22} =0 etc.

7.4 TYPES OF MATRIX

There are different types of matrices. We will discuss one by one.

(i) Row Matrix : Any 1x n matrix which has only one row and n column is called a row matrix.

For example,

A = [2, 5,- 3, 0, $\sqrt{3}$] is a row matrix order 1×5.

In general, B = $[a_{ij}]_{1\times n}$ is a row matrix of order 1x n

(ii)**Column Matrix :** Any m x1 matrix which has m rows and only one column is called a column matrix.

For example,

$$B = \begin{bmatrix} 6 \\ 0 \\ -3 \\ \sqrt{2} \end{bmatrix}$$ is a column matrix of order 4×1

In general, $B = [b_{ij}]_{m\times 1}$ is a column matrix of order $m\times 1$.

(iii) **Square Matrix :** A matrix whose numbers of rows is equal to the number of columns is called a square matrix.

For the matrix A= $[a_{ij}]_{m\times n}$, if m = n, then the matrix A is said to be square matrix of order m.

Example:

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ is a square matrix of order 2.

(b) $B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 5 & 0 & 1 \end{bmatrix}$ is a square matrix of order 3

In a square matrix $A = [a_{ij}]_{m \times m}$, the elements a_{11}, a_{22}, a_{33} ..... are called the **Diagonal Elements** and the line along which they lie is called the **Principal Diagonal** of the matrix.

(iv)**Diagonal Matrix :** A square matrix A is said to be a diagonal matrix if all its non-diagonal elements be zero.

Examples:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

are diagonal matrices of order 2 and 3 respectively.

(v)**Scalar Matrix :** A diagonal matrix (i.e. all non-diagonal elements being zero) where all the diagonal elements are equal is called a Scalar Matrix.

Examples :

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

are two scalar matrices of order 2 and 3 respectively.

Thus, the square matrix $A = [a_{ij}]_{m \times m}$ is a scalar matrix if

$a_{ij} = 0$, when $i \neq j$

$a_{ij} = k$(say) when $i = j$

(vi) **Unit (or Identity) Matrix :** A square matrix in which the diagonal elements are unity and non-diagonal elements are all zero is called a unit (or Identity)matrix.

Examples :

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Unit matrices are denoted by I.

I_2 and I_3 are Unit matrices of order 2 and 3 respectively.

Thus, the square matrix $A = [a_{ij}]_{m\times m}$ is a unit matrix if

a_{ij} = 0 when i ≠ j

a_{ij} = 1 when i = j

(vii) **Null Matrix or Zero Matrix :** A matrix of order m x n in which all the elements are zero is called a null matrix (a zero matrix). It is denoted by O.

Thus,

$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ are all zero matrices of order 2 x2 and 4x3

respectively.

(viii) **Upper Triangular Matrix and Lower Triangular Matrix:** Let A be a square matrix of order n.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n}\\ a_{21} & a_{22} & a_{23} & \dots & a_{2n}\\ a_{31} & a_{32} & a_{33} & \dots & a_{3n}\\ \dots & \dots & \dots & \dots & \dots\\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn}\end{bmatrix}$$

The diagonal elements of A are a_{11}, a_{12} ,….., a_{nn} i.e. a_{ij} for i= j.

All these elements of A above the principal diagonal are a_{12}

a_{13}……. a_{1n}, a_{23}……. a_{2n}..... etc. i.e. a_{ij} for i < j

All these elements of A below the principal diagonal are a_{21}, a_{31},

a_{32}, …….a_{n1}, a_{n2}, ……. i.e. a_{ij} for i > j.

If all a_{ij} for i > j be zeros in a square matrix A, then A is called **Upper Triangular matrix.**

Example : $A = \begin{bmatrix}3 & 1 & 2 & 6\\ 0 & 2 & -4 & 1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 5\end{bmatrix}$ is Upper Triangular Matrix

If all a_{ij} for i < j be zeros is a square matrix A, the A is called **Lower Triangular Matrix.**

Example: $A=\begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & -1 & 4 & 0 \\ -2 & 1 & 3 & 5 \end{bmatrix}$ is Lower Triangular Matrix

Remarks: A diagonal matrix is both upper and lower triangular.

7.5 EQUALITY OF MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if

(i) they are of the same size or order

(ii) each element of one matrix is equal to the corresponding element of the other matrix i.e. $a_{ij} = b_{ij}$ for all i and j.

If A and B are equal, then we write A = B.If A and B are not equal, then we write $A \neq B$.

Example :

1. If $A=\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}_{2\times2}, B=\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}_{2\times2}$ then A = B .

2. If $A=\begin{pmatrix} 1^2 & 3^2 \\ 2^2 & 4^2 \end{pmatrix}_{2\times2}, B=\begin{pmatrix} 1 & 9 \\ 4 & 16 \end{pmatrix}_{2\times2}$ then A = B.

3. If $A=\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}_{2\times2}, B=\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}_{2\times3}$ then $A \neq B$.

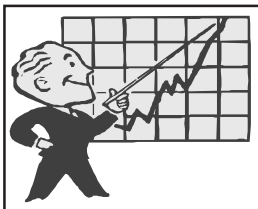
4. If $\begin{pmatrix} x & a \\ y & b \\ z & c \end{pmatrix}_{3\times2} = \begin{pmatrix} -3 & 0 \\ 1 & 2 \\ 4 & \sqrt{3} \end{pmatrix}_{3\times2}$

then, x = - 3, y = 1, z = 4, a = 0, b = 2, c = $\sqrt{3}$

Remarks: The inequality of two matrices arised due to the following reasons :

(i) This orders may not be equal.

(ii) Elements in the corresponding places may not be equal.



CHECK YOUR PROGRESS 1

Q 1: If a matrix has 10 elements what are the possible orders it can have?

Q 2: Find the values of a, b, c, d when

$$\begin{pmatrix} 2a+b & 5c-d \\ a-2b & 4c+d \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ -3 & 25 \end{pmatrix}$$

Q 3: Are the following matrices equal where

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 0 & 5 \\ 0 & 0 & 6 \end{pmatrix}, B = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

7.6 ADDITION AND SUBTRACTION OF MATRICES

Let $A = [a_{ij}]_{mxn}$ and $B = [b_{ij}]_{mxn}$ be two matrices of same order mxn. Then their sum(or difference) denoted by A+B (or A-B) is defined as another matrix $C= [c_{ij}]$ of the order m xn such that any element of C is the sum (or difference) of the corresponding element of A and B.

i.e. $c_{ij} = a_{ij}+b_{ij}$ for all i, j

Thus, $C =[c_{ij}] = [a_{ij}+b_{ij}]_{mxn}$

Example : If $A = \begin{pmatrix} 5 & 1 \\ 3 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & -5 \\ -2 & 1 \end{pmatrix}$

Then, $C = A+B = \begin{pmatrix} 5+2 & 1-5 \\ 3+(-2) & 0+1 \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ 1 & 1 \end{pmatrix}$

Similarly, $D=A-B = \begin{pmatrix} 5-2 & 1-(-5) \\ 3-(-2) & 0-1 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 5 & -1 \end{pmatrix}$

Remark : Two matrices A and B are said to be conformable for addition or subtraction if they are of the same order.

Negative of a Matrix : If A be a given matrix, then the negative of A denoted by –A is the matrix whose elements are the negative of the corresponding elements of A.

Example: If $A=\begin{pmatrix} 2 & -1 \\ 0 & 4 \\ -5 & 6 \end{pmatrix}$ then $-A=\begin{pmatrix} -2 & 1 \\ 0 & -4 \\ 5 & -6 \end{pmatrix}$

Properties of Matrix Addition:

The addition of matrices satisfies the following properties :

(i) **Commutative Law :** If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same order, say m xn, then A + B = B + A

Proof :

$A+B = [a_{ij}] + [b_{ij}]$

$= [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}]$ (Θ addition of numbers is commutative)

$= [b_{ij}] + [a_{ij}]$

$= B + A$

(ii)**Associative Law :** If $A = [a_{ij}]$, $B=[b_{ij}]$ and $C = [c_{ij}]$ are matrices of the same order mxn, then (A+B) + C = A + (B+C)

Proof: $(A+B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$= [a_{ij} + b_{ij}] + [c_{ij}]$

$= [(a_{ij} + b_{ij}) + c_{ij}]$

$= [a_{ij}] + [(b_{ij} + c_{ij})]$

$= [a_{ij}] + ([b_{ij}] + [c_{ij}])$

$= A + (B+C)$

i.e. $(i,j)^{th}$ element of (A+B)+C = $(i,j)^{th}$ element of A+(B+C)

(iii) **Existence of Additive Identity :** If O be the zero matrix of order $m \times n$ and $A=[a_{ij}]$ be an $m \times n$ matrix, then A+ O = O + A = A

Proof :

$A + O = [a_{ij}] + [0]$

$= [a_{ij} + 0] = [a_{ij}] = A$

and $O + A = [0] + [a_{ij}]$

$= [0+a_{ij}]$

$= [a_{ij}]$

$= A$

$\therefore$ A + O = O + A = A

Hence, the zero matrix of the type $m \times n$ plays the role of additive identity.

(iv) **Existence of Additive inverse :** If $A = [a_{ij}]$ be a matrix of order $m \times n$, then we have another matrix $-A = [-a_{ij}]$ such that

$A+(-A) = (-A) + A = 0$

Proof : $A + (-A) = [a_{ij}] + [-a_{ij}]$

$= [a_{ij} - a_{ij}]$

$= [0] = 0$

$(-A) + A = [-a_{ij}] + [a_{ij}]$

$= [-a_{ij} + a_{ij}]$

$= [0] = 0$

Hence, $A + (-A) = (-A) + A = 0$

Here, $-A$ acts as the additive inverse of A.

(v) **Subtraction of two Matrices :** If A and B be two matrices of order $m \times n$, then we define $A - B = A + (-B)$.

In other words, the difference $A - B$ is obtained by subtracting from each element of A the corresponding element of B.

(vi) **Cancellation Laws for addition of Matrices :** If A, B, C are three matrices of the same order say $m \times n$, then

$A + B = A + C \Rightarrow B = C$ [Left Cancellation law]

$B + A = C + A \Rightarrow B = C$ (Right Cancellation Law)

Proof : We have

$A + B = A + C$

$\Rightarrow -A + (A+B) = -A + (A+C)$ [Adding $-A$ to both sides]

$\Rightarrow (-A+A) + B = (-A+A) + C$ [Θ matrix addition is associative]

$\Rightarrow O + B = O + C$

$\Rightarrow B = C$

Similarly :

$B + A = C + A$

$\Rightarrow (B+A) + (-A) = (C+A) + (-A)$

$\Rightarrow B + (A - A) = C + (A - A)$

$\Rightarrow B + O = C + O$

$\Rightarrow B = C$

Example:(1) If $A=\begin{pmatrix} 5 & 0 \\ -3 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$, find A + B and A – B

Solution :

$$A+B=\begin{pmatrix} 5 & 0 \\ 3 & 1 \end{pmatrix}+\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 8 & 1 \\ 9 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} 8 & 1 \\ 9 & 3 \end{pmatrix}$$

$$A-B=\begin{pmatrix} 5 & 0 \\ 3 & 1 \end{pmatrix}-\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 5-3 & 0-1 \\ 3-6 & 1-2 \end{pmatrix}$$

$$=\begin{pmatrix} 2 & -1 \\ -3 & -1 \end{pmatrix}$$

(2) If $A=\begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}$, $B=\begin{pmatrix} 0 & 3 \\ 5 & -2 \end{pmatrix}$, $C=\begin{pmatrix} 3 & 5 \\ 2 & 0 \end{pmatrix}$

Show that (A+B) + C = A + (B+C)

Solution :

$$A+B=\begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}+\begin{pmatrix} 0 & 3 \\ 5 & -2 \end{pmatrix}=\begin{pmatrix} 1+0 & 0+3 \\ 4+5 & 3-2 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 3 \\ 9 & 1 \end{pmatrix}$$

$$(A+B)+C=\begin{pmatrix} 1 & 3 \\ 9 & 1 \end{pmatrix}+\begin{pmatrix} 3 & 5 \\ 2 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 1+3 & 3+5 \\ 9+2 & 1+0 \end{pmatrix}$$

$$=\begin{pmatrix} 4 & 8 \\ 11 & 1 \end{pmatrix}$$

Again, B+C = $\begin{pmatrix}0 & 3\\5 & -2\end{pmatrix}+\begin{pmatrix}3 & 5\\2 & +0\end{pmatrix}$

$= \begin{pmatrix}0+3 & 3+5\\5+2 & -2+0\end{pmatrix}$

$= \begin{pmatrix}3 & 8\\7 & -2\end{pmatrix}$

$\therefore A+(B+C)=\begin{pmatrix}1 & 0\\4 & 3\end{pmatrix}+\begin{pmatrix}3 & 8\\7 & -2\end{pmatrix}$

$= \begin{pmatrix}1+3 & 0+8\\4+7 & 3-2\end{pmatrix}=\begin{pmatrix}4 & 8\\11 & 1\end{pmatrix}$

$\therefore (A+B)+C=A+(B+C)$

(3) **Find the additive inverse of**

$A=\begin{pmatrix}2 & 3 & 1\\0 & -1 & 5\end{pmatrix}_{2\times3}$

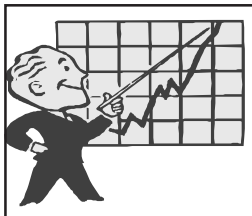
Solution:

A is a matrix of order 2×3.

– A will also be a matrix of order 2×3.

$-A=\begin{pmatrix}-2 & -3 & -1\\0 & -(-1) & -5\end{pmatrix}$

$=\begin{pmatrix}-2 & -3 & -1\\0 & 1 & -5\end{pmatrix}_{2\times3}$



CHECK YOUR PROGRESS 2

Q 1: Is it possible to define A+B when–

(i) A has 3 rows, 2 columns and B has 2 rows, 2 columns.

(ii) A has 3 rows, 3 columns and B has 3 rows, 3 columns.

(iii) A and B are square matrices of same order.

Q 2: What is the additive identity of 2x3 matrices ?

7.7 MULTIPLICATION OF A MATRIX BY A SCALAR

Multiplication of a matrix by a scalar is defined as follows:

If $A = [a_{ij}]_{m\times n}$ is a matrix of order $m\times n$ and k is a scalar, then

$kA = k[a_{ij}]_{m\times n} = [ka_{ij}]_{m\times n}$

i.e., (i, j)th element of kA is k times the (i,j) th element of A .

Example:

If $k = 3$ and $A = \begin{pmatrix} 2 & -1 \\ -2 & 3 \\ 0 & 1 \end{pmatrix}_{2\times3}$,then

$$3A = 3\begin{pmatrix} 2 & -1 \\ -2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3\times2 & 3\times(-1) \\ 3\times(-2) & 3\times3 \\ 3\times0 & 3\times1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -3 \\ -6 & 9 \\ 0 & 3 \end{pmatrix}_{3\times2}$$

and,

$$\frac{1}{2}A = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\times2 & \frac{1}{2}\times(-1) \\ \frac{1}{2}\times(-2) & \frac{1}{2}\times3 \\ \frac{1}{2}\times0 & \frac{1}{2}\times1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & \frac{3}{2} \\ 0 & \frac{1}{2} \end{pmatrix}_{3\times2}$$

PROPERTIES OF MULTIPLICATION OF A MATRIX BY A SCALAR :

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order m x n ; k and l are two scalars, then

(i) $k(A+B) = kA + kB$

(ii) $(k+l)A = kA + lA$

(iii) $k(lA) = (kl)A$

(iv) $(-k)A = -(kA) = k(-A)$

Proof of (i) :

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$

Then k(A+B) $=k([a_{ij}]+[b_{ij}])$

$= k([a_{ij}+b_{ij}])$

$=([ka_{ij}+kb_{ij}])$

$=[ka_{ij}+kb_{ij}]$

$= [ka_{ij}]+[kb_{ij}]$

$= k[a_{ij}]+k[b_{ij}]$

= kA + kB.

Example: If $A = \begin{pmatrix} 3 & 9 \\ 1 & 8 \end{pmatrix}_{2\times2}$, $B = \begin{pmatrix} 4 & 0 \\ 7 & 2 \end{pmatrix}_{2\times2}$

Then verify that 2(A+B) = 2A + 2B

Solution: We have $A + B = \begin{bmatrix} 3 & 9 \\ 1 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 7 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3+4 & 9+0 \\ 1+7 & 8+2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix}$$

$$2\ (A+B) = 2\begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix} = \begin{bmatrix} 2\times7 & 2\times9 \\ 2\times8 & 2\times10 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 18 \\ 16 & 20 \end{bmatrix}$$

Again, $2A+2B = 2\begin{bmatrix} 3 & 9 \\ 1 & 8 \end{bmatrix} + 2\begin{bmatrix} 4 & 0 \\ 7 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2\times3 & 2\times9 \\ 2\times1 & 2\times8 \end{bmatrix} + \begin{bmatrix} 2\times4 & 2\times0 \\ 2\times7 & 2\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 18 \\ 2 & 16 \end{bmatrix} + \begin{bmatrix} 8 & 0 \\ 14 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+8 & 18+0 \\ 2+14 & 16+4 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 18 \\ 16 & 20 \end{bmatrix}$$

2(A+B) = 2A + 2B verified.

Proof of (ii) Let A = $[a_{ij}]$, then

$$(k+l)A = (k+l)[a_{ij}]$$

$$= [ka_{ij} + la_{ij}]$$

$$= [ka_{ij}] + [la_{ij}]$$

$$= k[a_{ij}] + l[a_{ij}]$$

$$= kA + lA$$

Example: If k = 3, l = –2 and $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

Then verify that (k+l) A = kA + lA

$$(k+l)A = (3-2)\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} = (1)\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Again,

$$kA + lA = 3A - 2A = 3\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 \\ 3 & 0 \\ 6 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 4 \\ 2 & 0 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-6 & 6-4 \\ 3-2 & 0-0 \\ 6-4 & 3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$\therefore$ (k+l) A = kA + lA. Verified

Proof of (iii) : Let A = $[a_{ij}]$, then

$k(lA) = k(l[a_{ij}])$

$= k[l\, a_{ij}]$

$= [k(l\, a_{ij})]$

$= [(kl)\, a_{ij}]$

$= (kl)\, [a_{ij}] = (kl)\, A.$

Example: If k=4, l=2 and $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$, then

Verify that k(lA) = (kl)A.

Solution: $k(lA) = 4(2A) = 4 \begin{bmatrix} 2\times3 & 2\times2 \\ 2\times1 & 2\times0 \\ 2\times2 & 2\times1 \end{bmatrix}$

$= \begin{bmatrix} 4\times2\times3 & 4\times2\times2 \\ 4\times2\times1 & 4\times2\times0 \\ 4\times2\times2 & 4\times2\times1 \end{bmatrix}$

$= \begin{bmatrix} 8\times3 & 8\times2 \\ 8\times1 & 8\times0 \\ 8\times2 & 8\times1 \end{bmatrix}$

$= 8\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

= 8A

= (4×2) A = (kl)A. Verified

Proof of (iv): If $A = [a_{ij}]$ then

$(-k)\, A = [(-k)\, a_{ij}]$

$= [-(k\ a_{ij})]$

$= -[k\ a_{ij}] = -(kA)$

$$\text{Also } k(-A) = k[-a_{ij}]$$

$$= [k.-(a_{ij})]$$

$$= -[k\ a_{ij}]$$

$$= -(kA)$$

$$\therefore (-k)A = -(kA) = k(-A)$$

Example : If $k = 5$ and $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

Verify that $(-k)A = -(kA) = k(-A)$

Solution: $(-k)A = (-5)\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} (-5)\times 3 & -(5\times 2 \\ (-5)\times 1) & -(5\times 0) \\ -(5\times 2) & -(5\times 1) \end{bmatrix}$$

$$= -\begin{bmatrix} (5)\times 3 & (5)\times 2 \\ (5)\times 1 & (5\times 0) \\ (5\times 2) & (5\times 1) \end{bmatrix} = -(kA)$$

$$\text{Also, } k(-A) = 5\begin{bmatrix} -3 & -2 \\ -1 & 0 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times(-3) & 5\times(-2) \\ 5\times(-1) & 5\times 0 \\ 5\times(-2) & 5\times(-1) \end{bmatrix}$$

$$= -\begin{bmatrix} 5\times 3 & 5\times 2 \\ 5\times 1 & 5\times 0 \\ 5\times 2 & 5\times 1 \end{bmatrix}$$

$$= -5\begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} = -(kA) \quad \text{Verified.}$$

Example: If $A = \begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$, find 3A – 2B.

Solution: $3A - 2B = 3\begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 2 \end{bmatrix} - 2\begin{bmatrix} 4 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 9 & 6 \\ 3 & 12 & 6 \end{bmatrix} - \begin{bmatrix} 8 & 0 & -6 \\ 0 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0-8 & 9-0 & 6-(-6) \\ 3-0 & 12-2 & 6-4 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 9 & 12 \\ 3 & 10 & 2 \end{bmatrix}$$

7.8 MULTIPLICATION OF TWO MATRICES

Two matrices A and B are conformable for the product AB if and only if the number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$,

Then the product A,B,C is the matrix of order m×p

i.e. $AB = C = [c_{jk}]_{m\times p}$ where

$$C_{jk} = \sum_{j=1}^{n} a_{ij}\, b_{jk} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + \ldots\ldots\ldots\ldots + a_{in}\, b_{nk}$$

i.e. the (j, k) th element of the matrix C= AB is found by multiplying the corresponding element of the j-th row of A and the k-th column of B and then, adding the product.

The rule of multiplication is *row by column multiplication*.

Example: If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2\times 2}$

Then $AB = \begin{bmatrix} a_{11}\, b_{11} + a_{12}\, b_{21} & a_{11}\, b_{12} + a_{12}\, b_{22} \\ a_{21}\, b_{11} + a_{22}\, b_{21} & a_{21}\, b_{12} + a_{22}\, b_{22} \\ a_{31}\, b_{11} + a_{32}\, b_{21} & a_{31}\, b_{12} + a_{32}\, b_{22} \end{bmatrix}_{3\times 2}$

Note : 1) If the product AB exists, then it is not necessary that the product BA will also exist.

Example : $A = [2\ \ 3]_{1\times 2}$ and $B = \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix}_{2\times 3}$

Here, $AB = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 8+3 & 0+6 & 10+9 \end{bmatrix} = \begin{bmatrix} 11 & 6 & 19 \end{bmatrix}$

But $BA = \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \end{bmatrix}$ which does not exist

$\therefore AB \neq BA$

Matrix multiplication is not commutative:

2) In the case when both A and B are square matrices of the same order then also both AB and BA are defined but still $AB \neq BA$.

Example : If $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$

Then $AB = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2x3+1x4 & 2x5+1x1 \\ 0x3+2x4 & 0x5+2x1 \end{bmatrix}$

$= \begin{bmatrix} 10 & 11 \\ 8 & 2 \end{bmatrix}$

$BA = \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3x3+5x0 & 3x1+5x2 \\ 4x2+1x0 & 4x1+1x2 \end{bmatrix}$

$= \begin{bmatrix} 9 & 13 \\ 8 & 6 \end{bmatrix}$

It is seen that $AB \neq BA$

Example: If $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ -2 & 1 & 3 \end{bmatrix}$

Can you find AB and BA ?

Solution :

Here, A is of order 2×2

B is of order 2×3

∴ AB will be order 2×3

Again, B is of order 2×3

A is of order 2×2

∴ BA does not exist because the number of column of B is 3 and the number of rows of A is 2.

So, they are not conformable for the product BA.

PROPERTIES OF MATRIX MULTIPLICATION :

The multiplication of matrices possesses the following properties:

(1) The multiplication of matrices is not always commutative.

(a) Whenever AB exists, it is not necessary that BA should exist.

(b) Whenever AB and BA both exist, it is not always necessary that they should be matrices of same order.

(c) Whenever AB and BA exists and are of the same order, it is not necessary that AB = BA.

(d) In some cases, AB = BA.

(e) Matrix multiplication is associative i.e. A(BC) = (AB)C where A, B, C are matrices of order m×n, n×p, p×q respectively.

(f) Matrix multiplication is distribution w. r. t. addition of matrix .i.e, A(B+C) = AB+AC

Where, A, B, C are matrices of order m×n, n×p and n×p respectively.

(g) For every square matrix A, there exists the identity matrix I of same order of A such that AI = IA = A.

MATRIX MULTIPLICATION IS ASSOCIATIVE

If A, B, C are matrices of order m×n, n×p, p×q, then A(BC) = (AB) C

Proof :

Let A= $[a_{ij}]_{m\times n}$, B = $[b_{jk}]_{n\times p}$, C = $[c_{kl}]_{p\times q}$

Then AB = U = $[u_{ik}]_{m\times p}$ where

$$u_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$$

Also, $BC=V=[v_{jl}]_{n\times q}$ where $v_{jl} = \sum_{k=1}^{n} b_{jk} c_{kl}$

∴ A(BC) = W = $[w_{il}]_{m\times q}$ when (i, I)th element of A(BC)

$$= W_{il} = \sum_{j=1}^{n} a_{ij}\, v_{jl}$$

$$= \sum_{j=1}^{n}\left[a_{ij}\left\{\sum_{K=1}^{n} b_{jk}c_{kl}\right\}\right]$$

$$= \sum_{K=1}^{n}\left[\left\{\sum_{j=1}^{n} a_{ij}\, b_{kl}\right\}C_{kl}\right]$$

$$= \sum_{k=1}^{n} u_{ik}c_{kl}$$

= (i, j)th element of (AB) C

∴ A(BC) = (AB) C

MULTIPLICATION OF MATRICES IS DISTRIBUTIVE WITH RESPECT TO ADDITION OF MATRICES :

If A, B and C are matrices of order m×n, n×p, n×p respectively, then A(B+C) = AB + AC

Proof :

Let A=$[a_{ij}]_{m\times n}$, B =$[b_{jk}]_{n\times p}$, C=$[c_{jk}]_{nxp}$

We have B+C = $[b_{jk}+c_{jk}]_{n\times p}$

∴ (i, k)th element of A(B+C) = $\sum_{j=1}^{n} a_{ij}(b_{jk}+c_{jk})$

$$= \sum_{j=1}^{n} a_{ij}\, b_{jk} + \sum_{j=1}^{n} a_{ij}\, c_{jk}$$

= (i, k)th element of AB + (i, k)th element of AC

= (i, k)th element of AB +AC

Here, A(B+C) = AB + AC

Similarly, we can show that,

(B+C) D = BD + CD

where B, C and D are matrices of suitable types.

Example :

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$, $B = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}_{2\times 3}$

$C = \begin{bmatrix} 5 & 0 & 1 \\ 4 & 6 & 0 \end{bmatrix}_{2\times 3}$ then show that

A(B+C) = AB + AC

Solution :

$$B+C = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 1 \\ 4 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 1 & 3 \\ 4 & 8 & 1 \end{bmatrix}_{2\times 3}$$

A(B+C)

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \begin{bmatrix} 8 & 1 & 3 \\ 4 & 8 & 1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1\times 8+2\times 4 & 1\times 1+2\times 8 & 1\times 3+2\times 1 \\ 3\times 8+4\times 4 & 3\times 1+4\times 8 & 3\times 3+4\times 1 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 16 & 17 & 5 \\ 40 & 35 & 13 \end{bmatrix}$$

Again, $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\times 3+2\times 0 & 1\times 1+2\times 2 & 1\times 2+2\times 1 \\ 3\times 3+4\times 0 & 3\times 1+4\times 2 & 3\times 2+4\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & 4 \\ 9 & 11 & 10 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 4 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times5+2\times4 & 1\times0+2\times6 & 1\times1+2\times0 \\ 3\times5+4\times4 & 3\times0+4\times6 & 3\times1+4\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 12 & 1 \\ 31 & 24 & 3 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 3 & 5 & 4 \\ 9 & 11 & 10 \end{bmatrix} + \begin{bmatrix} 13 & 12 & 1 \\ 31 & 24 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 17 & 5 \\ 40 & 35 & 13 \end{bmatrix}$$

. $A(B + C) = AB + AC$

Example :

$$\text{If } A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}_{2\times2}, B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}_{3\times2}$$

$$C = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}_{2\times2}$$

Find A(BC) and (AB)C and show that A(BC) ≠ (AB) C.

Solution :

$$BC = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}_{3\times2} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}_{2\times2}$$

$$= \begin{bmatrix} 1\times1+3\times2 & 1\times2+3\times0 \\ 0\times1+2\times2 & 0\times2+2\times0 \\ -1\times1+4\times2 & -1\times2+4\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ 4 & 0 \\ 7 & 2 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 4 & 0 \\ 7 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times7+1\times4+(-1)\times7 & 1\times2+1\times0+(-1)\times(-2) \\ 2\times7+0\times4+3\times7 & 2\times2+0\times0+3\times(-2) \\ 3\times7+(-1)\times4+2\times7 & 3\times2+(-1)\times0+2\times(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 \\ 35 & -2 \\ 31 & 2 \end{bmatrix}$$

$$\text{and } AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+1\times0+(-1)(-1) & 1\times3+1\times2+(-1)4 \\ 2\times1+0\times0+3(-1) & 2\times3+0\times2+3\times4 \\ 3\times1+(-1)\times0+2\times(-1) & 3\times3+(-1)\times2+2\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times1+1\times2 & 2\times2+1\times0 \\ (-1)\times1+18\times2 & (-1)\times2+18\times0 \\ 1\times1+15\times2 & 1\times2+15\times0 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 35 & -2 \\ 31 & 2 \end{bmatrix}$$



CHECK YOUR PROGRESS 3

Q 1: Given

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ C = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix},\ D = \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

B(C+D) = BC + BD

(B+C) D = BD + CD

Q 2: For the matrices A,B,C put the numerical values of their orders, so that (A+B) C = AC + BC

Q 3: Find A and B if

$$A + B = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix},\ A - B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Q 4: Given $A = \begin{bmatrix} 0 & 1 \\ 21 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, Show that

$A^2 = B^2 = I$ (unit matrix)

Q 5: If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$

Q 6: If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then prove that $(A_\alpha)^3 = \begin{bmatrix} \cos 3\alpha & \sin 3\alpha \\ -\sin 3\alpha & \cos 3\alpha \end{bmatrix}$

7.9 TRANSPOSE OF A MATRIX

Let $A = [a_{ij}]_{m\times n}$. Then the n×m matrix obtained from A by changing the rows of A into columns and columns into rows is called the Transpose of matrix A and is denoted by A^I or A^T.

Therefore, A^I or A^T will have n-rows and m-columns.

Example :

If $A = \begin{bmatrix} 2 & 0 & 5 \\ 1 & 2 & 4 \end{bmatrix}_{2\times 3}$ Then,

$$A^I = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 5 & 4 \end{bmatrix}_{3\times 2}$$

SOME PROPERTIES OF TRANSPOSE OF MATRICES :

(i) The transpose of transpose of a matrix is the matrix itself i.e. if A is a matrix, then $(A^I)^I = A$

(ii) If A is a matrix f order mxn and k is a scalar, then $(kA)^I = kA^I$

(iii) If A and B are matrices of same order, then $(A+B)^I = A^I + B^I$

(iv) If A and B are two matrices conformable to multiplication, the $(AB)^I = B^I A^I$.

This law is known as "reversal Law for transpose".

Example:

If $A = \begin{bmatrix} 3 & 4 \\ 4 & 2 \\ 5 & 3 \end{bmatrix}$ then show that

$(A^I)^I = A$

Solution :

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & 3 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

$$(A')' = \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & 3 \end{bmatrix}$$

Hence, $(A')' = A$

Note : The reversal law for transpose can be extended for more than two matrices.

$(ABC)' = C'B'A'$

$(ABCD)' = D'C'B'A'$ etc.

Example :

If $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Show that (i) $(2A)' = 2A'$

(ii) $(A+B)' = A' + B'$

(iii) $(AB)' = B'A'$

Solution :

(i) $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

$$2A = \begin{bmatrix} 4 & 6 \\ 8 & 2 \end{bmatrix}, A' = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$$

$$\therefore (2A)' = \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix}, 2A' = 2\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix}$$

$\therefore \quad (2A)' = 2A'$

(ii) $A + B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 2+1 & 3+2 \\ 4+3 & 1+4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 5 \end{bmatrix}$$

$$(A+B)' = \begin{bmatrix} 3 & 7 \\ 5 & 5 \end{bmatrix}$$

Again, $A' + B' = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 2+1 & 4+3 \\ 3+2 & 1+4 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 5 & 5 \end{bmatrix}$$

Hence, $(A+B)' = A' + B'$

(ii) $AB = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 2\times1+3\times3 & 2\times2+3\times4 \\ 4\times1+1\times3 & 4\times2+1\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 16 \\ 7 & 12 \end{bmatrix} \text{ and } (AB)' = \begin{bmatrix} 11 & 7 \\ 16 & 12 \end{bmatrix}$$

Again, $B'A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\times2+3\times3 & 1\times4+3\times1 \\ 2\times2+4\times3 & 2\times4+4\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 7 \\ 16 & 12 \end{bmatrix}$$

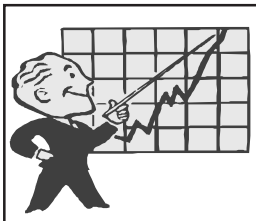
Hence, $(AB)' = B'A'$

Note :

i) $(kA)' = kA'$

ii) $(A+B)' = A'' + B'$

iii) $(AB)' = AB$ if $AB = BA$



CHECK YOUR PROGRESS 4

Q 1: If $A = \begin{bmatrix} -4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}_{3\times2}$ and $B = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}_{2\times3}$

Show that $(AB)' = B'A'$

Q 2: With suitable example show that $(A')' = A$

Q 3: With suitable example, show that $A' = -A$

7.10 SYMMETRIC AND SKEW-SYMMETRIC MATRIX

SYMMETRIC MATRIX :

A square matrix $A = [a_{ij}]$ is said to be symmetric if for all values of i and j, i.e, $a_{ij} = a_{ji}$

i.e. $(i,j)^{th}$ element is same as the $(j,i)^{th}$ element of A

For example

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

are symmetric matrix.

Theorem : The necessary and sufficient condition for a matrix A to be symmetric is that $A = A'$

Proof : Let $A=[a_{ij}]_{nxn}$ is a n-rowed square matrix.

Necessary : $a_{ij} = a_{ji}$

Now, A' which is the transpose of A is also a square matrix.

$\therefore$ $(i,j)^{th}$ element of A' =$(j, i)^{th}$ element of A

$= a_{ji}$ =$(i\ j)^{th}$ element of A

Hence A is symmetric.

Sufficient : Let $A' = A$

$\therefore$ $(i, j)^{th}$ elements of A = (j, i) th element of A'

= $(j,\ i)^{th}$ element of A

Hence A is symmetric.

Skew symmetric Metrix : A square matrix $A=[a_{ij}]$ is said to be skew-synmetric if for all value of i and j, $a_{ij} = - a_{ji}$

i.e. $(i,j)^{th}$ element is the negative of $(j,i)^{th}$ element of A

For example

$$\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$$

are skew symmetric matrix.

If A is skew-symmetric then –

$a_{ij} = -a_{ji}$ $\forall$ i, j

$\therefore$ $a_{ii} = - a_{ii}$ $\forall$ i

$= 2a_{ii} = 0$

$= a_{ii} = 0$

Thus the diagonal elements a_{11}, a_{22}, a_{33} ..... of a skew-symmetric matrix are all zero.

Theorem : Necessary and sufficient condition for a matrix A to be skew-symmetric is that. $A^{/} = -A$.

Prof : Necessary :

Let $A = [a_{ij}]_{nxn}$ be an n rowed skew symmetric matrix. Then $a_{ij} = -a_{ji}$

$\therefore$ (i,j) th element of $A^{/}$ =(j, i) th element of A

$= a_{ji}$

$= -a_{ij}$

= – (i,j)th element of A

$\therefore \quad A^{/} = -A$

Sufficiency part :

Let $A^{/} = -A$

$\therefore$ (i, j)th element of $A^{/}$ =–(i, j)th element of A

$=$ (j, i)th element of A = –(i, j)th element of A

$= a_{ji} = -a_{ij}$

$= a_{ij} = -a_{ji}$

Hence A is skew-symmetric.

Example 1:

If A is symmetric (skew symmetric) matrix then kA is also symmetrix (skew symmetric).

Note :

i) If A is a symmetric matrix, then $A^{/} = A$

ii) A is a skew-symmetric matrix then $A^{/} = -A$

Solution :

i) Let A be symmetric. Then $A^{/} = A$

Now $(kA)^{/} = kA^{/} = kA$

$=$ kA is also symmetric.

ii) Let A be skew symmetric. Then $A^{/} = -A$

Now, $(kA)^{/} = kA^{/} = k(-A) = -(kA)$

$=$ kA is also symmetric.

Example 2:

Show the matrix A+B is symmetric or skew-symmetric according as A and B are symmetric or skew-symmetric matrix.

Solution :

i) A and B be two symmetric matrics

$\therefore$ $A^{/}=A$, $B^{/}=B$

Now, $(A+B)^{/} = A^{/} + B^{/} = A+B$

$\therefore$ A+B is symmetric

ii) A and B be to skew-symmetric matrices

$\therefore A^{/} = -A, B^{/} = -B$

Now, $A^{/}+B^{/} = A^{/} + B^{/} = -A-B = -(A+B)$

$\therefore$ A+B is skew-symmetric.

Example 3:

If A and B are symmetric matices, then AB is symmetic if and only if A and B commute i.e. AB=BA

Solution : A and B are symmetric. To prove that $(AB)^{/} = AB$

$\therefore A^{/} = A, B^{/} = B$

Also, AB = BA

Now, $(AB)^{/} = B^{/}A^{/}$

$= BA$

$= AB$

$\therefore$ AB is symmetric.

Conversely, Let AB be symmetric. To prove that AB=BA

$\therefore (AB)^{/} = AB$ – (1)

Also $(AB)^{/} = B^{/}A^{/}$

$= BA$ – (2)

From (1) and (2), is equel

AB = BA Proved

Example 4:

If A is a n-squared matrix, then $AA^{/}$ and $A^{/}A$ are both symmetric matrices.

Solution :

we have

$(AA^{/})^{/} = (A^{/})^{/}A^{/}$

$= AA^{/}$ $\Theta (A^{/})^{/} = A$

$\therefore AA^{/}$ is symmetric

Also, $\left(A^{/}A\right)^{/} = A^{/}\left(A^{/}\right)^{/} = A^{/}A$

$A^{/}A$ is symmetric .

Example 5: Show that every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric metrix.

Solution : Let A be any square matrix.

We can write

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$
$$= P + Q$$

Where $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

Now,

$$P' = \left\{\frac{1}{2}(A + A')\right\}' = \frac{1}{2}(A + A')' \qquad \Theta\,(kA)' = kA'$$
$$= \frac{1}{2}\{A' + (A')'\}$$
$$= \frac{1}{2}(A' + A)$$
$$= \frac{1}{2}(A + A') = P$$

$\therefore$ P is symmetric matrics.

and

$$Q' = \left\{\frac{1}{2}(A - A')\right\}'$$
$$= \frac{1}{2}(A - A')'$$
$$= \frac{1}{2}\{A' - (A')'\}$$
$$= \frac{1}{2}(A' - A)$$
$$= -\frac{1}{2}(A - A')$$
$$= -Q$$

$\therefore$ Q is skew-symmetric matrix.

Uniqueness : To prove that the representation is unique,let us take another representation. A=R+S. where R is symmetric and S is skew-symmetric. To prove that R=P, S=Q.

$$A^{/} = (R+S)^{/} = R^{/} + S^{/}$$
$$= R - S$$

$$\therefore A + A^{/} = R + S + R - S = 2R$$

$$\Rightarrow R = \frac{1}{2}(A + A^{/})$$

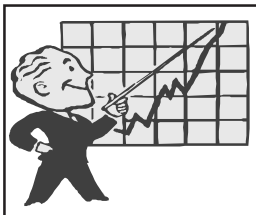
and

$$A - A^{/} = R + S - R + S = 2S$$

$$\Rightarrow S = \frac{1}{2}(A - A^{/})$$

Thus R=P and S=Q.

. The representation is unique.



CHECK YOUR PROGRESS – 5

Q 1: $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, find $AA^{/}$ and $A^{/}A$

Q 2: Let

$A = \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}$, then find $\frac{1}{2}(A + A^{/})$ and

$\frac{1}{2}(A - A^{/})$. show that $\frac{1}{2}(A + A^{/})$ is symmetric

$\frac{1}{2}(A - A^{/})$ is skew−symmetric.

Q 3: Show that $\begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$ is skew-symmetric matrix.

7.11 ADJOINT OF A SQUARE MATRIX

Let A= $\begin{bmatrix} a_{11} & a_{12.} \ldots & a_{1m} \\ a_{21} & a_{22.} \ldots & a_{2m} \\ \ldots\ldots\ldots\ldots\ldots & & \\ a_{m1} & a_{m2.} \ldots & a_{mm} \end{bmatrix} = [a_{ij}]_{mxn}$

The determinant of the matrix A,

$$|A| = \begin{bmatrix} a_{11} & a_{12.}\ldots & a_{1m} \\ a_{21} & a_{22.}\ldots & a_{2m} \\ \ldots\ldots\ldots\ldots\ldots\ldots & & \\ a_{m1} & a_{m2.}\ldots & a_{mm} \end{bmatrix}$$

Let the co-factors of the element a_{ij} be A_{ij} considering the matrix whose elements are the co-factors of the corresponding elements of A, we get

$$\Delta = \begin{bmatrix} A_{11} & A_{12.} & \ldots & A_{1m} \\ A_{21} & A_{22.} & \ldots & A_{2m} \\ . & \ldots\ldots\ldots\ldots\ldots\ldots & & \\ A_{m1} & A_{m2.}\ldots & & A_{mm} \end{bmatrix}$$

$$\Delta' = \begin{bmatrix} A_{11} & A_{21.} & \ldots & A_{m1} \\ A_{12} & A_{22.} & \ldots & A_{m2} \\ . & \ldots\ldots\ldots\ldots\ldots\ldots & & \\ A_{1m} & A_{2m.} . & .. & A_{mm} \end{bmatrix} = B(\text{say})$$

$B = \Delta'$ is called the adjoint of the matrix A and is denoted by adj A.

i.e. B = adj. A

Theorem: If A be any n-rowed square matrix, then

(adj A) A= A(adj A) = $|A| I_n$ where I_n is the n-rowed unit matrix.

Proof : Let A= $[a_{ij}]_{n\times n}$ be any n-rowed square matrix and

let adj A = B = $[b_{ij}]_{n\times n}$

Then b_{ij} = co-factor of a_{ij} in |A|

= A_{ji}

Since A and adj A are both n-rowed square matrices, therefore they are conformable for multiplication

i.e. Both A (adj A) and (adj A) A exist and are of the type n×n.

Now, (i, j)th element of A (adj A)

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$= \sum_{K=1}^{n} a_{ik}\ A_{jk}$$

$$= \begin{cases} 0 & \text{if } i \neq j \\ |A| & \text{if } i = j \end{cases}$$

In other words, all the elements of A(adj A) along the principal diagonal are equal to |A| and the non-diagonal elements are all zero.

$$\therefore A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 \ldots\ldots\ldots\ldots 0 \\ 0 & |A| & 0 \quad\quad 0 \\ 0 & 0 & |A| \ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & 0 & 0 \ldots\ldots\ldots |A| \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 0 & 0 \ldots\ldots\ldots 0 \\ 0 & 1 & 0 \ldots\ldots\ldots 0 \\ 0 & 0 & 1 \ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & 0 & 0 \ldots\ldots\ldots 1 \end{bmatrix}$$

$$= |A| \; I_n$$

Similarly, $(i, j)^{th}$ element of (adj A) A

$$= \sum_{K=1}^{n} b_{ik} a_{kj}$$

$$= \sum_{k=1}^{n} A_{ki} a_{kj}$$

$$= \begin{cases} 0 \text{ if } i \neq j; \\ |A| \text{ if } i = j \end{cases}$$

Therefore, (adj A) $A = |A| I_n$

Hence, A(adj A) = (adj A) A = $|A|I_n$ and $A^{-1} = \frac{1}{|A|}(\text{adj.A.})$.

Example : Find the adjoint of the matrix A

where $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix}$

Solution : We have $|A| = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix}$

Now, A_{11} = cofactor of $(1,1)^{th}$ element of |A|

$$= \begin{vmatrix} -1 & 4 \\ 0 & 1 \end{vmatrix} = -1 - 0 = -1$$

A_{12} = cofactor of (1,2)th element of |A|

$$= \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -(3-8) = 5$$

Similarly,

$$A_{13} = \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = -0-(-1)\,2 = 2$$

$$A_{21} = -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2$$

$$A_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, A_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = 4$$

$$A_{31} = \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} = 8,\ A_{32} = -\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4,\ A_{33} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -1-6 = -7$$

$$\therefore \text{adj } A = \begin{bmatrix} -1 & -2 & 8 \\ 5 & 1 & -4 \\ 2 & 4 & -7 \end{bmatrix}$$

Note :

(1) To find the Adj A, write down the determinant |A| and find the co-factor of various element of |A|. Replace each element in A by its co-factor and then take the transpose to get adj A.

(2) If A is a n-rowed square matrix, then A is said to be singular matrix if its determinant is zero i.e |A| = 0.

∴A(adj A) = (Adj A)A= |A| I_n= 0 (null matrix)

Hence, in this case all the elements of A(adj A) and (adj A)A will be zero's.

(3) If $|A| \neq 0$, then

$$|A\,\text{Adj}\,A| = \left\| |A|\,I_n \right\| = \left\| \begin{bmatrix} |A| & 0 & 0 \ldots. & 0 \\ 0 & |A| & 0 \ldots. & 0 \\ 0 & 0 & |A| \ldots & 0 \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots & \\ 0 & 0 & 0 \ldots. & |A| \end{bmatrix} \right\|$$

$$= |A|^n .$$

$$\Rightarrow |\text{Adj } A| = |A|^{n-1}$$

INVERSE OF A MATRIX:

Let A and B any n-squared matrices such that $AB = BA = I_n$

Then B is called inverse of A and is denoted by A^{-1} i.e. $B=A^{-1}$

Note : It is to be noted that every square matrix does not possess its inverse.

PROPERTIES OF THE INVERSE :

(a) The inverse of a matrix is unique.

Let A be n-squared matrix whose inverse exists.

Let B and C be two inverses of A.

Then $AB = BA = I_n$ —— (1)

$AC = CA = I_n$ —— (2)

For (1) $C(AB) = C\, I_n = C$ —— (3)

For (2) $(CA)\, B = I_n B = B$ —— (4)

Θ $C(AB) = (CA)\, B$

Θ $C = B$

which shows that an invertible matrix possesses a unique inverse.

(b) The necessary and sufficient condition for a square matrix A to possess an inverse is that A is non-singular i.e. $|A| \neq 0$

Proof: Necessary :

Let A be a square matrix of order n and B be its inverse

Then $AB = I_n$

$\therefore \; |AB| = |I_n|$

$= |A||B| = 1$

$\Rightarrow |A| \neq 0$

Sufficient :

Let $|A| \neq 0$, we define a matrix B so that $B = \frac{1}{|A|}(\text{adj.A.})$

Then $AB = A\,\frac{1}{|A|}\text{adj}\,A$

$= \frac{1}{|A|} A(\text{adj}\ A)$

$= \frac{1}{|A|} |A| I_n = I_n$

Similarly, $BA = \left(\frac{1}{|A|}\text{adj } A\right)A$

$$= \frac{1}{|A|}(\text{adj } A)\, A = \frac{1}{|A|}.|A|\, I_n = I_n$$

Thus, $AB = BA = I_n$.

Hence, A is invertible and B, is the inverse of A.

$$\therefore B = A^{-1} = \frac{1}{|A|}\text{adj } A$$

(c) **Reversal law for the inverse of a product.**

If A, B be two n-rowed non-singular, then AB is also non-singular and $(AB)^{-1} = B^{-1} A^{-1}$.i.e. the inverse of the product is the product of the inverses in the reverse order.

Proof : Let A and B be two non-singular matrices i.e. $|A| \neq 0$, $|B| \neq 0$,

So their inverses exist.

We have $|AB| = |A||B| \neq 0$

Hence, the matrix AB is also invertible .

Let C be the inverse of AB.

Let $A^{-1}$and B^{-1} be the inverses of A and B respectively and

$AA^{-1} = A^{-1}A = I_n$ and $B\ B^{-1} = B^{-1}B = I_n$

$\therefore\ C(AB) = (AB)\, C = I_n$

Choosing $C = B^{-1} A^{-1}$ we get

$C(AB) = (B^{-1} A^{-1})(AB)$

$= B^{-1} (A^{-1}A)B$

$= B^{-1}\ I_n\ B$

$= B^{-1}\ B$

$= I_n$ ... (1)

Again, $(AB)\, C = (AB)\, (B^{-1} A^{-1})$

$= A(BB^{-1})\, A^{-1}$

$= AI_nA^{-1}$

$= AA^{-1}$

$= I_n$ ... (2)

By (1) and (2)

$C(AB) = (AB)\, C = I_n$

Hence, C is the inverse of AB

or $B^{-1} A^{-1}$ is the inverse of AB

Thus, $(AB)^{-1} = B^{-1} A^{-1}$

(d) The operation of transposing and inverting are commutative

i.e. $(A')^{-1} = (A^{-1})'$

Proof :

Let A be a non-singular square matrix i.e. $|A| \neq 0$

$\therefore A^{-1}$ exists

We get $AA^{-1} = A^{-1}A = I_n$

$\therefore (AA^{-1})' = (A^{-1}A)' = I'_n$

$\Rightarrow (A^{-1})' A' = A'(A^{-1})' = I_n$

which shows that $(A^{-1})'$ is the inverse of A^{-1}.

Hence, $(A')^{-1} = (A^{-1})'$

> **Note:** If A, B are two n-rowed non-singular matrices.
>
> Then $(AB)^{-1} = B^{-1}A^{-1}$.

Example: Find the adjoint and inverse of the matrix $\begin{bmatrix} 0 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 0 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

$$\therefore |A| = \left|\begin{bmatrix} 0 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}\right|$$

$$= 0 + (-1)\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= (-1)(-1) + 3(6-1)$$

$$= 1 + 15 = 16 \neq 0$$

$\therefore A^{-1}$ exists

Now, Co-factors of the elements of 1st row are–

$$\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}, -\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \text{ i.e. } -2,\ 1,\ 5$$

Co-factors of the elements of 2nd row are –

$$-\begin{vmatrix}1 & 3\\ 2 & 0\end{vmatrix}, \begin{vmatrix}0 & 3\\ 1 & 0\end{vmatrix}, -\begin{vmatrix}0 & 1\\ 1 & 2\end{vmatrix} \text{ i.e. } 6, -3, 1$$

Co-factors of the elements of 3rd row are

$$\begin{vmatrix}1 & 3\\ 1 & 1\end{vmatrix}, -\begin{vmatrix}0 & 3\\ 3 & 1\end{vmatrix}, \begin{vmatrix}0 & 1\\ 3 & 1\end{vmatrix} \text{ i.e. } -2, 9, -3$$

Matrix formed by the corresponding co-factors of the elements of |A| is–

$$C = \begin{bmatrix}-2 & 1 & 5\\ 6 & -3 & 1\\ -2 & 9 & -3\end{bmatrix}$$

$\therefore$ adj A = transpose of C = $\begin{bmatrix}-2 & 6 & -2\\ 1 & -3 & 9\\ 5 & 1 & -3\end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$= \frac{1}{16}\begin{bmatrix}-2 & 6 & -2\\ 1 & -3 & 9\\ 5 & 1 & -3\end{bmatrix}$$

$$= \begin{bmatrix}\frac{4}{8} & \frac{3}{8} & \frac{-1}{8}\\ \frac{1}{16} & \frac{-3}{16} & \frac{9}{16}\\ \frac{5}{16} & \frac{1}{16} & \frac{-3}{16}\end{bmatrix}$$

Example: Find the inverse of the matrix $A = \begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$

Solution : $|A| = \begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$

$$= \begin{vmatrix}4 & 3\\ 3 & 4\end{vmatrix} - 3\begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix} + 3\begin{vmatrix}1 & 4\\ 1 & 3\end{vmatrix}$$

$$= (16 - 9) - 3(4 - 3) + 3(3 - 4)$$

$$= 7 - 3 - 3 = 7 - 6 = 1 \neq 0$$

Now, Co-factors of the elements of the 1st row are –

$\begin{vmatrix}4 & 3\\3 & 4\end{vmatrix}, -\begin{vmatrix}1 & 3\\1 & 4\end{vmatrix}, \begin{vmatrix}1 & 4\\1 & 3\end{vmatrix}$ i.e. 7, -1, -1

Co-factors of the elements of the 2nd row are –

$-\begin{vmatrix}3 & 3\\3 & 4\end{vmatrix}, \begin{vmatrix}1 & 3\\1 & 4\end{vmatrix}, -\begin{vmatrix}1 & 3\\1 & 3\end{vmatrix}$ i.e. 3, 1, 0

Co-factors of the elements of the 3rd row are –

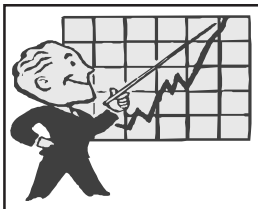
$\begin{vmatrix}3 & 3\\4 & 3\end{vmatrix}, -\begin{vmatrix}1 & 3\\1 & 3\end{vmatrix}, \begin{vmatrix}1 & 3\\1 & 4\end{vmatrix}$ i.e. -3, 0, 1

$$\therefore \text{adj } A = \begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$= \frac{1}{1}\begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}7 & -3 & 0\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$$



CHECK YOUR PROGRESS 6

Q 1: Find adj A if $A = \begin{bmatrix}0 & 1 & 2\\1 & 2 & 3\\3 & 10 & 1\end{bmatrix}$

Q 2: Find A^{-1} if $A = \begin{bmatrix}\cos\alpha & -\sin\alpha\\\sin\alpha & -\cos\alpha\end{bmatrix}$

Q 3: Show that if $|A| \neq 0$, then $|A^{-1}| = |A|^{-1}$

Q 4: If AB = BA, then prove that $A^{-1}B^{-1} = B^{-1}A^{-1}$

Q 5: If $|A| \neq 0$ and A is symmetric, show that A^{-1} is also symmetric.

Q 6: If A be a square matrix, then prove that $|\text{adj } A| = |A|^{n-1}$.

Q 7: If $|A| \neq 0$, then show that adj adj $A = |A|^{n-2}A$.



7.12 LET US SUM UP

- A rectangular array of mn numbers arranged in m-rows and n-columns and enclosed in square [ ] or a round bracket ( ), is called a matrix of order m×n (m by n).
- The numbers which constitute the matrix are called the elements of a matrix.
- The element which occurs in the i-th row and j-th column is called the (i,j) the element of the matrix and is denoted by a_{ij} .
- A matrix with exactly one row is called a row matrix. The order of a row matrix is of the type 1×n.
- A matrix with exactly one column is called a column matrix. The order of a column matrix is of the type m×1.
- A matrix A=$[a_{ij}]_{m\times n}$ is a square matrix if m=n.
- A square matrix a=$[a_{ij}]$ is a scalar matrix if $a_{ij} = \begin{cases} k & \text{where } i = j \\ 0 & \text{where } i \neq j \end{cases}$
- A square matrix A = $[a_{ij}]$ is a diagonal matrix if a_{ij} = 0 where i≠j.
- A square matrix A=$[a_{ij}]$ is a unit matrix if a_{ij} = 1 where i = j
  = 0 where i≠j ;
  A unit matrix of order n is denoted by I or I_n.
- A matrix in which all elements are zero is called a null matrix or zero matrix. A null matrix is denoted by O.
- A matrix is said to be upper triangular if a_{ij} = 0 when i>j.
- A matrix is said to be lower triangular if a_{ij} = 0 when i < j.
- Two matrix are said to be equal if and only of they are of same order and the corresponding elements are equal.
- If A and B are two matrix of same order. Then their sum or difference denoted by A + B ,A–B or B–A can be obtained.
- Matrix addition in commutative i.e. A+B = B+A.
- Matrix addition in associative i.e. (A+B) + C = A+(B+C)
- Existence of addition identity i.e. A+O = A = O +A

- Existence of additive inverse i.e. A+(–A) = (–A) + A
- Scalar multiplication of matrix i..e if (a) A= [a_{ij}] then kA= [ka_{ij}]

  (b) k(A+B) = kA+ kB (c) (k+l) A = kA + lA

  (d) k(lA) = (kl) A (e) (–k)A= k(–A)
- Matrix multiplication is not commutative i.e. AB ≠BA (in general)
- Matrix multiplication is associative i.e. (AB)C = A(BC)
- Matrix multiplication is distributive over addition i.e.

  (i) A(B+C) = AB + AC

  (ii) (A+B) C = AC + BC
- If A = [a_{ij}]$_{m\times n}$ then A^T=[b_{ij}]$_{n\times m}$ when $b_{ij} = a_{ij}$
- For matrices A and B of the same order $(A+B)^{/} = A^{/} + B^{/}$
- For matrix A and B conformable to multiplication

  $(AB)^{/} = B^{/}A^{/}$
- The matrix A is called symmetric if $A^{/}=A$
- The matrix A is called skew symmetric if $A^{/}= -A$
- Every square matrix A can uniquely be expressed as A = P+Q where

  P is a symmetric matrix and Q is a skew-symmetric matrix.
- For a square matrix A (adj A) A = A (adj A) = |A|I
- For a square non-singular matrix A, $A^{-1} = \frac{\text{adj A}}{|A|}$
- For two non-singular matrices A and B , $(AB)^{-1} = B^{-1}A^{-1}$.



7.13 ANSWER TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS 1

Answer Q No 1: Find all possible orders of a matrix with 10 elements.

$10 = 1\times 10$ $10 = 5\times 2$

$10 = 2\times 5$ $10 = 10\times 1$

Possible orders of matrices are 1×10, 2×5, 5×2 and 10×1.

Answer Q No 2: By equality of two matrices $\begin{pmatrix} 2a+b & 5c+d \\ a-2b & 4c+d \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ 3 & 25 \end{pmatrix}$

2a+b=4 5c–d=11

a–2b=–3 4c+d=25

Solving these we get

a = 1, b = 2, c = 4, d = 9

Answer Q No 3: For equality of two matrices, their orders and the corresponding elements must be same. Here, the order of the matrices are same. But (3,3)th element of $A \neq (3,3)^{th}$ element of B . $A \neq B$

CHECK YOUR PROGRESS 2

Answer Q No 1:

(i) It is not possible for define A + B, since A is of order 3×2 and B is of order 2×2.

(ii) It is possible to define A+B, since A is of order 3×3 and B is of order 3×3.

(iii) It is possible to define A+B, since A and B are square matrices of the same order.

Answer Q No 2:

$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the additive identity of 2 x3 matrices.

CHECK YOUR PROGRESS 3

Answer Q No 1 :

$$B(C+D) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left(\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 8 & 2 \\ 11 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1x8+2x11 & 1x2+2x6 \\ 3x8+4x11 & 3x2+4x6 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 14 \\ 68 & 30 \end{bmatrix}$$

$$BC+BD = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1x3+2x4 & 1x1+2x2 \\ 3x3+4x4 & 3x1+4x2 \end{bmatrix} + \begin{bmatrix} 1x5+2x7 & 1x1+2x4 \\ 3x5+4x7 & 3x1+4x4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 5 \\ 25 & 11 \end{bmatrix} + \begin{bmatrix} 19 & 9 \\ 43 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} 11+19 & 5+9 \\ 25+43 & 11+19 \end{bmatrix} = \begin{bmatrix} 30 & 14 \\ 68 & 30 \end{bmatrix}$$

$\therefore B(C + D) = BC + CD$

Again $(B+C)D = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \right) \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 1+3 & 2+1 \\ 3+4 & 4+2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4x5+3x7 & 4x1+3x4 \\ 7x5+6x7 & 7x1+6x4 \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 16 \\ 77 & 31 \end{bmatrix}$$

$$BD+CD = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1x5+2x7 & 1x1+2x4 \\ 3x5+4x7 & 3x1+4x4 \end{bmatrix} + \begin{bmatrix} 3x5+1x7 & 3x1+1x4 \\ 4x5+2x7 & 4x1+2x4 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 9 \\ 43 & 19 \end{bmatrix} + \begin{bmatrix} 22 & 7 \\ 34 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 19+22 & 9+7 \\ 43+34 & 19+12 \end{bmatrix} = \begin{bmatrix} 41 & 16 \\ 77 & 31 \end{bmatrix}$$

$\therefore (B + C)D = BD + CD$

Answer Q No 2:

If, A is of order 4×3

Then, B is of order 4×3

and C is of order 3×2

Answer Q No 3:

$$\text{Given, } A+B = \begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix},\ A-B=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$$

$$\therefore A+B+A-B=2A=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}+\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}7+3 & 0+0\\ 2+0 & 5+3\end{bmatrix}=\begin{bmatrix}10 & 0\\ 2 & 8\end{bmatrix}$$

$$A=\frac{1}{2}\begin{bmatrix}10 & 0\\ 2 & 8\end{bmatrix}=\begin{bmatrix}5 & 0\\ 1 & 4\end{bmatrix}$$

Again,

$$A+B-(A-B)=A+B-A+B=2B$$

$$=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}-\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}7-3 & 0-0\\ 2-0 & 5-3\end{bmatrix}=\begin{bmatrix}4 & 0\\ 2 & 2\end{bmatrix}$$

$$\therefore B=\frac{1}{2}\begin{bmatrix}4 & 0\\ 2 & 2\end{bmatrix}=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}$$

Answer Q No 4:

$$A^2=AA=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}0x0+1x1 & 0x1+1x0\\ 1x0+0x1 & 1x1+0x0\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

$$B^2=BB=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

$$=\begin{bmatrix}0x0+(-i)i & 0x(-i)+(-i)x0\\ ix0+0xi & ix(-i)+0x0\end{bmatrix}$$

$$=\begin{bmatrix}-i^2 & 0\\ 0 & -i^2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

Answer Q No 5:

$A^2-5A+7I=A.A-5A+7I$

$$= \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3x3+1(-1) & 3x1+1X2 \\ (-1)x3+2(-1) & (-1)x1+2x2 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Answer Q No 6:

$$A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$A_\alpha{}^2 = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & \cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha - \cos\alpha\sin\alpha & -\sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos2\alpha & \sin2\alpha \\ -\sin2\alpha & \cos2\alpha \end{bmatrix}$$

$$A_\alpha{}^3 = A_\alpha{}^2 A_\alpha = \begin{bmatrix} \cos2\alpha & \sin2\alpha \\ -\sin2\alpha & \cos2\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos2\alpha.\cos\alpha - \sin2\alpha.\sin\alpha & \cos2\alpha.\sin\alpha + \sin2\alpha.\cos\alpha \\ -\sin2\alpha.\cos\alpha - \cos2\alpha\sin\alpha & -\sin2\alpha.\sin\alpha + \cos2\alpha\cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\alpha+\alpha) & \sin(\alpha+2\alpha) \\ -\sin(2\alpha+\alpha) & \cos(2\alpha+\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos3\alpha & \sin3\alpha \\ -\sin3\alpha & \cos2\alpha \end{bmatrix}$$

CHECK YOUR PROGRESS 4

Answer Q No 6:

$$A' = \begin{bmatrix} -4 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix}_{2x3}, \quad B' = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}_{3x2}$$

$$AB = \begin{bmatrix} -4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4+6 & -12+3 & -16+6 \\ 1+2 & 3+1 & 4+2 \\ 0+4 & 0+2 & 0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -9 & -10 \\ 3 & 4 & 6 \\ 4 & 2 & 4 \end{bmatrix}$$

$$(AB)^{/} = \begin{bmatrix} 2 & 3 & 4 \\ -9 & 4 & 2 \\ -10 & 6 & 4 \end{bmatrix}$$

$$B^{/}A^{/} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -4 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4+6 & 1+2 & 0+4 \\ -12+3 & 3+1 & 0+2 \\ -16+6 & 4+2 & 0+4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ -9 & 4 & 2 \\ -10 & 6 & 4 \end{bmatrix}$$

$\therefore (AB)^{/} = B^{/}A^{/}.$

Answer Q No 2:

$$\text{Let } A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$$A^{/} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$$(A^{/})^{/} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = A$$

Answer Q No. 3 :

$$A=\begin{bmatrix} o & h & g \\ -h & o & f \\ -g & -f & o \end{bmatrix} \text{ is a skew symmetric matrix}$$

$$A^{/}=\begin{bmatrix} o & -h & -g \\ h & o & -f \\ g & f & o \end{bmatrix}=-\begin{bmatrix} o & h & g \\ -h & o & f \\ -g & -f & o \end{bmatrix}=-A$$

CHECK YOUR PROGRESS 5

Answer Q No. 1 :

$$A=\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix} \quad \therefore \quad A^{/}=\begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$AA^{/}=\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1x1+3x3+2x2 & 1x3+3x2+2x1 \\ 3x1+2x3+1x2 & 3x3+2x2+1x1 \end{bmatrix}$$

$$=\begin{bmatrix} 14 & 11 \\ 11 & 14 \end{bmatrix}$$

$$A^{/}A=\begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1x1+3x3 & 1x3+3x2 & 1x2+3x1 \\ 3x1+2x3 & 3x3+2x2 & 3x2+2x1 \\ 2x1+1x3 & 2x3+1x2 & 2x2+1x1 \end{bmatrix}$$

$$=\begin{bmatrix} 10 & 9 & 5 \\ 9 & 13 & 8 \\ 5 & 8 & 5 \end{bmatrix} \quad . \quad AA^{/}=A^{/}A$$

Answer Q No 2:

$$A=\begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix} \quad \therefore A'=\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

$$\frac{1}{2}(A+A') = \frac{1}{2}\left(\begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 1+1 & 6+2 & 3+4 \\ 2+6 & 8+8 & 5+1 \\ 4+3 & 1+5 & 7+7 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 8 & 7 \\ 8 & 16 & 6 \\ 7 & 6 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 4 & \frac{7}{2} \\ 4 & 8 & 3 \\ \frac{7}{2} & 3 & 7 \end{bmatrix}$$

$$\frac{1}{2}(A-A') = \frac{1}{2}\left(\begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 1-1 & 6-2 & 3-4 \\ 2-6 & 8-8 & 5-1 \\ 4-3 & 1-5 & 7-7 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -\frac{1}{2} \\ -2 & 0 & 2 \\ \frac{1}{2} & -2 & 0 \end{bmatrix}$$

Answer Q No 3: Let

$$A = \begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 3i & 4 \\ -3i & 0 & -8 \\ -4 & 8 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix} = -A$$

$\therefore$ A is Skew Symmetric

CHECK YOUR PROGRESS 6

Answer Q No 2:

$$|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$$

Co-factors of the elements of 1st row are –

$$\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}, \text{ i.e.} -1, 8, -5$$

Co-factors of the elements of 2nd row are –

$$-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}, \text{ i.e.} 1, -6, 3$$

Co-factors of the elements of 3rd row are –

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}, \text{ i.e.} -1, 2, -1$$

$$\therefore \text{adj } A = \begin{vmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{vmatrix}$$

Answer Q No 2:

$$|A| = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} \quad \therefore |A| = \cos^2\alpha + \sin^2\alpha = 1 \neq 0$$

. Co-factor of the elements of 1st row are

$\cos\alpha, -\sin\alpha$

Co-factor of the element of 2nd row are –

$-(-\sin\alpha), \cos\alpha$ i.e. $\sin\alpha, \cos\alpha$

$$\therefore \text{adj } A = \begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix}$$

$$\therefore A^{-1} = \frac{\text{adj}A}{|A|} = \begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix}$$

Answer Q No 3:

$|A| \neq 0 \quad \therefore A^{-1}$ exists

Now,

$$AA^{-1} = I$$

$$\Rightarrow |AA^{-1}| = |I|$$

$$\Rightarrow |A||A^{-1}| = 1$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|} = |A|^{-1}$$

Answer Q No 4:

Given that AB=BA

$$\therefore (AB)^{-1} = (BA)^{-1}$$

$$\Rightarrow B^{-1}A^{-1} = A^{-1}B^{-1}$$

Answer Q No 5:

$|A| \neq 0 \quad \therefore A^{-1}$ exists

also $A^{/}=A$

Now, $(A^{-1})^{/} = (A^{/})^{-1} = A^{-1}$

$\therefore A^{-1}$ is symmetric.

Answer Q No 6:

$$A\,adjA = |A|I_n$$

$$\therefore |A\,adjA| = ||A||I_n||$$

$$\Rightarrow |A||adj\,A| = |A|^n\,|I_n| \quad \Theta\,|kA| = k^n|A|$$

$$= |A|^n \qquad \Theta\,|I_n| = 1$$

$$\Rightarrow |adj\,A| = |A|^{n-1} \quad \text{if} \quad |A| \neq 0$$

If $|A| = 0$, then $|adj\ A| = 0 = |A|^{n-1}$

Hence the result.

Answer Q No 7:

$$|A| \neq 0$$

We have $A(adj\ A) = |A|I_n$

Replacing A by adj A ,we get

$$(adj\,A)(adj\,adj\ A) = |adj\,A|I_n$$

$$= |A|^{n-1}.I_n \qquad -(i)$$

Premultiplying both side of (i) by A

$$A\ (adj\,A)(adj\,adj\ A) = A\{|A|^{n-1}I_n\}$$

$$= |A|^{n-1}(AI_n)$$

$$= |A|^{n-1}A$$

$$\Rightarrow \{A(\text{adj } A)\}(\text{adjadj } A) = |A|^{n-1} A$$

$$\Rightarrow |A| I_n (\text{adj adj } A) = |A|^{n-1} A$$

$$\Rightarrow |A| (\text{adj adj } A) = |A|^{n-1} A$$

$$\Rightarrow \text{adj adj } A = |A|^{n-2} A \qquad [\text{Dividy both sides by} |A| \quad \Theta |A| \neq 0]$$



7.14 FURTHER READING

- G . Cullen, Charles. *Matrices and Linear Transformation*
- Eves, Howard. *Elementary Matrix Theory*
- N. Franklin, Joel. *Matrix theory*
- W. Lewis, David. *Matrix Theory*
- Bodeing, Ewald, *Matrix Calculus*
- Vatssa, B.S, *Theory of Matrices,* New Age International Publishers
- Kala, V.N. & Rana, Rajeshri. *Matrices*
- Narayan, Shanti & Mittal, A Text Book of Matrices



7.15 MODEL QUESTIONS

Q 1: In a Matrix $\begin{bmatrix} 4 & -3 & 19 & \sqrt{2} \\ 11 & 24 & -7 & -5 \\ 1 & \sqrt{3} & 0 & 1 \end{bmatrix}$

Find (i) Order of the matrix

(ii) The number of elements

(iii) Write the elements a_{23}, a_{12}, a_{32}, $a_{34,}$,$a_{44.}$

Q 2: If $A = \begin{bmatrix} 4 & 5 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 3 & -4 \end{bmatrix}$, find $A + B$ and $A - B$

Q 3: If $A = \begin{bmatrix} 3 & 2 \\ 0 & -2 \\ 1 & 0 \end{bmatrix}$, find $3A$ and $-5A$

Q 4: Find AB and BA if $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 5 & 3 \end{bmatrix}$

Q 5: What is the additive identity of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

Q 6: If $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ -1 & 3 & 5 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 \\ 1 & 3 \\ 0 & 2 \end{pmatrix},$

$C = \begin{pmatrix} 3 & 1 & 2 \\ -2 & 2 & 0 \end{pmatrix}$, find A(BC) and (AB)C and show that A(BC) = (AB) C

Q 7: Show that A(B+C) = AB + AC

where $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & 5 \\ -1 & 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & -3 \\ 6 & 0 & 4 \end{bmatrix}$

Q 8: If $A = \begin{pmatrix} 4 & 3 & 7 \\ 2 & 5 & 6 \end{pmatrix}$ show that $(A')' = A$

Q 9: If $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$, then show that $(AB)' = B'A'$

Q 10: If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then show that $A^4 = \begin{bmatrix} 1+2\times4 & -4\times4 \\ 4 & 1-2\times4 \end{bmatrix}$

Q 11: Find A^{-1} where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$

Q 12: Find the inverse of the matrix $\begin{pmatrix} 4 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

Q 13: If $A = \begin{bmatrix} 2 & -4 & -2 \\ 4 & 6 & 2 \\ 0 & 10 & -4 \end{bmatrix}$, find adj A

Q 14: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$, find A^{-1}

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UNIT 8: LIMIT AND CONTINUITY

UNIT STRUCTURE

8.1 Learning Objectives

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8.3 Limits

- 8.3.1 Finding Limits Analytically
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8.4 Continuity

- 8.4.1 Basic Definitions and Example
- 8.4.2 Algebra of Continuous Functions

8.5 Let Us Sum Up

8.6 Further Readings

8.7 Answer To Check Your Progress

8.8 Model Questions

8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define the concept of limit.
- evaluate limits by algebraic rules.
- explain the concept of continuity of a function.
- show that some well known functions are continuous.

8.2 INTRODUCTION

In the earlier unit, we have learnt about Set theory and functions.

In this unit we are going to discuss about limit, continuity, differentiation and Integration, which are again very important and fundamental in Mathematics.

The concept of limit is an abstract one. We will learn various algebraic rules to find limits at various points of well-known functions. We will also learn about left and right hand limits. We will also discuss about the continuity and differentiation of functions. Finally, we will discuss about the concept of integration. We will also discuss about two powerful methods of integration, namely, methods of substitution and the method of integration by parts.

8.3 LIMITS

We are already familiar with the concept of a real function and a real variable. The concept of limit is an abstract one. Consider the function $f(x) = \frac{x^2 - 9}{x - 3}$, which is defined at all points except $x = 3$. In the first column of following table, some values of x close to 3 are written and in the second column the corresponding values of $f(x)$ are written.

The value of the variable	The corresponding value of $f(x) = \frac{x^2 - 9}{x - 3}$
2.97	5.97
2.98	5.98
2.99	5.99
3.01	6.01
3.02	6.02
3.03	6.03

From the above table, we observe that if x is very close to 3, then $\frac{x^2 - 9}{x - 3}$ is very close to 6. It is also true that for all values of x close to 3, $\frac{x^2 - 9}{x - 3}$ is close to 6. We then say:

As x tends to 3, the function $\frac{x^2 - 9}{x - 3}$ tends to 6 or equivalently, we say: The function $\frac{x^2 - 9}{x - 3}$ has the limit 6 at the point 3.

In symbols, we write

$\lim_{x\to 3} \frac{x^2-9}{x-3} = 6.$

In general, by $\lim_{x\to a} f(x) = l$ we mean that as gets close to *a* then $f(x)$ gets close to a limit of *l*. This can be put in a more precise way as: $|f(x) - l|$ can be made arbitrarily small, by taking $|x - a|$ to be sufficiently small.

> The value of the function *f* at *a* is immaterial for finding the limit of *f* at *a*. Only the values that are close to *a* but not at *a* matter.

The following two results are simple consequences of the definition.

1. **Constant Rule:** $\lim_{x\to a} c = c$, that is, the limit of the constant function $f(x) = c$ as *x* tends to *a* is the value *c*.

2. **Identity Rule:** $\lim_{x\to a} x = a$, that is, the limit of the identity function $f(x) = x$ as *x* tends to *a* is the value *a*.

Uniqueness of Limit

> If $\lim_{x\to a} f(x)$ exists, it is unique. There cannot be two distinct numbers l_1 and l_2 such that when *x* tends to *a* the function *f (x)* tends to both l_1 and l_2.

8.3.1 Finding Limits Analytically

It is possible to find limits by using algebraic techniques. The limit behaves well with respect to the operations of addition, subtraction, multiplication by a constant, multiplication, division, etc. The most basic theorem governing analysis of limits is the Principal Limit Theorem.

Principal Limit Theorem:

Let *a* and *k* be real numbers, $n \geq 0$, and let *f* and *g* be functions with limits at *a* such that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$. Then

> 1. **Sum Rule:** $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = l + m.$
> 2. **Difference Rule:** $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = l - m.$
> 3. **Product Rule:** $\lim_{x\to a}[f(x) \bullet g(x)] = \lim_{x\to a} f(x) \bullet \lim_{x\to a} g(x) = l \bullet m.$

4. **Quotient Rule:** $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}=\frac{l}{m}$ provided $\lim_{x\to a} g(x)=m\neq 0.$

5. **Coefficient Rule:** $\lim_{x\to a} k\,f(x)=k\lim_{x\to a} f(x)=kl.$

6. **Power Rule:** $\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a} f(x)\right]^n=l^n.$

We can state the above rules in language. For example, Sum Rule could be expressed as **"The limit of a sum is the sum of the limits."**

8.3.2 Examples of Evaluation of Limits Using Various Rules

Example 1. Find $\lim_{x\to 2} x^2.$

Solution: $\lim_{x\to 2} x^2=\left(\lim_{x\to 2} x\right)^2$ (by Power Rule)

$= 2^2$ (by Identity Rule)

$= 4$

Example 2. Find $\lim_{x\to a} 2x^3+3x^2+4x+5.$

Solution:

$$\lim_{x\to a} 2x^3+3x^2+4x+5$$

$= 2\lim_{x\to a} x^3+3\lim_{x\to a} x^2+4\lim_{x\to a} x+\lim_{x\to a} 5$ (by Sum and Coefficient Rules)

$= 2\left(\lim_{x\to a} x\right)^3+3\left(\lim_{x\to a} x\right)^2+4\lim_{x\to a} x+\lim_{x\to a} 5$ (by Power Rule)

$= 2a^3+3a^2+4a+5.$ (by Identity and Constant Rules)

Remark: It is clear from the above example that for all polynomial functions, the limit at a point can be obtained by substituting that value for the variable *x*. But, for non-polynomial functions, this may not be true always. The next two examples show this fact.

Example 3. Find $\lim_{x\to 1}\frac{2x-1}{x^3-2}$.

Solution:

$$\lim_{x\to 1}\frac{2x-1}{x^3-2}=\frac{\lim_{x\to 1}(2x-1)}{\lim_{x\to 1}(x^3-2}$$ (by Quotient Rule)

$$=\frac{\lim_{x\to 1}2x-\lim_{x\to 1}1}{\lim_{x\to 1}x^3-\lim_{x\to 1}2}$$ (by Difference Rule)

$$=\frac{2\lim_{x\to 1}x-\lim_{x\to 1}1}{(\lim_{x\to 1}x)^3-\lim_{x\to 1}2}$$ (by Coefficient and Power Rules)

$$=\frac{2(1)-1}{(1)^3-2}$$ (by Constant and Identity Rules)

$= -1.$

Example 4. Find $\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2}$.

Solution: Since the denominator function x^2-3x+2 has the limit 0 at the point 1, that is, $\lim_{x\to 2}x^2-3x+2=0,$ so Quotient Rule cannot be applied here. Therefore, we have to resort to another method. Factorizing both the numerator and the denominator, we find that

$$\frac{x^2-4}{x^2-3x+2}=\frac{(x-2)(x+2)}{(x-1)(x-2)}.$$

When $x\neq 2,$ we can cancel the common factor *x – 2.*

Thus,

$$\frac{x^2-4}{x^2-3x+2}=\frac{x+2}{x-1}.$$

Therefore,

$$\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2}=\lim_{x\to 2}\frac{x+2}{x-1}=\frac{\lim_{x\to 2}x+\lim_{x\to 2}2}{\lim_{x\to 2}x-\lim_{x\to 2}1}=\frac{2+2}{2-1}=3$$



CHECK YOUR PROGRESS

Q 1: Write true (T) or false (F).

(a) $\lim_{x\to 1}(x-1)^2+5=5.$ (T / F)

(b) $\lim_{x\to 2}(x-2)(x+4)=0.$ (T/F)

(c) $\lim_{x\to 3}\frac{x^2-9}{x-3}=0.$ (T / F)

(d) $\lim_{x\to -1}\frac{x^2-1}{x+1}=2.$ (T/F)

8.3.3 Limitis by the Method of Substitution

Consider the example $\lim_{x\to 1}\frac{x^2-1}{x-1}$. We find that

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}=\lim_{x\to 1}(x+1)=2$$

The same limit can be evaluated with the help of substitution. If we substitute $y=x-1,$ i.e. $x=y+1,$ in the expression $\frac{x^2-1}{x-1},$ then $\frac{x^2-1}{x-1}=\frac{(y+1)^2-1}{y}=y+2.$ Also, note that when x tends 1, then y tends 0. Therefore,

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{y\to 0}(y+2)=2.$$

In general,

$$\boxed{\lim_{x\to a}f(x)=\lim_{h\to 0}f(a+h).}$$

Equivalently,

$$\boxed{\lim_{x\to a}f(x)=\lim_{y\to a+h}f(y),\quad where\ y=x+h.}$$

Example 5. Find $\lim_{x\to 1}\frac{(x-1)^4}{x^4 - 1}$.

Solution: We have

$$\lim_{x\to 1}\frac{(x-1)^4}{x^4-1}=\lim_{y\to 0}\frac{y^4}{(y+1)^4-1}=\lim_{y\to 0}\frac{y^4}{y^4+4y^3+6y^2+4y}.$$

$$=\lim_{y\to 0}\frac{1}{y^3+4y^2+6y+4}=\frac{1}{4}.$$

8.3.4 A Special Limit

Let us evaluate $\lim_{x\to a}\frac{x^n-a^n}{x - a}$, where n is a positive integer.

By the method of substitution, we find that

$$\lim_{x\to a}\frac{x^n-a^n}{x - a}=\lim_{h\to 0}\frac{(a+h)^n-a^n}{a + h - a}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[a^n+\binom{n}{1}a^{n-1}h+\binom{n}{2}a^{n-2}h^2+\ldots+h^n-a^n\right]$$

[By Binomial Theorem]

$$=\lim_{h\to 0}\left[\binom{n}{1}a^{n-1}+\binom{n}{2}a^{n-2}h+\ldots+h^{n-1}\right]=\binom{n}{1}a^{n-1}=na^{n-1}.$$

Thus,

$\lim_{x\to a}\frac{x^n-a^n}{x - a}=na^{n-1}$, where n is a positive integer.

The above formula remains true, when n is a rational number, provided a is positive.

Example 6. Find $\lim_{x\to 3}\frac{x^3-27}{x^2 - 9}$.

Solution:

$$\lim_{x\to 3}\frac{x^3-27}{x^2-9}=\lim_{x\to 3}\left[\frac{x^3-27}{x-3}\div\frac{x^2-9}{x-3}\right]=\lim_{x\to 3}\left[\frac{x^3-27}{x-3}\right]\div\lim_{x\to 3}\left[\frac{x^2-9}{x-3}\right].$$

$$=\lim_{x\to 3}\left[\frac{x^3-3^3}{x-3}\right]\div\lim_{x\to 3}\left[\frac{x^2-3^2}{x-3}\right]=3(3)^2\div 2(3)=27\div 6=\frac{9}{2}.$$

Example 7. If $\lim_{x\to 2}\frac{x^n-2^n}{x-2}=12$ and if n is a positive integer, find n.

Solution: We have $\lim_{x\to 2}\frac{x^n-2^n}{x-2}=n2^{n-1}$. According to the given question, $n2^{n-1}=12=3.2^2$. Therefore, $n = 3$.

8.3.5 Some Other Special Limits

The following limit rules of trigonometric and exponential functions are very useful.

1. **Sine Rule:** $\lim_{x\to 0}\frac{\sin x}{x}=1$, where x is measured in radian.
2. **Cosine Rule:** $\lim_{x\to 0}\cos x=1$, where x is measured in radian.
3. **Exponential Rule:** $\lim_{x\to 0}\frac{e^x-1}{x}=1$.

Example 8. Find $\lim_{\theta\to 0}\frac{\sin 3\theta}{\sin 4\theta}$.

Solution: We have

$$\lim_{\theta\to 0}\frac{\sin 3\theta}{\sin 4\theta}=\lim_{\theta\to 0}\left(\frac{\sin 3\theta}{3\theta}\cdot\frac{3\theta}{4\theta}\cdot\frac{4\theta}{\sin 4\theta}\right)=\lim_{\theta\to 0}\frac{\sin 3\theta}{3\theta}\cdot\lim_{\theta\to 0}\frac{3\theta}{4\theta}\cdot\lim_{\theta\to 0}\frac{4\theta}{\sin 4\theta}$$

$$=\lim_{3\theta\to 0}\frac{\sin 3\theta}{3\theta}\cdot\frac{3}{4}\cdot\frac{1}{\lim_{4\theta\to 0}\frac{\sin 4\theta}{4\theta}}=1\left(\frac{3}{4}\right)\div 1=\frac{3}{4}.$$

Example 9. Find $\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$.

Solution: We have

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta\cos\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\div\lim_{\theta\to 0}\frac{1}{\cos\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\div\frac{1}{\lim_{\theta\to 0}\cos\theta}=1.$$

Example 10. Find $\lim_{x\to 0}\frac{e^{4x}-1}{x}$.

Solution: We have

$$\lim_{x\to 0}\frac{e^{4x}-1}{x}=\lim_{x\to 0}\frac{e^{4x}-1}{4x}\cdot 4=4\lim_{y\to 0}\frac{e^y-1}{y}=4.$$

8.3.6 One Sided Limits

In this section we will discuss one sided limits.

The notation $\lim_{x\to a+} f(x) = l$ means that when is close x to a and greater than a then $f(x)$ is close to l.. Similarly, $\lim_{x\to a-} f(x) = l$ means that when is close x to a and less than a, then $f(x)$ is close to l.

$\lim_{x\to a+} f(x)$ is called the **right limit** of f at a and $\lim_{x\to a-} f(x)$ is called the **left limit** of f at a. Both these are called **one-sided limits** of f at a

Example11. Find $\lim_{x\to 0+} \frac{x}{|x|}$ and $\lim_{x\to 0-} \frac{x}{|x|}$.

Solutions: When x takes positive values, then $\frac{x}{|x|} = \frac{x}{x} = 1.$

Therefore, we have $\lim_{x\to 0+} \frac{x}{|x|} = 1.$

On the other hand, when takes negative values, then $\frac{x}{|x|} = \frac{x}{-x} = -1.$ So, $\lim_{x\to 0-} \frac{x}{|x|} = -1.$

If $\lim_{x\to a+} f(x) \neq \lim_{x\to a-} f(x)$ then $\lim_{x\to a} f(x)$ does not exist.

If $\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = l$ then $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = l.$

Example 12. Decide whether $\lim_{x\to 0} \frac{x}{|x|}$ exists or not.

Solution: From the solutions of Example 11, we note that $\lim_{x\to 0+} \frac{x}{|x|} = 1$ and $\lim_{x\to 0-} \frac{x}{|x|} = -1.$ Since the right limit and left limit are not equal, therefore $\lim_{x\to 0} \frac{x}{|x|}$ does not exist.



CHECK YOUR PROGRESS

Q 2: Write true (T) or false (F).

(a) If f is a real function such that $f(x) = f(-x)$ for all x, then $\lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x)$. (T /F)

(b) $\lim_{x\to 1-}[x] = 1$. (T / F)

(c) $\lim_{x\to 1+}[x] = 1$. (T / F)

Q 3: Find the following limits:

(i) $\lim_{x\to 2} \frac{x-2}{x+2}$ (ii) $\lim_{x\to 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$ (iii) $\lim_{x\to 3} \frac{e^x - e^3}{x-3}$

8.4 CONTINUITY

In this section, we will learn about the concept of continuity of a function. We will show that a large class of well known functions is continuous.

8.4.1 Basic Definitions and Examples

Let f be a real function and let a be in the domain of f If the limit of f at a is same as the value of f at a, then f is said to be continuous at a.

Thus, if f is to be continuous at a the following three conditions must be satisfied.

a) $f(a)$ is defined,

b) $\lim_{x\to a} f(x)$ exists,

c) $\lim_{x\to a} f(x) = f(a)$.

Example 13. Show that every constant function is continuous at all points.

Solution: Consider a constant function $f(x) = c$ and an arbitrary point a.

Then $f(a) = c$ and $\lim_{x \to a} f(x) = \lim_{x \to a} c = c$, and so, $\lim_{x \to a} f(x) = f(a)$.

Therefore, f is continuous at a

Example 14. Show that the identity function is continuous at all points.

Solution: Consider the identity function $f(x) = x$ and an arbitrary point *a.*

Then, we have $f(a) = a$ and $\lim_{x \to a} f(x) = \lim_{x \to a} x = a$.

Thus, $\lim_{x \to a} f(x) = f(a)$, which shows that f is continuous at 'a'.

Example 15. Show that the greatest integer function [x] is not continuous at 0.

Solution: Let $f(x) = [x]$.

Now, $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} [x] = 0$ and $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} [x] = -1$.

Thus, the right hand and left hand limits of f at 0 are not equal. Which means that $\lim_{x \to 0} f(x) = \lim_{x \to 0} [x]$ does not exist. Therefore, though $f(0) = [0] = 0$, f is not continuous at 0.

Some more results are listed in the following table.

(c) The function x^2 is continuous at 3.

(d) The function $\frac{x-4}{x-1}$ is continuous at 2.

(e) The function $\sin x$ is continuous at .

(f) The function $\sin x$ is continuous at 0.

(g) The function $\cos x$ is continuous at 0.

(h) The function e^x is continuous at 0.

(i) The function $\frac{x}{|x|}$ is not continuous at 0.

Definition:

A real function is said to be continuous in an open or closed interval if it is continuous at every point of the interval.

Remark: When a function f is considered on a closed interval $[a, b]$, then f is said to be continuous at the end point a if

$\lim_{x \to a+} f(x) = f(a)$. Similarly, *f* is said to be continuous at the end point *b* if $\lim_{x \to b-} f(x) = f(b)$.

8.4.2 Algebra of Continuous Functions

Theorem: Let *f* and *g* be two functions, continuous at *a* and let *c* be a real number. Then

a) $f + g$ is continuous at *a*

b) $f - g$ is continuous at *a*

c) *cf* is continuous at *a*

d) *fg* is continuous at *a*

e) $\frac{f}{g}$ is continuous at *a* provided $g(a) \neq 0$.

Proofs. Follow from the Principal Limit Theorem and the definition of continuity.

If *f* and *g* are continuous functions, that is, they are continuous at every point of their domains, then the above theorem can be generalized in the following way.

Theorem: If *f* and *g* are continuous functions, then

(a) $f + g$ is continuous,

(b) $f - g$ is continuous,

(c) cf is continuous at,

(d) fg is continuous at,

(e) $\frac{f}{g}$ is continuous, at those points where *g* does not take the value zero.

The statement (a) above means that the sum of two (or even more) continuous functions is continuous. Similarly, statement (b) means that the difference of two continuous functions is continuous, and so on.

Example 16. Show that the function $2x^2 + 3x + 4$ is continuous at all points.

Solution: We have already proved that the identity function *x* is continuous at all points. Thus, x^2 is continuous at all points being

the product of x and x. Again, the constant functions 2 and 4 are continuous at all points. Therefore, $2x^2$, $3x$ and 4 are continuous at all points. Finally, $2x^2 + 3x + 4$ is continuous at all points being the sum of three continuous functions.

Example 17. At What points is the function $\frac{x-2}{(x-3)(x-5)}$ continuous?

Solution: The given function is not defined at the points 3 and 5. The functions in the numerator, i.e., $x - 2$ and in the denominator, i.e., $(x-3)(x-5)$ are continuous at all points. Therefore, the function $\frac{x-2}{(x-3)(x-5)}$ is continuous at all those points where the denominator does not take the value zero. Thus, it is continuous at all points except at 3 and 5.



8.5 LET US SUM UP

- In this unit, we have learnt about the concept of limits and various algebraic rules to find limits at various points of well-known functions. We have also learnt about the left and right hand limits, and found that they need not be equal always.
- We have learnt about the continuity of functions. In summary, we can say that if the limit exists at a point and it is equal to the value of the function at that point then the function is continuous at that point.



8.6 FURTHER READING

1) Khanna V. K., Zameeruddin Qazi & Bhambri S.K.(1995). Business Mathematics, New Delhi, Vikas Publishing House Pvt Ltd. .

2) Hazarika P.L. Business Mathematics, New Delhi. S.Chand & Co.



8.7 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (a) T, (b) T, (c) F, (d) F

Ans to Q No 2: (a) T, (b) F, (c) T.

Ans to Q No 3: (i) 0 (ii) 3, (iii) e^3.

Ans to Q No 4: (a) 44, (b) No, (c) $a^x \log a$, (d) e^x

Ans to Q No 5: (a) T, (b) T, (c) T, (d) T, (e) T, (f) F, (g) T.

Ans to Q No 6:

Ans to Q No 7:

Ans to Q No 8:

(2) None,

Ans to Q No 9: (a) F, (b) T, (c) T.

Ans to Q No 10: (a) F, (b) F, (c)T,

Ans to Q No 11: $x \log x - x$

Ans to Q No 12: $\frac{x}{2} + \frac{\sin 2x}{4}$ and $\frac{x}{2} - \frac{\sin 2x}{4}$.



8.8 MODEL QUESTIONS

Q 1: Evaluate the following limits:

(a) $\lim_{x \to 3} \frac{x^2 - 7x + 12}{x^2 - 4x + 3}$ (b) $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$

(c) $\lim_{x \to 0} \frac{\sin 10x}{\sin 11x}$ (d) $\lim_{x \to k} \frac{x^{10} - k^{10}}{x^5 - k^5}$

Q 2: Suppose a function *f* is defined as follows.

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x > 0. \end{cases}$$

By evaluating the left and right hand limits, show that $\lim_{x \to 0} f(x) = 0.$

Q 3: If *f* is an odd function (i.e., $f(-x) = -f(x), \quad \forall x.$) and $\lim_{x \to 0} f(x)$ exists, then show that the value of this limit is zero.

Q 4: Decide whether the function given in Question 2 is continuous at $x = 0$ or not.

Q 5: A real function *f* is defined as follows.

$$f(x) = \begin{cases} \dfrac{\tan x}{x}, & x \neq 0 \\ \dfrac{3}{4}, & x = 0. \end{cases}$$

Show that *f* is continuous at 0.

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UNIT 9: DERIVATIVE OF FUNCTIONS

UNIT STRUCTURE

9.1 Learning objectives.

9.2 Introduction

9.3 Geometrical Interpretation of Derivative of a Function.

9.4 Derivative of Various Standard Functions.

9.5 Derivative of Sum and Difference of Functions.

9.6 Derivative of Product of Functions and Quotient Rule.

9.7 Derivative of Composite Functions.

9.10 Let Us Sum Up.

9.11 Answers to Check Your Progress

9.12 Further Reading

9.13 Model Questions.

9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- understand the geometrical meaning of $f'(x)$ or $\frac{dy}{dx}$
- evaluate derivative of algebraic, trigonometrical, exponential and logarithm functions.
- determine derivatives of composite functions (function of functions).
- evaluate derivative of implicit functions.

9.2 INTRODUCTION

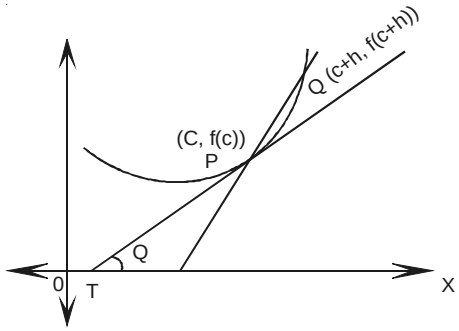
We have already introduced the concept of derivative in unit 1. In this unit , we discuss algebra of derivative including derivative of certain standard function. The notion of differentiation is essentially concerned with the rate of charge of a dependent variable with respect to an independent variable. In this unit, we also discuss the nature of the function whose numerator and denominator both approach to zero at a particular point (Indeterminate form). It is called indeterminate form. Some indeterminate forms are $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, 1^{∞}, 0^{0} ,∞^{0} etc.

9.3 GEOMETRICAL INTERPRETATION OF DERIVATIVE

Let y=f(x) be a function. Let P (c, f(c)) and Q (c+h, f(c+h)) be two neighbouring points on the graph of y=f(x).

Slope of Chord PQ $= \frac{f(c+h)-f(c)}{(c+h)-c}$

$= \frac{f(c+h)-f(c)}{h}$



When Q – P along the curve, the chord PQ becomes tangent PT at P. Thus, the slope of the chord PQ becomes slope of the tangent at P when Q – P, i.e. h – 0

∴ Slope of the tangent at P= $\lim_{h-0} \frac{f(c+h)-f(c)}{h}$

= $f'(c)$ (by definition)

Also, if θ is the angle made by tangent at P with x - axis, then slope ot the tangent at P = $\tan\theta$.

Thus, $f'(c)=\tan\theta = \lim_{h-0} \frac{f(c+h)-f(c)}{h}$

Note :- If tangent is parallel to x-axis, then $f'(c)=0$.

9.4 DERIVATIVES OF VARIOUS STANDARD FUNCTIONS

1. $\frac{d}{dx}(k) = 0$, k is any constant
2. $\frac{d}{dx}(x) = 1$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x . \cot x$

5. $\frac{d}{dx}(\log_e x)=\frac{1}{x}$

7. $\frac{d}{dx}(\sin x) = \cos x$

8. $\frac{d}{dx}(\cos x) = -\sin x$

9. $\frac{d}{dx}(\tan x) = \sec^2 x$

Proof :

1. Let $y = f(x) = k$

$\therefore f(x+h) = k$

By definition, $\frac{dy}{dx} = \lim_{h-0} \frac{f(x+h)-f(x)}{h}$

$$= \frac{d}{dx}(k)= \lim_{h-0} \frac{k-k}{h}$$

$$= 0$$

2. Let $y = f(x) = x$

$\therefore f(x+h) = x+h$

By definition, $\frac{dy}{dx} = \lim_{h-0} \frac{f(x+h)-f(x)}{h}$

$$= \frac{d}{dx}(x)= \lim_{h-0} \frac{x+h-x}{h}$$

$$= \lim_{h\to 0} 1$$

$$= 1$$

3. Let $y = f(x) = x^n$

$\therefore f(x+h) = (x+h)^n$

By definition,

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}\left(x^n\right)= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

Case I : Let n be a positive integer

$$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{\left(x^n + {}^n c_1 x^{n-1}h + {}^n c_2 x^{n-2}h^2 + ... + h^n\right) - x^n}{h}$$

$$= \lim_{h\to 0}\left({}^{n}c_{1}\, x^{n-1} + {}^{n}c_{2}\, x^{n-2}h + \ldots + h^{n-1}\right)$$

$$= {}^{n}c_{1}\, x^{n-1}$$

$$= nx^{n-1}$$

Case II : Let n be a negative integer or a fraction

$$\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{x^n\left(1+\frac{h}{x}\right)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[x^n\left\{1 + n.\frac{h}{x} + \frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2 + \ldots\infty\right\} - x^n\right]$$

$$= \lim_{h\to 0}\left[x^n\left\{n.\frac{1}{x} + \frac{n(n-1)}{2!}.\frac{h}{x^2} + \ldots\infty\right\}\right]$$

$$= x^n.n.\frac{1}{x}$$

$$= nx^{n-1}$$

∴ for all rational n, $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

4. Let $y = f(x) = e^x$

∴ $f(x+h) = e^{x+h}$

By definition,

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}\left(e^x\right) = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x.e^h - e^x}{h}$$

$$= e^x \lim_{h\to 0}\frac{e^h - 1}{h}$$

$$= e^x.1$$

$$= e^x$$

5. Let $y = f(x) = \log_e x$

∴ $f(x+h) = \log_e(x+h)$

By definition,

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(\log_e x) = \lim_{h\to 0} \frac{\log_e(x+h) - \log_e x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\log_e\left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\log\left(1+\frac{h}{x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3}....\infty\right]$$

$$= \lim_{h\to 0} \left[\frac{1}{x} - \frac{h}{2x^2} + \frac{h^2}{3x^3}...\infty\right]$$

$$= \frac{1}{x}$$

7. Let y = f(x) = sin x

. f(x+h) = sin (x+h)

By definition,

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\frac{x+h+x}{2}\sin\frac{x+h-x}{2}}{h}$$

$$= \lim_{h\to 0} \left[2\cos\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{h}\right]$$

$$= \lim_{h\to 0} . \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

= cos x.1

= cosx

8. Let y = f(x) = cosx

. f(x+h) = cos (x+h)

By definition,

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(\cos x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{-2\sin\frac{x+h+x}{2}\sin\frac{x+h-x}{2}}{h}$$

$$= \lim_{h\to 0} \left[-2\sin\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{h}\right]$$

$$= \left(\lim_{h\to 0} -\sin\left(x+\frac{h}{2}\right)\right).\left(\lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)$$

$= -\sin x.$

9. Let $y = f(x) = \tan x$

. $f(x+h) = \tan(x+h)$

By definition $\left\{\cos\left(x+\frac{h}{2}\right)\right\}$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(\tan x) = \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0} \frac{\sin(x+h-x)}{h\cos(x+h)\cos x}$$

$$= \left(\lim_{h\to 0} \frac{\sinh}{h}\right).\left(\lim_{h\to 0} \frac{1}{\cos(x+h)\cos x}\right)$$

$$= 1.\frac{1}{\cos x.\cos x}$$

$= \sec^2 x$

10. Let $y = f(x) = \cot x$

. $f(x+h) = \cot(x+h)$

By definition,

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(\cot x) = \lim_{h\to 0} \frac{\cot(x+h) - \cot x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right]$$

$$= \lim_{h\to 0} \frac{\sin x \cos(x+h) - \cos x \sin(x+h)}{h \sin(x+h)\sin x}$$

$$= \lim_{h\to 0} \frac{\sin\{x-(x+h)\}}{h \sin(x+h)\sin x}$$

$$= \lim_{h\to 0} \frac{\sin(-h)}{h \sin(x+h)\sin x}$$

$$= -\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\sin(x+h).\sin x}\right)$$

$$= -1.\ \frac{1}{\sin x.\sin x}$$

$$= -\ \text{cosec}^2 x$$

11. Let $y = f(x) = \sec x$

$\therefore\ f(x+h) = \sec(x+h)$

By definition,

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(\sec x) = \lim_{h\to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}.\frac{\cos x - \cos(x+h)}{\cos(x+h)\cos x}$$

$$= \lim_{h\to 0} \frac{1}{h}.\frac{2\sin\frac{x+x+h}{2}\ \sin\frac{x+h-x}{2}}{\cos(x+h)\cos x}$$

$$= \lim_{h\to 0}\left[\sin\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}.\frac{1}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0} \sin\left(x+\frac{h}{2}\right).\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\cdot\frac{1}{\lim_{h\to 0}\{\cos(x+h).\cos x\}}$$

$$= \sin x.1.\frac{1}{\cos x.\cos x}$$

$$= \frac{1}{\cos x}.\frac{\sin x}{\cos x}$$

$$= \sec x.\tan x.$$

12. Let $y = f(x) = \operatorname{cosec} x$

$\therefore f(x+h) = \operatorname{cosec}(x+h)$

By definition,

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(\operatorname{cosec} x) = \lim_{h\to 0}\frac{\operatorname{cosec}(x+h)-\operatorname{cosec} x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$= \lim_{h\to 0}\frac{\sin x-\sin(x+h)}{h.\sin(x+h)\sin x}$$

$$= \lim_{h\to 0}\frac{2\cos\frac{x+x+h}{2}\sin\frac{x-x-h}{2}}{h\sin(x+h)\sin x}$$

$$= \lim_{h\to 0}\left[\frac{2.\cos\left(x+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{h\sin(x+h)\sin x}\right]$$

$$= \lim_{h\to 0}\left[-\frac{\cos\left(x+\frac{h}{2}\right)}{\sin(x+h).\sin x}.\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]$$

$$= -\lim_{h\to 0} \frac{\cos\left(x+\frac{h}{2}\right)}{\sin(x+h).\sin x} . \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= -\frac{\cos x}{\sin x.\sin x}.1$$

$$= -\operatorname{cosec} x \cot x$$

9.5 THE DERIVATIVE OF SUM AND DIFFERENCE OF FUNCTIONS

Theorem 2.5.1 If f_1 and f_2 are two differentiable functions on the interval I, and f be their sum function so that $f = f_1 + f_2$, Then f is differentiable and

$$\frac{d}{dx} f = \frac{d}{dx} f_1 + \frac{d}{dx} f_2$$

Proof : Let x be any point of I and x+h be a neighbouring point.

Then, $f(x) = f_1(x) + f_2(x)$

and $f(x+h) = f_1(x+h) + f_2(x+h)$

$\therefore f(x+h)-f(x) = f_1(x+h) + f_2(x+h) - f_1(x) - f_2(x)$

$$\Rightarrow \frac{f(x+h)-f(x)}{h} = \frac{f_1(x+h)-f_1(x)}{h} + \frac{f_2(x+h)-f_2(x)}{h}$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f_1(x+h)-f_1(x)}{h} + \lim_{h\to 0} \frac{f_2(x+h)-f_2(x)}{h} \quad -(1)$$

Θ f_1 and f_2 are differential in I

$$\therefore \frac{df_1}{dx} = \lim_{h\to 0} \frac{f_1(x+h)-f_1(x)}{h}$$

and $\frac{df_2}{dx} = \lim_{h\to 0} \frac{f_2(x+h)-f_2(x)}{h}$ exist.

$$\therefore (1) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x)$$

This shows that f is differentiable

and $\frac{d}{dx} f(x) = \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x)$

i.e., $\frac{d}{dx} f = \frac{d}{dx} f_1 + \frac{d}{dx} f_2$

Theorem 2.5.2 If f_1 and f_2 are two differentiable functions on the interval I,

and f be their difference function so that $f=f_1-f_2$ then f is differentiable and

$$\frac{d}{dx}f = \frac{d}{dx}f_1 - \frac{d}{dx}f_2$$

Proof : Try yourself.

ILLUSTRATIVE EXAMPLE :

Example 1. Find the derivatives of the following with respect to x.

a) $x(1+x)^2$ b) $b\log_e x+x^3+\sqrt{x}$

c) $4\log_{10}x$ d) $\sqrt[3]{1+3\log_a x+3(\log_a x)^2+(\log_a x)^3}$

Solution :

a) $\frac{d}{dx}\{x(1+x)^2\}=\frac{d}{dx}\{x(1+2x+x^2)\}$

$$= \frac{d}{dx}\left(x+2x^2+x^3\right)$$

$$= \frac{d}{dx}(x)+2\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(x^3\right)$$

$= 1 + 2(2x) + 3x^2$

$= 1 + 4x + 3x^2$

b) $\frac{d}{dx}\left(b\log_e x+x^3+\sqrt{x}\right)=b\frac{d}{dx}(\log_e x)+\frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$

$$= b.\frac{1}{x}+3x^2+\frac{1}{2}x^{\frac{1}{2}-1}$$

$$= \frac{b}{x}+3x^2+\frac{1}{2\sqrt{x}}$$

c) $\frac{d}{dx}(4\log_{10}x)=\frac{d}{dx}\left(\frac{4\log_e x}{\log_e 10}\right)$

$$= \frac{4}{\log_e 10}\frac{d}{dx}(\log_e x)$$

$$= 4\log_{10}e\frac{1}{x}$$

$$= \frac{4\log_{10}e}{x}$$

d) $\frac{d}{dx}\sqrt[3]{1+3\log_a x+3\left(\log_a x\right)^2+\left(\log_a x\right)^3} = \frac{d}{dx}\left\{\left(1+\log_a x\right)^3\right\}^{\frac{1}{3}}$

$$= \frac{d}{dx}\left\{1+\frac{\log_e x}{\log_e a}\right\} = \frac{d}{dx}(1)+\frac{1}{\log_e a}\ \frac{d}{dx}\left(\log_e x\right)$$

$$= 0+\frac{1}{\log_e a}.\frac{1}{x} = \frac{1}{x}\log_a e$$

Example 2 : Find the derivatives of y with respect to x for the following

a) $y = \frac{9}{\sin x} - 3\cot x + a^x$ b) $y = \tan x - e^x + 4$

c) $y = 3\log x + x^7 - 4\sec x$

Solution :

a) $y = \frac{9}{\sin x} - 3\cot x + a^x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{9}{\sin x} - 3\cot x + a^x\right)$$

$$= 9\frac{d}{dx}(\operatorname{cosec} x) - 3\frac{d}{dx}(\cot x) + \frac{d}{dx}(a^x)$$

$$= 9(-\operatorname{cosec} x.\cot x) - 3(-\operatorname{cosec}^2 x) + a^x \log_e a$$

$$= -9\operatorname{cosec} x.\cot x + 3\operatorname{cosec}^2 x + a^x \log_e a$$

b) $y = \tan x - e^x + 4$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\tan x - e^x + 4\right)$$

$$= \frac{d}{dx}(\tan x) - \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}(4)$$

$= \sec^2 x - e^x$

c) $y = 3\log x + x^7 - 4\sec x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(3\log x + x^7 - 4\sec x\right)$$

$$= 3\frac{d}{dx}(\log x) + \frac{d}{dx}\left(x^7\right) - 4\frac{d}{dx}(\sec x)$$

$$= \frac{3}{x} + 7x^6 - 4\sec x \tan x$$

Example 3 : Find the point on the curve $y = x^2 - 3x$ at which the tangent is parallel to the x–axis.

Solution : Given $y = x^2 - 3x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^3 + 2x\right)$$

= 2x–3

For tangents parallel to x – axis, we have

$$\frac{dy}{dx} = 0 \Rightarrow 2x - 3 = 0 \Rightarrow x = \frac{3}{2}$$

$$\therefore y = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) = \frac{9}{4} - \frac{9}{2} = \frac{-9}{4}$$

$$\therefore \text{The point is } \left(\frac{3}{2}, \frac{-9}{4}\right)$$



CHECK YOUR PROGRESS - 1

Q 1: Find the derivative of the following functions with respect to x.

a) $\frac{x^3 + x^2 - x + 2}{x}$ b) $\sin x + e^x - \frac{1}{x^4}$

c) cosecx $+a^x$ d) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 + 2$

e) $\log(a^2 x^3 e^x)$ f) $3\sin^{-1}x + \cos x - a^x + 1$

Q 2: Prove that no tangent to the curve $y = x^3 + 2x$ is parallel to the x–axis.

9.6 DERIVATIVE OF PRODUCT OF FUNCTIONS AND QUOTIENT RULE

Theorem 2.6.1 : Let u(x) and v(x) be two differentiable functions. Then the product u(x) v(x) is also differentiable and

$$\frac{d}{dx}\big(u(x)v(x)\big) = u(x)v'(x) + v(x)u'(x)$$

Proof : Let x be any point in the domain, and x+h be a neighbouring point.

f(x)=u(x)v(x)

∴ f(x+h) = u(x+h)v(x+h)

∴ f(x+h)–f(x) =u(x+h)v(x+h)–u(x)v(x)

= u(x+h)v(x+h)–u(x+h)v(x)+u(x+h)v(x)–u(x)v(x)

= u(x+h) {v(x+h)–v(x)} + v(x) {u(x+h)–u(x)}

$$\therefore \frac{f(x+h) - f(x)}{h} = u(x+h).\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h}$$

$$\therefore \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}u(x+h).\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}+v(x)\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

$$=u(x).\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}+v(x).\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} \quad \ldots (1)$$

Θ u and v are differentiable.

$$\therefore u'(x)=\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

and $v'(x)=\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}$

$$\therefore (1)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=u(x)v'(x)+v(x)u'(x)$$

Thus f is differentable at x and

$f'(x)=u(x)v'(x)+v(x)u'(x)$

i.e., $\frac{d}{dx}(f(x))=u(x)v'(x)+v(x)u'(x)$

i.e., $\frac{d}{dx}u(x)v(x)=u(x)v'(x)+v(x)u'(x)$.

Corollary : If u, v, w are three differentable functions , then

$$\frac{d}{dx}(u(x)v(x)w(x))=u'(x)v(x)w(x)+v'(x)u(x)w(x)+w'(x)u(x)v(x)$$

Theorem 2.6.2 : Let u(x) and v(x) be two differentiable functions in the same interval, then the quotient function $\frac{u(x)}{v(x)}$ is also differentiable and

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right)=\frac{v(x)\frac{d}{dx}(u(x))-u(x)\frac{d}{dx}(v(x))}{(v(x))^2}$$

$$=\frac{v(x)u'(x)-u(x)v'(x)}{(v(x))^2}$$

Proof : Let $f(x)=\frac{u(x)}{v(x)}$

$$\therefore f(x+h)=\frac{u(x+h)}{v(x+h)}$$

$$\therefore f(x+h)-f(x)=\frac{u(x+h)}{v(x+h)}-\frac{u(x)}{v(x)}$$

$$=\frac{v(x)u(x+h)-u(x)v(x+h)}{v(x+h)v(x)}$$

$$\Rightarrow \frac{f(x+h)-f(x)}{h} = \frac{v(x)u(x+h)-v(x)u(x)+v(x)u(x)-u(x)v(x+h)}{h.v(x+h)v(x)}$$

$$= \frac{v(x)[u(x+h)-u(x)]+u(x)[v(x)-v(x+h)]}{h.v(x+h)v(x)}$$

$$= \frac{v(x).\frac{[u(x+h)-u(x)]}{h} - u(x)\frac{[v(x+h)-v(x)]}{h}}{v(x+h)v(x)}$$

$$\therefore \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{v(x).\lim\limits_{h\to 0}\frac{u(x+h)-u(x)}{h} - u(x).\lim\limits_{h\to 0}\frac{v(x+h)-v(x)}{h}}{v(x).v(x)} \quad \ldots (1)$$

Θ u(x) and v(x) are differentiable

$$\therefore u'(x) = \lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

and $v'(x) = \lim\limits_{h\to 0}\frac{v(x+h)-v(x)}{h}$

$$\therefore (1) = \lim_{h-0}\frac{f(x+h)-f(x)}{h} = \frac{v(x)u'(x)-u(x)v'(x)}{\{v(x)\}^2}$$

Thus, f(x) is differentiable at x and

$$f'(x) = \frac{v(x)u'(x)-u(x)v'(x)}{\{v(x)\}^2}$$

$$\text{i.e. } \frac{d}{dx}(f(x)) = \frac{v(x)u'(x)-u(x)v'(x)}{\{v(x)\}^2}$$

$$\text{i.e. } \frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v(x)u'(x)-u(x)v'(x)}{(v(x))^2}$$

Illustrative examples :

Example 1 : Find the derivatives of the following functions with respect to x.

a) x^2e^x b) $e^x\log x$ c) $x^3e^x\sin x$

Solution : a) $\frac{d}{dx}(x^2e^x) = e^x\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(e^x)$

$= e^x.2x + x^2.e^x$

$= xe^x(2+x)$

b) $\frac{d}{dx}(e^x\log x) = \log x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(\log x)$

$= \log x.e^x + e^x.\frac{1}{x}$

$$= e^x \left(\log x + \frac{1}{x}\right)$$

c) $\frac{d}{dx}\left(x^3 e^x \sin x\right) = e^x \sin x \frac{d}{dx}\left(x^3\right) + x^3 \sin x \frac{d}{dx}\left(e^x\right) + x^3 e^x \frac{d}{dx}(\sin x)$

$= e^x \sin x.3x^2 + x^3 \sin x.e^x + x^3 e^x \cos x$

$= x^2 e^x(3\sin x + x \sin x + x \cos x)$

Example 2 : Find the derivatives of the following with respect to x.

a) $\frac{\cos x}{\log_a x}$ b) $\frac{x^n}{\log_e x}$ c) $\frac{x^2}{e^x}$

Solution : a) $\frac{d}{dx}\left(\frac{\cos x}{\log_a x}\right) = \frac{\log_a x \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\log_a x)}{(\log_a x)^2}$

$$= \frac{\log_a x(-\sin x) - \cos x.\frac{d}{dx}\left(\frac{\log_e x}{\log_e a}\right)}{(\log_a x)^2}$$

$$= \frac{-\sin x \log_a x - \cos x.\frac{1}{\log_e a}.\frac{1}{x}}{(\log_a x)^2}$$

$$= \frac{-\sin x \log_a x - \cos x \log_a e}{x(\log_a x)^2}$$

b) $\frac{d}{dx}\left(\frac{x^n}{\log_e x}\right) = \frac{\log_e x \frac{d}{dx}\left(x^n\right) - x^n \frac{d}{dx}(\log_e x)}{(\log_e x)^2}$

$$= \frac{\log_e x.nx^{n-1} - x^n.\frac{1}{x}}{(\log_e x)^2}$$

$$= \frac{nx^{n-1}\log_e x - x^{n-1}}{(\log_e x)^2}$$

c) $\frac{d}{dx}\left(\frac{x^2}{e^x}\right) = \frac{e^x \frac{d}{dx}\left(x^2\right) - x^2 \frac{d}{dx}\left(e^x\right)}{\left(e^x\right)^2}$

$$= \frac{e^x.2x - x^2.e^x}{(e^x)^2}$$

$$= \frac{xe^x(2-x)}{(e^x)^2}$$

$$= \frac{x(2-x)}{e^x}$$

Example 3 : Find the derivatives of the following functions with respect to x.

a) $3x^2+x^3\log x$ b) $\frac{x^2+x-1}{x^2 - x + 1}$

c) $\frac{\sec x}{x + e^x}$ d) $x^3\sin x + \frac{e^x+1}{\log x}$

Solution : a) $\frac{d}{dx}(3x^2+x^3\log x) = \frac{d}{dx}(3x^2)+\frac{d}{dx}(x^3\log x)$

$$= 3\frac{d}{dx}(x^2) + \log x\frac{d}{dx}(x^3) + x^3\frac{d}{dx}(\log x)$$

$$= 3(2x) + \log x\,(3x^2) + x^3.\frac{1}{x}$$

$$= 6x + 3x^2\log x + x^2$$

b) $\frac{d}{dx}\left(\frac{x^2+x-1}{x^2-x+1}\right) = \frac{(x^2-x+1)\frac{d}{dx}(x^2+x-1)-(x^2+x-1)\frac{d}{dx}(x^2-x+1)}{(x^2-x+1)^2}$

$$= \frac{(x^2-x+1)(2x+1)-(x^2+x-1)(2x-1)}{(x^2-x+1)^2}$$

$$= \frac{2x^3-2x^2+2x+x^2-x+1-2x^3-2x^2+2x-x^2-x+1}{(x^2-x+1)^2}$$

$$= \frac{-4x^2+2x+2}{(x^2-x+1)^2}$$

c) $\frac{d}{dx}\left(\frac{\sec x}{x+e^x}\right) = \frac{(x+e^x)\frac{d}{dx}(\sec x) - \sec x\frac{d}{dx}(x+e^x)}{(x+e^x)^2}$

$$= \frac{(x+e^x).\sec x\tan x - \sec x(1+e^x)}{(x+e^x)^2}$$

$$= \frac{\sec x\{(x+e^x)\tan x - 1 - e^x\}}{(x+e^x)^2}$$

d) $\frac{d}{dx}\left(x^3 \sin x + \frac{e^x+1}{\log x}\right)$

$$= \frac{d}{dx}(x^3 \sin x) + \frac{d}{dx}\left(\frac{e^x+1}{\log x}\right)$$

$$= \sin x \frac{d}{dx}(x^3) + x^3 \frac{d}{dx}\sin x + \frac{\log x \frac{d}{dx}(e^x+1) - (e^x+1)\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \sin x.3x^2 + x^3.\cos x + \frac{\log x.(e^x+0) - (e^x+1).\frac{1}{x}}{(\log x)^2}$$

$$= x^2(3\sin x + x\cos x) + \frac{e^x(x\log x) - e^x - 1}{x(\log x)^2}$$

Example 4 : If $y = \frac{x}{x+5}$, prove that $x\frac{dy}{dx} = y(1-y)$

Solution : We have $y = \frac{x}{x+5}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{x+5}\right)$$

$$= \frac{(x+5)\frac{d}{dx}(x) - x\frac{d}{dx}(x+5)}{(x+5)^2}$$

$$= \frac{(x+5).1 - x(1+0)}{(x+5)^2}$$

$$= \frac{x+5-x}{(x+5)^2}$$

$$\therefore x\frac{dy}{dx} = \frac{5x}{(x+5)^2} \qquad \text{... (1)}$$

Also, $y(1-y) = \frac{x}{x+5}\left(1 - \frac{x}{x+5}\right)$

$$= \frac{x}{x+5}.\frac{x+5-x}{x+5}$$

$$= \frac{5x}{(x+5)^2} \qquad \ldots (2)$$

From (1) and (2), we get

$$x\frac{dy}{dx} = y(1-y)$$



CHECK YOUR PROGRESS – 2

Q 1: Find the derivatives of the following functions with respect to x.

a) $\frac{2x+3}{3x+4}$ b) $\frac{1+\tan x}{1-\tan x}$ c) $\frac{x}{a^2+x^2}$

d) $7xa^x + \log x^x$ e) $3\log x \tan x - x^3e^x+2$

f) $\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$ g) $\log (x^3 e^{4x} a^x)$

Q 2: If $y= \frac{\sin x}{1+\cos x}$ prove that $\frac{dy}{dx} = \frac{1}{1+\cos x}$

Q 3: If $f(x) = \frac{x-2}{2\sqrt{x}}$, find $f'(4)$.

9.7 Derivative of Composite Functions

If $y=\phi(t)$ and $t=f(x)$, then $y=\phi(f(x))$ is a composite function. Examples of composite functions are *cos 2x, sinx², log cosx, e^{sinx}* etc.

Theorem 2.7.1 : Let $y = \phi(t)$ when $t=f(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

Proof : Let k be the increment in t corresponding to the increment h in x.

$\Theta\ t = f(x) \qquad \ldots (1)$

$\therefore\ t+k = f(x+h) \qquad \ldots (2)$

(2)–(1) $\Rightarrow$ $k = f(x+h) - f(x) \qquad \ldots (3)$

Also, $h \to 0$ gives $t+k \to t$ or $k \to 0$

Now, $\phi(f(x)) = \phi(t)$

$\Rightarrow\ \phi(f(x+h)) = \phi(t+k)$

$\therefore\ \phi(f(x+h)) - \phi(f(x)) = \phi(t+k) - \phi(t)$

$$= \frac{\phi(f(x+h)) - \phi(f(x))}{h} = \frac{\phi(t+k) - \phi(t)}{k} \cdot \frac{k}{h}$$

$$= \frac{\phi(t+k) - \phi(t)}{k} \cdot \frac{f(x+h) - f(x)}{h}, \text{ by (3)}$$

Taking limit on both sides, we get

$$\lim_{h\to 0} \frac{\phi(f(x+h)) - \phi(f(x))}{h} = \lim_{k\to 0} \frac{\phi(t+k) - \phi(t)}{k} \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(\phi(f(x))) = \frac{d}{dt}(\phi(t)) \cdot \frac{d}{dx}(f(x))$$

$$\Rightarrow \frac{d}{dx}(\phi(t)) = \frac{d}{dt}(y) \cdot \frac{d}{dx}(t)$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Generalisation of above theorem :

Let y=f(u), u=g(t), t=h(x) then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dt} \cdot \frac{dt}{dx}$$

Thus, if the composite function y is a function of many functions, then

$$\frac{dy}{dx} = \frac{d(1^{st} \text{ function})}{d(2^{nd} \text{ function})} \cdot \frac{d(2^{nd} \text{ function})}{d(3^{rd} \text{ function})} \cdots \frac{d(\text{last function})}{dx}$$

Above is also known as **chain Rule**.

Illustrative examples :

Example 1 : Find the derivatives of the following functions with respect to x.

a) $\cos x^3$ b) $e^{\sin x^2}$ c) $\sec^3 3x$.

Solution : a) Let $t = x^3$

$$\therefore \frac{dt}{dx} = 3x^2$$

$$\therefore \frac{d}{dx}(\cos x^3) = \frac{d}{dx}(\cos t)$$

$$= \frac{d}{dt}(\cos t) \cdot \frac{dt}{dx}$$

$$= -\sin t \cdot 3x^2$$

$$= -3x^2 \sin x^3$$

b) Let $y = e^{\sin x^2}$

$= e^{\sin t}$, where $t = x^2$

$= e^u$, where u = sint

By chain rule

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dt}.\frac{dt}{dx}$$

$$= \frac{d}{du}(e^u)\frac{d}{dt}(\sin t).\frac{d}{dx}(x^2)$$

$= e^u \cos t.2x$

$= 2xe^{\sin t}.\cos t$

$= 2xe^{\sin x^2}\cos x^2$

c) Let $y = \sec^3 3x$

$= \sec^3 u$, where u = 3x

$= t^3$, where t = secu

By chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{du}.\frac{du}{dx}$$

$$= \frac{d}{dt}(t^3)\frac{d}{du}(\sec u).\frac{d}{dx}(3x)$$

$= 3t^2 \sec u \tan u.3$

$= 9\sec^2 u.\ \sec u \tan u$

$= 9\sec^3 3x \tan 3x.$

Example 2 : Find the derivatives of the following functions with respect to x

a) $y = \log(\sqrt{x} + \frac{1}{\sqrt{x}})$ b) $y=\log \log \log x^2$

c) $y=\cos^{-1}x^2$ d) $y=a \cot^{-1}(m \tan^{-1} bx)$

Solution : a) Let $t = \sqrt{x} + \frac{1}{\sqrt{x}}$

$\therefore\ y=\log t$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$= \frac{d}{dt}(\log t).\frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$$

$$= \frac{1}{t}\left(\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}\right)$$

$$= \frac{1}{\sqrt{x} + \frac{1}{\sqrt{x}}}.\frac{x-1}{2x\sqrt{x}}$$

$$= \frac{\sqrt{x}}{x+1}.\frac{x-1}{2x\sqrt{x}}$$

$$= \frac{x-1}{2x(x+1)}$$

b) $y = \log \log \log x^2$

Let $v = \log \log x^2$

$w = \log x^2$

$t = x^2$

$$\therefore \frac{dy}{dx} = \frac{dy}{dv}.\frac{dv}{dw}.\frac{dw}{dt}.\frac{dt}{dx}$$

$$= \frac{d}{dv}(\log v).\frac{d}{dw}(\log w).\frac{d}{dt}(\log t).\frac{d}{dx}(x^2)$$

$$= \frac{1}{v}.\frac{1}{w}.\frac{1}{t}.2x$$

$$= \frac{2x}{\log\log x^2.\log x^2.x^2}$$

$$= \frac{2}{x(\log\log x^2)(\log x^2)}$$

c) $y = \cos^{-1}x^2$

$= \cos^{-1}u$,where $u = x^2$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$= \frac{d}{du}\left(\cos^{-1}u\right)\frac{d}{dx}\left(x^2\right)$$

$$= \frac{-1}{\sqrt{1-u^2}}.2x$$

$$= \frac{-2x}{\sqrt{1-x^4}}$$

d) $y = a\cot^{-1}(m\tan^{-1}bx)$

$= a\cot^{-1}(m \tan^{-1} u)$, where $u = bx$,

$= a \cot^{-1} v$, where $v = m\tan^{-1}u$,

$= aw$, where $w = \cot^{-1} v$.

By chain rule, we get,

$$\frac{dy}{dx} = \frac{dy}{dw}.\frac{dw}{dv}.\frac{dv}{du}.\frac{du}{dx}$$

$$= \frac{d}{dw}(aw).\frac{d}{dv}\left(\cot^{-1} v\right).\frac{d}{du}\left(m\tan^{-1} u\right).\frac{d}{dx}(bx)$$

$$= a.1.\frac{-1}{1+v^2}.\frac{m}{1+u^2}.b.1$$

$$= \frac{-mab}{\left\{1+\left(m\tan^{-1} bx\right)^2\right\}\left(1+b^2x^2\right)}$$

Example 3 : Prove that the derivative of an even function is always an odd function.

Solution : Let f be an even function.

Then f(x) = f(–x)

$$\therefore \frac{d}{dx} f(x) = \frac{d}{dx} f(-x)$$

$$\Rightarrow f'(x) = f'(-x)\frac{d}{dx}(-x)$$

$$= f'(-x).(-1)$$

$$= -f'(-x)$$

$\therefore$ $f'(-x) = -f'(x)$

$=$ f' is an odd function.

Example 4 : If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$,prove that

$$\left(1-x^2\right)\frac{dy}{dx} = xy + 1$$

Solution : $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ ... (1)

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-x^2}.\frac{1}{\sqrt{1-x^2}} - \sin^{-1} x.\frac{1}{2\sqrt{1-x^2}}(-2x)}{1-x^2}$$

$$= \frac{1+x.\frac{\sin^{-1} x}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{1+xy}{1-x^2}$$

$$\therefore (1-x^2)\frac{dy}{dx} = xy + 1$$



CHECK YOUR PROGRESS – 3

Q 1: Find the derivatives of the following functions with respect with respect to x.

a) y = log (sec x + tan x)

b) y = sin(cos x)

c) $y = \log\left(x + \frac{1}{x}\right)$

Q 2: Find $\frac{dy}{dx}$ of the following.

a) $y = \log\left(x + \sqrt{a^2 + x^2}\right)$

b) $y = e^{ax^2+bx+c}$

c) $y = \log(e^{3x}.x^3)$

d) $y = x^3 \sin 2x \,.\, \log x$

Q 2: Prove that the derivative of an odd function is always an even function.



9.10 LET US SUM UP

- $\frac{dy}{dx}$ at a point on a given curve y = f(x) represents the gradient of the tangent line at that point.
- For tangents parallel to x–axis, $\frac{dy}{dx} = 0$,
- $\frac{d}{dx}(k) = 0 \quad (k = \text{constant})$

  $\frac{d}{dx}(x^n) = nx^{n-1}$

  $\frac{d}{dx}(e^x) = e^x$

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x.\tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x.\cot x$$

- $\frac{d}{dx}(f_1 \pm f_2) = \frac{df_1}{dx} \pm \frac{df_2}{dx}$
- $\frac{d}{dx}(uv) = v.\frac{du}{dx} + u.\frac{dv}{dx}$
- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} + u.\frac{dv}{dx}}{v^2}, v \neq 0$
- If y = f(u), u = g(t), t = h(x), then

  $$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dt}.\frac{dt}{dx}$$
- If u and v are two functions of x such that u=f(x) and v = g(x), then

  $$\frac{du}{dv} = \frac{\text{Derivative of u with respect to x}}{\text{Derivative of v with respect to x}}$$

9.11 ANSWERS TO CHECK YOU PROGRESS

Check your progress - 1

Ans to Q No 1: a) $\frac{d}{dx}\left(\frac{x^3 + x^2 + x + 2}{x}\right) = \frac{d}{dx}\left(\frac{x^3}{x} + \frac{x^2}{x} + \frac{x}{x} + \frac{2}{x}\right)$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(x) - \frac{d}{dx}(1) + 2\frac{d}{dx}(x^{-1})$$

$= 2x + 1 - 0 + 2(-1)x^{-2}$

$= 2x + 1 - \frac{2}{x^2}$

b) $\frac{d}{dx}\left(\sin x + e^x - \frac{1}{x^4}\right) = \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x) - \frac{d}{dx}(x^{-4})$

$= \cos x + e^x - (-4)x^{-4-1}$

$= \cos x + e^x + \frac{4}{x^5}$

c) $\frac{d}{dx}(\cos ecx + a^x) = \frac{d}{dx}(\cos ecx) + \frac{d}{dx}(a^x)$

$= -\text{cosecxcotx} + a^x \log_e a$

d) $\frac{d}{dx}\left\{\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 + 2\right\} = \frac{d}{dx}\left(x + \frac{1}{x} + 2 + 2\right)$

$= \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1}) + \frac{d}{dx}(4)$

$= 1 + (-1)x^{-1-1} + 0$

$= 1 - \frac{1}{x^2}$

e) $\frac{d}{dx}\{\log(a^2x^3e^x)\} = \frac{d}{dx}(\log a^2 + \log x^3 + \log e^x)$

$= \frac{d}{dx}(2\log a + 3\log x + x\log e)$

$= \frac{d}{dx}(2\log a) + 3\frac{d}{dx}(\log x) + \frac{d}{dx}(x) \quad (\because \log e = 1)$

$= 0 + \frac{3}{x} + 1 = 1 + \frac{3}{x}$

f) $\frac{d}{dx}(3\sin^{-1}x + \cos x - a^x + 1)$

$= 3\frac{d}{dx}(\sin^{-1}x) + \frac{d}{dx}(\cos x) - \frac{d}{dx}(a^x) + \frac{d}{dx}(1)$

$= \frac{3}{\sqrt{1-x^2}} + (-\sin x) - a^x \log_e a + 0$

$= \frac{3}{\sqrt{1-x^2}} - \sin x - a x \log_e a$

Ans to Q No 2: Given, $y = x^3+2x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^3 + 2x\right)$$

$$= \frac{d}{dx}\left(x^3\right) + 2\frac{d}{dx}(x)$$

$$= 3x^2 + 2$$

$$\neq 0 \text{ for } x \in \Re$$

. There does not exist any value of x for which $\frac{dy}{dx} = 0$

. No tangent to the curve $y = x^3+2x$ will be parallel to x–asis.

Check your progress - 2

Ans to Q No 1: a) $\frac{d}{dx}\left(\frac{2x+3}{3x+4}\right) = \frac{(3x+4)\frac{d}{dx}(2x+3) - (2x+3)\frac{d}{dx}(3x+4)}{(3x+4)^2}$

$$= \frac{(3x+4)(2+0) - (2x+3)(3+0)}{(3x+4)^2}$$

$$= \frac{(6x+8) - (6x+9)}{(3x+4)^2}$$

$$= \frac{-1}{(3x+4)^2}$$

b) $\frac{d}{dx}\left(\frac{1+\tan x}{1-\tan x}\right) = \frac{(1-\tan x)\frac{d}{dx}(1+\tan x) - (1+\tan x)\frac{d}{dx}(1-\tan x)}{(1-\tan x)^2}$

$$= \frac{(1-\tan x)(\sec^2 x) - (1+\tan x)(-\sec^2 x)}{(1-\tan x)^2}$$

$$= \frac{\sec^2 x(1-\tan x+1+\tan x)}{(1-\tan x)^2}$$

$$= \frac{2\sec^2 x}{(1-\tan x)^2}$$

c) $\frac{d}{dx}\left(\frac{x}{a^2+x^2}\right) = \frac{\left(a^2+x^2\right)\frac{d}{dx}(x) - x\frac{d}{dx}\left(a^2+x^2\right)}{\left(a^2+x^2\right)^2}$

$$= \frac{\left(a^2+x^2\right)1 - x(2x)}{\left(a^2+x^2\right)^2}$$

$$= \frac{a^2 + x^2 - 2x^2}{\left(a^2 + x^2\right)^2}$$

$$= \frac{a^2 - x^2}{\left(a^2 + x^2\right)^2}$$

d) $\frac{d}{dx}\left(7xa^x + \log x^x\right) = \frac{d}{dx}\left(7xa^x + x\log x\right)$

$$= 7\frac{d}{dx}\left(xa^x\right) + \frac{d}{dx}\left(x\log x\right)$$

$$= 7\left[a^x \frac{d}{dx}(x) + x\frac{d}{dx}\left(a^x\right)\right] + \left[\log x \frac{d}{dx}(x) + x\frac{d}{dx}(\log x)\right]$$

$$= 7\left[a^x.1 + x.a^x \log_e a\right] + \left[\log x.1 + x.\frac{1}{x}\right]$$

$$= 7a^x(1 + x\log_e a) + \log x + 1$$

e) $\frac{d}{dx}\left[3\log x \tan x - x^3e^x + 2\right]$

$$= 3\frac{d}{dx}(\log x \tan x) - \frac{d}{dx}\left(x^3e^x\right) + \frac{d}{dx}(2)$$

$$= 3\left[\tan x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\tan x)\right]$$

$$-\left[e^x \frac{d}{dx}\left(x^3\right) + x^3\frac{d}{dx}\left(e^x\right)\right] + 0$$

$$= 3\left[\tan x.\frac{1}{x} + \log x.\sec^2 x\right] - \left[e^x.3x^2 + x^3.e^x\right]$$

$$= 3\left[\frac{\tan x}{x} + \log x \sec^2 x\right] - x^2e^x[3 + x]$$

f) $\frac{d}{dx}\left[\frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}}\right] = \frac{\left(\sqrt{a} - \sqrt{x}\right)\frac{d}{dx}\left(\sqrt{a} + \sqrt{x}\right) - \left(\sqrt{a} + \sqrt{x}\right)\frac{d}{dx}\left(\sqrt{a} - \sqrt{x}\right)}{\left(\sqrt{a} - \sqrt{x}\right)^2}$

$$= \frac{\left(\sqrt{a} - \sqrt{x}\right)\left(0 + \frac{1}{2\sqrt{x}}\right) - \left(\sqrt{a} + \sqrt{x}\right)\left(0 - \frac{1}{2\sqrt{x}}\right)}{\left(\sqrt{a} - \sqrt{x}\right)^2}$$

$$= \frac{\frac{1}{2\sqrt{x}}\left(\sqrt{a}-\sqrt{x}+\sqrt{a}+\sqrt{x}\right)}{\left(\sqrt{a}-\sqrt{x}\right)^2}$$

$$= \frac{2\sqrt{a}}{2\sqrt{x}\left(\sqrt{a}-\sqrt{x}\right)^2}$$

$$= \frac{\sqrt{a}}{\sqrt{x}\left(\sqrt{a}-\sqrt{x}\right)^2}$$

g) $\frac{d}{dx}\left[\log\left(x^3 e^{4x} a^x\right)\right] = \frac{d}{dx}\left[\log x^3 + \log e^{4x} + \log a^x\right]$

$$= \frac{d}{dx}\left[3\log x + 4x\log e + x\log a\right]$$

$$= 3\frac{d}{dx}(\log x) + 4\frac{d}{dx}(x) + \log a\frac{d}{dx}(x) \quad [\Theta\ \log e = 1]$$

$$= \frac{3}{x} + 4 + \log a$$

Ans to Q No 2: $y = \frac{\sin x}{1+\cos x}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right)$$

$$= \frac{(1+\cos x)\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x + 1}{(1+\cos x)^2}$$

$$= \frac{1}{1+\cos x}$$

Ans to Q No 3: Given, $f(x) = \frac{x-2}{2\sqrt{x}}$

$$= \frac{\sqrt{x}}{2} - \frac{1}{\sqrt{x}}$$

$$\therefore \quad f'(x) = \frac{d}{dx}\left(\frac{\sqrt{x}}{2} - \frac{1}{\sqrt{x}}\right)$$

$$= \frac{1}{2}\frac{d}{dx}\left(x^{\frac{1}{2}}\right) - \frac{d}{dx}\left(x^{-\frac{1}{2}}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2}x^{\frac{1}{2}-1} - \left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right)$$

$$= \frac{1}{4\sqrt{x}} + \frac{1}{2x\sqrt{x}}$$

$$\therefore \quad f'(4) = \frac{1}{4(2)} + \frac{1}{2(4)(2)}$$

$$= \frac{1}{8} + \frac{1}{16}$$

$$= \frac{3}{16}$$

Check your progress - 3

Ans to Q No 1: a) y = log(secx+tanx)

= logt where t = secx + tanx

By chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{d}{dt}(\log t)\frac{d}{dx}(\sec x + \tan x)$$

$$= \frac{1}{t}(\sec x \tan x + \sec^2 x)$$

$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x.$$

b) y = sin(cosx)

= sint ,where t = cosx

By chain rule –

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{d}{dt}(\sin t)\frac{d}{dx}(\cos x)$$

= cost (–sinx) = – cos(cosx). sinx

c) $y = \log\left(x+\frac{1}{x}\right) = \log t$,where $t=x+\frac{1}{x}$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$= \frac{d}{dt}(\log t)\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$= \frac{1}{t}\left(1-\frac{1}{x^2}\right)$$

$$= \frac{1}{x+\frac{1}{x}}\left(1-\frac{1}{x^2}\right)$$

$$= \frac{x}{x^2+1}.\frac{x^2-1}{x^2}$$

$$= \frac{x^2-1}{x(x^2+1)}$$

Ans to Q No 2: a) $y = \log\left(x+\sqrt{a^2+x^2}\right)$

$=$ y = log t, where $t= x+\sqrt{a^2+x^2}$

By chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$= \frac{d}{dt}(\log t).\frac{d}{dx}\left(x+\sqrt{a^2+x^2}\right)$$

$$= \frac{1}{t}\left[1+\frac{1}{2\sqrt{a^2+x^2}}.2x\right]$$

$$= \frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}\left(x+\sqrt{a^2+x^2}\right)}$$

$$= \frac{1}{\sqrt{a^2+x^2}}$$

b) $y = e^{ax^2+bx+c}$

$=$ $y = e^t$, where $t = ax^2+bx+c$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(e^t\right).\frac{d}{dx}\left(ax^2 + bx + c\right)$$

$$= e^t (2ax+b)$$

$$= e^{\left(ax^2+bx+c\right)}.(2ax + b)$$

c) $y = \log (e^{3x}.x^3)$

$= 3x\log e + 3\log x$

$= 3x+3\log x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(3x + 3\log x) = 3 + \frac{3}{x}$$

d) $y = x^3 \sin 2x \log x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^3 \sin 2x \log x\right)$$

$$= \frac{d}{dx}\left(x^3\right)\sin 2x \log x + x^3 \log x \frac{d}{dx}(\sin 2x) + x^3 \sin 2x \frac{d}{dx}(\log x)$$

$$= 3x^2 \sin 2x \log x + x^3 \log x.\cos 2x.2 + x^3 \sin 2x.\frac{1}{x}$$

$$= 3x^2 \sin 2x \log x + 2x^3 \log x.\cos 2x + x^2 \sin 2x$$

Ans to Q No 3: Let f be an odd function.

Then $f(-x) = -f(x)$

$$\therefore \frac{d}{dx} f(-x) = \frac{d}{dx}\{-f(x)\}$$

$$= f'(-x)\frac{d}{dx}(-x) = -f'(x)$$

$$= f'(-x).(-1) = -f'(x)$$

$$= -f'(-x) = -f'(x)$$

$$= f'(-x) = f'(x)$$

. f is an even function.



9.12 FURTHER READING

1) Sharma, A.K.,Text Book of Differential Calculus for B.A,B.Sc, B.Com,I.A.S.,P.C.S., New Delhi,Discovery Publishing House,2004,1st edition.

2) Habib, Amber.,The calculus of finance, Hyderabad,Universities Press India Pvt. Ltd.2011,1st edition.

3) Gupta, Sudhir.,Basics of Differentential Calculus,New Delhi,Anmol publication Pvt. Ltd,2011, 1st edition.

4) Maron, I.A. Problems in calculus of one variable,New Delhi,CBS publishers & Distributors Pvt Ltd.1993,1st edition.



9.13 MODEL QUESTIONS

Q 1: Differentiate the following with respect to x

(a) $\frac{1}{3x+5}$ (b) $\sqrt{ax+b}$ (c) $\sqrt{\sec x}$ (d) $\sqrt{\log x}$ (e) $\frac{x^4+8x+5}{x}$

(f) $2e^{3x}+\tan x-\cos 2x+9\sin^{-1}x$ (g) $\left(\sqrt{x}+1\right)\left(\frac{1}{x}+\frac{1}{\sqrt{x}}\right)$

Q 2: Find the differential coefficient of the following with respect to x

(a) $\frac{1-x^2}{1+x^2}$ (b) $\left(a+bx^2\right)^{\frac{1}{2}}$ (c) $\log\left(x+\sqrt{a^2+x^2}\right)$

(d) $\log\left(x+2+\sqrt{x^2+4x+1}\right)$

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UNIT 10: APPLICATION OF DERIVATIVES

UNIT STRUCTURE

10.1 Learning objectives

10.2 Introduction

10.3 Maximum and minimum of a function

10.3.1 Methods for finding maximum and minimum

10.5 Application of maxima and minima

10.6 Let us sum up

10.7 Answers to check your progress.

10.8 Further Readings

10.9 Model Questions

10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- find maximum and minimum value of a function
- determine the points on a given curve where extremum exist, if any

11.2 INTRODUCTION

In this unit, we shall discuss maximum and minimum of a function. Also, we also discuss the methods of finding maximum and minimum values of a function. Lastly, we will discuss application of maxima and minima.

11.3 MAXIMUM AND MINIMUM OF A FUNCTION

A function f(x) is said to have a maximum at x=a if f(a) is greater than every other value assumed by f(x) in the immediate neighbourhood of x=a.

This means that f(a) > f(a+h) and f(a) > f(a–h) for a sufficiently small positive h. (See fig. 4.4)

Similarly, a function f(x) is said to have a minimum at x=b if f(b) is smaller than every other value assumed by f(x) in the immediate neighbourhood of x=b.

This means that

$f(b) < f(b+h)$

and $f(b) < f(b-h)$

for a sufficiently small positive h (See fig 4.5).

Geometrically, a function f continuous at a point has a maximum or a minimum at that point according as the graph of f has a peak or a cavity at that point.

Note : i) The maximum and minimum values of a function are also known as local (relative) maxima or local (relative) minma as these are the greatest and least values of the function relative to some neighbourhood of the point in the question. The term extremum is also used for both a maximum and minimum value.

ii) A local maximum value at a point may be less than a local minimum value at another point.

iii) A function can have several maximum and minimum values. The maximum and minimum values of a continuous function occur alternatively.

iv) At maximum, the function f(x) changes from increasing to decreasing state and at minimum the function f(x)changes from decreasing to increasing state.

10.3.1 Methods of finding maximum or minimum

Method I :

1) Find $\frac{dy}{dx}$

2) Solve the equation $\frac{dy}{dx}=0$ for real values of x. Let these values be a, b, c etc.

3) Consider x=a. Study the sign of $\frac{dy}{dx}$ for values of a slightly less than a and slightly greater than a.

4) If $\frac{dy}{dx}$ changes sign from positive to negative, then f(x) is maximum at x=a.

If $\frac{dy}{dx}$ changes sign from negative to positive, then f(x) is minimum at x=a.

If $\frac{dy}{dx}$ does not change sign, then x=a is a point neither of maximum nor of minimum.

Method II :

Steps I and 2 are same as in method I.

3. Find $\frac{d^2y}{dx^2}$ and put values. Consider x=a.

i) If $\frac{d^2y}{dx^2}<0$ at x=a, then f(x) is maximum at x=a.

ii) If $\frac{d^2y}{dx^2}>0$ at x=a, then f(x) is minimum at x=a.

iii) If $\frac{d^2y}{dx^2}=0$ but $\frac{d^3y}{dx^3}\neq 0$ at x=a, then there is neither maximum nor minimum at x=a.

Illustrative Examples

Example 2 : Find x for which f(x)=x(5–x) is the maximum. What is the maximum value of f(x) ?

Solution : Given, f(x)=x(5–x)

$= f(x) = 5x - x^2$

$\therefore f'(x)= 5 - 2x$

For on extremum, we have

$f'(x)=0$

$= 5-2x=0$

$= x=\frac{5}{2}$

Also, $f''(x) = \frac{d}{dx}(5-2x)$

$= -2$

$f''\left(\frac{5}{2}\right)= -2 < 0$

$\therefore$ f(x) is maximum at x= $\frac{5}{2}$ and the maximum value

$$= f\left(\frac{5}{2}\right)$$

$$= \frac{5}{2}\left(5-\frac{5}{2}\right)$$

$$= \frac{25}{4}$$

Example 3 : Find the values of Maximum and Minimum of $y = 2x^3 + 3x^2 - 36x + 10$.

Solution : Given $y = 2x^3 + 3x^2 - 36x + 10$

$$= \frac{dy}{dx} = 6x^2 + 6x - 36$$

$$= 6\left(x^2 + x - 6\right)$$

For a maximum or minimum

$$\frac{dy}{dx} = 0$$

$$= 6\left(x^2 + x - 6\right) = 0$$

$$\Rightarrow 6(x+3)(x-3) = 0$$

$$= x = 2 \text{ or } x = -3$$

Now, $\frac{d^2y}{dx^2} = 12x + 6$

At $x = 2$, $\frac{d^2y}{dx^2} = 12(2) + 6 = 30 > 0$

At $x = -3$, $\frac{d^2y}{dx^2} = 12(-3) + 6 = -30 < 0$

Thus y is maximum at the point $x = -3$ and is minimum at the point $x = 2$.

. Minimum value of y $= 2(2)^3 + 3(2)^2 - 36(2) + 10$

$$= 16 + 12 - 72 + 10$$

$$= -38$$

and Maximum value of y $= 2(-3)^3 + 3(-3)^2 - 36(-3) + 10$

$$= -52 + 27 + 108 + 10$$

$$= 93.$$

Example 4 : Show that the function $y = x^3 - 6x^2 + 9x - 8$ has a

maximum value at $x=1$ and a minimum value at $x=3$.

Solution : Given $y=x^3-6x^2+9x-8$

$$\Rightarrow \frac{dy}{dx}=3x^2-12x+9$$

$$=3\left(x^2-4x+3\right)$$

For maximum and minima, $\frac{dy}{dx}=0$

$$\Rightarrow 3\left(x^2-4x+3\right)=0$$

$$= 3(x-1)(x-3)=0$$

$$= x=1,3$$

Now, $\frac{d^2y}{dx^2}=2x-4$

At $x=1$, $\frac{d^2y}{dx^2}=-2<0$

At $x=3$, $\frac{d^2y}{dx^2}=2>0$

Thus y is maximum at $x=1$ and the maximum value is

$=(1)^3-6(1)^2+9(1)-8$

= -4

y is minimum at $x=3$ and the minimum value is

$=(3)^3-6(3)^2+9(3)-8$

$=-8$



CHECK YOUR PROGRESS – 3

Q 1: Find the maximum or minimum values of the following functions.

a) $2x^2-6x+3$ b) $x+\frac{1}{x}$

c) $4x^3-15x^2+12x-1$

10.4 APPLICATION OF MAXIMA AND MINIMA TO PROBLEMS

The theory of maxima and minima finds applications in practical life. For example, one has to find out the relation between the radius and height of a conical tent which may have a deifinite capacity and at the same time use minimum amount of cloth. In solving such type of problems we find outthe function whose maximum or minimum has to be found out. If it involves two variables, then by eliminating one variable from the given data, the function is reduced to one having only one variable.

Example 1 : The sum of two numbers is constant. Prove that their product is maximum when each number is half of their sum.

Solution : Let the two numbers be x and y. Then x+y=constant = c, say

$= y = c–x \quad ... (1)$

Let P be the product of the numbers. Then

$P= xy$

$= P = x(c–x)$, by (1)

$= P = cx–x^2$

$\therefore \frac{dP}{dx} = c - 2x$

For maximum or minimum value of P, we have

$\frac{dP}{dx} = 0 \quad \Rightarrow c - 2x = 0$

$\Rightarrow x = \frac{c}{2}$

Now, $\frac{d^2P}{dx^2} = \frac{d}{dx}(c - 2x)$

$= –2 < 0$

. P is maximum when $x= \frac{c}{2}$

Also, when $x=\frac{c}{2}$, we get from (1) $y = \frac{c}{2}$

Hence P is maximum when $x=y=\frac{c}{2}$

i.e., the product is maximum when each number is half of their sum.



CHECK YOUR PROGRESS – 4

Q 1: Find two numbers whose sum is 20 and product is maximum.



10.5 LET US SUM UP

- $\frac{dy}{dx}$ represents the rate of change of y with respect to x.
- The values of x obtained on solving $f'(x)=0$ are called stationary or critical points. A function have local maximum or minimum at these points according at $f''(x)<0$ or $f''(x)>0$ respectively.



10.6 ANSWERS TO CHECK YOUR PROGRESS

Check your progress - 2

Ans to Q No 1: (a) Let $f(x) = 2x^2–6x+3$

$\therefore f'(x)=4x–6$

For maximum or minimum, we have

$f'(x)=0$

$\Rightarrow 4x–6=0$

$\Rightarrow x = \frac{3}{2}$

Now, $f''(x) = \frac{d}{dx}(4x–6)$

$\Rightarrow f''(x) = 4$

$\Rightarrow f''\left(\frac{3}{2}\right)= 4 > 0$

$\therefore$ f(x) is minimum when $x=\frac{3}{2}$

and the minimum value is

$= f\left(\frac{3}{2}\right)$

$= 2\times\frac{9}{4}-6\times\frac{3}{2}+3$

$= \frac{9-18+6}{2}$

$= -\frac{3}{2}$

Ans to Q No 1: (b) Let $f(x) = x+\frac{1}{x}$

$\therefore f'(x)=1-\frac{1}{x^2}$

For maximum or minimum, we have

$f'(x) = 0$

$\Rightarrow 1-\frac{1}{x^2}=0$

$= x^2 - 1=0$

$= (x+1)(x-1)=0$

$= x= -1, x=1$

Now, $f''(x)=\frac{d}{dx}\left(1-\frac{1}{x^2}\right)$

$= 0 - \frac{-2}{x^3}$

$= \frac{2}{x^3}$

$\therefore f''(-1)=-2<0$

= f(x) is maximum at x=–1, and the maximum value is f(–1)$=-1+\frac{1}{-1}$=–2.

Again, $f''(1)=2 > 0$

. f(x) is minimum at x=1 and the minimum value is f(1)=1+1=2

Ans to Q No 1: (c) Let $f(x) =4x^3-15x^2+12x-1$

= $f'(x)=12x^2-30x + 12$

For maximum or minimum, we have $f'(x) =0$

= $12x^2 - 30x + 12 = 0$

= $6(2x^2-5x+2) = 0$

$= (2x-1)(x-2)=0$

$= x=\frac{1}{2}, \quad x=2$

Now, $f''(x) = \frac{d}{dx}\left(12x^2 - 30x + 12\right)$

$= 24x-30$

$\therefore f''\left(\frac{1}{2}\right) = 12 - 30$

$= -18 < 0$

$=$ f(x) is maximum when $x=\frac{1}{2}$ and the maximum value is

$= f\left(\frac{1}{2}\right)$

$= 4\times\frac{1}{8} - 15\times\frac{1}{4} + 12\times\frac{1}{2} - 1$

$= \frac{1}{2} - \frac{15}{4} + 5$

$= \frac{2 - 15 + 20}{4}$

$= \frac{7}{4}$

Also, $f''(2)=24 \times 2 - 30$

$= 18 > 0$

$=$ f(x) is minimum when x=2 and the minimum value is

= f(2)

= 4 x 8 – 15 x 4 + 12 x 2 – 1

= 32– 60 + 24 – 1

= 56 – 61

= – 5

Check your progress - 3

Ans to Q No 1: Let the two numbers be x and y.

According to question,

x+y = 20

$=$ y = 20–x ... (1)

Let P be the product of the numbers

$\therefore$ P = xy

$=$ P = x(20–x), using (1)

$=$ P =20x – x^2

$\therefore \frac{dP}{dx} = 20 - 2x$

For maximum or minimum of P, we have

$\frac{dP}{dx} = 0$

$=$ 20–2x = 0

$=$ x = 10

Now, $\frac{d^2P}{dx^2} = \frac{d}{dx}(20 - 2x)$

= –2 < 0

$\therefore$ P is maximum when x=10

Also, x=10 gives y=10

$\therefore$ The numbers are 10 and 10.



10.7 FURTHER READING

1) Sharma,A.K.,Text Book of Differential Calculus for B.A,B.Sc,B.Com,I.A.S.,P.C.S.,New Delhi,Discovery Publishing House,2004,1st edition.

2) Habib,Amber.,The calculus of finance,Hyderabad,Universities Press India Pvt.Ltd.2011,1st edition.

3) Gupta,Sudhir.,Basics of Differentential Calculus,New Delhi,Anmol publication Pvt. Ltd,2011,1st edition.

4) Maron,I.A. Problems in calculus of one variable,New Delhi,CBS publishers & Distributors Pvt Ltd.1993,1st edition.



10.8 MODEL QUESTIONS

Q 1: Find the points at which the following functions have maximum and minimum values :

(a) $y = 9x^3 - 45x^2 + 48x + 11$ (b) $y = 11 - 12x + 6x^2 - x^3$

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UNIT 11: INTEGRATION AND METHODS OF INTEGRATION

UNIT STRUCTURE

12.1 Learning Objectives

12.2 Introduction

12.3 Intregration of Simple Functions

12.4 Properties of Indefinite Integrals

12.5 Methods of Integration

12.5.1 Integration by Substitution

12.5.2 Integration by Parts

12.6 Let Us Sum Up

12.7 Further Reading

12.8 Answers to check your progress

12.9 Model Questions

11.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- understand about integration of simple functions from the view point of inverse of differentiation
- know the properties of indefinite integrals
- understand the various methods of integrations viz
  – method of substitution
  – integration by parts

11.2 INTRODUCTION

The development of integral calculus can be viewed from two different directions which are related to solving of problems of the following two types.

(i) to find the function when its derivative is known

(ii) to find the area bounded by the curve (or graph) of a function under certain given conditions.

The process related to the first category leads to the development of "indefinite integration" and the other leads to "definite integration".

The word "integration" literally means "summation". It is, in fact, the process of finding the "limit of a sum" of a certain number of elements, as the number of elements tends to infinity and each of the elements becomes infinitesimally small.

In this unit, we shall confine ourselves to the study of indefinite integral including some process of integration.

11.3 INTEGRATION OF SIMPLE FUNCTIONS

We have already learnt how to find the derivative *of* a functional of defined in an interval. If $y = f(x)$ is a function then its derivative is denoted by $f'(x)$ or $\frac{dy}{dx}$. Now, here in this unit, we will learn how to find *f* whenever f' is given. Integration is the inverse process of differentiation, i.e., to find a function when the differential coefficient is given.

Let us consider the following examples.

We know that (i) $\frac{d}{dx}\left(\frac{x^4}{4}\right) = x^3$

and (ii) $\frac{d}{dx}(\sin x) = \cos x$

Here we say that, in case of (i) $\frac{x^4}{4}$ is an antiderivative (or an integral or primitive) of x^3. Similarly, in (ii) $\sin x$ is an antiderivative of $\cos x$.

Let us take two other examples :

$$\text{(iii)}\ \frac{d}{dx}\left(\frac{x^4}{4} + C\right) = x^3$$

$$\text{and (iv)}\ \frac{d}{dx}(\sin x + C) = \cos x$$

Here C is any real number

It is evident from (iii), that $\frac{x^4}{4} + C$ is also an antiderivative of x^3 and in (iv) $\sin x + C$ is also an antiderivative of $\cos x$. Therefore, we have understood that the antiderivative of a function is not unique.

Definition : The set of functions that could possibly have the given function as a derivative are called "antiderivatives" (or integrals or primitives) of the function.

More generally, if there is a function $\phi(x)$ defined in an interval I such that $\frac{d}{dx}[\phi(x)] = f(x),\ x \in I$, then for any real number C $\frac{d}{dx}[\phi(x) + C] = f(x),\ x \in I$. Thus $\phi(x) + C$ denotes the family of anti-derivatives of $f(x)$. We introduce the symbol $\int f(x)dx$ which will represent the whole class of anti-derivatives, i.e. $\int f(x)dx = \phi(x) + C$.

The symbol $\int f(x)dx$ is read as the indefinite indegral of $f(x)$ with respect to *x*. The process of finding the anti-derivatives is known as "integration". The sign '$\int$' is called the integral sign. The symbol '$\int$' is an elongated S which is taken from the first letter of 'sum'.

Some Basic Results

Based upon the above definition and also on some standard derivatives, we obtain the following integration formulae.

	Derivatives	**Integrals (or anti derivatives)**
1.	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ (n \neq -1)$
2.	$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x dx = \sin x + C$
3.	$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x dx = -\cos x + C$
4.	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x dx = \tan x + C$
5.	$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + C$
6.	$\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$	$\int \text{cosec}^2 x dx = -\cot x + C$

7. $\frac{d}{dx}(\text{cosec}x) = -\text{cosec}x \cot x$ $\quad \int \text{cosec}x \cot x dx = -\text{cosec}x + C$

8. $\frac{d}{dx}(e^x) = e^x$ $\quad \int e^x dx = e^x + C$

Examples :

Example 1 : Evaluate (i) $\int x^4 dx$ (ii) $\int \frac{1}{\sqrt{x}} dx$ (iii) $\int 3^x dx$

Solution : (i) $\int x^4 dx = \frac{x^{4+1}}{4+1} + C = \frac{x^5}{5} + C$ (using formula 1)

(ii) $\int \frac{1}{\sqrt{x}} dx = \int x^{-½} dx = \frac{x^{-½+1}}{-½+1} + C = 2\sqrt{x} + C$

(iii) $\int 3^x dx = \frac{3^x}{\log_e 3} + C$ (using formula 9)

11.4 PROPERTIES OF INDEFINITE INTEGRAL

P–1 Differentiation and integration are inverse processes of each other in the sense of the results given below–

$\frac{d}{dx}\int f(x)dx = f(x)$ and $\int f'(x)dx = f(x) + C$, C is a constant.

Proof : Let $\phi(x)$ be any antiderivative of $f(x)$ i.e. $\frac{d}{dx}[\phi(x)] = f(x)$

Then $\int f(x)dx = \phi(x) + C$

$\therefore \frac{d}{dx}\int f(x)dx = \frac{d}{dx}[\phi(x) + C] = \frac{d}{dx}[\phi(x)] + \frac{d}{dx}(C) = f(x)$

(first part is proved)

Similarly, we know that $f'(x) = \frac{d}{dx} f(x)$

Hence $\int f'(x)dx = f(x) + C$, C is any arbitrary constant.

P-2 Two identical integrals with the same derivative lead to the same family of curves and hence both are equivalent.

Proof : Let $\frac{d}{dx}\int f(x)dx = \frac{d}{dx}\int g(x)dx$

Then $\frac{d}{dx}\int f(x)dx - \frac{d}{dx}\int g(x)dx = 0$

or, $\frac{d}{dx}[\int f(x)dx - \int g(x)dx] = 0$

or, $\int f(x)dx - \int g(x)dx = C$

or, $\int f(x)dx = \int g(x)dx + C$

It is customarily expressed as $\int f(x)dx = \int g(x)dx$ without mentioning the constant of integration.

P-3 $\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

Proof : By property P-1, We have

$$\frac{d}{dx}\int[f(x)+g(x)]dx = f(x)+g(x) \quad \text{... (a)}$$

Also we know that

$$\frac{d}{dx}[\int f(x)dx + \int g(x)dx] = \frac{d}{dx}\int f(x)dx + \frac{d}{dx}\int g(x)dx$$

$$= f(x)+g(x) \quad \text{... ... ...(b)}$$

In view of property P-2, from (a) and (b) we can write

$$\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

Corollary

(A) $\int[f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$

(B) We can generalize property P-3 as below -

$$\int[f(x)+g(x)+h(x)+....+\phi(x)]dx$$

$$= \int f(x)dx + \int g(x)dx + \int h(x)dx + ... + \int \phi(x)dx$$

P-4 For any real number K,

$$\int K f(x)dx = K\int f(x)dx$$

Proof : We know that $\frac{d}{dx}\int K f(x)dx = K f(x)$, by property P-1

Also $\frac{d}{dx}\left[K\int f(x)dx\right] = K\frac{d}{dx}\int f(x)dx = K f(x)$

So by property P-2

$$\int K f(x)dx = K\int f(x)dx$$

Illustrative Examples

Example 1 : Evaluate (i) $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$

(ii) $\int\left(x^{-3/2} - 4e^x + \cos x\right)dx$ (iii) $\int\left(2x^2 + 4\sec^2 x - \sqrt[3]{x}\right)dx$

Solutions : (i) $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$

$$= \int \left(x + 2 + \frac{1}{x} \right) dx$$

$$= \int x dx + \int 2 dx + \int \frac{1}{x} dx \qquad \text{(by property P-3)}$$

$$= \frac{x^{1+1}}{1+1} + C_1 + 2\int dx + \log x + C_2$$

$$= \frac{x^2}{2} + C_1 + 2x + C_3 + \log x + C_2$$

$$= \frac{x^2}{2} + 2x + \log x + C \qquad (\text{Here } C_1 + C_2 + C_3 = C)$$

(ii) $\int \left(x^{-3/2} - 4e^x + \cos x \right) dx$

$$= \int x^{-3/2} - dx - \int 4e^x dx + \int \cos x \, dx$$

$$= \frac{x^{-\frac{3}{2}+1}}{\frac{-3}{2}+1} + C_1 - 4\int e^x + \sin x + C_2$$

$$= \frac{x^{-1/2}}{\frac{-1}{2}} + C_1 - 4e^x + C_3 + \sin x + C_2$$

$$= -2x^{-1/2} - 4e^x + \sin x + C$$

(iii) $\int \left(2x^2 + 4\sec^2 x - \sqrt[3]{x} \right) dx$

$$= \int 2x^2 dx + \int 4\sec^2 x \, dx - \int \sqrt[3]{x} \, dx$$

$$= 2\int x^2 dx + 4\int \sec^2 x \, dx - \int x^{1/3} dx$$

$$= \frac{2}{3}x^3 + C_1 + 4\tan x + C_2 - \frac{x^{4/3}}{4/3} + C_3$$

$$= \frac{2}{3}x^3 + 4\tan x - \frac{3}{4}x^{4/3} + C$$

Example 2 : Evaluate (i) $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$

(ii) $\int \frac{3x^4 + 7x + 5}{2\sqrt{x}} dx$ (iii) $\int \frac{1 + \sin x}{\cos^2 x} dx$

Solution : (i) $\int \frac{x^3 - x^2 + x - 1}{x - 1}\, dx$

$$= \int \frac{(x-1)(x^2+1)}{(x-1)} dx$$

$$= \int (x^2 + 1) dx$$

$$= \frac{x^3}{3} + x + C$$

(ii) $\int \frac{3x^4 + 7x + 5}{2\sqrt{x}}\, dx$

$$= \int \left(\frac{3}{2} x^{7/2}\, dx + \frac{7}{2} x^{1/2} + \frac{5}{2} x^{-1/2} \right) dx$$

$$= \frac{3}{2} . \frac{2}{9} x^{9/2} + \frac{7}{2} . \frac{2}{3} x^{3/2} + \frac{5}{2} . \frac{2}{1} x^{1/2} + C$$

$$= \frac{1}{3} x^{9/2} + \frac{7}{3} x^{3/2} + 5\sqrt{x} + C$$

(iii) $\int \frac{1 + \sin x}{\cos^2 x}\, dx$

$$= \int \frac{1}{\cos^2 x}\, dx + \int \frac{\sin x}{\cos^2 x}\, dx$$

$$= \int \sec^2 dx + \int \tan x \sec x\, dx$$

$$= \tan x + \sec x + C$$

Example 3 : Write an antiderivative of $\cos 2x$.

Solution : We know that $\frac{d}{dx}(\sin 2x) = 2\cos 2x$

or $\cos 2x = \frac{1}{2} \frac{d}{dx}(\sin 2x) = \frac{d}{dx}\left(\frac{1}{2} \sin 2x \right)$

$\therefore$ antiderivative of $\cos 2x$ is $\frac{1}{2} \sin 2x + C$.



CHECK YOUR PROGRESS -1

Q 1: Find an antiderivative of each of the followings–

(i) $\frac{1}{x}, (x \neq 0)$ (ii) $\sin 3x$ (iii) $(ax - b)^2$

Q 2: Find the following integrals

(i) $\int x^3\left(x+\frac{1}{x}\right)^2 dx$ (ii) $\int(\cos x+\sin x)^2 dx$

(iii) $\int\left(\frac{2\cos x}{5\sin^2 x}+\frac{1}{3\cos^2 x}\right)dx$ (iv) $\int\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$

(v) $\int\left(e^{x\log a}+e^{a\log x}+e^{a\log a}\right)dx$ (vi) $\int\sqrt{1+\cos 2x}\,dx$

(vii) $\int\frac{(1+x)^2}{\sqrt{x}}\,dx$ (viii) $\int\left(\frac{x}{m}+\frac{m}{x}+x^m+m^x\right)dx$

Q 3: Evaluate (i) $\int\left(x^4+x^2+1\right)d\left(x^2\right)$ (ii) $\int\frac{x^4+x^2+1}{x^2-x+1}dx$

(iii) $\int\frac{x^{-1/3}+\sqrt{x}+2}{\sqrt[3]{x}}\,dx$

Q 4: If the derivative of $f(x)$ w.r.t. x is $\frac{1}{x}+\frac{1}{1-x^2}$ and $f(1)=\pi/4$, then find $f(x)$.

11.5 METHODS OF INTEGRATION

In our earlier discussion, we have considered integrals of those functions which are readily obtainable from derivatives of some standard functions. This method is based on inspection, i.e. it is dependent on the search of a function F whose derivative is *f* which lead us to the integrals of *f.* But integrals of certain functions cannot be obtained directly if they are not in one of the standard forms. Hence we need to develop some other techniques or methods for finding the integrals by reducing the given functions to standard forms. Some of the methods are– (i) Integration by substitution, (ii) Integration by parts.

11.5.1 Integration by Substitution

The method of evaluating an integral by reducing it to standard form by a proper substitution is called “integration by substitution”.

Suppose $\phi(x)$ is a differentiable function. Then to evaluate integrals of the form $\int f(\phi(x))\phi'(x)dx$, we substitute $\phi(x)=t$ which

then gives $\phi'(x)dx = dt$. This substitution reduces the above integral to the form $\int f(t)dt$.

If now $\int f(t)dt$ is the standard form then we can find the integrals. Finally we replace t by x with the help of the above substitution.

It is important to guess what will be the useful substitution. Usually we make a substitution for a function whose derivative also occurs in the integrand.

A few examples are given below to illustrate the method.

Illustrative Examples

Example 1 : Evaluate $\int (ax+b)^n dx$

Solution : Let $ax + b = t$, then a $\frac{dx}{dt} = 1$

or $dx = \frac{1}{a}dt$

$$\therefore \int (ax+b)^n dx = \int t^n \frac{1}{a} dt = \frac{1}{a}\int t^n dt = \frac{1}{a}\frac{1}{n+1}t^{n+1} + C$$

$$= \frac{1}{a(n+1)}(ax+b)^{n+1} + C$$

Example 2 : Evaluate $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$

Solution : Let $\sqrt{x} = t$, then $\frac{1}{2\sqrt{x}}dx = dt$, or $\frac{1}{\sqrt{x}}dx = 2dt$

Now $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = \int 2\sin t\, dt = 2(-\cos t) + C = -2\cos\sqrt{x} + C$

Example 3 : Evaluate $\int \tan^4\theta \sec^2\theta d\theta$

Solution : Let $\tan\theta = t$, then $\sec^2\theta d\theta = dt$

Now $\int \tan^4\theta \sec^2\theta d\theta = \int t^4 dt = \frac{1}{5}t^5 + C = \frac{1}{5}\tan^5\theta + C$

Example 4 : Evaluate $\int \frac{10x^9 + 10^x \log_e^{10}}{10^x + x^{10}}\, dx$

Solution : Let $10^x + x^{10} = t$, then $(10^x \log_e 10 + 10x^9)dx = dt$

Now $\int \frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}} dx = \int \frac{1}{t} dt = \log t + C = \log\left(10^x + x^{10}\right) + C$

A Few Special Results

1. $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$

Proof : Let $f(x) = z$, then $f'(x)dx = dz$

Therefore, $\int \frac{f'(x)}{f(x)} dx = \int \frac{dz}{z} = \log|z| + C = \log|f(x)| + C$

This result can be applied in case of the followings

(a) $\int \tan x \, x = \log|\sec x| + C$ (b) $\int \cot x \, dx = \log|\sin x| + C$

(c) $\int \sec x \, dx = \log|\sec x + \tan x| + C$

(d) $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C$

Proof (a) : $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx$

$= -\log|\cos x| + C = \log|\sec x| + C$

Proof (c) : $\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

$= \log|\sec x + \tan x| + C$

N.B. : (b) and (d) are left for the learners



CHECK YOUR PROGRESS- 2

Q 1: Find the integrals

1. $\int \frac{x}{9 - 4x^2} dx$ 2. $\int (4x+2)\sqrt{x^2 + x + 1} \, dx$

3. $\int \frac{dx}{x\sqrt{x^2 - a^2}}$ 4. $\int x\sqrt{x^2 + 5} \, dx$ 5. $\int \frac{dx}{x^2\sqrt{1 - x^2}}$

6. $\int \frac{\cos x \, dx}{(a + b\sin x)^2}$ 7. $\int \frac{e^{2x} - 1}{e^{2x} + 1} dx$ 8. $\int \frac{(x+1)(x + \log x)^2}{x} dx$

9. $\int \frac{3x - 1}{\sqrt{3x^2 - 2x + 7}} dx$ 10. $\int \frac{e^x(1 + x)}{\cos^2(xe^x)} dx$ 11. $\int \frac{1}{x^2(x^4 + 1)^{3/4}} dx$

12. $\int \frac{x^2+5x+2}{x+2}\,dx$ 13. $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\,dx$ 14. $\int \cos^4 x\,dx$

15. $\int \sin^3 x\,dx$ 16. $\int \cos^2 x\,dx$ 17. $\int \frac{2x+5}{x^2+5x-7}\,dx$

18. $\int \frac{1}{\sin(x-a)\sin(x-b)}\,dx$ 19. $\int \frac{dx}{\sin(x-a)\cos(x-b)}$

20. $\int \sin x \sin 2x \sin 3x\,dx$ 21. $\int \frac{x+2}{(x+1)^2}\,dx$

22. $\int \frac{3x+1}{(3x^2+2x+1)^3}\,dx$

11.5.2 Integration by Parts

An important integration formula can be had from the derivative of the product of two functions.

We know from differential calculus that for two differentiable functions $f(x)$ and $q(x)$,

$$\frac{d}{dx}\{f(x).q(x)\} = f(x).\frac{d}{dx}\{q(x)\} + q(x).\frac{d}{dx}\{f(x)\}$$

Integrating both sides w.r.t. x we get

$$f(x).q(x) = \int\left[f(x).\frac{d}{dx}\{q(x)\} + q(x).\frac{d}{dx}\{f(x)\}\right]dx$$

$$= \int f(x)\frac{d}{dx}\{q(x)\}dx + \int q(x).\frac{d}{dx}\{f(x)\}dx$$

or $$\int f(x)\frac{d}{dx}\{q(x)\}dx = f(x).q(x) - \int q(x).\frac{d}{dx}\{f(x)\}dx \qquad ...(1)$$

Now let us take $f(x)=u$ and $\frac{d}{dx}\{q(x)\}=v$.

Then $q(x)=\int v dx$

If we substitute there in (1) then we get

$$\int uvdx = u\int vdx - \int\left\{\frac{d}{dx}(u).\int vdx\right\}dx$$

The above formula can be stated as below—

"The integral of the product of two functions = 1st function × integral of the 2nd function – integral of [differential coefficient of 1st function × integral of the 2nd function]"

Note : While applying the rule of integration by parts for the product of two functions, it is very important to choose the first and the second function very carefully. As an aid to the learners, we can choose the first function as the function that comes first in the word "ILATE". Here the letters stand for as given below—

I – Inverse trigonometric function

L – Logarithmic function

A – Algebraic function

T – Trigonometic function

E – Exponential function.

ILLUSTRATIVE EXAMPLES

Example 1 : Evaluate $\int x^2 e^x dx$

Solution : $\int x^2 e^x dx = x^2 \int e^x dx - \int \left\{ \frac{d}{dx}(x^2) \int e^x dx \right\} dx$

$= x^2 e^x - \int 2x e^x dx$

Here we have taken
x^2 = 1st function
e^x = 2nd function

$= x^2 e^x - 2\left[x \int e^x dx - \int \left\{ \frac{d}{dx}(x) . \int e^x dx \right\} dx \right]$

$= x^2 e^x - 2\{ xe^x - \int 1.e^x dx \} = x^2 e^x - 2(xe^2 - e^x) + C$

$= e^x (x^2 - 2x + 2) + C$

Example 2 : Evaluate $\int x \sin 3x \, dx$

Solution : $\int x \sin 3x \, dx = x \int \sin 3x \, dx - \int \left\{ \frac{d}{dx}(x) . \int \sin 3x \, dx \right\} dx$

$= x\left(-\frac{\cos 3x}{3} \right) - \int 1.\left(-\frac{\cos 3x}{3} \right) dx$

$= -\frac{x}{3} \cos 3x + \frac{1}{3} \int \cos 3x \, dx$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{3}\frac{\sin 3x}{3} + C$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$$

Example 4 : $\int \log x\, dx$

Solution : $\int \log x\, dx = \int \log x.1\, dx$

Now take 1st function = log x

2nd function = 1 and proceed.

Similarly in case of $\int \sin^{-1} x\, dx$ we take $\sin^{-1} x$ as first function and 1 as second function.

ILLUSTRATIVE EXAMPLES

Example 1: Evaluate $\int e^x \cos x\, dx$

Solution : Let $I = \int e^x \cos x\, dx$

$$= e^x \int \cos x\, dx - \int \left\{ \frac{d}{dx}(e^x) . \int \cos x\, dx \right\} dx$$

$$= e^x \sin x - \int e^x \sin x\, dx$$

$$= e^x \sin x - \left[e^x \int \sin x\, dx - \int \left\{ \frac{d}{dx}(e^x) . \int \sin x\, dx \right\} dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$= e^x (\sin x + \cos x) - I$$

$\Rightarrow 2I = e^x (\sin x + \cos x)$

$\therefore I = \frac{1}{2} e^x (\sin x + \cos x) + C$



CHECK YOUR PROGRESS - 4

Q 1: Evaluate the followings

1. $\int x^2 \cos x\, dx$ 2. $\int x^3 \log 2x\, dx$

3. $\int x^2 e^{-x} dx$ 4. $\int \cos \sqrt{x}\, dx$ 5. $\int e^x \sin x\, dx$ 6. $\int e^x \sin x \sin 2x\, dx$

7. $\int e^x (\cos x + \sin x) dx$ 8. $\int \sin(\log x) dx$



11.6 LET US SUM UP

- Differentiation and integration are inverse processes of one another
- We may also define integration as a process of finding the limit of a sum [The method is discussed in the next unit].
- The set of functions that could possibly have in given functon as a derivative are called anti derivatives (or integrats or primitives) of a function.
- Based on the results of some standard derivatives, we obtain the formula of some standard simple Integration.
- There are different methods for determination of integrals.Some of the methods are (i) Integration by substitution, (ii) Integration by parts.
- A few important forms of integrals

  (i) $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$

  (ii) $\int \{f(x)\}^n f'(x) dx = \frac{1}{n+1}[f(x)]^{n+1} + C, \quad n \neq -1$

  (iii) $\int f'(ax+b) dx = \frac{1}{a} f(ax+b) + C$

- An important formula for product of two functions is

  $$\int uv\, dx = u\int v dx - \int \left\{ \frac{d}{dx}(u).\int v dx \right\} dx$$



11.7 FURTHER READING

1) *Integral Calculus,* Kishan, Hari, Atlantic Publishers & Distributors. New Delhi -110027.

2) *Integral Calculus*, Dhami, H.S. New age International Publishers, New Delhi - 110002.

3) *Text book of Integral Calculus*, Sharma, A.K. Discovery Publishing House, New Delhi -110002.



11.8 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1: (i) We know that $\frac{d}{dx}(\log x)=\frac{1}{x}, x>0$

and $\frac{d}{dx}(\log(-x))=\frac{1}{-x}(-1), x<0$

Combining above ,we get $\frac{d}{dx}(\log|x|)=\frac{1}{x}, x\neq 0$

Therefore, $\int\frac{1}{x}dx=\log|x|$ is anti-derivative of $\frac{1}{x}$.

(ii) We know that $\frac{d}{dx}(\cos 3x)=-3\sin 3x$

$\Rightarrow \sin 3x=-\frac{1}{3}\frac{d}{dx}(\cos 3x)$

$\therefore \sin 3x=\frac{d}{dx}\left(-\frac{1}{3}\cos 3x\right)$

Therefore, anti -derivative of $\sin 3x$ is $-\frac{1}{3}\cos 3x$.

(iii) We know that $\frac{d}{dx}(ax-b)^3=3a(ax-b)^2$

$\Rightarrow (ax-b)^2=\frac{1}{3a}\frac{d}{dx}(ax-b)^3$

$\therefore (ax-b)^2=\frac{d}{dx}\left\{\frac{1}{3a}(ax-b)^3\right\}$

Therefore, anti -derivative of $(ax-b)^2$ is $\frac{1}{3a}(ax-b)^3$.

Ans to Q No 2:

(i) $\int x^3\left(x^2+\frac{1}{x^2}+2x.\frac{1}{x}\right)dx=\int\left(x^5+x+2x^3\right)dx$

$=\frac{x^6}{6}+\frac{x^2}{2}+2\frac{x^4}{4}+C=\frac{x^6}{6}+\frac{x^2}{2}+\frac{x^4}{2}+C$.

(ii) $\int(\cos x+\sin x)^2 dx=\int(\cos^2 x+\sin^2 x+2\sin x\cos x)dx$

$=\int(1+\sin 2x)dx$

$=x-\frac{\cos 2x}{2}+C$.

(iii) $\int\left(\frac{2\cos x}{5\sin^2 x}+\frac{1}{3\cos^2 x}\right)dx=\int\left(\frac{2}{5}\cot x\cos ecx+\frac{1}{3}\sec^2 x\right)dx$

$=\frac{2}{5}\int\cot x\cos ecxdx+\frac{1}{3}\int\sec^2 xdx$

$=\frac{2}{5}(-\cos ecx)+\frac{1}{3}\tan x+C$.

(iv) $\int\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx=\int\left(\frac{\sin^2 x}{\sin^2 x\cos^2 x}-\frac{\cos^2 x}{\sin^2 x\cos^2 x}\right)dx$

$=\int(\sec^2 x-\cos ec^2 x)dx$

$=\tan x + \cot x + C$

(v) $\int(e^{x\log a}+e^{a\log x}+e^{a\log a})dx=\int(e^{\log a^x}+e^{\log x^a}+e^{\log a^a})dx$

$=\int(a^x+x^a+a^a)dx$

$=\int a^x dx+\int x^a dx+\int a^a dx$

$=\frac{a^x}{\log a}+\frac{x^{a+1}}{a+1}+a^a.x+C$.

(vi) $\int\sqrt{1+\cos 2x}\,dx=\int\sqrt{2\cos^2 x}dx$, $\cos 2x = 2\cos^2 x - 1$

$=\sqrt{2}\int\cos xdx$

$=\sqrt{2}\sin x + C$.

(vii) $\int\frac{(1+x)^2}{\sqrt{x}}dx=\int\frac{1+2x+x^2}{\sqrt{x}}dx$

$=\int\left(\frac{1}{\sqrt{x}}+2\sqrt{x}+x^{\frac{3}{2}}\right)dx$

$=\int x^{-\frac{1}{2}}dx+2\int x^{\frac{1}{2}}dx+\int x^{\frac{3}{2}}dx$

$=\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+2.\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}+C$

$$= 2x^{\frac{1}{2}} + \frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + C$$

(viii) $\int\left(\frac{x}{m} + \frac{m}{x} + x^m + m^x\right)dx = \frac{1}{m}\int x dx + m\int\frac{1}{x}dx + \int x^m dx + \int m^x dx$

$$= \frac{1}{m}.\frac{x^2}{2} + m\log|x| + \frac{x^{m+1}}{m+1} + \frac{x^m}{\log m} + C$$

Ans to Q No 3: (i) Let $x^2 = t$ .Then $\int\left(x^4 + x^2 + 1\right)d\left(x^2\right) = \int\left(t^2 + t + 1\right)dt$

$$= \frac{t^3}{3} + \frac{t^2}{2} + t + C$$

$$= \frac{x^6}{6} + \frac{x^4}{2} + x^2 + C$$

(ii) $\int\frac{x^4 + x^2 + 1}{x^2 - x + 1}dx = \int\frac{\left(x^2+1\right)^2 - x^2}{x^2 - x + 1}dx$

$$= \int\frac{\left(x^2 + 1 + x\right)\left(x^2 + 1 - x\right)}{x^2 - x + 1}dx$$

$$= \int\left(x^2 + x + 1\right)dx = \frac{x^3}{3} + \frac{x^2}{2} + x + C\,.$$

(iii) $\int\frac{x^{-1/3} + \sqrt{x} + 2}{\sqrt[3]{x}}\,dx = \int\frac{x^{-\frac{1}{3}} + x^{\frac{1}{2}} + 2}{x^{\frac{1}{3}}}dx$

$$= \int\left(\frac{x^{-\frac{1}{3}}}{x^{\frac{1}{3}}} + \frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}} + \frac{2}{x^{\frac{1}{3}}}\right)dx$$

$$= \int\left(x^{-\frac{2}{3}} + x^{\frac{1}{2}-\frac{1}{3}} + 2x^{-\frac{1}{3}}\right)dx$$

$$= \int\left(x^{-\frac{2}{3}} + x^{\frac{1}{6}} + 2x^{-\frac{1}{3}}\right)dx$$

$$= \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + \frac{x^{\frac{1}{6}+1}}{\frac{1}{6}+1} + 2\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C$$

$$= \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + \frac{x^{\frac{7}{6}}}{\frac{7}{6}} + 2\frac{x^{\frac{2}{3}}}{\frac{2}{3}} + C$$

$$= 3x^{\frac{1}{3}} + \frac{6}{7}x^{\frac{7}{6}} + 3x^{\frac{2}{3}} + C\,.$$

Ans to Q No 4: We have $f(x) = \int f'(x)dx = \int\left(\frac{1}{x} + \frac{1}{1+x^2}\right)dx$

$$= \int\frac{1}{x}dx + \int\frac{1}{x^2 + 1}dx = \log|x| + \tan^{-1}x + C$$

When $x = 1$, we have $f(x) = \frac{\pi}{4}$

$$\therefore \frac{\pi}{4} = \log|1| + \tan^{-1}1 + C$$

$$\Rightarrow \frac{\pi}{4} = \frac{\pi}{4} + C \;=\; C = 0$$

$$\therefore f(x) = \log|x| + \tan^{-1}x\,.$$

CHECK YOUR PROGRESS – 2

Ans to Q No 1: Let $9 - 4x^2 = t$

$$\therefore\; -8xdx = dt$$

$$\int\frac{x}{9 - 4x^2}dx = -\frac{1}{8}\int\frac{1}{t}dt \;= -\frac{1}{8}\log|t| + C$$

$$= -\frac{1}{8}\log|9 - 4x^2| + C$$

Ans to Q No 2: Let $x^2 + x + 1 = t \;\;\therefore (2x+1)dx = dt$

$$\int(4x+2)\sqrt{x^2 + x + 1}dx$$

$$= \int 2\sqrt{t}dt = 2\int\sqrt{t}dt = 2\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{4}{3}\left(x^2 + x + 1\right)^{\frac{3}{2}} + C$$

Ans to Q No 3: We put $x = a\sec\theta \Rightarrow \frac{x}{a} = \sec\theta \Rightarrow \theta = \sec^{-1}\frac{x}{a}$

$$dx = a\sec\theta\tan\theta d\theta$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta\sqrt{a^2\sec^2\theta - a^2}}$$

$$= \int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta\sqrt{a^2\tan^2\theta}}$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + C$$

$$= \frac{1}{a}\sec^{-1}\frac{x}{a} + C$$

Ans to Q No 4: Let $x^2 + 5 = t$

$= 2xdx = dt \Rightarrow xdx = \frac{1}{2}dt$

$$\int x\sqrt{x^2+5}\,dx = \frac{1}{2}\int t^{\frac{1}{2}}dt = \frac{1}{2}\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{1}{2}\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C \quad = \frac{1}{3}t^{\frac{3}{2}} + C$$

$$= \frac{1}{3}\left(x^2+5\right)^{\frac{3}{2}} + C$$

Ans to Q No 5: Let $x = \sin\theta = dx = \cos\theta d\theta$

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = \int \frac{\cos\theta d\theta}{\sin^2\theta\sqrt{1-\sin^2\theta}}$$

$$= \int \frac{\cos\theta d\theta}{\sin^2\theta\cos\theta}$$

$$= \int \cos ec^2\theta d\theta$$

$$= -\cot\theta + C$$

$$= -\cot\left(\sin^{-1}x\right) + C$$

Ans to Q No 6: Let $a + b\sin x = t = b\cos x dx = dt$

$$\cos x dx = \frac{1}{b}dt$$

$$\int \frac{\cos x\,dx}{(a+b\sin x)^2} = \int \frac{\frac{1}{b}dt}{t^2} = \frac{1}{b}\int \frac{dt}{t^2}$$

$$= \frac{1}{b}.\frac{t^{-2+1}}{-2+1} + C$$

$$= -\frac{1}{b}.\frac{1}{t} + C$$

$$= -\frac{1}{b}.\frac{1}{a+b\sin x} + C$$

Ans to Q No 7: Dividing numerator and denominator by e^x, we have

$$\frac{\frac{e^{2x}-1}{e^x}}{\frac{e^{2x}+1}{e^x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Let $e^x + e^{-x} = t$

$\therefore (e^x - e^{-x})dx = dt$

$$\int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log|e^x + e^{-x}| + C$$

Ans to Q No 8:

$$\frac{(x+1)(x+\log x)^2}{x} = \left(\frac{x+1}{x}\right)(x+\log x)^2 = \left(1+\frac{1}{x}\right)(x+\log x)^2$$

Let $x+\log x = t \quad \therefore \left(1+\frac{1}{x}\right)dx = dt$

$$\int \left(1+\frac{1}{x}\right)(x+\log x)^2 dx = \int t^2 dt$$

$$= \frac{t^3}{3} + C$$

$$= \frac{1}{3}(x+\log x)^3 + C$$

Ans to Q No 9: Let $3x^2 - 2x + 7 = t \Rightarrow (6x-2)dx = dt$

$$\Rightarrow 2(3x-1)dx = dt$$

$$\Rightarrow (3x-1)dx = \frac{1}{2}dt$$

$$\int \frac{3x-1}{\sqrt{3x^2-2x+7}}\,dx = \int \frac{\frac{1}{2}}{\sqrt{t}}dt = \frac{1}{2}\int t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \sqrt{t} + C$$

$$= \sqrt{3x^2-2x+7} + C$$

Ans to Q No 10: Let $e^x x = t \Rightarrow (e^x x + e^x.1)dx = dt$

$$\Rightarrow e^x(x+1)dx = dt$$

$$\therefore \int \frac{e^x(1+x)}{\cos^2(e^x x)}dx = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\,dt = \tan t + C$$

$$= \tan(e^x x) + C$$

Ans to Q No 11: Multiplying and dividing by x^{-3}, we have

$$\frac{x^{-3}}{x^2.x^{-3}(x^4+1)^{\frac{3}{4}}} = \frac{x^{-3}(x^4+1)^{-\frac{3}{4}}}{x^2.x^{-3}} = \frac{(x^4+1)^{-\frac{3}{4}}}{x^5(.x^4)^{-\frac{3}{4}}}$$

$$= \frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$$

$$= \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$$

Let $\frac{1}{x^4} = t \Rightarrow -\frac{4}{x^5}dx = dt \Rightarrow \frac{1}{x^5}dx = -\frac{dt}{4}$

$$\therefore \int \frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx = \int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}dx$$

$$= -\frac{1}{4}\int (1+t)^{-\frac{3}{4}}dt$$

$$= -\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right]+C$$

$$= -\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}}+C$$

$$= -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$$

Ans to Q No 12: $\int \frac{x^2+5x+2}{x+2}\,dx = \int \frac{(x+3)(x+2)}{x+2}dx = \int (x+3)dx$

$$= \frac{x^2}{2}+3x+C$$

Ans to Q No 13: $\frac{1}{\sqrt{x+a}+\sqrt{x+b}} = \frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\left(\sqrt{x+a}+\sqrt{x+b}\right)\left(\sqrt{x+a}-\sqrt{x+b}\right)}$

$$= \frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\left(\sqrt{x+a}\right)^2-\left(\sqrt{x+b}\right)^2}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)} = \frac{1}{a-b}\left(\sqrt{x+a}-\sqrt{x+b}\right)$$

$$\therefore \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx = \frac{1}{a-b}\left\{\int \sqrt{x+a}dx - \int \sqrt{x+b}dx\right\}$$

$$= \frac{1}{a-b}\left\{\int \sqrt{u}du - \int \sqrt{v}dv\right\}$$ where $x+a=u$ and $x+b=v$

$\therefore dx = du$ and $dx = dv$

$$= \frac{1}{a-b}\left\{\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{v^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right\} + C$$

$$= \frac{1}{a-b}\left\{\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{3}v^{\frac{3}{2}}\right\} + C$$

$$= \frac{2}{3(a-b)}\left\{(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right\} + C$$

Ans to Q No 14: $\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2$

$$= \frac{1}{4}\left[1 + \cos^2 2x + 2\cos 2x\right]$$

$$= \frac{1}{4}\left[1 + \left(\frac{1+\cos 4x}{2}\right) + 2\cos 2x\right]$$

$$= \frac{1}{4}\left[1 + \frac{1}{2} + \frac{\cos 4x}{2} + 2\cos 2x\right]$$

$$= \frac{1}{4}\left[\frac{3}{2} + \frac{\cos 4x}{2} + 2\cos 2x\right]$$

$$\therefore \int \cos^4 x dx = \int\left(\frac{3}{8} + \frac{\cos 4x}{8} + \frac{\cos 2x}{2}\right)dx$$

$$= \frac{3}{8}x + \frac{\sin 4x}{32} + \frac{\sin 2x}{4} + C$$

Ans to Q No 15: From the identity $\sin 3x = 3\sin x - 4\sin^3 x$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$\therefore \int \sin^3 x dx = \frac{3}{4}\int \sin x dx - \frac{1}{4}\int \sin 3x dx$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$$

Ans to Q No 16: From the identity, $\cos 2x = 2\cos^2 x - 1$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\int \cos^2 x dx = \frac{1}{2}\int (1+\cos 2x)dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x dx$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

Ans to Q No 17: Let $x^2 + 5x - 7 = t = (2x+5)dx = dt$

$$\int \frac{2x+5}{x^2+5x-7}\,dx = \int \frac{dt}{t} = \log t + C$$

$$= \log\left(x^2+5x-7\right) + C$$

Ans to Q No 18:

We have $\int \frac{1}{\sin(x-a)\sin(x-b)}dx = \frac{1}{\sin(a-b)}\int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}dx$

$$= \frac{1}{\sin(a-b)}\int \{\cot(x-a) - \cot(x-b)\}dx$$

$$= \frac{1}{\sin(a-b)}\{\log|\sin(x-a)| - \log|\sin(x-b)|\} + C$$

$$= \cos ec(a-b)\left\{\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|\right\} + C.$$

Ans to Q No 19:

$$\int \frac{1}{\sin(x-a)\cos(x-b)}dx = \frac{1}{\cos(a-b)}\int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos(x-a)\cos(x-b)+\sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int \{\cot(x-a) + \tan(x-b)\}dx$$

$$= \frac{1}{\cos(a-b)}[\log|\sin(x-a)| - \log|\cos(x-b)|] + C$$

$$= \frac{1}{\cos(a-b)}\log\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C.$$

Ans to Q No 20: We have $\int \sin x \sin 2x \sin 3x dx = \frac{1}{2}\int (2\sin 2x \sin x)\sin 3x dx$

$$= \frac{1}{2}\int (\cos x - \cos 3x)\sin 3x dx$$

$$= \frac{1}{4}\int (2\sin 3x \cos x - 2\sin 3x \cos 3x) dx$$

$$= \frac{1}{4}\int (\sin 4x + \sin 2x - \sin 6x) dx$$

$$= \frac{1}{4}\left[-\frac{\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6}\right] + C.$$

Ans to Q No 21: $\int \frac{x+2}{(x+1)^2} dx = \int \frac{(x+1)+1}{(x+1)^2} dx$

$$= \int \frac{1}{x+1} dx + \int \frac{1}{(x+1)^2} dx$$

$$= \int \frac{1}{z} dz + \int \frac{1}{z^2} dz \text{ put } x+1 = z \;.\; dx = dz$$

$$= \log|z| + \frac{z^{-2+1}}{-2+1} + C$$

$$= \log|z| - \frac{1}{z} + C$$

$$= \log|x+1| - \frac{1}{x+1} + C$$

Ans to Q No 22: Let $3x^2 + 2x + 1 = t \Rightarrow (6x+2)dx = dt$

$$\Rightarrow (3x+1)dx = \frac{1}{2}dt$$

$$\int \frac{3x+1}{(3x^2+2x+1)^3} dx = \frac{1}{2}\int \frac{dt}{t^3} = \frac{1}{2}\int t^{-3} dt$$

$$= \frac{1}{2}\frac{t^{-3+1}}{-3+1} + C$$

$$= \frac{1}{2}.\frac{t^{-2}}{-2} + C \quad = -\frac{1}{4}.\frac{1}{t^2} + C$$

$$= -\frac{1}{4}.\frac{1}{(3x^2+2x+1)^2} + C.$$

CHECK YOUR PROGRESS – 3

Ans to Q No 1: $\int x^2 \cos x dx = x^2 \sin x - \int 2x \sin x dx$

$= x^2 \sin x - 2\left[x(-\cos x) - \int (-\cos x)dx\right]$

$= x^2 \sin x + 2x\cos x - 2\sin x$

$= (x^2 - 2)\sin x + 2x\cos x$

Ans to Q No 2: $\int x^3 \log 2x dx = \log 2x \int x^3 dx - \int \left(\frac{d}{dx}\log 2x . \int x^3 dx\right)dx$

$= \log 2x \frac{x^4}{4} - \int \frac{1}{2x}.2.\frac{x^4}{4}dx$

$= \frac{x^4 \log 2x}{4} - \frac{1}{4}\int x^3 dx$

$= \frac{x^4 \log 2x}{4} - \frac{1}{4}.\frac{x^4}{4} + C$

$= \frac{x^4 \log 2x}{4} - \frac{x^4}{16} + C$.

Ans to Q No 3: $\int x^2 e^{-x} dx = x^2 \int e^{-x} dx - \int \left(\frac{d}{dx}x^2 \int e^{-x} dx\right)dx$

$= x^2(-e^{-x}) - \int 2x(-e^{-x})dx$

$= -x^2 e^{-x} + 2\int x e^{-x} dx$

$= -x^2 e^{-x} + 2\left[x(-e^{-x}) - \int 1.(-e^{-x})dx\right]$

$= -x^2 e^{-x} + 2(-xe^{-x} + e^{-x})$

$= -x^2 e^{-x} - 2xe^{-x} + 2e^{-x}$

Ans to Q No 4: We substitute $x = t^2$,We get $dx = 2tdt$

$\therefore \int \cos\sqrt{x} dx = 2\int t\cos t dt$

$= 2\left[t\sin t - \int \sin t dt\right]$

$= 2\left[t\sin t + \cos t\right] + C$

$= 2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + C$

Ans to Q No 5: Let $I = \int e^x \sin x dx = e^x(-\cos x) + \int e^x \cos x dx$

$= -e^x \cos x + e^x \sin x - \int e^x \sin x dx$

$= I = -e^x \cos x + e^x \sin x - I$

$\Rightarrow 2I = e^x(\sin x - \cos x)$

$$\therefore I = \frac{e^x}{2}(\sin x - \cos x)$$

Ans to Q No 6: Let $I = \int e^x \sin x \sin 2x dx$

$$= \frac{1}{2}\int e^x (\cos x - \cos 2x)dx$$

$$= \frac{1}{2}\int e^x \cos x dx - \int e^x \cos 2x dx$$

$$= \frac{1}{2}e^x \frac{(\cos x + \sin x)}{1^2 + 1^2} - \frac{1}{2}e^x \frac{(\cos 3x + 3\sin 3x)}{1^2 + 3^2} + C$$

$$= \frac{1}{4}e^x(\cos x + \sin x) - \frac{1}{20}e^x(\cos 3x + 3\sin 3x) + C$$

Ans to Q No 7: We have $\int e^x(\sin x + \cos x)dx = \int e^x \sin x dx + \int e^x \cos x dx$

$$= \sin x e^x - \int \cos x e^x dx + \int \cos x e^x dx$$

$$= \sin x e^x + C$$

Ans to Q No 8: Let $\log x = t$,then $x = e^t = dx = e^t dt$

$$\therefore I = \int (\sin t)e^t dt = -e^t \cos t - \int e^t(-\cos t)dt$$

$$= -e^t \cos t + \int e^t \cos t dt$$

$$= -e^t \cos t + \left[e^t \sin t - \int e^t \sin t dt\right]$$

$$= I = -e^t \cos t + e^t \sin t - I$$

$$= 2I = -e^t \cos t + e^t \sin t$$

$$\Rightarrow I = \frac{e^t}{2}(\sin t - \cos t) + C$$

Hence, $\int \sin(\log x)dx = \frac{x}{2}\{\sin(\log x) - \cos(\log x)\} + C$



11.9 MODEL QUESTIONS

A. Evaluate

1. $\int \frac{x^4}{x^2 + 1}dx$ 2. $\int \left(1 + \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} + \frac{5}{x\sqrt{x^2-1}} + 3^x\right)dx$

3. $\int \frac{4^x + 3^x}{5^x}dx$

B. Integrate the following–

1. $\int \frac{\cos 2x + x + 1}{x^2 + 2x + \sin 2x} dx$ 2. $\int \frac{1 - \sin 2x}{x + \cos^2 x} dx$ 3. $\int \frac{x + 1}{x(x + \log^x)} dx$

4. $\int (4x + 2)\sqrt{x^2 + x + 1} dx$ 5. $\int \cos^5 x \, dx$ 6. $\int \sin^3 x \cos^5 x \, dx$

C. Evaluate

1. $\int x^2 e^{-x} dx$ 2. $\int x^2 \sin^2 x \, dx$ 3. $\int \log_{10}^x dx$ 4. $\int \log_e^x dx$

5. $\int e^{-x} dx \cos x \, dx$ 6. $\int e^{ax} dx \cos x \, dx$ 7. $\int 2^x \sin x \, dx$

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UNIT 12 : LENEAR PROGRAMMING PROBLEM (LPP)

UNIT STRUCTURE

12.1 Learning Objectives

12.2 Introduction

12.3 Meaning

12.4 Conditions for Using LPP Technique

12.5 Basic Assumptions of LPP

12.6 Areas of applications of LPP

12.7 Limitations of LPP

12.8 General Linear Programming Problem

12.9 Formation of a LPP of Two Variable

12.10 Some Definitions

12.11 Graphical Method of Solution of LPP

12.12 Let Us Sum Up

12.13 Further Readings

12.14 Answers To Check Your Progress

12.15 Model Questions

12.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- understand the meaning of linear programming problem (LPP)
- know the various terms associated with LPP
- know about the various field of application of LPP
- draw the graph of linear inequalities
- learn how to find the feasible region of a LPP of two variable from the graph
- obtain the optimal solution of an LPP of two variable from the graph.

12.2 INTRODUCTION

The central theme of economic theory and management science is to optimise the use of scarce resources at their disposal, which include

machine, man-power, money, raw materials, warehouse space etc. There are several theoretical tools to accomplish this purpose in both the stream. But such tools are not adequate for treating a complex economic problem with several alternatives each with its own restrictions and limitations. It is for tackling such problems that the use of linear programming has been found to be most useful.

12.3 MEANING

The word programming means planning and refers to the process of determining a particular plan of action from amongst several alternative.

The word linear means that the relations involved among two or more variable in a particular problem are linear (i.e. the indices of all the variables involved must have unit power only). Linear Programming is a mathematical technique for determining the optimal allocation of resources and obtaining a particular objective (i.e.) cost minimization or profit mazimization) when there are alternative uses of resources like land, labour, capital, materials, machines etc.

Note : (i) The variables involved in a LPP are called decision variables.

(ii) The linear functions which is to be optimized is called Objective Function.

(iii) The inequalities or equations involving the decision variables of an LPP which describe certain restrictions (or conditions) under which objective function is to be optimized are called constraints. Signs $\geq, =, \leq$ are used in constraints.

12.4 CONDITIONS FOR USING LPP TECHNIQUE

LPP technique may be used for optimization problems subject to the fulfillment of following conditions :

1. There must be a well-defined objective function relating to profit, cost or quantity produced.
2. There must be a set of linear constraints relating to the attainment of the objective.

3. All decision variables must assume non-negative values.
4. There must be an alternative course of action leading to alternative solutions for the purpose of choosing the best one.

12.5 BASIC ASSUMPTIONS OF LPP

The following four assumptions are made in the LPP :

1. Proportionality : It is assumed that the proportionality exists in the objective function and in the constraints.

2. Certainity : All the co-efficients in the objective function and constraints are completely known with certainty and do not change during the period being stated.

3. Divisibility : This assumption states that the decision variables can take any non-negative values i.e. fractional values of the decision variables are permitted.

4. Additivity : We assume additivity of all activities which means that the total activities equal the sum of all activities.

12.6 AREAS OF APPLICATIONS OF LPP

Historically, the field of linear programming was developed by George B. Dantzig and his associates during World War II and was first used by U.S. Air Force as an aid in decision making. Today it has been applied to a wide variety of problems. Some of these are :

(i) Manufacturing Problem : To find the number of items of each type that should be manufactured so as to maximize the profit subject to production restrictions imposed by limitation on the use of machinery, law material and labour.

(ii) Transportation Problem : To find the least costly way of transportation from the warehouses to customers.

(iii) Investment Problem : To find the amount to be invested in variety of shares and debentures so as to maximize retern on investment.

(iv) Advertising Media Problem : To find the optimum allocation of advertisements in various media subject to total advertising budget to maximize effective number of customers.

(v) Diet Problem : To find the minimum requirement of nutrients subject to availability of foods and their prices.

(vi) Production Problem : To decide the production schedule that could satisfy future demands (seasonal or otherwise) and minimize cost in face of fluctuating rates and storage (inventory) cost.

(vii) Job Assigning Problem : To assign job to workers for mazimum effectiveness and optimum results subject to restrictions of wages and other cost.

12.7 LIMITATIONS OF LPP

Inspite of having wide applicability of LPP, it has certain limitations also. Some of which are as follows :

(i) In real life situation, many objective function and constraints can not be expressed linearly.

(ii) In practical situations, it may not be possible to state all the coefficients in the objective function and constraints with certainity.

(iii) Linear programming will fail to give a solution if management have multiple goals instead of one goal.

12.8 GENERAL LINEAR PROGRAMMING PROBLEM

The mathematical statement (model) of the general form of a LPP may be written as follows :

Optimize (Maximize or Minimize)

$$Z = C_1x_1 + C_2x_2 + \ldots\ldots\ldots + C_nx_n$$

Subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n \; (\leq =\geq)\; b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n \; (\leq =\geq)\; b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots\ldots + a_{mn}x_n \; (\leq =\geq)\; b_m \end{array}\right\} - (1)$$

and $x_1, x_2, \ldots\ldots\ldots\ldots x_n \geq 0$ – (2)

Where (i) $x_1, x_2, \ldots\ldots x_n$ are the variables whose values are to be determined are called decision variable.

(ii) The linear function z which is to be mazimized or minimized is called objective function.

(iii) The inequalities (1) are called constraints.

(iv) The set of inequalities (2) are known as non-negative restrictions.

(v) The constants aij (i = 1, 2, ... m, j = 1, 2, ... n) are called technological coefficients.

(vi) The constants bi (i = 1, 2, ... m) are called availability parameters.

(vii) The expression ($\leq, =, \geq$) means that only one of the relationship would hold for a particular constraint.

Note : Here we will restrict our discussion on LPP of two variables only.

12.9 FORMATION OF A LPP OF TWO VARIABLE

The following steps are to be followed in the formulation of LPP involving two variables :

Step I : Identify the two decision variables and denote them by x and y. We have to determine the value of x and y.

Step II : Identify the objective function and express it as a linear function of decision variables x and y i.e. as $z = c_1x + c_2y$ where c_1 and c_2 are constants. We have to maximize z as profit (or production) or to minimize z as cost.

Step III : Identify all constraints and express them as linear inequality/ equations in terms of x and y by using the given conditions.

Step IV : Impose non-negativity restrictions on x and y i.e. take () as decision variables can not be negative.

Example 1: A company manufactures two types of showpiece made of plywood. SHow piece of type A requires 5 minutes each for cutting and 10 minutes each for assembling. Showpiece of type B requires 8 minutes for cutting and 8 minutes each for assembling. There are 3 hours and 20 minutes available for cutting and 4 hours available for assembling. The

profit is Rs. 10 for each showpiece of type A and Rs. 12 for that of type B. Give a mathematical formulation of the above LPP in terms of mazimizing profit.

Solution : Let x = Number of showpiece of type A

y = number of showpiece of type B

The information supplied in the given problem may be summerised in the tabular form as follows :

Type of Showpiece	Number of Showpiece	Manufacturing Time (In minute)		Profit per unit (Rs.)
		Cutting	Assembling	
A	x	5	10	10
B	y	8	8	12
Total available time		3 hours 20 minutes=200 minutes	4 hours = 240 minutes	

From the above table, we observe that as our objective is to maximize profit,

The objective function is

Z = 10x + 12y.

We oberve that the total time (in minutes) required in the cutting machine for x numbers of showpiece of type A and y numbers of showpiece of type B is 5x + 8y. Thus the limitation of 3 hours 20 minutes (= 200 minutes) at most on cutting machine is expressed by the constraint.

$5x = 8y \leq 200$

Similarly for the assembling constraint we get

$10x + 8y \leq 240$

Again, the number of showpiece manufactured cannot be negative

$\therefore \quad x \geq 0, \quad y \geq 0$

Thus th emathematical formulation of the given LPP is stated below:

Mazimize Z = 10x + 12y

Subject to constraints

$5x + 8y \leq 200$

$10x + 8y \leq 240$

$x, y \geq 0.$

Example 2 : A diet is to contain at lead 400 units of carbohydrats, 500 units of fat and 300 units of protein. Two foods are available : F_1 which costs Rs. 2 per unit, F_2 which costs Rs. 4 per unit. A unit of food F_1 contains 10 units of carbohydrate, 20 units of fat and 15 units of protein. A unit of food F_2 contains 25 units of carbohydrate, 10 units of fat and 20 units of protein. Find the minimum cost for a diet that consists of a mixture of these two foods and also the minimum nutrition requirement formulate the problem as a LPP.

Solution :

Let x = Number of units of food F_1 to be purchased

y = Number of units of food F_2 to be purchased.

The information supplied in the given problem may be summerised in the tabular form as follows :

Type of food	Number of units of food	Nutrients			Cost per unit (Rs.)
		Carbohydrate	Fat	Protein	
F_1	x	10	20	15	2
F_2	y	25	10	20	4
Minimum requirement		400	500	300	

Here we obserue that our objective is to minimize cost–

The objective function is Z = 2x + 4y.

The total amount of carbohydrates from foods F_1 and F_2 is 10x + 25y and this is required to be at least the minimum amount of carbohydrates required, namely 400 units. Therefore, we must have $10x + 25y \geq 400$

Similarly, for the fat constraint, we get $20x + 10y \geq 500$

For the protein constraint, we get $15x + 20y \geq 300$

Again, units of food manufactured can not be negative.

$\therefore \quad x \geq 0, \; y \geq 0$

These the mathematical formulation of the given LPP is stated below.

Minimize Z = 2x + 4y

Subject to constraints

$10x + 25y \geq 400$

$20x + 10y \geq 500$

$15x + 20y \geq 300$

$x \geq y \geq 0$

12.10 SOME DEFINITIONS

It is appropriate at this stage to give some definitions pertaining to solution of LPP.

Definition 1 : A set of values of the decision variables which satisfy the constraints of general LPP is called a solution to the LPP.

Definition 2 : Any solution to the general LPP which also satisfies the non-negative restrictions of the problem is called a **feasible solution.** The set of all feasible solutions constitutes what is called the **feasible region.**

Definitions 3 : Any feasible solution which optimizes the objective function of a general LPP is called an **optimum solution.**

12.11 GRAPHICAL METHOD OF SOLUTION OF LPP

Now, we are going to descuss the graphical method for solving LPP that involve only two variables. This method is based on a theorem. called **extreme point theorem,** which states as follows :

Extreme Point Theorem : an optimum solution to a LPP, if it exists, occurs at one of the corner (or extreme) points of the area of feasible region.

Steps for Graphical Method of Solution.

Step I : In this step, we have to draw the graph of each linear constraint (generally expressed as linear equality). The graph of a lineae inequality in two variables x and y is the set of all points (x, y) for which the inequality holds. In this step we have to determine the area of **feasible region.** Feasible region is the area which contains the set of all points that simultaneously satisfy all constraints. (including non-negative constraints)

Note : The procedure for graphing a linear inequality is as follows.

Step (i): Graph the corresponding linear equation, a line. This is done by finding any two points on the line and drawing a straight line through them. For convenience, we choose the two points so that each is on azis (i.e. either of the two coordinates is zero)

Step (ii): Choose a point P not on the line. Normally, this point is taken as the origin as long as it does not lie on the line. Test P to see whether P satisfies the inequality.

Step (iii): It the coordinate of the chosen point P satisfy the inequality, then all points on the same side of the line as the point P satisfy the inequality. It the coordinate of the point P do not satisfy the inequality, then all points on the opposite side of the line from P satisfy the inequality.

Step II: Note the coordinate of each corner point of the feasible region and substitute the coordinates of the corner points into the objective function.

Step III: The optimum solution occurs in a mazimization case at the corner point yielding the largest value of the objective function and in a minimization case at the corner point yielding the smallest value of the objective function.

Note : The method of finding the feasible region is illustrated by an example given below :

$2x + 3y \leq 18$ – (1)

$2x + y \leq 12$ – (2)

$x \geq 0, y \geq 0$

First let us draw th egraph of the corresponding equations.

$2x + 3y = 18$ – (3)

$2x + y = 12$ – (4)

For (3) : When $x = 0, y = 6$

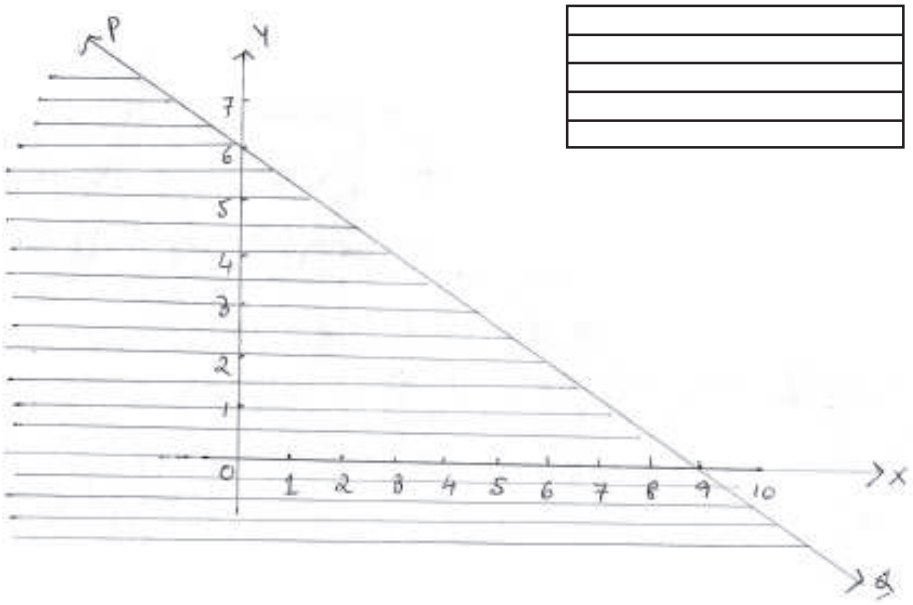
When $y = 0, x = 9$

. Graph of (3) is the line PQ obtained by joining (0, 6) and (9, 0).

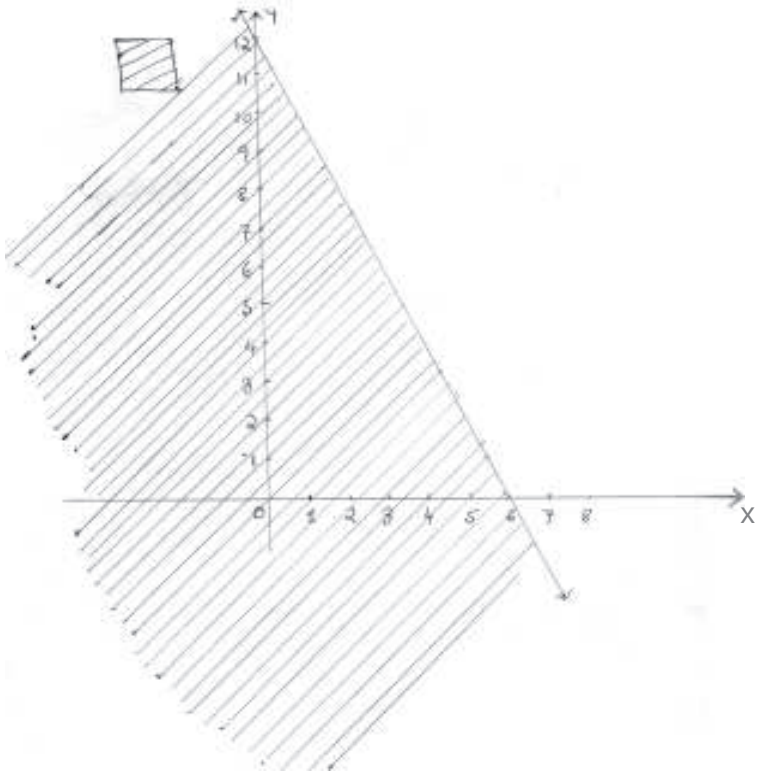
To see which side of th eline satisfies the inequality, the coodinates of th origin are substitited into the inequality.

$2(0) + 3(0) = 0 \leq 18$. Which is true.

Therefore all points on the same side of the line as the origin satisfy the inequality. The graph of the inequality is the shaded region shown by

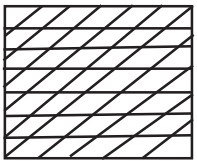


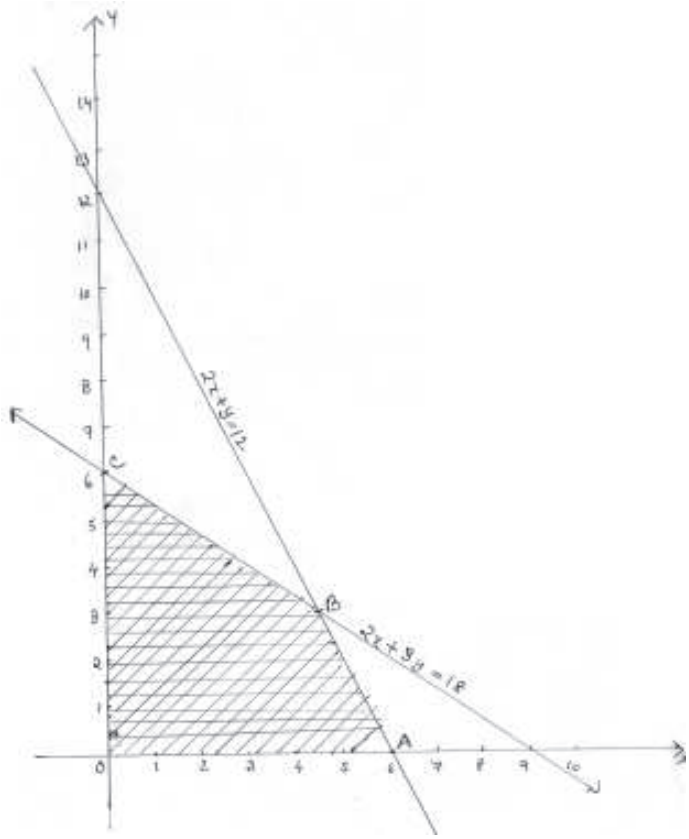
Similarly, the graph of th einequality 2x + 2y ≤ 12 is the shaded region shown by



The inequalities $x \geq 0$ and $y \geq 0$ are satisfied by the fact that we confine the graph to the first quadrant only.

Hence the feasible region of the above system will be OABCO shown by–





Example 3 : Maximize Z = 3x + 4y

Subject to constraints

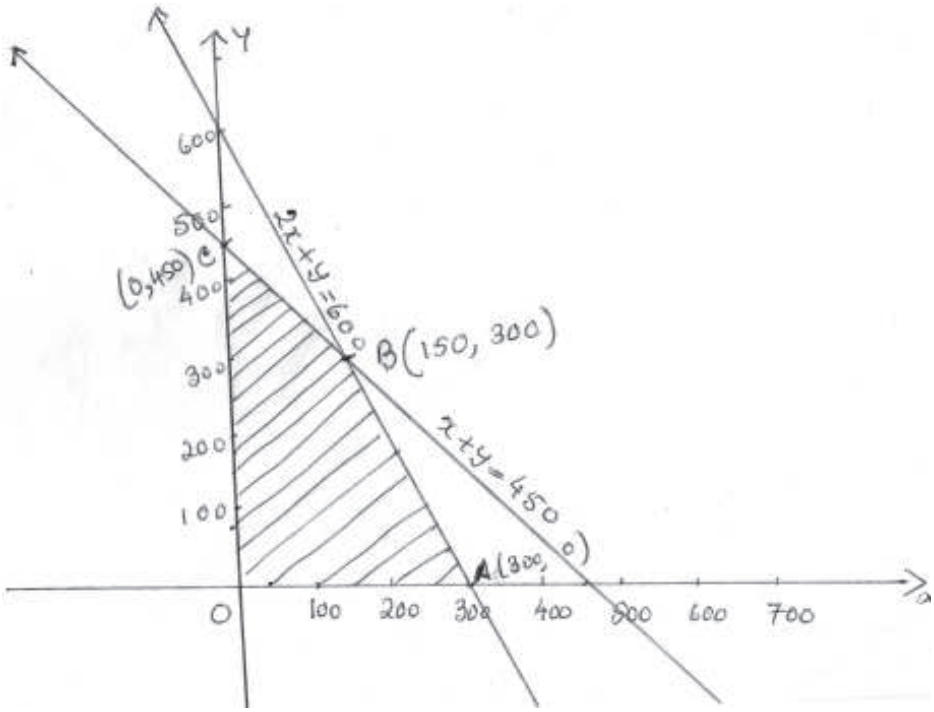
$x + y \leq 450$

$2x + y \leq 600$

$x \geq 0$ and $y \geq 0$

Solution :

First, we find the feasible region which consists of the set of all points whose coordinates simultaneously satisfy all constraints including the non-negative restrictions. Applying the techniques illustrated above, the feasible region for the system of inequalities is OABCO as shown in the diagram below



The corner points of the feasible region are 0(0, 0), A(300, 0), B(150, 300) and C(0, 450). Note that the coordinates of the point B have been obtained by solving x + y = 450 and 2x + y = 600 simultaneously for x and y. Now the values of the objective function at these corner points are summerized below

Corner points	Coordinates	Value of the objective function z = 3x + 4y
O	(0, 0)	Z = 3 x 0 + 4 x 0 = 0
A	(300, 0)	Z = 3 x 300 + 4 x 0 = 900
B	(150, 300)	Z = 3 x 150 + 4 x 300 = 1650
C	(0, 450)	Z = 3 x 0 + 4 x 450 = 1800

Hence maximum value of z is 1800 and it is attained at the vertes C(0, 450) i.e. Maximum value of the objective function is 1800 when x = 0 and y = 450

Example 4 : Minimize z = 2x + 3y

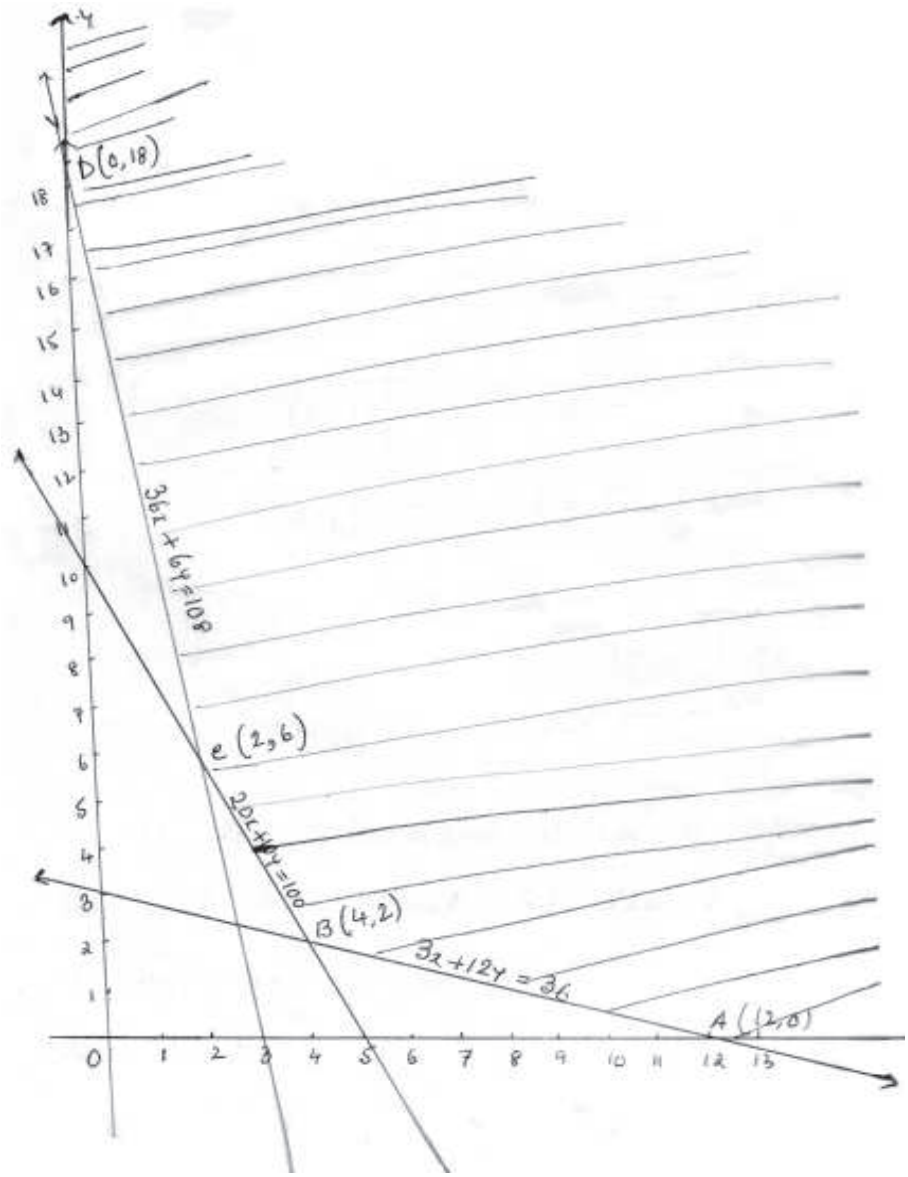
Subject to $36x + 6y \geq 108$

$3x + 12y \geq 36$

$20x + 10y \geq 100$

$x, y \geq 0$

Solution : First, we find the feasible region which consists of the set of all points whose coordinates simultaneously satisfy all the constraints including the non-negative restrictions. The feasible region for the system of inequalities is YDCBAX as shown in the diagram below.



The corner points of the feasible region are D(0, 18), C(2, 6), B(4, 2) and A(12, 0). Note that the coordinate of C is obtained by solving 36x + 6y = 108 and 20x + 10y = 100 and that of B by solving 20x + 10y = 100 and 3x + 12y = 36 simultaneously for x and y. Now, the values of the objective function at these corner points are summerized below.

Corner points	Coordinates	Value of the objective function z = 2x + 3y
D	(0, 18)	Z = 2 x 0 + 3 x 18 = 54
C	(2, 6)	Z = 2 x 2 + 3 x 6 = 22
B	(4, 2)	Z = 2 x 4 + 3 x 2 = 14
A	(12, 0)	Z = 2 x 12 + 3 x 0 = 24

So, the minimum value of Z is 14 and it is attained at the vertes B(4, 2) i.e. minimum value of Z is 14 when x = 4 and y = 2.

Example 5 : Maximize $Z = 10x_1 + 15x_2$

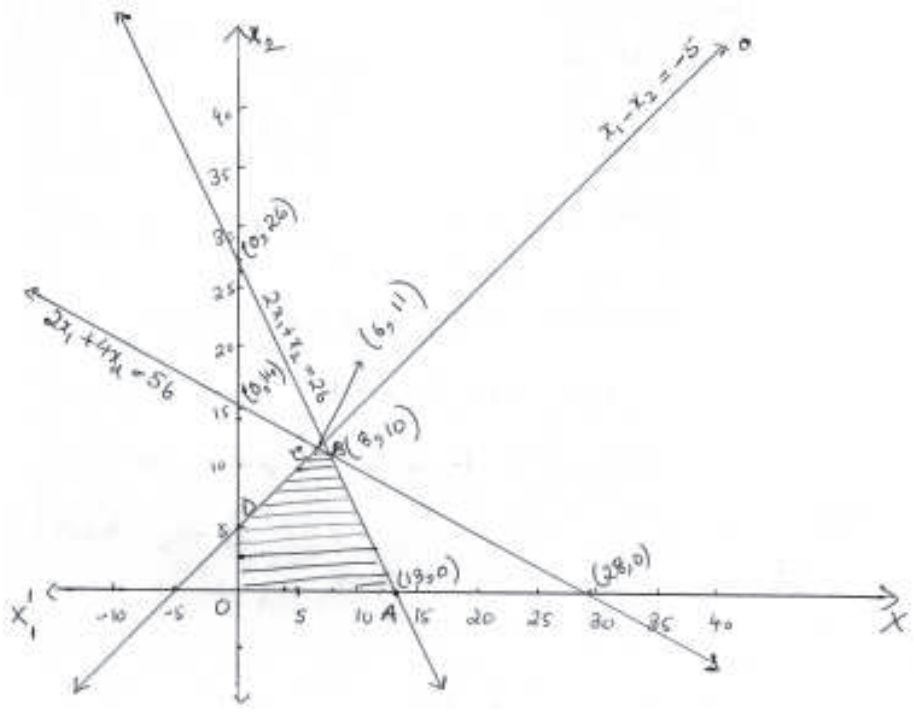
Subject to $2x_1 + x_2 \leq 26$

$2x_1 + 4x_2 \leq 56$

$x_1 - x_2 \geq -5$

$x_1, x_2 \geq 0.$

Subject : First, we find the feasible region which consists of the set of all points whose coordinates simultaneously satisfy all the costraints including the non-negative restrictions. The feasible region for th esystem of inequalities is OABCDO as shown in the diagram below.



The corner points of the feasible region are A(13, 0), B(8, 10), C(6, 11) and D(0, 5). Note that coordinates of B is obtained by solving $2x_1 + x_2$

= 26 and $2x_1 + 4x_2 = 56$ and that of C is obtained by solving $2x_1 + 4x_2 = 56$ and $x_1 – x_2 = –5$ simultaneously for x_1 and x_2. Now the values of the objective function at these corner points are sumerized below.

Point	Coordinates	Value of the objective function $z = 10x_1 + 15x_2$
A	(13, 0)	Z = 10 x 8 + 15 x 10 = 130
B	(8, 10)	Z = 10 x 8 + 15 x 10 = 230
C	(6, 11)	Z = 10 x 6 + 15 x 11 = 225
D	(0, 5)	Z = 0 + 15 x 5 = 75

So, the maximum value of z is 230 and it is attained at the vertex B(8, 10) i.e. maximum values of z is 230 when $x_1 = 8$ and $x_2 = 10$.



CHECK YOUR PROGRESS

Q 1: What do you mean by linear programming problem?

Q 2: Define feasible solution, optimum solution and feasible region.

Q 3: Say whether the following statements are true (T) or fasle (F)

(i) In a LPP, all parameters are assumed to be constant.

(ii) See the variables in the solution of LPP are either positive or negative.

(iii) Production planning is one of the application areas of LPP.

(iv) Linear programming is a lichnique for finding the best uses of an organization’s manpower, money and machinery.

Q 4: Fill in the blanks.

(i) The variables involved in a LPP are called ..... variable.

(ii) In a Linear Programming, the linear function to be optimized is called ..... function.

(iii) In a Linear Programming, all relationship among decision variables must be .....

(iv) Most of the constraints in an LPP are expressed as .....

(v) ..... of the decision variables is one of the assumptions of a LPP.

(vi) Linear Programming is used to allocate .... to activities so as to optimize the value of the objective function.



12.12 LET US SUM UP

In this unit, we have learnt–

- The meaning of LPP.
- Formulation of a LPP.
- Various terms associated with a LPP.
- About the basic assumptions, limitations and importance of LPP.
- Graphical solution of a LPP involving two variables.



12.13 FURTHER READING

1) Thukral, Dr. J. K. *Mathematics for Business Studies,* Mayoor Paperbacks.

2) Bhowel M. K. *Fundamentals of Business Mathematics,* Asian Books Private Limited.

3) Ghosh, R. K. and Saha, S. *Business Mathematics and Statistics,* New Central Books Agency (P) Ltd.

4) Sharma, J. K. *Quantativs Methods, Theory and Application,* Macmillon Publishers India Ltd.



12.14 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 3: (i) T (ii) F (iii) T (iv) T.

Ans to Q No 4: (i) decision (ii) objective (iii) linear (iv) inequality (v) certainity (vi) scarce resource.



12.15 MODEL QUESTIONS

Q 1: What do you understand by Linear Programming ? What are its major assumptions ?

Q 2: What are the requirements of an LPP ?

Q 3: Write a note on the limitations of linear programming ?

Q 4: Discuss the scope of linear programming in solving Business Problem.

Q 5: Choose the correct alternatives

(a) A constraint in an LP model restricts.
- (i) value of objective function.
- (ii) value of decision variables.
- (iii) use of the available resources.
- (iv) all of the above.

(b) The distingushing feature of a LPP is
- (i) the relationship among all variables is linear.
- (ii) value of decision variables is non-negative.
- (iii) it has single objective function and constraints.
- (iv) all of the above.

(c) Which of the following is an assumption of a LPP.
(i) divisibility (ii) certainity (iii) additivity (iv) all the above.

(d) Which of the following is not the characteristic of LPP.
- (i) resources must be linited.
- (ii) only one objective function.

(iii) all the relationship involved must be linear.

(iv) the problem must be of minimization type.

(e) Non-negativity condition in an LPP implics

(i) a positive co-efficient of variables in the objective function.

(ii) non-negative value of resources.

(iii) a positive co-efficient of variables in any constraint.

(iv) none of the above.

Q 6: Solve the following LPP graphically.

(i) Maximize $Z = 40x + 80y$

Subject to $2x + 3y \leq 48$

$x \leq 15$

$y \leq 10$

$x, y \geq 0$

(ii) Maximize $Z = 5x + 6y$

Subject to $2x + 3y \leq 18$

$2x + y \leq 12$

$3x + 3y \leq 24$

$x, y \geq 0$

(iii) Maximize $Z = 20x + 10y$

Subject to $x + 2y \leq 40$

$3x + y \geq 30$

$3x + 3y \geq 60$

$x, y \geq 0$

(iv) Maximize $Z = 10x_1 + 15x_2$

Subject to $15x_1 + 25x_2 \leq 180$

$x_1 + x_2 \leq 10$

$x_1 \geq 2, \ x_2 \geq 3$

$x_1, x_2 \geq 0$

(v) Maximize $Z = x + 5y$

Subject to $x + 4y \leq 12$

$x + y \geq 2$

$x \leq 8$

$x, y \geq 0$

(vi) Maximize $Z = 8x_1 + 5x_2$

Subject to $x_1 \leq 150$

$x_2 \leq 250$

$2x_1 x_2 \leq 500$

$x_1, x_2 \geq 0$

Q 7: A firm produces three products. There products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines all given in the following table.

Machine	Time Per Unit (minutes)			Machine Capacity
	Product I	Product II	Product III	(Minutes/day)
M_1	2	3	2	440
M_2	4	–	3	470
M_3	2	5	–	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product I, II and III are Rs. 4, Rs. 3 and Rs. 6 respectively. It assumed that all the items produced are consumed in the market. Formulate the problem as LPP.

Q 8: A manufacturer produces two different types of products A and B. Each product requires three different operations grinding, assembling and testing. Each unit of product A requires 15, 20 and 10 minute to grinding, assembling and testing respectively, where as each unit of product B requires 7.5, 40 and 45 minutes for these three operations. respectively. The production run calls for at least 7.5 hours of grinding time, at least 20 hours of assembling time and at least 15 hours of testing time. Each unit of product A costs Rs. 60 and each unit of product B costs Rs. 90 to manufacture, determine the number of units of each product the firm should produce to minimize the cost.

Q 9: A company produces two types of pens, Hightech and Smoothflow. Pen Hightech is of superior quality and smoothflow is of lower quality. Profits from Hightech and Smoothflow are Rs. 3 per pen respectively Raw materials required for each pen of Hightech type is twice as that of pen of Smoothflow Raw materials required for each pen of Hightech

type is twice as that of pen of Smoothflow type. The supply of raw materials is sufficient only for 1000 pens of Smoothflow type. Pen hightech requires a special clip and 400 such clips are available per day. Smooth, flow pen requires a particular type of clip and 700 such clips are available per day.

Formulate the alove information as an LPP and solve it graphically.

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