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MATHEMATICAL METHODS FOR ECONOMIC ANALYSIS

SEMESTER - I

ECONOMICS

BLOCK - 1

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COURSE INTRODUCTION

This course introduces a learner to the mathematical methods of economic analysis. The course has three purposes. First, to introduce learners to the mathematical concepts and methods that are central to modern economics. Second, to give a set of economic applications of the mathematical methods. Third, to develop the ability of the learners to formulate logical arguments with the degree of precision. Economic analysis often uses quantitative methods when reviewing specific information in an economy. Quantitative methods are mathematical or statistical calculations that provide economists with indicators for comparing the current economic analysis to those of previous periods. Economists often use various types of mathematics to ensure their personal judgments, inferences or theories are supported by meaningful calculations. The study of mathematical methods is also useful for forecasting and in research activities.

We divide the study of mathematical methods of economic analysis into 15 units, which are distributed in two blocks.

BLOCK INTRODUCTION

The first block of the course on mathematical methods of economic analysis has 8 units. Unit I introduces the learners to different concepts of matrix algebra. Operations on matrices like addition, subtraction and multiplication are dealt here. Unit 2 deals with matrix inversion and the operations of determinants. Unit 2 also explains how simultaneous equations can be solved by matrix inverse method and by Cramer's rule. This unit deals with partial equilibrium model and the equilibrium of an economy with a goods market and a money market. Different concepts of differentiation are dealt with in unit 3. The unit deals with the rate of change and derivative and the economic interpretation of derivative. The basic rules of differentiation with single explanatory variable and with rules of differentiation involving two or more functions of the same explanatory variable are explained here. Economic applications in elasticity of demand and supply, cost, revenue, consumption and production functions are explained in unit 4. Unit 5 deals with partial and total differentiation and its concept and rules. Unit 6 explains comparative static analysis of market model. Application of Euler's theorem and mathematic derivation of the shape of indifference curve and isoquants are explained in this unit. Unit 7 explains relative and absolute extrema and the criteria for relative extrema. Unit 8 deals with cost minimization, profit maximization and maximization of tax revenue. Taxation and monopoly output are also discussed in this unit.

While going through a unit, you will notice some along-side boxes, which have been included to help you to know some of the difficult, unseen terms. Again, we have included some relevant concepts

in "LET US KNOW" along with the text. And, at the end of each section, you will get "CHECK YOUR PROGRESS" questions. These have been designed to self-check your progress of study. It will be better if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with "ANSWERS TO CHECK YOUR PROGRESS" given at the end of each unit.

UNIT 1: MATRIX ALGEBRA

UNIT STRUCTURE

1.1 Learning Objectives

1.2 Introduction

1.3 Matrices and Vectors (General Notation)

1.4 Basic Matrix Operations

1.4.1 Addition and subtraction of Matrices

1.4.2 Matrix Multiplication

1.5 Types of Matrices

1.6 Transpose, Norms and Rank of Matrix

1.7 System of Equations in Matrix form

1.8 Let Us Sum up

1.9 Further Reading

1.10 Answer to Check Your Progress

1.11 Model Questions

1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to–

- state the meaning, operations and types of Matrics
- understand Determinants and their properties
- explain Matrix inversion and their applications in solving simultaneous equations system and other economic models
- apply Cramer's rule to solve the simultaneous system of equations
- solve of linear market & national income model
- find solution of the problems related to equilibrium in goods market and money market.

1.2 INTRODUCTION

Matrix algebra is a significant mathematical tool to deal with the simultaneous linear equations system. Especially, for the situations where the number of unknowns to be determined is more than three or more. We can easily determine the value of unknown if in the system we have to

determine only one or two or three variable with ordinary algebra. But if the number of unknown to be determine in more, it becomes almost unsolvable to find the value with simple algebraic operation. For such cases, the technique of matrix can be used. In Economics we have to deal with such situations, when number of variables is more. To address such models or variables, we have to just arrange those in linear form and then to matrix form which make it easier to solve.

1.3 MATRICES AND VECTORS (GENERAL NOTATION)

Rectangular arrangements of elements into rows and columns is called matrix. In other words, a matrix can be defined as the rectangular array of elements in the form of rows and columns. Numbers arranged in that particular form are enclosed by the notation [ ] or ( ). For example,

$$A = \begin{bmatrix} 9 & 2 \\ 3 & 7 \end{bmatrix}_{2 \times 2}$$

The numbers arranged in rows and columns are called elements (9, 2, 3, and 7 are elements in the above matrix). There is significance of the placing of the elements. For example 9 in the above matrix is placed in the 1st row if we count row wise. It is also 1st element if we count it column wise. Again, element 2 is placed 2nd in the 1st row, but placed in the 2nd column of the matrix. If we denote element by a_{ij} where suffix *i* represents the row number and *j* represents the column number, then, $a_{11} = 9$, $a_{12} = 2$, $a_{21} = 3$ and $a_{22} = 7$. As we have written 2 X 2 outside the bracket in suffix form in the order of the matrix. The number of rows and columns is called the order of the matrix. The above matrix is of order 2 X 2, i.e. there are two numbers of rows and two numbers of columns in the matrix. Order of the matrix has significance in the matrix operation. Now let us consider a generalised form of a matrix with m x n order

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots\ldots\ldots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots\ldots & a_{mn} \end{bmatrix}_{m\times n}$$

The above matrix is m by n order matrix i.e. here number of rows is m and number of column is n. For each element, the first subscript represents the row number while 2nd subscript represents the column number.

Vector: Vectors are special kind of matrix which has only one row or one column. A matrix having only one row is called a row vector. For example,

$V = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$

On the other hand, a matrix having only one column is called a column vector. For example,

$$C = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

Equal matrices and equivalent matrices: Matrices are said to be equal matrices if and only if they have equal dimension and each elements in corresponding locations are same or equal value. That is, matrices A and B are equal iff $a_{ij} = b_{ij}$. They are equivalent if they have same dimension or order, need not necessarily all corresponding elements are same. Thus all equal matrices are equivalent. Let us consider the following

$$A=\begin{pmatrix} 2 & 4 \\ 3 & 5 \\ 2 & 1 \end{pmatrix}, \text{ and } B=\begin{pmatrix} 2 & 4 \\ 3 & 5 \\ 0 & 4 \end{pmatrix}$$

A and *B* are not equal matrices, but they are equivalent as both are 3x2 order matrices. It is to be noted that in case of equal matrices, the each and every elements of one matrix must be equal with respective elements of the same location. Unlike the equality of sets, mere containing same elements don't mean equality of matrices.

1.4 BASIC MATRIX OPERATIONS

Let us proceed to the process how algebraic operations such as additions or subtractions of two or more matrices, multiplications, divisions, etc are performed.

1.4.1 Addition and subtraction of Matrices

Two matrices can be added or subtracted if and only if both have the same order or dimension which will give a new matrix with same dimension. If A = $\lfloor a_{ij} \rfloor$ and B = $\lfloor b_{ij} \rfloor$, then A + B = $\lfloor a_{ij} + b_{ij} \rfloor$ and A – B = $\lfloor a_{ij} + b_{ij} \rfloor$



Let us know

Addition of matrices satisfies the following properties:

- For two matrices A and B having same dimension, A + B = B + A
- For three matrices A, B and C having same dimension, (A + B) + C = A + (B + C)
- A + A = 2A

Example 1 : If $A = \begin{pmatrix} 2 & 4 \\ -3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 3 & 3 \end{pmatrix}$

Then, $A + B = \begin{pmatrix} 2+1 & 4+4 \\ -3+3 & 5+3 \end{pmatrix}$

$= \begin{pmatrix} 3 & 8 \\ 0 & 8 \end{pmatrix}$

Example 2:

If $A = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 6 \\ 5 & 8 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 1 \\ 6 & 0 \end{pmatrix}$,

Then we can show that (A + B) + C = A + (B + C)

$$A+B=\begin{pmatrix}4+4 & 3+6\\ 2+5 & 1+8\end{pmatrix}=\begin{pmatrix}8 & 9\\ 7 & 9\end{pmatrix}$$

L.H.S=

$$(A+B)+C=\begin{pmatrix}8+2 & 9+1\\ 7+6 & 9+0\end{pmatrix}=\begin{pmatrix}10 & 10\\ 13 & 9\end{pmatrix}$$

Again, B + C = $\begin{pmatrix}4+2 & 6+1\\ 5+6 & 8+0\end{pmatrix}=\begin{pmatrix}6 & 7\\ 11 & 8\end{pmatrix}$

R.H.S = $A(B+C)=\begin{pmatrix}4+6 & 3+7\\ 2+11 & 1+8\end{pmatrix}=\begin{pmatrix}10 & 10\\ 13 & 9\end{pmatrix}$

Therefore, , i.e. LHS = RHS

1.4.2 Matrix Multiplication

Multiplication of matrices is not as simple as simple numbers. We may consider the procedure of multiplication of matrices in two cases – multiplication of a matrix by a scalar, and multiplication of two matrices.

In case of multiplication of a matrix by a scalar, we have to multiply each element of the matrix by the scalar. Let us consider the consider the following matrix,

$A=\begin{pmatrix}7 & 3\\ 2 & 9\end{pmatrix}$, if we have to multiply the matrix by a scalar, say 3, then,

3A = $3\begin{pmatrix}7 & 3\\ 2 & 9\end{pmatrix}=\begin{pmatrix}7x3 & 3x3\\ 2x3 & 9x3\end{pmatrix}=\begin{pmatrix}21 & 9\\ 6 & 27\end{pmatrix}$

Two matrices A and B are conformable for multiplication if the number of columns of the first matrix (Matrix A) is equal to the number of rows of the second matrix (matrix B). In other words, if a matrix A is of order *m x n* and another matrix B of order *p x q*, then AB is

defined if $n = p$. The order of the new matrix will be m x n. Thus in case of multiplication of two matrices, the placing or the ordered pair of the matrices, i.e. which will come first and which second is very important. Matrices A and B may be conformation for AB, but may not be for BA. And also AB may not necessary to be equal to BA. Process of multiplication is also different from ordinary number multiplication.

For two matrices, A and B, to find the AB, we have to multiply the row of first matrix with the column of 2nd matrix. To get the first row first element of the resultant matrix, we have to multiply the first row of the first matrix with first column of the second matrix. Similarly, to get the 2nd element of first row, and second column of the resultant matrix, we have to multiply the first row of the first matrix with the second column of the second matrix and so on for the various elements first row of the new matrix.

Then, we have to multiply second row with first column of the second matrix, which will give the first element of second row of the new matrix. And then again second row of first matrix with second column of second matrix and so on for the respective elements of second row of the resultant matrix. Let us consider two matrices,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

Order of A is 2x2 and that of B is 2x3, therefore the order of the new matrix AB (say C) will be 2x3.

Procedure of multiplication is as follows.

$$AB = C = \begin{bmatrix} c_{11} & c_{12} & c_{12} \\ c_{12} & c_{22} & c_{23} \end{bmatrix}, \text{ where,}$$

$C_{11} = a_{11}b_{11}+a_{12}b_{21}$ $c_{12} = a_{11}b_{12}+a_{12}b_{22}$ $c_{12} = a_{11}b_{13}+a_{12}b_{23}$

$C_{21} = a_{21}b_{11}+a_{22}b_{21}$ $c_{22} = a_{21}b_{12}+a_{22}b_{22}$ $c_{23} = a_{21}b_{13}+a_{22}b_{23}$

Example 3: Multiply A and B if,

$$A=\begin{bmatrix}2 & 5\\ 3 & 4\end{bmatrix} \text{and} B=\begin{bmatrix}1 & 6\\ 3 & 2\end{bmatrix}$$

Solution:

AB is conformable for multiplication and new matrix will be of 2x2 order.

$$AB=\begin{bmatrix}2 & 5\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 6\\ 3 & 2\end{bmatrix}=\begin{bmatrix}2\times1+5\times3 & 2\times6+5\times2\\ 3\times1+4\times3 & 3\times6+4\times2\end{bmatrix}=\begin{bmatrix}17 & 22\\ 15 & 26\end{bmatrix}$$

Example 4: If $A=\begin{bmatrix}2 & 3 & 0\\ 5 & 1 & 2\end{bmatrix} \text{and} B=\begin{bmatrix}4 & 1\\ 2 & 3\\ 1 & 5\end{bmatrix}$, find AB. Also calculate the BA.

Solution: The matrices A and B are conformable for finding AB and the new matrix will be of 2x2 order.

$$AB\begin{bmatrix}2 & 3 & 0\\ 5 & 1 & 2\end{bmatrix}\begin{bmatrix}4 & 1\\ 2 & 3\\ 1 & 5\end{bmatrix} = \begin{bmatrix}2\times4+3\times2+0\times1 & 2\times1+3\times3+0\times5\\ 5\times4+1\times2+2\times1 & 5\times1+1\times3+2\times5\end{bmatrix}$$

$$= \begin{bmatrix}8+6+0 & 2+9+0\\ 20+2+2 & 5+3+10\end{bmatrix}$$

$$= \begin{bmatrix}14 & 11\\ 24 & 28\end{bmatrix}$$

BA is also defined, and the new matrix for BA will be of 3x3 order.

$$BA=\begin{bmatrix}4 & 1\\ 2 & 3\\ 1 & 5\end{bmatrix}\begin{bmatrix}2 & 3 & 0\\ 5 & 1 & 2\end{bmatrix} = \begin{bmatrix}4\times2+1\times5 & 4\times3+1\times1 & 4\times0+1\times2\\ 2\times2+3\times5 & 2\times3+3\times1 & 2\times0+3\times2\\ 1\times2+5\times5 & 1\times3+5\times1 & 1\times0+5\times2\end{bmatrix}$$

$$= \begin{bmatrix} 8+5 & 12+1 & 0+2 \\ 4+15 & 6+3 & 0+6 \\ 2+25 & 3+5 & 0+10 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 13 & 2 \\ 19 & 9 & 6 \\ 27 & 8 & 10 \end{bmatrix}$$

1.5 TYPES OF MATRICES

There are different types of matrices depending upon the structure and nature of particular matrix. Some of those are highlighted below.

- **Null matrix:** A matrix having its all the elements zero is called a null matrix. Null matrix represents zero as in case of ordinary numbers. For example,

$$\emptyset = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- **Square matrix:** A matrix which has equal number of rows and column is called square matrix. For example,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2x2}$$

- **Diagonal matrix:** A square matrix whose all diagonal elements are scalar number and all off-diagonal elements are zero is called diagonal matrix. Following matrix A is a diagonal matrix;

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- **Scalar matrix:** A diagonal matrix having all elements of principal diagonal are a common scalar and off-diagonal elements are zero, then such matrix is called scalar matrix. For example,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- **Identity Matrix:** A Square matrix if it's all the diagonal elements are 1 and all the off-diagonal elements are zero, then it is called an identity matrix. For example,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is to be note that all the identity and scalar matrices are special types of diagonal matrix. If we pre multiply or post multiply any matrix A, by an identity matrix, it gives the original matrix A and for such cases, AI = IA.

- **Triangular matrix:** If all the elements in one side of the principal diagonal are zero such matrices are called Triangular matrix.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 5 & 0 \\ 6 & 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 6 & 6 & 4 & 0 \\ 9 & 2 & 8 & 3 \end{bmatrix}$$

- **Symmetric matrix:** If a matrix contains such elements that interchange of rows and columns gives the same matrix, then it is called symmetric matrix. Symmetric matrices are also needs to be square matrix. For example,

$$A = \begin{bmatrix} 2 & 6 & 3 \\ 6 & 5 & 7 \\ 3 & 7 & 4 \end{bmatrix}$$

- **Idempotent matrix:** A symmetric matrix that reproduces itself if it is multiplied by itself is called idempotent matrix. That is, if A is symmetric matrix and A.A = A, then A is an idempotent matrix.



Let us know:

If the dimension of a matrix is too large, we can make it part into sub matrices. Such matrices are called partitioned matrix

1.6 TRANSPOSE, NORMS AND RANK OF MATRIX

- **Transpose of a matrix:**

If we interchange the rows and columns of a matrix A, it is called the transpose of matrix and denoted by A'. If A is m x n order matrix, will be n x m order matrix. For the matrix,

In case of transpose of matrix, we may note the following

- Transpose of a transposed matrix gives the original matrix. That is (A')' = A
- If k is a scalar, then (kA)' = kA'
- For sum of the matrices A and B, (A + B)' = A' + B'
- Similarly, for product of two matrices A and B, (AB)' = B'A'

- **Norms of a matrix:**

The largest column sum of a matrix is called the norms of the matrix. For example if we consider the following matrices

$$A = \begin{bmatrix} 4 & 2 & 7 \\ 2 & 5 & 8 \\ 1 & 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 2 & 5 \\ 3 & 1 & 6 & 2 \\ 1 & 4 & 4 & 1 \\ 2 & 2 & 0 & 3 \end{bmatrix}$$

The norms of the matrix A is 18 which is the largest column sum (3rd column) and that of matrix B is 12 which the largest column sum (3rd column)

- **Rank of a matrix:**

Rank of a matrix is the order of the largest square sub-matrix whose determinant value is not zero. It is the maximum number of linearly

independent rows or columns of a matrix. For example, let us consider the following matrix,

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 5 & 1 \\ 6 & 4 & 3 \end{bmatrix}$$

Though the order of the matrix is 3 x 3, the determinant value of the 3 x 3 order matrix will be zero because first column is the multiple of 3rd column and therefore rank of the matrix will be less than 3. Now let us proceed for the value of 2 x 2 order sub determinant.

Sub determinant $A_1 = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix} = 20 - 4 = 16$ which is not equal to zero. Therefore the rank of the matrix is 2 which is the order of non zero sub determinant within the matrix A.

- **Trace of a matrix:**

Trace of a matrix is the sum of the elements of principal diagonal of the matrix. For example,

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the trace of the matrix is given by,

$Tr(A) = a_{11} + a_{22} + a_{33}$

- **Singular and Non-singular matrix:**

A matrix whose determinant value is zero is called singular matrix and whose value is not equal to zero is called non-singular matrix.

- **Adjoint of a matrix:**

Adjoint of a matrix is the transposed cofactor matrix of a matrix. If A is a matrix such that,

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and the cofactor matrix of A is $A = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$,

then the Adjoint of the matrix A will be,

$$\text{Adj } (A) = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$



CHECK YOUR PROGRESS:

Q 1 Given the matrix

$A = \begin{bmatrix} 3 & 2 & 0 \\ 4 & 1 & 3 \\ 2 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 0 & 1 \\ 2 & 2 & 5 \end{bmatrix}$

Find AB and BA.

Q 2 Given, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 \\ 6 & 7 \end{bmatrix}$ show that

(i) $(A + B)' = A' + B'$

(ii) $(AB)' = B'A'$

Q 3 If, $A = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix}$

Find A'.

1.7 SYSTEM OF EQUATIONS IN MATRIX FORM

Let us consider the following system of equation where x_1 and x_2 are two unknowns to be determined.

$a_{11}X_1 + a_{12}X_2 = b_1$

$a_{21}X_1 + a_{22}X_2 = b_2$

The above two equations can be expressed in matrix form as,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Or AX = B, Where, A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \times = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Or $X = A^{-1}B$

This is the matrix inversion method for solving simultaneous equations system. As shown in above two equations and two unknowns, same can be expanded for higher number of unknowns with same number of equations. If we consider the general form of such systems with n number of linear equation for n number of variables as,

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ...+ a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ...+ a_{2n}x_n = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ...+ a_{3n}x_n = b_3$

.....

$A_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + ...+ a_{nn}x_n = b_n$

The above also can be expressed in matrix form as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & ..... & .....a_{1n} \\ a_{21} & a_{21} & a_{21} & ..... & .....a_{2n} \\ a_{31} & a_{32} & a_{33} & ..... & .....a_{3n} \\ ..... & ..... & ..... & ..... & .. \\ ..... & ..... & ..... & ..... & .. \\ a_{m1} & a_{m2} & a_{m3} & ..... & .....a_{mn} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ \vdots \\ x_{n1} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

Or AX = B

Or $X = A^{-1}B$

(Now, we can solve this equation in Matrix form by applying either inverse method or Cramer's rule. We will discuss about the inverse method and Cramer's rule of Matrix solution in the next unit.)

Let us take an example of three equations as,

$2x_1 - 4x_2 + 3x_3 = 3$

$4x_1 - 6x_2 + 5x_3 = 2$

$-2x_1 + x_2 - x_3 = 1$

If we want to find the values of x_1, x_2, and x_3 with matrix method, let us arrange the above system of equations into matrix form as,

$$\begin{bmatrix} 2 & -4 & 3 \\ 4 & -6 & 5 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{Let, } A = \begin{bmatrix} 2 & -4 & 3 \\ 4 & -6 & 5 \\ -2 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{Therefore, } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = X = A^{-1}B$$



1.8 LET US SUM UP

- Rectangular arrangements of elements into rows and columns is called matrix. The numbers arranged in rows and columns are called elements. The number of rows and columns is called the order of the matrix.
- Vectors are special kind of matrix which has only one row or one column. A matrix having only one row is called a row vector. A matrix having only one column is called a column vector.
- Two matrices are said to be equal if they have equal dimension and each elements in corresponding locations are same or equal.
- Two matrices are conformable for multiplication if the number of columns of the first matrix is equal to the number of rows of the second matrix.

- A matrix which has equal number of rows and column is called square matrix, having all elements zero is called null matrix. A Square matrix if it's all the diagonal elements are 1 and all the off-diagonal elements are zero, then it is called an identity matrix.
- Determinants are some scalar number associated with square matrix which gives particular value against the matrix. The minor of an element of a determinant is the sub-determinant of one lower order which is obtained by deleting the row and column to which it belongs to. When we multiply the minor by $(-1)^{i+j}$ it is called the cofactor of the particular element.
- We can find the inverse of a matrix if it is a non-singular square matrix.
- Rank of a matrix is the order of the largest square sub-matrix whose determinant value is not zero. It is the maximum number of linearly independent rows or columns of a matrix.



1.9 FURTHER READING

1) Mehta, Madnani (1996). *Mathematics for Economics*. S. Chand & Sons.
2) Chiang A.C., Wainwrite K (2013). *Fundamental Methods of Mathematical Economics.* McGraw Hill Education (India) Pvt. Ltd.
3) Monga G.S. (2003). *Mathematics & Statistics for Economics*. Vishal Publishing House Pvt. Ltd.
4) Baruah S (2012). *Basic Mathematics & Its Application in Economics*. Macmillan Publishers India Ltd.



1.10 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q. No. 1 : $AB=\begin{bmatrix}14 & 3 & 8\\ 18 & 10 & 24\\ 18 & 8 & 21\end{bmatrix}$, $BA=\begin{bmatrix}14 & 9 & 9\\ 14 & 10 & 3\\ 24 & 16 & 21\end{bmatrix}$

Ans to Q. No. 3 : $A^{/}=\begin{bmatrix}2 & 4\\ 3 & 0\\ 2 & 1\end{bmatrix}$

1.11 MODEL QUESTIONS

Q1. Distinguish between equality of matrices and equivalence of matrices.

Q2. Given the matrices A = A = $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 5 & 1 \\ 6 & 4 & 3 \end{bmatrix}$ and B = A = $\begin{bmatrix} 3 & 4 & 2 \\ 2 & 1 & 3 \\ 3 & 4 & 0 \end{bmatrix}$,

Prove that k (A + B) = kA + kB

Q3. Given A = A = $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, B = A = $\begin{bmatrix} 0 & 3 \\ 1 & 6 \end{bmatrix}$, and C = $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$,

Prove that (A + B + C)' = A' + B' + C' and (ABC)' = C'B'A'

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UNIT 2: DETERMINANTS

UNIT STRUCTURE

2.1 Learning Objectives

2.2 Introduction

2.3 Determinants and Properties of Determinants

2.4 Expansion by Cofactors

2.5 Inverse of a Matrix : Definition, Properties and Technique of finding

2.6 Cramer's Rule

2.7 Solution of Partial Equilibrium Market Model, National Income Model, Goods Market and Money Matket Equilibrium

2.7.1 Solution of Linear Partial Equilibrium Market Model

2.7.2 Solution of National Income Model

2.7.3 Equilibrium of the Economy with Good Market and Money Market

2.8 Let us Sum up

2.9 Further Reading

2.10 Answer to Check Your Progress

2.11 Model Question

2.1 LEARNING OBJECTIVES

After going through this unit, you will be able to–

- solve simultaneous equations using matrix inversion method
- solve simultaneous equations using Cramer's rule
- apply Matrix Algebra to solve various market & national income Models
- apply Cramer's rule to solve various market & national income Models.

2.2 INTRODUCTION

Matrix algebra is useful to solve the linear simultaneous equation systems. We can use the matrix inversion method, augmented matrix

method or another method known as Cramer's Rule to deal with such system of equations where there is equal number of equations with the number of unknowns to be determined.

2.3 DETERMINANTS AND PROPERTIES OF DETERMINANTS

Each square matrix has its corresponding determinant. Determinants are some scalar number associated with square matrix which gives particular value against the matrix and symbolised by vertical bars as |A|. Finding the value of determinant has different procedures for different order or the matrix.

For a given matrix A, having 2x2 order as $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, determinant value is calculated as,

$$|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

However, finding the determinant value of 3x3 or more order matrix is little complex. Let us first deal with the matrix of 3x3 order.

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{23} - a_{23}a_{32}) - a_{12}(a_{21}a_{23} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example 1 : Solve the determinant $|A|=\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

Solution:

$$|A| = 5\times 3 - 7\times 2 = 1$$

Example 2 : Given the matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & -1 \\ 4 & 1 & 5 \end{bmatrix}$

Solution:

$$|A| = \begin{vmatrix} 1 & 3 & 2 \\ 0 & 2 & -1 \\ 4 & 1 & 5 \end{vmatrix} =$$

$$= 1\begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} - 3\begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} + 2\begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix}$$

$= 1\,(10 + 1) - 3\,(0 + 4) + 2\,(0 - 8)$

$= -17$

Some properties of determinants

Property I: If row and columns of a determinant are interchanged, the value of the determinant remains unchanged.

For example if $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Now let us interchange the rows and columns as $A = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$, then

$$|A| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ab - bc$$

Property II: If any two rows (or columns) of a determinant are interchanged, the sign of value of the determinant changes. That is,

$$\text{if } |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ then } A_1 = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \text{ or } A_2 = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

For example,

If we have $|A| = \begin{vmatrix} 3 & 0 & 2 \\ 2 & 1 & 2 \\ 4 & 2 & 3 \end{vmatrix} = 3(3-4) - 0(6-8) + 2(4-4) = -3$

Now let us interchange the 1st and 2nd row. Thus

$$|B| = \begin{vmatrix} 2 & 1 & 2 \\ 3 & 0 & 2 \\ 4 & 2 & 3 \end{vmatrix} = 2(0-4) - 1(9-8) + 2(6-0) = 3$$

Property III: If the elements of a determinant are multiplied by k, the value of the determinant will also be multiplied by k. That is,

$$\begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property IV: If any row (or column) or a multiple of any row (or column) is added to or subtracted from another row (or column), the value of the determinant remains unchanged. For example,

$$A = \begin{vmatrix} 2 & -1 & 5 \\ 3 & 0 & -2 \\ -5 & -2 & 1 \end{vmatrix} = 2(0-4) + 1(3-10) + 5(-6+0) = -8 - 7 - 30 = -45$$

Now let us multiply the first column by 2 and then subtract from 2nd column as,

$$A_1 = \begin{vmatrix} 2 & (-1-4) & 5 \\ 3 & (0-6) & -2 \\ -5 & (-2+10) & 1 \end{vmatrix} = \begin{vmatrix} 2 & -5 & 5 \\ 3 & -6 & -2 \\ -5 & 8 & 1 \end{vmatrix} = 2(-6+16) + 5(3-10) + 5(24-30)$$

$$= 20 - 35 - 30$$

$$= -45$$

Property V: If any two rows (or columns) of a determinant are equal, the value of the determinant is zero. For example,

$$A = \begin{vmatrix} a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{vmatrix} = a_1(a_2c_3 - a_3c_2) - a_1(a_2c_3 - a_3c_2) + a_1(a_2a_3 - a_2a_3)$$

$$= a_1a_2c_3 - a_1a_3c_2 - a_1a_2c_3 + a_1a_3c_2 + a_2a_3a_1 - a_2a_3a_1$$

$$= 0$$

Property VI: If all the elements of any row or column of a matrix is zero, the value of the determinant is zero.

Property VII: If one row (or column) of a determinant is multiple of any row (or column), the value of the determinant will be zero. For example,

$$A = \begin{vmatrix} 2a & 2b \\ a & b \end{vmatrix} = 2ab - 2ab = 0$$



Check your progress:
Solve the determinants:

Q1. $\begin{vmatrix} 2 & -2 & 3 \\ 1 & 4 & -3 \\ 3 & 4 & 0 \end{vmatrix}$

Q2. $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$

Q3. $\begin{vmatrix} 5 & 6 & 1 \\ 2 & 3 & 0 \\ 7 & -3 & 0 \end{vmatrix}$

Q4. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

2.4 EXPANSION BY COFACTORS

Before we go for the finding the process of inverse of a matrix, it is essential to know about the minor and cofactor of a matrix or a determinant. The minor of an element of a determinant is the sub-determinant of one lower order which is obtained by deleting the row and column to which it belongs to. If the determinant is given by

$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, the minors of the elements are –

Minor of a_{11} is given by $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Minor of a_{12} is given by $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Minor of a_{13} is given by $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Minor of a_{21} is given by $M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

Minor of a_{22} is given by $M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

Minor of a_{23} is given by $M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

And so on.....

The Cofactor of an element is the product of $(-1)^{i+j}$ and the respective minor. Thus the cofactor of the element a_{11} is given by $C_{11} = (-1)^{1+1}M_{11} = M_{11}$. Similarly,

Cofactor of a_{12} is given by $C_{12} = (-1)^{1+2}M_{12} = -M_{12}$

Cofactor of a_{13} is given by $C_{13} = (-1)^{1+3}M_{13} = M_{13}$ .....and so on. Thus for the matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactor matrix $= \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

2.5 INVERSE OF A MATRIX : DEFINITION, PROPERTIES AND TECHNIQUE OF FINDING

Definition :

Let A and B any n-squared matrices such that $AB = BA = I_n$

Then B is called inverse of A and is denoted by A^{-1} i.e. $B = A^{-1}$

If A is a non-singular matrix, then the inverse of the matrix exists and it is denoted by A^{-1}. Inverse of a matrix A is defined as

$$A^{-1} = \frac{Adj(A)}{|A|}$$

Properties of the Inverse:

- The inverse of a matrix is unique.

  Let A be n-squared matrix whose inverse exists.

  If possible,let B and C be two inverses of A.

  Then $AB = BA = I_n$ (1)

  $AC = CA = I_n$ (2)

  For (1) $C(AB) = C\,I_n = C$ (3)

  For (2) $(CA)B = I_nB = B$ (4)

  $\because$ C(AB) = (CA)B

  $\because$ C = B

  which shows that an invertible matrix possesses a unique inverse.

- The necessary and sufficient condition for a square matrix A to possess inverse is that A is non-singular i.e. $|A| \neq 0$

Proof: Necessary: Let A be a square matrix of order n and B be its inverse.

Then $AB = I_n$

$\therefore |AB| = |I_n|$

$\Rightarrow |A||B| = 1$

$\Rightarrow |A| \neq 0$

Sufficient: Let $|A| \neq 0$, we define a matrix B so that $B = \frac{1}{|A|}(\text{adj.A.})$

Then $AB = A\frac{1}{|A|}\text{adj } A$

$= \frac{1}{|A|} A(\text{adj } A)$

$= \frac{1}{|A|}|A| I_n = I_n$

Similarly, $BA = \left(\frac{1}{|A|}\text{adj } A\right)A$

$= \frac{1}{|A|}(\text{adj } A)\, A = \frac{1}{|A|}.|A| I_n = I_n$

Thus, $AB = BA = I_n$

Hence, A is invertible and B, is the inverse of A.

$\therefore B = A^{-1} = \frac{1}{|A|}\text{adj } A$

- **Reversal Law for the Inverse of a Product:** If A, B be two n-rowed non-singular matrices,

  then AB is also non-singular and $(AB)^{-1} = B^{-1} A^{-1}$

  i.e. the inverse of the product is the product of the inverses in the reverse order.

  Proof: Let A and B be two non-singular matrices i.e. $|A| \neq 0$, $|B| \neq 0$,

  So their inverses exist.

  We have $|AB| = |A||B| \neq 0$

  Hence, the matrix AB is also invertible .

  Let A^{-1} and B^{-1} be the inverses of A and B respectively and

  $AA^{-1} = A^{-1}A = I_n$ and $B\ B^{-1} = B^{-1}B = I_n$

  Choosing $C = B^{-1} A^{-1}$ we get

$C(AB) = (B^{-1} A^{-1})(AB)$

$= B^{-1} (A^{-1}A)B$

$= B^{-1} I_n B$

$= B^{-1} B$

$= I_n$ (1)

Again, $(AB) C = (AB) (B^{-1} A^{-1})$

$= A(BB^{-1}) A^{-1}$

$= AI_nA^{-1}$

$= AA^{-1}$

$= I_n$ (2)

By (1) and (2) $C(AB) = (AB)C = I_n$

Hence, C is the inverse of AB

or $B^{-1} A^{-1}$ is the inverse of AB

Thus, $(AB)^{-1} = B^{-1} A^{-1}$

- The operation of transposing and inverting are commutative

  i.e. $(A')^{-1} = (A^{-1})'$

  Proof: Let A be a non-singular square matrix i.e. $|A| \neq 0$

  $\therefore$ A^{-1} exists

  We get $AA^{-1} = A^{-1}A = I_n$

  $\therefore (AA^{-1})' = (A^{-1}A)' = I'_n$

  $\Rightarrow (A^{-1})' A' = A' (A^{-1})' = I_n$

  which shows that $(A^{-1})'$ is the inverse of A'.

  Hence, $(A')^{-1} = (A^{-1})'$

Note: If A, B are two n-rowed non-singular matrices.

Then $(AB)^{-1} = B^{-1}A^{-1}$.

Technique of Finding Inverse of Matrix :

Let us understand the technique of finding inverse of matrix with some examples :

Example 3: Find the inverse of the matrix $A = \begin{bmatrix} 2 & 6 \\ 3 & 8 \end{bmatrix}$

Solution: Given, $A = \begin{bmatrix} 2 & 6 \\ 3 & 8 \end{bmatrix}$

Let us find the determinant value of the matrix A,

$|A| = \begin{bmatrix} 2 & 6 \\ 3 & 8 \end{bmatrix}$ = 16 – 18 = - $2 \neq 0$, therefore the A^{-1} exists.

Now, Cofactor matrix of A = $\begin{bmatrix} 8 & -3 \\ -6 & 2 \end{bmatrix}$

Therefore, Adj (A) = $\begin{bmatrix} 8 & -6 \\ -3 & 2 \end{bmatrix}$

Therefore, $A^{-1} = \frac{Adj(A)}{|A|} = \frac{\begin{bmatrix} 8 & -6 \\ -3 & 2 \end{bmatrix}}{|-2|} = \begin{bmatrix} -4 & 3 \\ 3/2 & -1 \end{bmatrix}$

Example 4: If A = $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 0 \end{bmatrix}$ find A^{-1}

Solution:

$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 0 \end{vmatrix}$ = 1 (0 – 12) – 0(0 – 3) + 1 (8 – 1) = - 4 $\neq$ 0, therefore the A^{-1} exists.

Now, the Cofactor matrix of A = $\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$

$= \begin{bmatrix} -12 & 3 & 7 \\ 4 & -1 & -4 \\ -1 & -1 & 1 \end{bmatrix}$

Therefore, the Adj (A) $= \begin{bmatrix} -12 & 4 & -1 \\ 3 & -1 & -1 \\ 7 & -4 & 1 \end{bmatrix}$

Therefore, $A^{-1} = \frac{\begin{bmatrix} -12 & 4 & -1 \\ 3 & -1 & -1 \\ 7 & -4 & 1 \end{bmatrix}}{-4} = \begin{bmatrix} 3 & -1 & 0.25 \\ -0.75 & 0.25 & 0.25 \\ -1.75 & 1 & -0.25 \end{bmatrix}$

Note that $AA^{-1} = I$, that is, if we pre-multiply the inverse of a matrix by itself, we get an identity matrix.

Example 5: Find the value of x_1, x_2, & x_3 by using matrix inverse method.

$2x_1 - 4x_2 + 3x_3 = 3$

$4x_1 - 6x_2 + 5x_3 = 2$

$-2x_1 + x_2 - x_3 = 1$

Solution :

To find the values of x_1, x_2, and x_3 with matrix method, let us arrange the above system of equations into matrix form as,

$$\begin{bmatrix} 2 & -4 & 3 \\ 4 & -6 & 5 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Let, $A = \begin{bmatrix} 2 & -4 & 3 \\ 4 & -6 & 5 \\ -2 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} and\ B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Therefore, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = X = A^{-1}B$

Now, to find A^{-1},

$$|A| = \begin{vmatrix} 2 & -4 & 3 \\ 4 & -6 & 5 \\ -2 & 1 & -1 \end{vmatrix} = 2\ (6 - 5) + 4\ (-4 + 10) + 3\ (4 - 12) = 2 + 24 - 24 = 2$$

$$\text{The cofactor matrix of A} = \begin{bmatrix} \begin{vmatrix} -6 & 5 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 4 & 5 \\ -2 & -1 \end{vmatrix} & \begin{vmatrix} 4 & -6 \\ -2 & 1 \end{vmatrix} \\ -\begin{vmatrix} -4 & 3 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ -2 & 1 \end{vmatrix} \\ \begin{vmatrix} -4 & 3 \\ -6 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 2 & -4 \\ 4 & -6 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -6 & -8 \\ -1 & 4 & 6 \\ -2 & 2 & 4 \end{bmatrix}$$

$$\text{Therefore, Adj (A)} = \begin{bmatrix} 1 & -1 & -2 \\ -6 & 4 & 2 \\ -8 & 6 & 4 \end{bmatrix}$$

$$\text{Therefore, } A^{-1} = \begin{bmatrix} 1 & -1 & -2 \\ -6 & 4 & 2 \\ -8 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & -1 \\ -3 & 2 & 1 \\ -4 & 3 & 2 \end{bmatrix}$$

$$\text{Therefore, } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & -1 \\ -3 & 2 & 1 \\ -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & -1-1 \\ -9 & +4+1 \\ -12+6+2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 \\ -4 \\ -4 \end{bmatrix}$$

Thus, $x_1 = -0.5$, $x_2 = -4$, and $x_3 = -4$

We can also solve same system of equation with the help of augmented matrix method, where the coefficient matrix is transformed into the diagonal form by suitable operations. But this method though is shorter, not so simple. A very simpler method for finding the value of variables with matrix algebra is Cramer's Rule.



Check your progress:
Find the inverse of the following matrices

$$\textbf{Q5.}\quad A=\begin{vmatrix} 0 & 4 & 2 \\ -1 & 6 & 1 \\ 3 & -2 & -5 \end{vmatrix}$$

$$\textbf{Q6.}\quad A=\begin{vmatrix} 1 & 1 & -3 \\ 2 & 5 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

2.6 CRAMER'S RULE

Let us consider a three variables three equation system of equation as,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Let us arrange the above system of equations as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}(A)}{|A|}$$

$$\text{Again, Adj (A)} = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

$$\text{Therefore, } A^{-1} = \frac{1}{|A|}\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

$$\text{Therefore, } \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Now, for X_1= $\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{32} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = c_{11}b_1 + c_{21}b_2 + c_{31}b_3$

$$= b_1 \begin{vmatrix} a_{22} & a_{23} \\ a_{22} & a_{23} \end{vmatrix} - b_2 \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + b_3 \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

(If we find the determinant value of this, we will get the above process)

$$\text{Let, } \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = |A_1|$$

$$\text{Thus, } x_1 = \frac{|A_1|}{|A|}$$

Similarly, we can find that,

$$X_2 = \frac{|A_2|}{A} \text{ Where } |A_2| = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \text{ and}$$

$$X_3 = \frac{|A_2|}{A}, \text{Where } |A_3| = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

From the above process, it can be easily observed that, for finding X_1, we have to just replace first column of the coefficient matrix by the constant column vector and to find the determinant value to find the A_1. Then if we divide A_1 determinant by the determinant of coefficient matrix, we can have the value of x_1. Similarly, if we 2nd column of the coefficient matrix is replaced by constant column vector and follow the same procedure, we get the value of x_2. Similarly by replacing 3rd column of coefficient matrix by constant vector, and follow the same procedure, we get x_3. The same procedure can be followed for a generalised system of equations of n order. An example will make clear the rule.

Example 6: Solve by Cramer's Rule

x + 6y - z = 10
2x + 3y + 3z = 17
3x - 3y - 2z = -9

Solution: Let us arrange the equation in matrix form

$$\begin{bmatrix} 1 & 6 & -1 \\ 2 & 3 & 3 \\ 3 & -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 17 \\ -9 \end{bmatrix}$$

A X B

Therefore, $$X = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 10 & 6 & -1 \\ 17 & 3 & 3 \\ -9 & -3 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 6 & -1 \\ 2 & 3 & 3 \\ 3 & -3 & -2 \end{vmatrix}} = \frac{10(-6+9)-6(-34+27)-1(-51+27)}{1(-6+9)-6(-4-9)-1(-6-9)} = \frac{96}{96} = 1$$

$$Y = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 10 & -1 \\ 2 & 17 & 3 \\ 3 & -9 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 6 & -1 \\ 2 & 3 & 3 \\ 3 & -3 & -2 \end{vmatrix}} \frac{1(-34+27)-10(-4-9)-1(-18+51)}{1(-6+9)-6(-4-9)-1(-6-9)} = \frac{192}{96} = 2$$

$$Z = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 6 & -10 \\ 2 & 3 & 17 \\ 3 & -3 & -9 \end{vmatrix}}{\begin{vmatrix} 1 & 6 & -1 \\ 2 & 3 & 3 \\ 3 & -3 & -2 \end{vmatrix}} \frac{1(-27+51)-6(-18+51)+10(-6-9)}{1(-6+9)-6(-4-9)-1(-6-9)} = \frac{288}{96} = 3$$



Check your progress:

Solve the following using matrix inversion method/ Cramer's Rule

Q7. $3x + 2y = 7$

$5x - y = 3$

Q8. $x_1 - x_3 = 4$

$2x_2 - x_3 = 6$

$x_1 + x_2 = 10$

2.7 SOLUTION OF PARTIAL EQUILIBRIUM MARKET MODEL, NATIONAL INCOME MODEL, GOODS MARKET AND MONEY MATKET EQUILIBRIUM

2.7.1 Solution of linear partial equilibrium market model

Let us consider a simple linear partial equilibrium market model as,

$Q_d = a - bP$ (Demand function, where a and b are intercept and slope of the demand function)

$Q_s = - c + dP$ (Supply function, where c and d are intercept and slope of the supply function)

$Q_d = Q_s$ (Equilibrium condition)

To find the equilibrium price and output, Instead of ordinary algebra, we may solve this market model with matrix algebra. For this, let us rearrange the above system of equation as,

$$Q_d + bP = a$$
$$Q_s - dP = -c$$
$$Q_d - Q_s = 0$$

Or,

$$Q_d + 0 \times Q_s + bP = a$$
$$0 \times Q_d + Q_s - dP = - c$$

$Q_d - Q_s + 0 \times P = 0$

Now we can arrange the above system in matrix form as,

$$\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} = \begin{bmatrix} a \\ -c \\ 0 \end{bmatrix}$$

A X B

i.e. AX = B

Now,

$$|A| = \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix} = 1(0-d) - 0(0+d) + b(0-1) = -b - d = -(b+d)$$

Using Cramer's Rule,

$$Q_d = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} a & 0 & b \\ -0 & 1 & -d \\ 0 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix}} = \frac{a(0-d) - 0(0+0) + b(c+0)}{-(b+d)} = \frac{-ad+bc}{-(b+d)} = \frac{ad-bc}{b+d}$$

$$Q_s = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & a & b \\ 0 & -c & -d \\ 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix}} = \frac{1(0+0) - a(0+d) + b(0+6)}{-(b+d)} = \frac{-ad+bc}{-(b+d)} = \frac{ad-bc}{b+d}$$

$$P=\frac{|A_3|}{|A|}=\frac{\begin{vmatrix}1 & 0 & a\\ 0 & 1 & -c\\ 1 & -0 & 0\end{vmatrix}}{\begin{vmatrix}1 & 0 & b\\ 0 & 1 & -d\\ 1 & -1 & 0\end{vmatrix}}=\frac{1(0-c)-0(0+c)+a(0-1)}{-(b+d)}=\frac{-(a+c)}{-(b+d)}=\frac{a+c}{b+d}$$

Thus, equilibrium output and price are,

$$Q_d=Q_s=\frac{ad-bc}{b+d} \text{ and } P=\frac{a+c}{b+d}$$

We can also use the matrix inversion method to get the same result.

2.7.2 Solution of National Income Model

Let us consider a simple national income model

$Y = C + I_0 + G_0$

$C = a + bY$

Where Y is national income, C is consumption, I_0 is autonomous investment and G_0 is the Government Expenditure. In the Consumption function, a denotes the autonomous consumption and b is the marginal propensity to consume. To find the equilibrium national income (Y) and consumption (C), we can use the matrix algebra. For that, let us rearrange the above system of equations as

$Y - C = I_0 + G_0$

$-bY + C = a$

In matrix form,

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}\begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

That is $A = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}, X = \begin{bmatrix} Y \\ C \end{bmatrix}$, and $B = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix} = 1 - b$$

$$Y = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} I_0 + G_0 - 1 \\ a \quad 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{I_0 + G_0 + a}{1 - b}$$

$$C = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & I_0 + G_0 \\ -b & a \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{b(I_0 + G_0) + a}{1 - b}$$

2.7.3 Equilibrium of the economy with Good Market and Money Market

Another application of matrix algebra we can see in an economy made up of two sectors: real good sector and the monetary sector which can be presented as below,

Goods market

$Y = C + I$ ..... (i)

$C = \alpha + \beta Y$ ($\alpha > 0$; $0 < \beta < 1$) ..... (ii)

$I = a + br$ ($a > 0$; $0 < b < 1$) ..... (iii)

Where Y is national income, C is consumption, I is investment, r is rate of interest. α, β, a and b are behavioural constants with appropriate functions.

Where Y is national income, C is consumption, I is investment, r is rate of interest. α, β, a and b are behavioural constants with appropriate functions.

Money Market

$M_d = M_s$ ..... (iv)

$M_d = \gamma + \delta Y + \lambda r$ (where $\gamma, \delta > 0; 0 < b < 1$) ..... (v)

$Ms = M_0$ ..... (vi)

Where, M_d and M_s denote demand for and supply of money respectively. δ, γ, and λ are constants. In money market equilibrium, $M_s = M_0$ indicates that money supply is constant.

Substituting (ii) and (iii) in (i)

$Y = (\alpha + \beta Y) + a + br$

Or, $Y - \beta Y - br = a + \alpha$

Or, $(1 - \beta)Y - br = a + \alpha$ .....(vii)

Again, substituting (v) and (vi) in (iv)

$\gamma + \delta Y + \lambda r = M_0$

Or, $\delta Y + \lambda r = M_0 - \gamma$ .....(viii)

Now, Let us arrange equations (vii) and (viii) in matrix form.

$$\begin{bmatrix} 1-\beta & -b \\ \delta & \lambda \end{bmatrix} \begin{bmatrix} y \\ r \end{bmatrix} = \begin{bmatrix} a+ & \alpha \\ M_0 - \gamma \end{bmatrix}$$

$$\text{Here, } A = \begin{bmatrix} 1-\beta & -b \\ \delta & \gamma \end{bmatrix}, X = \begin{bmatrix} y \\ r \end{bmatrix}, \text{and } B \begin{bmatrix} a+\alpha \\ M_0 - \gamma \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1-\beta & -b \\ \delta & \gamma \end{bmatrix} = \gamma(1-\beta) + b\delta$$

Using Cramer's Rule,

$$y = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} a+\alpha & -b \\ M_0-\gamma & \lambda \end{vmatrix}}{\begin{vmatrix} 1-\beta & -b \\ \delta & \lambda \end{vmatrix}} = \frac{\lambda(a+\alpha)+b(M_0-\gamma)}{\lambda(1-\beta)+b\delta}$$

$$r = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1-\beta & a+\alpha \\ \delta & M_0-\gamma \end{vmatrix}}{\begin{vmatrix} 1-\beta & -b \\ \delta & \gamma \end{vmatrix}} = \frac{(1-\beta)(M_0-\gamma)-\delta(a+\alpha)}{\lambda(1-\beta)+b\delta}$$

Thus, we can find the equilibrium national income and rate of interest by using the matrix algebra we may use any one of Cramer's Rule and matrix inversion method for solving such models.



2.8 LET US SUM UP

- Matrix algebra is useful to solve the linear simultaneous equation systems
- Cramer's Rule can also be used to deal with such system of equations where there is equal number of equations with the number of unknowns to be determined.
- Using Matrix Inversion method and Cramer's rule, solution can be found of different market models and goods and money market.
- These two methods can also solve national income models
- Solution of Market Models, National Income Model, Goods market and Money market equilibrium can also be found by using Cramer's rule.



2.9 FURTHER READING

1) Mehta, Madnani (1996). *Mathematics for Economics*. S. Chand & Sons.
2) Chiang A.C., Wainwrite K (2013). *Fundamental Methods of Mathematical Economics.* McGraw Hill Education (India) Pvt. Ltd.

3) Monga G.S. (2003). *Mathematics & Statistics for Economics*. Vishal Publishing House Pvt. Ltd.

4) Baruah S (2012). *Basic Mathematics & Its Application in Economics*. Macmillan Publishers India Ltd.

2.10 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q. No. 1: 27

Ans to Q. No. 2: 2

Ans to Q. No. 3: 27

Ans to Q. No. 4: $3abc - a^3-b^3-c^3$

Ans to Q. No. 5:
$$\begin{bmatrix} 0.70 & -0.40 & 0.20 \\ 0.05 & 0.15 & 0.05 \\ 0.40 & -0.30 & -0.10 \end{bmatrix}$$

Ans to Q. No. 6:
$$\begin{bmatrix} 7 & -11 & 16 \\ -3 & 5 & -7 \\ 1 & -2 & 3 \end{bmatrix}$$

Ans to Q. No. 7: x=1, y=2

Ans to Q. No. 8: $x_1=6$, $x_2=4$, $x_3=2$

2.11 MODEL QUESTIONS

1. Solve the following system of equation by matrix inversion

$$2x_1 + x_2 + 3x_3 = 15$$
$$x_1 - 2x_2 + 5x_3 = 13$$
$$4x_1 + 3x_2 - x_3 = 11$$

2. Given the market model

$$D = 50 - 2P$$
$$S = -10 + 3P$$
$$D = S$$

Find the equilibrium price and output using matrix algebra.

*** ***** ***

UNIT 3: DIFFERENTIATION

UNIT STRUCTURE

3.1 Learning Objectives

3.2 Introduction

3.3 Rate of Change and Derivative

3.4 Economic Interpretation of derivative

3.5 Rules of differentiation with one explanatory variable

3.6 Rules of differentiation involving two or more functions of the same explanatory variable

3.7 Rules of differentiation involving functions of different variables

3.8 Second and higher order differentiation

3.8.1 Interpretation of relation between 1st order and 2nd order derivative

3.8.2 Maximum and minimum values of a function

3.9 Let us sum up

3.10 Further reading

3.11 Answers to Check Your Progress

3.12 Model Questions

3.1: LEARNING OBJECTIVES:

After going through this unit, you will be able to:

- define rate of change and derivative
- understand the economic interpretation of derivative
- find the derivative of a function
- apply the rules of differentiation.

3.2: INTRODUCTION

In case of functional relation between two or more variables, it is important to know the rate of change in the dependent variable when there is change in the independent variable. Especially when the change is to be measured in case of a continuous function and changes keep changing

along the movement, it becomes essential to examine about very small change. For such movements in the variables, the technique of differential calculus is very useful tool. Thus, differentiation is a mathematical tool to measure the rate of change in dependent variable with respective change in the independent variable. For a function,

Y = f (X), where Y is dependent variable and X is independent or explanatory variable.

If we are to see the change in Y as X changes, we may use the differentiation and it is symbolically denoted as $\frac{\Delta y}{\Delta x}$ or $\frac{dy}{dx}$ or f'(X). Thus, differentiation may be defined as the ratio of change in dependent variable to change in independent variable.

In economics the techniques of differentiation is useful mostly in finding the marginal functions from total functions such as Marginal revenue from Total revenue, Marginal cost from Total cost, elasticity of demand, slope of the demand curve, in optimization to find maximum profit or minimum cost, etc. Graphically, since it represents the slope at a particular point of the curve representing the function, the result of derivative shows the direction of movement of the curve. The negative value of the differential coefficient indicates an inverse relationship between the dependent and independent variable and vice versa. Again, the differential coefficient zero means zero slope of the curve which indicates the particular point is either maximum or minimum point of the curve. Thus, in different field of economics and geometry or numerical science, the technique of differentiation is applied.

3.3 RATE OF CHANGE AND DERIVATIVE

Differentiation technique is useful in finding the rate of change in dependent variable with respect to change in the independent variable especially when we have to examine the average change for a function having changing sloped curve. It is possible to find the rate of change with

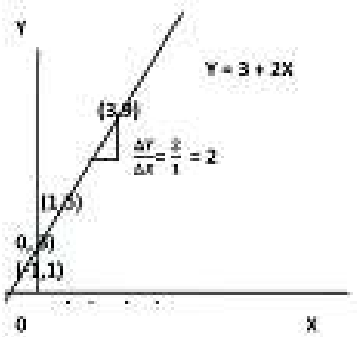
simple algebra when the curve is linear or in other words, the rate is constant throughout the curve. For example if the function is given by,

$Y = f(X) = 3 + 2X$

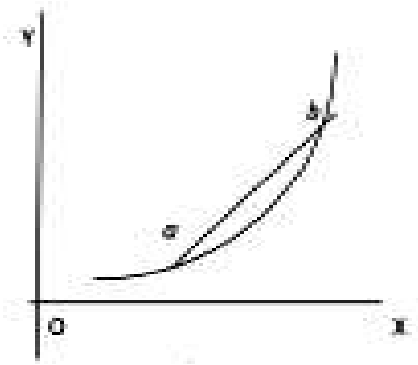
For the above function, Y values corresponding to X values are:

X	0	1	2	3	-1
Y	3	5	7	9	1

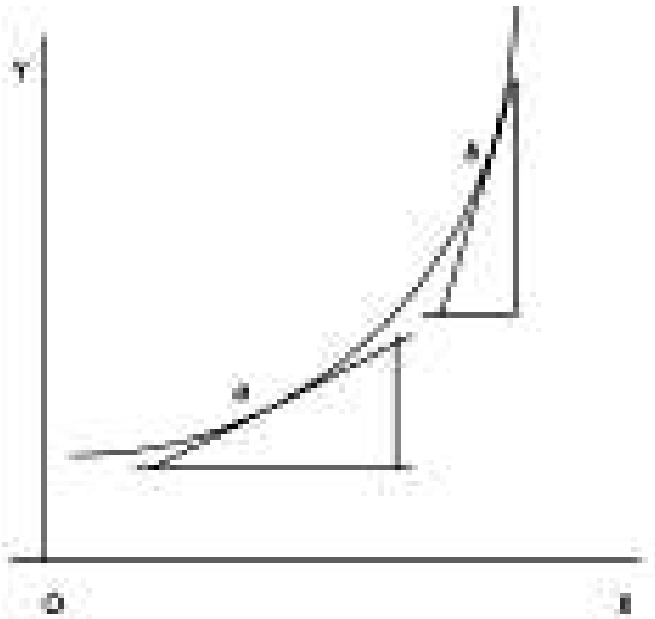
If we plot the values on graph,



Here rate of change $\frac{\Delta y}{\Delta x} = \frac{2}{1} = 2$ for all the points. However, if the curve is not linear, slope of the curve is different at different points such as a and b in the figure.



The slope at a particular point is the slope of the tangent drawn for the particular point as shown in the figure below. There are infinite numbers of points throughout the curve where slopes will be different. To draw tangent for each point and averaging for all is difficult task.



Therefore, for a non linear curve shown above if we have to find the rate of change while moving from the point a to b, we may take the help of limit assuming the distance as much as minimum and hence taking the limit $\Delta x \to 0$, we can find the dY/dX, which is the derivative of Y with respect to X.

For the function, Y = f (X), ---------- (1)

Let us assume as a result of change in X by Δ X, there is a change in Y by Δ Y. therefore,

$$Y + \Delta Y = f(X + \Delta X)$$
$$\Delta Y = f(X + \Delta X) - Y$$
$$\Delta Y = f(X + \Delta X) - f(X)$$

Now, dividing both sides by Δ X,

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Taking limit $\lim_{\Delta x \to 0}$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(X + \Delta x) - f(X)}{\Delta x}$$

However, with some proven rules, we can find the values of $\frac{dy}{dx}$ without taking the limit which will be discussed in the coming sub section of the chapter. Though there are number of rules of differentiation for algebraic treatment, all are not relevant in economics, e.g. rules or techniques applying trigonometry are not very much relevant in the field of economics. Therefore, only the normal algebraic and simple logarithmic rules will be focussed in the chapter.

3.4 ECONOMIC INTERPRETATION OF DERIVATIVE

Derivatives are used in various aspects of economics. Since it is the rate of change in dependent variables with respect independent variable, the slope of the curve is given by the derivatives. Derivatives help us in findings various marginal function given the total function such as MR from TR, MC from TC, MPC from Consumption function, slope and curvature of a curve, etc. That is Any Marginal function is the value of derivative of the total function. For example,

$$MR = \frac{d}{dx}(TR)$$

$$\text{Similarly, } MC = \frac{d}{dx}(TC)$$

The technique of differentiation can also be used to find the optimum values. With first order and higher order differentiation, it is possible to know the movement of the AC, MC, MU, etc. And then by some simple analysis, the point of maxima or minima can be determined which helps in finding profit maximising output, price, maximum profit, conversely the cost minimising output, input combination, etc. The technique of differentiation is also used in analysing and measuring elasticity which have great

implication in the advanced economic theory. At macro level, the technique of differentiation can be applied for getting the marginal propensity to consume (MPC) from consumption function, MPS from saving function, and also the marginal propensity to import, export ect. Moreover, to examine the impact of rate of taxation on the tax revenue of the Government, differentiation techniques are used widely.

3.5 RULES OF DIFFERENTIATION WITH ONE EXPLANATORY VARIABLE

In the following we are discussing the rules of differentiation with one explanatory variable:

- **Derivative of a constant:**

Derivative of a constant is zero. If the function is given by

$Y = f(X) = c$, where c is a constant

$$\frac{dy}{dx} = f'(X) = \frac{d}{dx}(C) = 0$$

It indicates that Y = c irrespective of value of X. It can be proved as follows:

If $Y = f(X) = c$

Let the values of X is a, then,

$Y = f(a) = c$. By the definition of derivative

$$\frac{dy}{dx} = f'(a) = \lim_{x \to a} \frac{f(a) - f(X)}{x - a}$$

$$= \lim_{x \to a} \frac{c - c}{x - a}$$

$$= 0$$

Example 1: If the function is given by,

$Y = f(x) = 20$

$$\frac{dy}{dx} = \frac{d}{dx}(20) = 0$$

- **Power function Rule:**

If a function is defined as, $Y = f(X) = X^n$, the derivative will be

$\frac{dy}{dx} = f'(X) = nX^{n-1}$ (power of the variable will be coefficient and new power of the variable will be obtained by subtracting 1 from the original power).

For example,

If $Y = X^6$,

$$\frac{dy}{dx} = f'(X) = \frac{d}{dx}(X^6) = 6X^{6-1} = 6X^5$$

Example 2:

$Y = 2x$

$$dy/dx = \frac{d}{dx}(2x)$$

$$= 2.1x^{1-1}$$

$$= 2x^0 = 2$$

LET US KNOW



Proof of the power function Rule of differentiation:

If $y = f(x) = x^n$, then , $\frac{dy}{dx} = nx^{n-1}$

Basic Rule of differentiation is given by, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(X + \Delta x) - f(X)}{\Delta x}$

$Y = x^n$

$$\Rightarrow Y + \Delta y = (X + \Delta x)^n$$
$$\Rightarrow \Delta y = (X + \Delta x)^n - Y$$
$$\Rightarrow \Delta y = (X + \Delta x)^n - X^n$$

Dividing both sides by Δx and talking limit Δx tends to 0, we get,

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(X+\Delta x)^n-x^n}{\Delta x}$$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{x^n+nx^{n-1}.\Delta x+\frac{n(n-1)}{2!}x^{n-2}.(\Delta x)^2+.....+(\Delta x)^n-x^n}{\Delta x}$$

{Using Binomial expansion for $(x+\Delta x)^n$}

$$=\lim_{\Delta x\to 0}\frac{nx^{n-1}.\Delta x+\frac{n(n-1)}{2!}x^{n-2}.(\Delta x)^2+.....+(\Delta x)^n}{\Delta x}$$

$$=\lim_{\Delta x\to 0}nx^{n-1}+\frac{n(n-1)}{2!}x^{n-2}.(\Delta x)+.....+(\Delta x)^{n-1}$$

Taking the limiting value of $\Delta x\to 0$

$$\frac{dy}{dx}=nx^{n-1}+\frac{n(n-1)}{2!}x^{n-2}.(0)+.....+(0)^{n-1}$$

$$\frac{dy}{dx}=nx^{n-1}$$

- **Logarithmic function Rule:**

If $y=\log x$

$dy/dx=1/x$

$$\Rightarrow y+\Delta y=\log(x+\Delta x)$$
$$\Rightarrow \Delta y=\log(x+\Delta x)-y$$
$$\Rightarrow \Delta y=\log(x+\Delta x)-\log x$$
$$=\log\frac{x+\Delta x}{x}$$
$$=\log\left(1+\frac{\Delta x}{x}\right)$$

$$=\frac{\Delta x}{x}-\frac{\left(\frac{\Delta x}{x}\right)^2}{2}+\frac{\left(\frac{\Delta x}{x}\right)^3}{3}+.....\{\ Since\ \log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+.....\ \}$$

Now dividing both sides by Δx

$$\frac{\Delta y}{\Delta x}=\frac{1}{x}-\frac{\Delta x}{2x^2}+\frac{\Delta x^2}{3x^3}+.....$$

Taking limit $\Delta x\to 0$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{0}{2x^2} + \frac{0^2}{3x^3} + \ldots\ldots\ldots\ldots$$
$$= \frac{1}{x} - 0 + 0 \ldots\ldots\ldots$$

$or,\quad \frac{dy}{dx} = \frac{1}{x}$

- **Exponential function Rule:**

If the function is given by,

$Y = e^x$

$dy/dx = e^x$

Proof

$$y = e^x$$
$$y + \Delta y = e^{(x+\Delta x)}$$
$$\Delta y = e^{(x+\Delta x)} - y$$
$$\Delta y = e^{(x+\Delta x)} - e^x$$
$$= e^x (e^{\Delta x} - 1)$$
$$= e^x (1 + \Delta x + \frac{\Delta x^2}{2} + \ldots\ldots\ldots\ldots\ldots - 1)$$
$$= e^x (\Delta x + \frac{\Delta x^2}{2} + \ldots\ldots\ldots\ldots\ldots ..)$$

Dividing both sides by Δx and taking lim $\Delta x \to 0$

$$\frac{dy}{dx} = e^x \lim_{\Delta x \to 0} (1 + \frac{\Delta x}{2} + \ldots\ldots\ldots)$$
$$= e^x (1 + 0 + \ldots\ldots\ldots)$$

$\text{Or,}\quad \frac{dy}{dx} = e^x$

In case of exponential function, coefficient of the independent variable which is in the exponent will come in normal coefficient. That is,

$\text{If}\quad y = e^{ax}, \text{then}$
$dy/dx = ae^{ax}$

Example 3 :

$$y = e^{-2x}$$

$$dy/dx = \frac{d}{dx}(e^{-2x}) = -2e^{-2x}$$



CHECK YOUR PROGRESS

Find the derivatives of the following:

Q1. $y = \sqrt{x}$

Q2. $y = 4x^{-5}$



LET US KNOW

Differential coefficient $\frac{dy}{dx}$ or $f'(x)$ means the limit of the ratio of change in dependent variable Δy to change in independent variable Δx. Thus, $\frac{dy}{dx} = \lim_{x \to 0} \frac{\Delta y}{\Delta x}$. The value of differential coefficient can indicate the nature of the function.

- If $\frac{dy}{dx} > 0$, the curve rises from left to right and $\frac{dy}{dx} < 0$, the curve is falling, $\frac{dy}{dx} = 0$, the curve has zero slope.
- More the value of $\frac{dy}{dx}$, more will be the slope of the curve whether it is downwards or upwards sloping.
- If $\frac{dy}{dx} = 0$, the curve is a horizontal straight line to the x axis.
- If $\frac{dy}{dx} = \infty$, the curve is vertical to the x-axis.
- If at a particular point, $\frac{dy}{dx} = 0$, it means the point is optimum, either maximum, or minimum, which again indicative by the result of the value of second order differentiation.

3.6 RULES OF DIFFERENTIATION INVOLVING TWO OR MORE FUNCTIONS OF THE SAME EXPLANATORY VARIABLE

In the following we are discussing the rules of differentiation involving two or more functions of the same explanatory variable.

- **Sum difference Rule:**

When two or more functions with same independent variable are stated as sum or subtraction, then derivative of the sum or difference are also the sum or difference of the individual derivatives. That is,

If y = U+V+W, then

$$dy/dx = \frac{d}{dx}(U) + \frac{d}{dx}(v) + \frac{d}{dx}(W)$$

Example: 4 $y = 1 + 2x - x^2$, Find dy/dx.

Solution:

Given, $y = 1 + 2x - x^2$

$$\begin{aligned} dy/dx &= \frac{d}{dx}(1) + \frac{d}{dx}(2x) + \frac{d}{dx}(x^2) \\ &= 0 + 2 + 2x \\ &= 2 + 2x \end{aligned}$$

- **Product Rule of differentiation:**

When a function is given by the product of two separate functions having same variable, such that, Y = U.V,

The derivative of such function is obtained as,

$$\frac{dy}{dx} = U\ \frac{d}{dx}(V) + V\ \frac{d}{dx}(U)$$

For example, let us suppose a function is given by,

$$Y = (4x^2 + 2x)(8x^3 + 3x^2)$$

$$\frac{dy}{dx} = (4x^2 + 2x)\ \frac{d}{dx}(8x^3 + 3x^2) + (8x^3 + 3x^2)\ \frac{d}{dx}\ (4x^2 + 2x)$$

$$= (4x^2 + 2x)(24x^2 + 6x) + (8x^3 + 3x^2)(8x + 2)$$

Example : 5 Differentiate the following

(i) $Y = X^2(x-3)$

(ii) $Y = (2x + 5)(3x^2)$

Solution:

(i) Given,

$Y = X^2(x-3)$

$$\frac{dy}{dx} = x^2\ \frac{d}{dx}(x-3) + (x - 3)\ \frac{dy}{dx}\ (x^2)$$

$$= x^2.\ 1 + (x - 3)(2x)$$

$$= x^2 + 2x^2 - 6x$$

$$= 3x^2 - 6x$$

(ii) Given,

$Y = (2x + 5)(3x^2)$

$$\frac{dy}{dx} = (2x + 5)\ \frac{d}{dx}(3x^2) + 3x^2\ \frac{d}{dx}(2x + 5)$$

$$= (2x + 5)6x + 3x^2.\ (2 + 0)$$

$$= 12x^2 + 30x + 6x^2$$

$$= 18x^2 + 30x$$

- **Quotient Rule of differentiation:**

If two functions with same independent variables are related in such a way that one is divided by another as,

$Y = U/V$, then

$$\frac{dy}{dx} = \frac{v\frac{d}{dx}(U) - U\frac{d}{dx}(V)}{v^2}$$

Example: 6 Find $\frac{dy}{dx}$ of the following

(i) $y = \frac{x+2}{2x+3}$

(ii) $y = \frac{3x+4}{e^x+2}$

Solution:

(i) Given, $Y = \frac{x+2}{2x+3}$

$$\frac{dy}{dx} = \frac{(2x+3)\frac{d}{dx}(x+2) - (x+2)\frac{d}{dx}(2x+3)}{(2x+3)^2}$$

$$= \frac{(2x+3)1 - (x+2)2}{(2x+3)^2}$$

$$= \frac{-1}{(2x+3)^2}$$

(ii) Given, $Y = \frac{3x+4}{e^x+2}$

$$\frac{dy}{dx} = \frac{(e^x+2)\frac{d}{dx}(3x+4) - (3x+4)\frac{d}{dx}(e^x+2)}{(e^x+2)^2}$$

$$= \frac{(e^x+2)(3) - (3x+4)e^x}{(e^x+2)^2}$$

$$= \frac{(3e^x+6) - e^x 3x - 4e^x}{(e^x+2)^2}$$

$$= \frac{-e^x - e^x 3x + 6}{(e^x+2)^2}$$

3.7 RULES OF DIFFERENTIATION INVOLVING FUNCTIONS OF DIFFERENT VARIABLES

In the following we are discussing the rules of differentiation involving functions of different variables –

- **Chain Rule of differentiation (Differentiation of the function of a function):**

When the dependent variable, say Y, is a function of U and U is function of X, for such types of functions, derivative of y with respect to x, that is $\frac{dy}{dx}$ is given by,

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Similarly, If Y = f(U), U= f(V) and v= f(X), then,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

Example: 7

(i) $Y = (3x + 2)^4$

(ii) $Y = e^{5x^2+2x+2}$

Solution:

(i) Given, $y = (3x + 2)^4$

Let us suppose, $U = 3x + 2$

Therefore, $y = U^4$

$\frac{dy}{dU} = 4U^3$, Again $\frac{dU}{dx} = 3$

Using Chain Rule of differentiation,

$$\frac{dy}{dx} = \frac{dy}{dU} \cdot \frac{dU}{dx}$$

$= 4U^3 . 3$

$= 12U^3$

$= 12(3x + 2)^3$

(ii) $Y = e^{5x^2+2x+2}$

Let us suppose, $U = 5x^2 + 2x + 2$

Therefore, $y = e^U$

$\frac{dy}{du} = e^U$, again $\frac{du}{dx} = 10x + 2$

Using Chain Rule of differentiation,

$$\frac{dy}{dx} = \frac{dy}{dU} \cdot \frac{dU}{dx}$$

$= e^u(10x + 2)$

$= (e^{5x^2 + 2x + 2})(10x + 2)$

Chain Rule of differentiation is mainly useful when the function is sum or subtracted within bracket and then whole expressions are powered with more than two or three (e.g. y = y = $(3 + 2x^2)^4$ or the expression is within the root (e.g. y = $\sqrt{x^2 + 2}$) or logarithms.

- **Differentiation of an Implicit function:**

Sometimes the function may be in such a form that it is not explicitly indicated which is dependent and which is independent variable such that F(x, y) = 0. Such functions are called implicit function. How the derivative $\frac{dy}{dx}$ is obtained will clear from the below example.

Given the function, $x^2 - y^2 + 3x = 5y$

Differentiating both sides with respect to x, we get,

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) + \frac{d}{dx}(3x) = \frac{d}{dx}(5y)$$

$$\Rightarrow 2x - 2y\frac{dy}{dx} + 3 = 5\frac{dy}{dx}$$

$$\Rightarrow 5\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x + 3$$

$$\Rightarrow \frac{dy}{dx}(2y + 5) = 2x + 3$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x + 3}{2y + 5}$$

Differentiation of implicit function can also be found with the help of total differential as follows:

$$\frac{dy}{dx} = \frac{\delta F}{\delta x}.dx + \frac{\delta F}{\delta x}dy = \frac{d}{dx}(0)$$

$$= \frac{Fx}{Fy}$$

Example: 8 $2x^2 - x^2y + 10 = 0$

$Fx = 4x - 2xy$

$Fy = -x^2$

Therefore, $\frac{dy}{dx} = -\frac{4x - 2xy}{-x^2}$

Now, $2x^2 - x^2y + 10 = 0$

$\Rightarrow x^2y = 2x^2 + 10$

$\Rightarrow y = \left(\frac{2x^2 + 10}{x^2}\right)$

Putting the value of y

$$\frac{dy}{dx} = \frac{4x - 2x(\frac{2x^2 + 10}{x^2})}{-x^2}$$

$$= \frac{4x^2 - 4x^2 - 20}{x^3}$$

$$= -20/x^3$$



CHECK YOUR PROGRESS

Q3. If $y = \frac{2x+5}{x^2-3}$ then find $\frac{dy}{dx}$

Q4. If $y = \log(5 - 2x + 3x^2)$ then find $\frac{dy}{dx}$

3.8 SECOND AND HIGHER ORDER DIFFERENTIATION

The process of differentiation has been discussed so far i.e. $\frac{dy}{dx}$ is the process of examining the rate of change in dependent variable as a result of change in independent variable. However, if the result of the differentiation at the first instant is carried forward and again we take the same procedure of differentiation, we get the higher order differentiation. The 2nd order derivative is the derivative of the derivative of the function and denoted as f''(x) or $\frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2}$

Example : 9 Given the function, $y = 5x^4 + 2x^3$

1st order derivative, $f'(x) = \frac{dy}{dx} = \frac{d}{dx}(5x^4 + 2x^3)$

$$= 20x^3 + 6x^2$$

2nd order derivative, $f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(20x^3 + 6x^2)$

$$= 60x^2 + 12x$$

3rd order derivative, $f'''(x) = \frac{d^3y}{dx^3} = \frac{d}{dx}(\frac{d^2y}{dx^2}) = \frac{d}{dx}(60x^2 + 12x)$

$$= 120x + 12$$

4th order derivative, $f''''(x) = \frac{d^4y}{dx^4} = \frac{d}{dx}(\frac{d^3y}{dx^3}) = \frac{d}{dx}(120x + 12)$

$$= 120$$

5th order derivative, $f'''''(x) = \frac{d}{dx}(120) = 0$

3.8.1 Interpretation of relation between 1st order and 2nd order derivative

As we earlier discussed, the first order differentiation f'(x) measures the rate of change of the function f(x). Similarly, the 2nd order derivatives measure the rate of change of f'(x). In other words, first order derivative measures the rate of change in dependent variable as a result of an infinitesimal change in the independent variable. If we put the function graphically, the derivative of the function gives the slope of the curve drawn from the function. If dy/dx is positive, we can say that there is direct or positive relationship between the variables and therefore the slope of the curve for the function will be negative and vice versa. Thus, first order derivative says us the slope of the curve. On the other hand, 2nd order derivative tells us about the change in the slope of the curve or the curvature of the curve. Henceforth the 2nd order derivative of the linear function is zero indicating no change in the slope for a linear function or a straight line. However, for functions of two or more degree, 2nd or higher order derivatives measure the movement of change in the slope of the curve. That is,

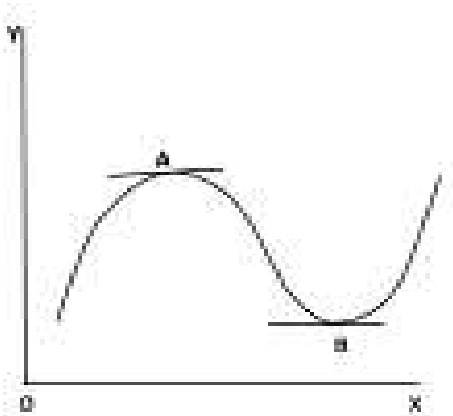
If $\frac{d^2y}{dx^2} > 0$, slope of the curve tends to increase or in other words, the rate of change is positive or increasing.

If $\frac{d^2y}{dx^2} < 0$, slope of the curve tends to decreasing indicating rate of change to be decreasing.

Thus 2nd order derivatives have significance in finding relative minima or maxima which is identified as the point where the curve changes its direction. These will be discussed in the economic application in details.

3.8.2 Maximum and minimum values of a function

Let us consider the points A and B in the following figure. At point A as well as B, the slope of the curve is zero, which indicates that both the points are extreme points, i.e. the function is either maximum or minimum at those points. However, If we move from the left from right, the curve moves downwards after the point A, but the curve moves upwards after the point B. That means, if we see the rate of change of the slope, after A, it has negative movement, while after B, it is positive. That is, $\frac{d^2y}{dx^2} < 0$, after A and $\frac{d^2y}{dx^2} > 0$, after B. From the figure, it can be observe that point A is maximum point of the curve while B is the minimum.



Thus, using 1st and 2nd order derivative technique we can put the conditions for maximum and minimum values of a function as,

1. $\frac{dy}{dx} = 0,$ known as the 1st order condition (both for maximization and minimizations)

2. $\frac{d^2y}{dx^2} < 0$ for maximization) and $\frac{d^2y}{dx^2} > 0$ for minimization

First order condition is also known as the necessary condition and 2nd order condition is known as the sufficient condition. After applying the first order condition, we can just say that the point will be either maximum or minimum point of the function, but it cannot be said whether it will be the maximum or minimum point. This can be said after using the second order or sufficient condition. It is to be noted that in certain cases ($\frac{d^2y}{dx^2} = 0$. For such cases, though the 1st order condition is satisfied, the function is neither maximised nor minimised. Such points are called the point of inflexion.

Example: 10 Find the maximum or minimum value of the function $y = 24x - 2x^2$

Solution: Condition for extreme value of the function, let us first apply the first order condition which requires,

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{d}{dx}(24x - 2x^2) = 0$$

$$\Rightarrow 24 - 4x = 0$$

$$\Rightarrow > x = 6$$

But still we cannot say whether x = 6 will maximize or minimize the function. For that let us verify the second order condition.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(24 - 4x)$$

$$= -4 < 0$$

The second order condition fulfils the sufficient condition for maximization. Therefore, x = 6 will maximize the function. Putting x = 6 in the function we can have the maximum value of the function as,

$y = 24 \text{ X } 6 - 2 \text{ X } 6^2 = 144 - 72 = 72$

LET US KNOW



- A linear function's 2nd order derivative is equal to zero i.e. the variable exhausted with the 1st order derivative. Similarly, for a qubic function, the variable exhausts with 3rd order derivative and so on.
- If the f''(x) is positive, the curve is concave, if f''(x) is negative, then the curve is convex.
- F'(x) = 0 means slope of the curve is zero and hence the point will either be a relative maxima or minima, but whether it will be maxima or minima, that can be known with 2nd order derivative



3.9 LET US SUM UP

- Differentiation is the rate of change in the dependent variable to a change in independent variable.
- By the process of differentiation we can observe the nature of functional relation between two variables, i.e. whether the dependent variable is increasing or decreasing function of the independent variable.

- Graphically, derivative represents the slope at a particular point of the curve representing the function; the result of derivative shows the direction of movement of the curve. The negative value of the differential coefficient indicates an inverse relationship between the dependent and independent variable and vice versa. Again, the differential coefficient zero means zero slope of the curve which indicates the particular point is either maximum or minimum point of the curve.

- Differentiation of a constant is zero, derivative of logarithmic function is 1/x and derivative of an exponential function produces the function again. However the coefficient of the exponent along with the sign is reproduced in the coefficient of the function.

- In case functions where there is power with independent variable, differential coefficient is obtained as: power of the variable will be coefficient and new power of the variable will be obtained by subtracting 1 from the original power).

- Differentiation of sum/difference of two or more functions with same variable is the sum/difference of individual derivatives of the functions.

- Derivative of the function of a function is obtained with the help of chain rule.

- Finding the Derivative of the derivative is called higher order differentiation. In other words, if we differentiate the value of the first differential, it is called 2^{nd} order derivative.

- First order differentiation measures the slope of the curve for the function while the 2^{nd} order differentiation measures the curvature of the curve for the polynomial function. For a linear function, 2^{nd} order differentiation is zero.

- The techniques of differentiation are widely used for finding out the marginal functions from a given total function. In economics, derivation

of Marginal revenue, marginal cost, marginal propensity to consume and save, etc are derived from the total revenue function, total cost function, consumption function or savings functions. Different elasticities such as price elasticities, income elasticities, etc. Which are the percentage change in demand as a result of change in the price or income are calculated with the help of differentiation. 2nd order differentiation technique is used in profit maximization, or tax revenue maximization or cost minimization.

- Two conditions for optimizations in terms of differentiation are: 1. $\frac{dy}{dx} = 0$, known as the 1st order condition (both for maximization and minimizations); and 2. $\frac{d^2y}{dx^2} < 0$ for maximization) and $\frac{d^2y}{dx^2} > 0$ for minimization.

- In certain cases $\frac{d^2y}{dx^2} = 0$. For such cases, though the 1st order condition is satisfied, the function is neither maximised nor minimised. Such points are called the point of inflexion.



3.10 FURTHER READING

1. Mehta, Madnani. (1996). *Mathematics for Economics*. S. Chand & Sons

2. Chiang, A.C. & Wainwrite, K. (2013). *Fundamental Methods of Mathematical Economics*. McGraw Hill Education (India) Pvt. Ltd.

3. Monga, G.S. (2003). *Mathematics & Statistics for Economics*. Vishal Publishing House Pvt. Ltd.

4. Baruah, S. (2012). *Basic Mathematics & Its Application in Economics*. Macmillan Publishers India Ltd.

5. Patri, D & Patri, D.N. (2004). *Quantitative Methods for Economic Analysis*. Kalyani Publishers.

3.11 ANSWERS TO CHECK YOUR PROGRESS:

Ans to Q1. $\frac{1}{2\sqrt{x}}$

Ans to Q2. $-\frac{20}{x^6}$

Ans to Q3. $\frac{-(2x^2+10x+6)}{(x^2-3)^2}$

Ans to Q4. $\frac{-2+6x}{5-2x+3x^2}$

3.12 MODEL QUESTIONS

1. Find the derivatives of the following

   (i) $Y = e^{ax}$ (iv) $y = 2x^2 + 3x$

   (ii) $y = 1/x^4$ (v) $y = x + 1/x$

   (iii) $y = \sqrt[2]{x}$ (vi) $(x - 2)^2$

2. Differentiate the following

   (a) $y = \frac{(1-x)^2}{x^2}$ (c) $Y = (x^2 + 1)(3x^2 - 2x^3)$

   (b) $Y = \text{Log } x\,(2x^2 + 3)$ (d) $y = \frac{x^{1/2} - 2}{x^{1/2}}$

3. Use Chain rule of differentiation to solve the following

   (i) $Y = (3 + 2x^2)^5$

   (ii) $Y = e^{\log(2x + 3)}$

4. Find Marginal functions from the following total functions

   (i) $TC = 60 - 12x + 2x^2$

   (ii) $C = 10Q - 4Q^3 + 3Q^2$

(iii) TR = 400Q - 2Q2 + 3Q3

5. Find second and higher order derivative of the following:

   (a) $y = 5x^3 - 3x^2 + 2x + 10$

   (b) $y = \frac{e^{2x}}{10x^2}$

   (c) log $(2x^2 + 5x + 10)$

6. If xy = a + bx, show that

   $$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

7. Find the maximum and minimum values of the following functions:

8. (i) $y = x^3 - 3x + 1$

   (ii) 4x + 1/x

**** ***

UNIT 4: ECONOMIC APPLICATIONS OF DERIVATIVE

UNIT STRUCTURE:

4.1 Learning Objectives

4.2 Introduction

4.3 Elasticity of demand and supply

4.4 Derivative as the Marginal function

4.5 Relation between Average cost and Marginal cost

4.6 Effect of Tax and subsidy on equilibrium price and output

4.7 Effect of Investment on National Income: Multiplier effect on Income and consumption

4.8 Let us Sum Up

4.9 Further Reading

4.10 Answer to Check Your Progress

4.11 Model Questions

4.1 LEARNING OBJECTIVES:

After going through this unit, you will be able to –

- calculate the price elasticity of demand and supply, Income elasticity of demand
- derive the marginal functions from total functions
- derive equilibrium price and output under different markets
- understand the impact of tax and subsidies on pricing, output and profit of a firm.

4.2 INTRODUCTION

Technique of differentiation can be applied in different fields of economics. In economics the techniques of differentiation is useful mostly in finding the marginal functions from total functions such as Marginal revenue from Total revenue, Marginal cost from Total cost, elasticity of demand, Utility analysis, in optimization to find maximum profit or minimum cost, impact of tax and subsidies in pricing, output determination and profit

of firm, etc. Since it is the rate of change in dependent variables with respect to independent variable, the slope of the curve is given by the derivatives. Derivatives help us in findings various marginal function given the total function such as MR from TR, MC from TC, MPC from Consumption function, slope and curvature of a curve, etc. Moreover in this we may see that this mathematical tool can be used to show the relationship among various economic functions such as relationship between MR and AR, relationship among elasticity of demand with revenue functions, relationship between MC and AC, etc. we can also show the effect of tax and subsidy on price and output of different market forms.

Now let us examine these applications of derivatives in economic problems one by one.

4.3 ELASTICITY OF DEMAND AND SUPPLY

Derivative is useful in finding the value of elasticity with respect to different economic variables such as price elasticity of demand, elasticity of supply, income elasticity, etc. We can also show some significant relationship of elasticity with average revenue and marginal revenue. The inference that we derived theoretically, we can prove those numerically which we will discuss here.

- **Elasticity of demand:**

As we know, the price elasticity of demand is the degree of responsiveness of quantity demanded to its price and defined as the ratio of proportionate change in quantity demanded to the proportionate changes in its price. That is,

$$E_d = \frac{\text{Proportionate change in quantity demanded}}{\text{Proportionate change in price}}$$

$$= \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}} = \frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q}$$

Where Q is the quantity demanded, ΔQ is change in quantity demanded, P is price and ΔP is the change in prices. Here $\frac{\Delta Q}{\Delta P} \cdot \frac{p}{Q}$ is the average elasticity of demand over the range of price from P to P + ΔP. But since we have to determine the point elasticity of demand at a particular price, ΔP is to be made as small as it tends to zero, which is mathematically termed as the limit $\Delta P \to 0$. That is,

$$E_d = \lim_{\Delta P \to 0} \left(\frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q}\right)$$

$$= \frac{dQ}{dP} \cdot \frac{P}{Q}$$



LET US KNOW

Generally, elasticity of demand is negative for normal commodity since demand and price moves in opposite direction. That is, $e_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$. But the negative sign is often ignored by putting the modulus sign and considered only the absolute value as, $|E_d| = \frac{dQ}{dP} \cdot \frac{p}{Q}$

This is to be noted that $\frac{dQ}{dP} \cdot \frac{p}{Q}$ can also be written as $E_d = \frac{\frac{\Delta Q}{dP}}{\frac{Q}{P}}$,

where, $\frac{dQ}{dP}$ is the marginal demand and $\frac{Q}{P}$ is the average demand.

Therefore, we can also define price elasticity of demand as,

$$E_d = \frac{\text{Marginal demand}}{\text{Average demand}}$$

Example 1 : If the demand function is given by Q = 7 - 2P, find elasticity of demand when price is Rs. 2.

Solution: We know,

$$e_d = \frac{dQ}{dP}.\frac{p}{Q}$$

$$\text{Now, } \frac{dQ}{dP} = \frac{d}{dP}(7-2P)$$
$$= -2$$

$$\text{Therefore, } e_d = -2\left(\frac{p}{7-2P}\right), \text{at } p = 2$$

$$E_d = \frac{-4}{7-4} = -\frac{4}{3}$$

- **Elasticity of supply:**

As we can find demand elasticity from a given demand function, supply elasticity at a particular price also can be calculated by the formula as,

$$E_s = \frac{dQ}{dP}.\frac{P}{Q},$$

Here the Q denotes the quantity supplied.

Example 2: Find e_s for the supply function $Q_s = 5 + 2p^2$ for P = 3

Solution: Elasticity of supply is given by,

$$e_s = \frac{dQ}{dP}.\frac{P}{Q}$$

$$\text{Now, } \frac{dQ}{dP} = 4P$$

$$\text{Therefore, } e_s = 4P.\frac{p}{5+2P^2} = \frac{4P^2}{5+2P^2}$$

$$\text{At } P = 3 \quad e_s = \frac{4\times 3^2}{5+2\times 3^2} = \frac{36}{23}$$

Let us consider a demand curve which is a rectangular hyperbola such that,

$$xp^{-a} = b$$

or $$x = bp^a$$

$$\frac{dx}{dP} = abp^{a-1}$$

$$= \frac{a}{P}bp^a$$

Now, $$e_d = \frac{dx}{dP}.\frac{p}{x}$$

$$= \frac{a}{p}bp^a\frac{p}{x}$$

$$= \frac{ax}{P}.\frac{p}{x}$$

$$= a$$



CHECK YOUR PROGRESS

Q1. Find the elasticity of demand from the following demand function

a. Q = 200 - 2P , when P = 25

Q2. Find elasticity of supply from the following supply function Q = 100 + 7P when P = 2.

- **Relationship between AR, MR and Elasticity of demand:**

We have already studied the relation between AR, MR and elasticity which was established through diagram. Here we can establish the relation with the product rule of differentiation.

We know that total revenue (R) is the product of average revenue (AR) and price (P). That is,

$R = PQ$

The Marginal Revenue (MR) is the differentiation of Total revenue (R). i.e.

$MR = \frac{d}{dQ}(R)$

Now, $$\frac{d}{dQ}(R) = \frac{d}{dQ}(P.Q.) = P\frac{d}{dQ}(Q) + Q\frac{d}{dQ}(P)$$

$$MR = p + Q\frac{d}{dQ}(p) \quad .....(1)$$

Assuming at any level of output, MR lies below AR.

$$\text{Therefore}, p + Q\frac{d}{dQ}(P) < P \qquad (\text{since } AR = P)$$

$$\text{Therefore}, Q\frac{d}{dQ}(P) < 0, \text{which indicates that}$$

$$\frac{d}{dQ}(P) < 0$$

Or, $$\frac{d}{dQ}(AR) < 0$$

That is, to hold the fact that MR is less AR, then AR must be falling. Alternatively, we can say that when AR is declining, MR lies below AR. Now, from the equation (1) let us further extend the relationship between revenue functions with elasticity

$$\text{We have, MR.} = P + Q\frac{d}{dQ}(P)$$

$$= P\{1 + \frac{Q}{P}\frac{dP}{dQ}\}$$

$$= P\{1 + \frac{1}{\frac{P}{Q}\frac{dQ}{dP}}\}$$

$$\text{Therefore,} \quad MR = P\{1 + \frac{1}{e}\}$$

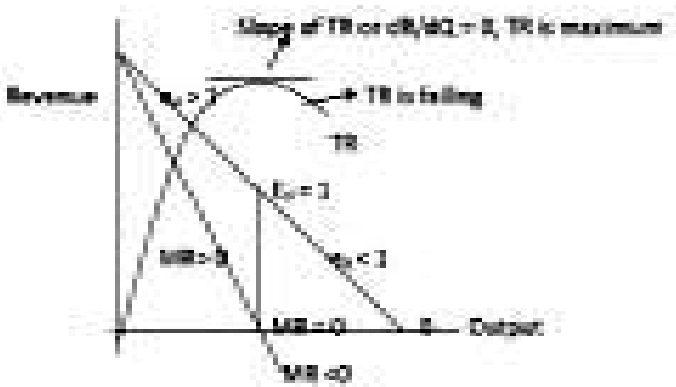
Since elasticity of demand is generally negative,

$$MR = P\{1 - \frac{1}{e}\}$$

$$\text{Or,} \quad MR = AR\{1 - \frac{1}{e}\}$$

From the above equation, the following relations can be established

- If ed > 1, MR >0, indicating TR is increasing with increased output.
- If ed = 1, MR = 0, indicating slope of TR is zero which means TR is maximum
- If ed < 1, MR < 0, indicating TR falls with increase in output.



The above relation can be further extended which may be simpler in mathematical treatment as,

$$MR = AR\{1 - \frac{1}{e}\}$$

Or, $$\frac{MR}{AR} = 1 - \frac{1}{e}$$

Or, $$\frac{1}{e} = 1 - \frac{MR}{AR}$$

Or, $$= \frac{AR - MR}{AR}$$

Therefore, $$e_d = \frac{AR}{AR - MR}$$

It can be observe that when AR = MR, $e_d = \infty$

Conversely, if AR = P = 0, e_d must be zero. And when e_d =1, MR = ½ AR

Thus from given TR or AR (need not to mention that AR function is also the demand function) function, we can find the elasticity of demand with the above relation.

Example:3 If the demand function is given by P = 42 - 3Q; find the output at which TR will be maximum and what will it be.

Solution: Given, P = 42 - 3Q

$TR = P.Q = 42Q - 3Q^2$

We know, TR is maximum when MR = 0

$$Now, MR = \frac{d}{dQ}(TR) = \frac{d}{dQ}(42Q - 3Q^2) = 0$$

$\Rightarrow 42 - 6Q = 0$

$\Rightarrow 6Q = 42$

Therefore, $Q = 7$

That is Q = 7 will maximise the TR. Putting Q = 7 in the TR function,

$TR = 62 \times 7 - 3 \times 7^2 = 147$

4.4 DERIVATIVE AS THE MARGINAL FUNCTION

We already have discussed that if we take the derivative of the total function, we get the marginal function. That is, any Marginal function is the value of derivative of the total function. For example, $MR = \frac{d}{dx}(TR)$, Similarly, $MC = \frac{d}{dx}(TC)$.

Here we state some instances how marginal cost functions, revenue functions, marginal propensity to consume and save functions, etc are derived from the respective total functions.

- **Marginal cost and Marginal revenue:**

Marginal cost is the derivative of TC function and MR is derivative of TR function with respect to output, i.e. $MC = \frac{d}{dx}(TC)$ and MR = $MR = \frac{d}{dx}(TR)$

Example : 4 Given the short run cost function as,

$C = 3Q^2 + 7Q + 12$

Find MC, TFC, TVC and AC functions.

Solution:

We know, $MC = \frac{d}{dQ}(TC)$

Therefore, $MC = \frac{d}{dQ}(3Q^2 + 7Q + 12) = 6Q + 7$

TFC = TC - TVC.

In the above TC function, variable is Q, hence TVC = $3Q^2 + 7Q$

Therefore, TFC = $3Q^2 + 7Q + 12 - (3Q^2 + 7Q) = 12$

$$AC = \frac{C}{Q} = \frac{3Q^2 + 7Q + 12}{Q} = 3Q + 7 + 12/Q$$

- **Marginal propensity to Consume (MPC) and marginal propensity to save (MPS):**

MPC is the derivative of consumption function(c) and MPS is the derivative of savings function(s) with respect to income. That is,

$$MPC = \frac{d}{dY}(C) \text{ and } MPS \frac{d}{dY}(S)$$

Example 5: Given the consumption function

$$C = 1000 - \frac{5000}{3 + Y}$$

Solution:

$$MPC = \frac{d}{dy}(C) = \frac{d}{dy}(1000 - \frac{5000}{3 + Y})$$

$$0 + \frac{5000}{(3 + Y)^2} = \frac{5000}{(3 + Y)^2}$$

$$\text{When } Y = 97, MPC = \frac{5000}{(3+97)^2} = \frac{5000}{10000} = 0.5$$

MPS = 1 - MPC

Therefore, MPS = 1 - 0.5 = 0.5.

- **Marginal Utility and Marginal product:**

Marginal utility is the derivative of Total utility function with respect to quantity consumed. That is $MU = \frac{d}{dQ}(TU)$. Similarly, Marginal product of a factor is the derivative of total product with respect to quantity of the factor employed. That is $MP = \frac{d}{dL}(TP)$ (labour is the variable factor)

Example : 6 Given the utility function,

$U = aQ^b$

$MU = \frac{d}{dQ}(aQ^b) = abQ^{b-1}$

Marginal physical product is different from the marginal revenue product. Marginal Revenue Product (MRP) is the product of marginal physical product and marginal revenue earned from an additional unit of the factor. Or it is the derivative of total revenue function with respect to umber of factor employed i.e. $MRP = \frac{dR}{dL}$. For the factor L,

$$MR_L = \frac{d}{dQ}(R) \text{ and } MP_L = \frac{d}{dL}(Q)$$

$$MRP_L = MR \times MPP_L$$

$$\frac{dR}{dL} = \frac{dR}{dQ} \times \frac{dQ}{dL}$$

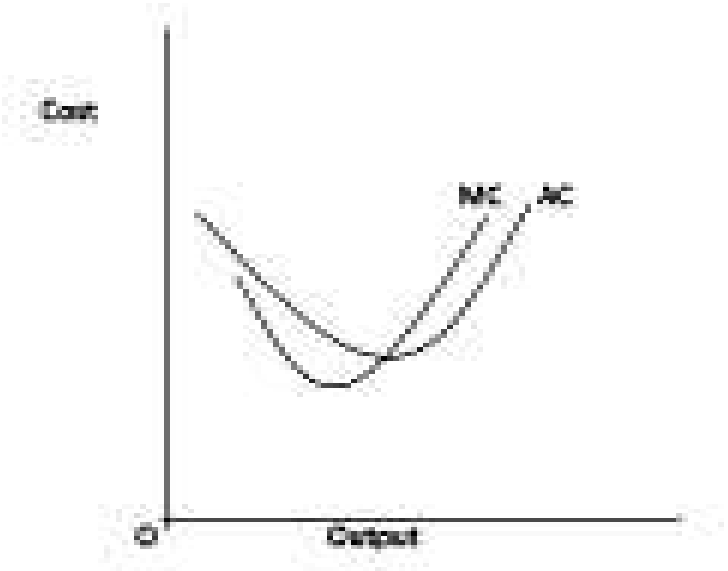
4.5 RELATION BETWEEN AVERAGE COST AND MARGINAL COST

We know the following relationship between AC and MC for a U-shaped AC curve:

(i) When AC is falling, MC lies below the AC

(ii) When AC is rising, MC is higher than AC

(iii) When AC is minimum, AC = MC. i.e. MC cuts AC from below.



We can establish the above facts with the help of quotient rule of differentiation. Let us consider AC as a function of quantity produced. We know,

$$AC = \frac{c}{Q} \quad \text{.....(1)}$$

where C is total cost and Q is output produced.

The slope of AC is given by the derivative of the AC function. That is,

$$\frac{d}{dQ}\left(\frac{c}{Q}\right)$$

When AC is failing, $\frac{d}{dQ}\left(\frac{C}{Q}\right) < 0$

Or, $$\left\{\frac{Q\frac{d}{dQ}(C) - C\frac{d}{dQ}(Q)}{Q^2}\right\} < 0$$

Or, $$\left\{\frac{Q\frac{d}{dQ}(C) - C}{Q^2}\right\} < 0$$

Or, $$\frac{1}{Q^2}\{Q\frac{d}{dQ}(C)-C\}<0$$

Or, $$\left\{\frac{1}{Q}\frac{d}{dQ}(C)-\frac{C}{Q^2}\right\}<0$$

Or, $$\frac{1}{Q}\{\frac{d}{dQ}(C)-\frac{C}{Q}<0\}$$

Or, $$\left\{\frac{d}{dQ}(C)-\frac{C}{Q}\right\}<0$$

Or, $$MC < AC \quad (\text{Since}\frac{d}{dQ}(C)=MC \text{and}\frac{C}{Q}=AC)$$

Thus it can be established that when Slope of AC is negative, MC lies below AC.

Now let us consider when AC is rising.

When AC is rising, $\frac{d}{dQ}(\frac{C}{Q})>0$

Or, $$\left\{\frac{Q\frac{d}{dQ}(C)-C\frac{d}{dQ}(Q)}{Q^2}\right\}>0$$

Or, $$\left\{\frac{Q\frac{d}{dQ}(C)-C}{Q^2}\right\}>0$$

Or, $$\frac{1}{Q^2}\{Q\frac{d}{dQ}(C)-C\}>0$$

Or, $$\left\{\frac{1}{Q}\frac{d}{dQ}(C)-\frac{C}{Q^2}\right\}>0$$

Or, $$\frac{1}{Q}\{\frac{d}{dQ}(C)-\frac{C}{Q}\}>0$$

Or, $\frac{d}{Q}(C) - \frac{C}{Q} > 0$

Or, MC > AC

Similarly we can show that when slope of AC is zero, i.e. $\frac{d}{dQ}(\frac{C}{Q}) = 0$, AC = MC. It is to be noted that since AC is U-shaped, it is minimum, when $\frac{d}{dQ}(\frac{C}{Q}) = 0$. Therefore we can say that MC becomes equal to AC when AC is minimum.

Example 7:

The total cost function is given by,

$C = Q^3 - 4Q^2 + 15Q$

State the AC and MC relation from the above cost function.

Solution:

$AC = C/Q = Q^2 - 4Q + 15$, and

$MC = \frac{d}{dQ}(C) = \frac{d}{dQ}(Q^3 - 4Q^2 + 15Q)$

$= 3Q^2 - 8Q + 15$

Now, let us consider AC is falling, i.e.

$\frac{d}{dQ}(AC) = \frac{d}{dQ}(Q^2 - 4Q + 15) < 0$

$\Rightarrow 2Q - 4 < 0$

$\Rightarrow Q < 2$

That is If $Q < 2$, $MC < AC$.

If $Q > 2$, $MC > AC$

And If $Q = 2$ $MC = AC$

(Readers may verify the fact putting Q = 2 in both AC and MC functions to get the actual value and check whether at Q = 2, MC = AC or not.)

Example 8: Given the total cost function as

$$C = \frac{1}{3}Q^3 - 3Q^2 + 9Q$$

Find the level of output where AC is minimum and also find the minimum AC.

Solution: Given the TC function,

$$C = \frac{1}{3}Q^3 - 3Q^2 + 9Q$$

$$AC = \frac{C}{Q} = \frac{1}{3}Q^2 - 3Q + 9$$

And $$MC = \frac{d}{dQ}(\frac{1}{3}Q^3 - 3Q^2 + 9Q)$$

$$= Q^2 - 6Q + 9$$

AC is minimum, when, MC = AC

Putting the values, => $Q^2 - 6Q + 9 = \frac{1}{3}Q^2 - 3Q + 9$

$$\Rightarrow \frac{2}{3}Q^2 = 3Q$$

$$\Rightarrow Q = \frac{9}{2}$$

Thus $Q = \frac{9}{2}$ will minimise the AC

At $Q = \frac{9}{2}$, the value of AC will be,

$$AC = \frac{1}{3}(\frac{9}{2})^2 - 3(\frac{9}{2}) + 9 = \frac{9}{4}$$

We can see that, at Q = $\frac{9}{2}$, MC = $\frac{9}{4}$

(We will get the same result if we apply the minimisation rule i.e. 1st order condition and 2nd order condition of minimisation)

LET US KNOW



Similar relation holds good for AVC while talking about short run cost function. That is, AVC is also U shaped curve and MC = AVC when AVC is minimum. For falling portion of AVC, MC lies below AVC, while for rising portion of AVC, MC lies above AVC. Readers may try to prove this with same procedure.

4.6 EFFECT OF TAX AND SUBSIDY ON EQUILIBRIUM PRICE AND OUTPUT

The nature and also the impact of tax is different in different market. In case of competitive market, the market equilibrium is attained by equality of demand and supply. But when a per unit tax is imposed, the seller has to get the price after deduction of tax while the buyer pays the same price as before. Therefore the equilibrium quantity is adjusted according to new supply function. Let us suppose a simple linear market model as,

$D = a - bp$ .....(1)

$S = -c + d(p-t)$ .....(2)

$D = S$ .....(3)

Putting (1) and (2) in (3), we get,

$a - bp = -c + d(p - t)$

or, $a - bp = -c + dp - dt$

or, $bp + dp = a + c + dt$

or, $p = \frac{a+c+dt}{b+d}$

or, $P = \frac{a+c}{b+d} + \frac{d}{b+d}t$ ..... (4)

This is the price after imposition of tax. Putting equation for in the demand function, we can get the post tax equilibrium output as,

$$Q = a - b(\frac{a+c}{b+d} + \frac{d}{b+d}t)$$

- **Tax yield in competitive market:**

We show the effects of changes in the rate of tax on tax yield (sales tax or excise tax) with the help of derivative techniques. Let us again consider above market model where due to imposition of tax, supply price the supplier receives becomes (p - t).

Total tax yield T is the product of rate of tax t and output sold. Thus,

$$T = t.Q$$

$$= t\{a - b(\frac{a+c}{b+d} + \frac{d}{b+d}t)\}$$

$$= at - \frac{b(a+c)}{b+d}t + \frac{bd}{b+d}t^2$$

$$= \frac{at(b+d) - b(a+c)t}{b+d} + (\frac{bd}{b+d}t^2)$$

$$= \frac{(ab+ad-ab-bc)t}{b+d} + \frac{bd}{b+d}t^2$$

$$\text{Or, } T = \frac{(ab-bc)}{b+d}t + \frac{bd}{b+d}t^2$$

To see the effect of change in rate of tax on total tax, let us differentiate T w.r.t. t,

$$\frac{dT}{dt} = \frac{d}{dt}\{\frac{(ad-bc)t}{b+d} + \frac{bd}{b+d}t^2\}$$

$$= \frac{(ad-bc)}{b+d} + \frac{2bdt}{b+d}$$

$\frac{dT}{dt}$ will be positive, i.e. increase in rate of tax will increase the tax yield if ad - bc > 2bdt and vice versa.

Example 9 : Given the market model,

$Q_d = 20 - 3P$

$Q_s = - 10 + 2 (P - t)$

$Q_d = Q_s$

Evaluate the effect of change in rate of tax on tax yield.

Solution: Given,

$Q_d = 20 - 3P$ ..... (1)

$Q_s = - 10 + 2 (P - t)$ ..... (2)

$Q_d = Q_s$ ..... (3)

Putting (1) and (2) in (3)

$20 - 3P = - 10 + 2 (P - t)$

Or, $5P = 30 + 2t$

Or, $P = \frac{30 + 2t}{5}$ ..... (4)

Putting the equilibrium price in the demand function,

$$Q = 20 - 3(\frac{30 + 2t}{5})$$

$$= 20 - \{\frac{90 + 6t}{5}\}$$

$$= \frac{10 - 6t}{5}$$

Tax yield (T) is given by,

$T = t.Q$

$$T = t\{\frac{10-6t}{5}\}$$

$$= \frac{10t-6t^2}{5}$$

Now,

$$\frac{dT}{dt} = \frac{d}{dt}(\frac{10t-6t^2}{5})$$

$$= \frac{10-12t}{5}$$

$$= 2-2.4t$$

Since t is a positive fraction, the value of $\frac{dT}{dt} > 0$. Thus t and T will move in the same direction.

Case of subsidy is just opposite to the tax. That is in case of subsidy, it makes addition to the supplier price, and hence, subsidy is added to the price in supply function or deducted from price of the demand function.

- **Effect of Tax and subsidy on monopoly output:**

The imposition of tax may or may not affect the equilibrium price and output depending on the nature of tax. If Govt. imposes per unit of tax such as excise tax, sales tax, etc, it will affect the equilibrium combination of price and output. However, if it is a lumsum tax or profit tax which is imposed on the total profit, it will reduce the profit by increasing the cost, but the equilibrium combination will be unaffected. Let us suppose the Revenue and cost function of a monopolist are given by

$R = R(Q)$ and $C = C(Q)$

Equilibrium condition for the monopolist is given by MC = MR and

$\frac{d}{dQ}$ (MC) > 0.

Now, MR = R' (Q) and MC = C' (Q)

In equilibrium,

R' (Q) = C' (Q) and C'' (Q) > 0

Now let us suppose a lum sum tax is imposed by the Govt. which adds to the cost of the firm. Therefore, the cost function becomes,

C = C (Q) + T

Now the MC = $\frac{d}{dQ}$ C(Q) + $\frac{d}{dQ}$ (T) = C' (Q) + 0 = C' (Q)

Thus the equilibrium condition remains same and hence the price output combination too. Now let us examine the case when the tax is imposed on per unit sold. When per unit of tax is imposed total tax imposed (T) is rate of tax multiplied by output sold (Q), i.e. tQ. Thus the total cost function becomes,

C = C (Q) + tQ

Therefore, MC = $\frac{d}{dQ}$ {C (Q) + tQ}

= C' (Q) + t

For equilibrium,

R' (Q) = C' (Q) + t.

It is seen that MC is more after imposition of tax. Therefore, the price to be imposed by the monopolist will also be more while output will be reduced. A numerical example will make it clearer.

Example 10 :

Demand and cost functions of a monopolist are given as,

$P = 20 - 4Q$ and $C = 4Q$.

If a tax of 0.5 per unit of output is imposed, (i) evaluate the effect of tax on equilibrium price and output. (iii) Also find the total tax revenue of the Govt.

Solution: Given,

$P = 20 - 4Q$ and $C = 4Q$.

Equilibrium output is determined when MR = MC

Now, before tax,

$TR = P.Q = (20 - 4Q).Q = 20Q - 4Q^2$

Therefore, $MR = \frac{d}{dQ}(20Q - 4Q^2) = 20 - 8Q$, and

$MC = \frac{d}{dQ}(C) = \frac{d}{dQ}(4Q) = 4$

In equilibrium, $20 - 8Q = 4$

Or, $Q = 2$

Putting Q = 2 into demand function

$P = 20 - 4x2 = 12$

Thus before tax, equilibrium price and output are 12 and 2 respectively.

After tax, the cost function,

$C^* = 4Q + 0.5Q$

Therefore, $MC^* = 4 + 0.5 = 4.5$

For equilibrium output,

$20 - 8Q = 4.5$

Or, $Q = 1.94$

Putting $Q = 1.94$, $P = 20 - 4x1.94 = 12.24$

Thus it is seen that after imposition of tax, price is increased while output is decreased.

Total tax revenue (T) is given by,

$T = t.Q.$ we have $t = 0.5$ and Q after tax $= 1.94$

Therefore, $T = 0.5 \times 1.94 = 0.97$.

The case of subsidy is opposite to the case of tax. While imposition of tax increases the price and reduce the quantity; while provision of subsidy will reduce the cost and hence reduce the equilibrium price with increased output.

4.7 EFFECT OF INVESTMENT ON NATIONAL INCOME: MULTIPLIER EFFECT ON INCOME AND CONSUMPTION

If we increase the investment, income increases by the multiplier times which is defined as,

$K = 1/s$, where s is MPS. This

To show the effect of investment on income, let us consider the following National Income Model

$Y = C + I_0 + G_0$ ..... (1)

Where Y is national Income, C is consumption, I_0 is Investment and G_0 Government expenditure.

$C = A + cY \quad 1 > c > 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Where A is autonomous investment and c is MPC.

Putting (2) in (1)

$$Y = A + cY + I_0 + G_0$$

Or, $$Y - cY = A + I_0 + G_0$$

Or, $$(1 - c)\, Y = A + I_0 + G_0$$

$$Y = \frac{A + I_0 + G_0}{(1 - C)}$$

To see the effect of change in investment on national income, let us differentiate Y with respect to I_0

$$\frac{dY}{dI} = \frac{1}{(1-c)} = \frac{1}{s}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Where s = 1 - c, s is MPS

Thus we can see that as a result of increase in investment income will increase by $\frac{1}{s}$ times, which is called investment multiplier. From equation (3) it can also be seen that, multiplier value of multiplier varies directly with MPC and inversely with MPC.



CHECK YOUR PROGRESS

Q3. Given the total cost function $C=1000+100q-10q^2+\frac{1}{3}q^3$, find the marginal cost function and the slope of the marginal cost function.

Q4. A consumer has a utility function $U = \alpha Q^{\beta}, \alpha > 0, 0 < \beta < 1$, then find the slop of marginal utility.



4.8 LET US SUM UP

- Derivatives help us in findings various marginal function given the total function such as MR from TR, MC from TC, MPC from Consumption function, slope and curvature of a curve, etc. Moreover in this we may see that this mathematical tool can be used to show the relationship among various economic functions such as relationship between MR and AR, relationship among elasticity of demand with revenue functions, relationship between MC and AC, etc.

- We can also show the effect of tax and subsidy on price and output of different market forms. It is also indicative about which types of tax affects the equilibrium and which types don't.

- With differentiation technique, we can also define elasticity as, $E = -\frac{\text{Marginal function}}{\text{Average function}}$

- One of the functional relationship of elasticity of demand with AR and MR is that,

  $$MR = AR\{1-\frac{1}{e}\}$$

  From the above equation, the following relations can be established

  If $e_d > 1$, MR >0, indicating TR is increasing with increased output.

  If $e_d = 1$, MR = 0, indicating slope of TR is zero which means TR is maximum

  If $e_d < 1$, MR < 0, indicating TR falls with increase in output.

- Marginal cost is the derivative of TC cost function and MR is derivative of TR with respect to output, i.e. MC = $\frac{d}{dx}$(TC) and MR = $\frac{d}{dx}$(TR)

- With Chain Rule of differentiation we can express total revenue function as the variable factor of production (labour L) as, $\frac{dR}{dL} = \frac{dR}{dQ} \times \frac{dQ}{dL}$

  Where $\frac{dR}{dL}$ is MRP_L, $\frac{dR}{dQ}$ = MR and $\frac{dQ}{dL}$ = MPP_L

- When slope of AC is zero, i.e. $\frac{d}{dQ}(\frac{C}{Q})$ = 0, AC = MC. It is to be noted that since AC is U-shaped, it is minimum, when $\frac{d}{dQ}(\frac{C}{Q})$ = 0. Therefore we can say that MC becomes equal to AC when AC is minimum.

- The imposition of tax may or may not affect the equilibrium price and output depending on the nature of tax. If Govt. imposes per unit of tax such as excise tax, sales tax, etc, it will affect the equilibrium combination of price and output. However, if it is a lumsum tax or profit tax which is imposed on the total profit, it will reduce the profit by increasing the cost, but the equilibrium combination will be unaffected.

4.9 FURTHER READING

1) Mehta, Madnani (1996). *Mathematics for Economics*. S. Chand & Sons.
2) Chiang A.C., Wainwrite K (2013). *Fundamental Methods of Mathematical Economics.* McGraw Hill Education (India) Pvt. Ltd.
3) Monga G.S. (2003). *Mathematics & Statistics for Economics*. Vishal Publishing House Pvt. Ltd.
4) Baruah S (2012). *Basic Mathematics & Its Application in Economics*. Macmillan Publishers India Ltd.

4.10 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q. No. 1 : .33

Ans to Q. No. 2 : .12

Ans to Q. No. 3 : $MC=100-20q+q^2$

Slope of MC = -20+2q

Ans to Q. No. 4 : Slope of MU= $\alpha\beta(\beta-1)Q^{\beta-2}$

4.11 MODEL QUESTIONS

Q1. Find the elasticity of demand if D = 25 - 4P + P2, when

(i) P = 8 (ii) P = 4.

Comment on the nature of the commodity at each level of prices.

Q2. If the demand function is given by $P = (4 - 5Q)^2$, for what value of Q the elasticity of demand will be unity?

Q3. Given the production function and the revenue functions as,

$Q = 2.5L^2$ and $R = 120Q - 0.2Q^2$, find MRP_L, when 10 labours are employed.

Q4. The Utility function is given by

$U = \alpha Q^{\beta}$ where $\alpha > 0$ and $0 < \beta < 1$

Evaluate whether marginal utility is diminishing or not.

Q5. Given the demand function

P = 10 - 2Q. Find MR and elasticity of demand. Also state the nature of the commodity.

Q6. Given the cost function

$C = Q^3 - 10Q^2 + 12Q + 75$. Find the output at which MC is equal to AVC.

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UNIT 5: PARTIAL DIFFERENTIATION

UNIT STRUCTURE

5.1 Learning Objectives

5.2 Introduction

5.3 The concept of partial differentiation

5.4 Rules of partial differentiation

5.5 Second order partial derivatives

5.6 Differentials and total differentials

5.7 Total Derivatives

5.8 Derivatives of Implicit Function

5.9 Let Us Sum Up

5.10 Further Reading

5.11 Answer to Check Your Progress

5.12 Model Questions

5.1. LEARNING OBJECTIVES:

After going through this unit, you will be able to–

- define partial differentiation
- learn the rules of partial differentiation
- define total differentiation
- find out the derivative of an implicit function.

5.2 INTRODUCTION

In the earlier unit on differentiation, we are confronted with only one Independent variable which effect the dependent variable. Such as quantity demanded as a function of income (Y)

But, in economics it is seen that variable depends on a number of independent variable,e.g. quality demand is a function of income (Y), price of the product (P) price of the related(PR) goods, and say, the size of the family (S). Such that -

$$Q_d = f(Y, P, PR, S)$$

These independent variables have positive and negative effects on the dependent variable Q_d. In order to trace out the change in total quantity demand Q_d due to change in one independent variable say price(P), assuming all other independent variables [income(Y), price of related goods (PR) and sign of the family (S)] be constant, we use the differentiation technique and the resultant change in quantity demanded is known as partial derivatives. It is called "partial derivatives" became the effect of independent variable on dependent variable is partially analyzed, assuming the other independent variables as unchanged.

But, when all the independent variables change simultaneously then to study about the total net effect of independent variable on dependent variable, we use the total differentiation technique. Here, the partial derivatives of each independent variables are added together to get the total net effect on dependent variable.

5.3 THE CONCEPT OF PARTIAL DIFFERENTIATION

In order to define the concept of partial derivative, let us consider a function having 'n' independent variables -

$y= f(x_1, x_2, x_3,\ldots\ldots\ldots\ldots\ldots\ldots, x_n)$ where are all independent variables, i.e., each can vary itself without affecting the others. Now assuming a change in the value of x1 with all other independent variables $(x_2, x_3, \ldots\ldots\ldots\ldots, x_n)$ remaining unchanged, will bring about a corresponding changes in the value of y. If Δx_1, denotes the change in the value of x_1, the new value of x_1 becomes $(x_1 + \Delta x_1)$. If the corresponding change in the value of y is denoted by Δy, the value of y will change to $(y+ \Delta y)$.

$$y+\Delta y = f(x_1+\Delta x_1, x_2, x_3\ldots\ldots\ldots\ldots x_n)$$

or $$\Delta y = f(x_1+\Delta x_1, x_2, x_3\ldots\ldots\ldots\ldots x_n) - y$$

or $$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1+\Delta x_1, x_2, x_3, \ldots\ldots\ldots, x_n) - y}{\Delta x_1}$$

$\frac{\Delta y}{\Delta x_1}$ represents the rate of change in y with respect to a change in the value of x_1 assuming other independent variables x_2, x_2…… x_n to be constant.

Now, the limit of the quotient $\frac{\Delta y}{\Delta x1}$ is called the "partial derivatives" of y with respect to x_1, where the limiting value of Δx_1 is equal to zero.

It is denoted as-

$$\frac{\partial y}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1}$$

In the same way, the partial derivatives of y with respect to the other independent variables can be defined.

The whole process of finding the partial derivative is called partial differentiaton .

Partial derivatives are assigned distinctive symbols. Instead of using the letter 'd' as used in the ordinary derivative, the symbol ∂ (delta) is employed to indicate partial derivative such as $\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}$... .....$\frac{\partial y}{\partial x_n}$ etc

We can also use the symbols f_1, f_2,……,fn etc. to denote partial derivative where the subscripts indicates which independent variable is being allowed to vary.

5.4 RULES OF PARTIAL DIFFERENTIATION

The rules of partial differentiation are almost similar to the ordinary differentiation. The only difference is when we differentiate the function with a particular variable then the rest of independent variables are considered to be constant. The treatment of the constants in partial differentiation is the same as the case of ordinary differentiation.

Let us know, how to use the partial differentiation -

Example :1

Given, y= f (x_1, x_2) = $2x_1^2 + X_1X_2 + 3x_2^2$ find out $\frac{\partial y}{\partial x_1}$ and $\frac{\partial y}{\partial x_2}$.

Solution:

In the process of finding $\frac{\partial y}{\partial x_1}$, we treat x_2 as a constant. So the additive constant ($3x_2^2$) will vanish and the multiplicative constant (x_1x_2) will be retained.

$$\therefore f_1 = \frac{\partial y}{\partial x_2} = \frac{\partial y}{\partial x_1}(2x_1^2 + x_1x_2 + 3x_2^2)$$

= $4x_1 + x_2 + 0$ (since x_2=constant)

= $4x_1 + x_2$

Similarly, in finding $\frac{\partial y}{\partial x_2}$, x_1 will be considered as constant. Thus,

$$f_2 = \frac{\partial y}{\partial x_2} = \frac{\partial y}{\partial x_2}(2x_1^2 + x_1x_2 + 3x_2^2)$$

= $x_1 + 6x_2$ (since x_1 = constant)

From the above results it is found that partial derivatives are also the functions of x_1 and x_2. Thus, we can write -

$$f_1 = \frac{\partial y}{\partial x_1} = f_1(X_1, X_2)$$

$$f_2 = \frac{\partial y}{\partial x_2} = f_2(X_1, X_2)$$

Example 2 :

$$\text{Given, } Y = f(X_1X_2) = (x_1^2 + 5)(2X_1 - x_2^2). \text{Find } f_1 \text{ and } f_2.$$

Solution:

Since y is the product of two functions, the partial derivatives can be derived by using product rule of differentiation -

$$\frac{\partial y}{\partial x_1} = f_1 = (x_1^2 + 5)\frac{\partial y}{\partial x_1}(2X_1 - x_2^2) + (2X_1 - x_2^2)\frac{\partial y}{\partial x_1}(x_1^2 + 5)$$
$$= (x_1^2 + 5)(2 - 0) + (2X_1 - x_2^2)(2X_1 - 0) \quad \text{(Since } X_2 \text{ is a constant)}$$
$$= 2x_1^2 + 10 + 4x_1^2 - 2X_1x_2^2$$
$$= 6x_1^2 - 2X_1x_2^2 + 10$$

Similarly,

$$\frac{\partial y}{\partial x_2} = f_2 = (x_1^2 + 5)\frac{\partial y}{\partial x_2}(2X_1 - x_2^2) + (2X_1 - x_2^2)\frac{\partial y}{\partial x_2}(x_1^2 + 5)$$
$$= (x_1^2 + 5)(0 - 2X_2) + (2X_1 - x_2^2) \quad \text{(Since } X_1 \text{ is a constant)}$$
$$= 2x_1^2X_2 - 10x_2$$

Example 3 :

Given, $y = \log (50 + x_1^2 + 2x_1x_2 + x_2^4)$, find $\frac{\partial y}{\partial x_1}$ and $\frac{\partial y}{\partial x_2}$.

Solution :

Here we use the modified version of chain rule of differentiation.

Let, u= log (u)

$$\therefore \frac{\partial y}{\partial x_1} = \frac{\partial y}{\partial u} \times \frac{\partial u}{\partial x_1}$$
$$= \frac{\partial}{\partial u}\log(u) \times \frac{\partial}{\partial x_1}(50 + x_1^2 + 2X_1X_2 + x_2^4)$$
$$= \frac{1}{u} \times (0 + 2X_1 + 2X_2 + 0) \qquad (\text{since } X_2 = \text{constant})$$
$$= \frac{2x_1 + 2x_2}{50 + x_1^2 + 2x_1x_2 + x_2^4}$$

Similarly,

$$\frac{\partial y}{\partial x_2} = \frac{\partial y}{\partial u} \times \frac{\partial u}{\partial x_2}$$

$$= \frac{\partial}{\partial u}\log(u) \times \frac{\partial}{\partial x_2}(50 + x_1^2 + 2X_1X_2 + x_2^4)$$

$$= \frac{1}{u} \times (0 + 0 + 2X_2 + 4X_2^3) \qquad \text{(since } X_2 = \text{constant)}$$

$$= \frac{2x_2 + 4x_2^3}{50 + x_1^2 + 2x_1x_2 + x_2^4}$$



Check Your Progress:

Q1. Given, y= $3x_1 + \frac{x_1}{x_2} + 10x_2^2$, find f_1 and f_2.

Q2. Given, y= $(3x_1^2 - 2x_2)(x_1^2 + 2x_2^2)$, find f_1 and f_2.

5.5 SECOND ORDER PARTIAL DERIVATIVES

In case of differentiation, when we differentiate Y= f(x), then we get f'(x) or $\frac{dy}{dx}$ which is also a function of x.

Similarly, in case of a function having more than one independent variable-

Y= $f(x_1, x_2)$, the partial derivatives $f_1 = \frac{\partial y}{\partial x_1}$ and $f_2 = \frac{\partial y}{\partial x_2}$ are also functions of x_1 and x_2. This implies that $\frac{\partial y}{\partial x_1}$ and $\frac{\partial y}{\partial x_2}$ can be differentiated partially again with respect to x_1 and x_2 to yield second order partial derivatives. So in case of a function-

$$y = f(X_1, X_2)$$

the first order partial derivative $\frac{\partial y}{\partial x_1} = f_1$ can be further differentiated partially with respect to x_1 to give-

$$\frac{\partial y}{\partial x_1}\left(\frac{\partial y}{\partial x_1}\right) = \frac{\partial^2 y}{\partial x_1^2} = f_{11}$$

Here, the f_{11} measures the rate of change of f_1 with respect to x_1 assuming x_2 to be constant.

Similarly, the second order partial derivative of the function Y= f(x_1, x_2) with respect to x_2 is defined as-

$$\frac{\partial y}{\partial x_2}\left(\frac{\partial y}{\partial x_2}\right) = \frac{\partial^2 y}{\partial x_2^2} = f_{22}$$

Here, f_{22} measures the rate of change of f_2 with respect to x_2 while x_1 is assumed to be constant.

Again when f_1 is differentiated partially further with respect to x_2 or when f_2 is differentiated partially further with respect to x_1; both the second order partial derivation are known as "cross- second order partial derivatives" and it is denoted by-

$$\frac{\partial y}{\partial x_2}\left(\frac{\partial y}{\partial x_1}\right) = \frac{\partial^2 y}{\partial x_1 \partial x_2} = f_{12}$$

$$\frac{\partial y}{\partial x_1}\left(\frac{\partial y}{\partial x_2}\right) = \frac{\partial^2 y}{\partial x_2 \partial x_1} = f_{21}$$

Let us know, how to use the second order partial derivatives-

Example: 4 :

Given, y= f(x_1 x_2) = $2x_1^2 + x_1x_2 + 3x_2^2$. Find f_{12} and f_{21}.

Solution:

Here, y= $2x_1^2 + x_1x_2 + 3x_2^2$

$\therefore$ The first condition requires

$$\frac{\partial y}{\partial x_1} = \frac{\partial}{\partial x_1}(2x_1^2 + x_1x_2 + 3x_2^2)$$

$$= 4X_1 + x_2 + 0 \qquad (\because X_2 = \text{constant})$$

$$= 4X_1 + x_2 \qquad \text{..... ....(1)}$$

Similarly,

$$\frac{\partial y}{\partial x_2} = \frac{\partial}{\partial x_2}(2x_1^2 + x_1x_2 + 3x_2^2)$$

$$= 0 + x_1 + 6X_2 \qquad (\because X_1 = \text{constant})$$

$$= X_1 + 6x_2$$

Now, the second order condition requires-

$$f_{11} = \frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_1}\right)$$

$$= \frac{\partial}{\partial x_1}(4x_1 + x_2) \qquad \text{[Putting the value of } \frac{\partial y}{\partial x_1} \text{ from equation (1)]}$$

= 4+ 0 *($\because$ x_2= constant)*

= 4

$$f_{12} = \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_1}\right)$$

$$= \frac{\partial}{\partial x_2}(4x_1 + x_2) \qquad \text{[from equation (1)]}$$

$$= 0 + 1 \qquad (\because 4x_1 = 0)$$

$$= 1$$

$$f_{21} = \frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_2}\right) = \frac{\partial^2 y}{\partial x_2 x_1}$$

$$= \frac{\partial}{\partial x_1}(x_1 + 6x_2)$$

$$= 1 + 0 \qquad (\because x_2 = \text{constant})$$

$$= 1$$

$$f_{22} = \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_2}\right) = \frac{\partial^2 y}{\partial x_2^2}$$

$$= \frac{\partial}{\partial x_2}(x_1 + 6x_2)$$

$$= 0 + 6$$

$$= 6$$

$\therefore$ *The second order partial derivatives are-*

$f_{11} = 4$ $\quad$ $f_{21} = 1$

$f_{12} = 1$ $\quad$ $f_{22} = 6$

Note: The second order partial derivatives can be extended to a function having more than two independent variable, say-

$$y = f(x_1, x_2, x_3)$$

In case of three independent variable , we have nine second order partial derivatives-

$$f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, \text{ and } f_{33}$$

As for example-

Example 5 **:**

Given, y= f(x_1, x_2, x_3) = $10 + 2x_1^2x_2 + 3x_2^2x_3^2$ *. Find all nine second order partial derivatives.*

Solution :

In the function,

$$y = 10 + 2x_1^2x_2 + 3x_2^2x_3^2$$

$$f_1 = \frac{\partial y}{\partial x_1} = \frac{\partial}{\partial x_1}(10 + 2x_1^2x_2 + 3x_2^2x_3^2)$$

$= 4x_1x_2 + 0$ [$\because$ $x_2 = x_3$= constant]

$=4x_1x_2$

$$f_2 = \frac{\partial y}{\partial x_2} = \frac{\partial}{\partial x_2}(10 + 2x_1^2x_2 + 3x_2^2x_3^2)$$

$= 2x_1^2 + 6x_2x_3^2$ [$\because X_1 = X_3 = \text{constant}$]

$$f_3 = \frac{\partial y}{\partial x_3} = \frac{\partial}{\partial x_3}(10 + 2x_1^2x_2 + 3x_2^2x_3^2)$$

$= 6x_2^2x_3$ [$\because X_1 = X_2 = \text{constant}$]

$$f_{11} = \frac{\partial}{\partial x_1}(f_1)$$

$$= \frac{\partial}{\partial x_1}(4X_1X_2)$$

$$= 4X_2$$

$$f_{12} = \frac{\partial}{\partial x_2}(f_1)$$

$$= \frac{\partial}{\partial x_2}(4X_1X_2)$$

$$= 4X_1$$

$$f_{13} = \frac{\partial}{\partial x_3}(f_1)$$

$$= \frac{\partial}{\partial x_3}(4X_1X_2)$$

$$= 0$$

$$f_{21} = \frac{\partial}{\partial x_1}(f_2)$$

$$= \frac{\partial}{\partial x_1}(2x_1^2 + 6X_2x_3^2)$$

$$= 4X_1$$

$$f_{22} = \frac{\partial}{\partial x_2}(f_2)$$

$$= \frac{\partial}{\partial x_2}(2x_1^2 + 6X_2x_3^2)$$

$$= 6X_3^2$$

$$f_{23} = \frac{\partial}{\partial x_3}(f_2)$$

$$= \frac{\partial}{\partial x_3}(2x_1^2 + 6X_2x_3^2)$$

$$= 12X_2X_3$$

$$f_{31} = \frac{\partial}{\partial x_1}(f_3)$$

$$= \frac{\partial}{\partial x_1}(6x_2^2X_3)$$

$$= 0$$

$$f_{32} = \frac{\partial}{\partial x_2}(f_3)$$

$$= \frac{\partial}{\partial x_2}(6x_2^2 X_3)$$

$$= 12X_2X_3$$

$$f_{33} = \frac{\partial}{\partial x_3}(f_3)$$

$$= \frac{\partial}{\partial x_3}(6x_2^2 X_3)$$

$$= 6x_2^2$$

∴ The results are-

$f^{11} = 4x_2$ $f_{12} = 4x_1$ $f_{13} = 0$ $f_{21} = 4x_1$ $f_{22} = 6x_3^2$

$f_{23} = 12\, x_2x_3$ $f_{31} = 0$ $f_{32} = 12\, x_2x_3$ $f_{33} = 6x_3^2$

5.6 DIFFERENTIALS AND TOTAL DIFFERENTIALS

Suppose, we have a function-

$y = f(x_1, x_2)$

In case of partial differentiation, f_1 measures the rate of change of y with respect to a small change in the value of x_1 assuming x_2 to be constant. Similarly, f_2 represents the rate of change in y with respect to a small change in the value of x_2 holding x_1 to be constant.

When both x_1 and x_2 change simultaneously the resultant total change in y is known as total differential.

Before explaining the concept of total differential, let us first explain about the concept of differentials.

Differentials:

Let us consider a function y= f(x), where an arbitrary change in the value of x (denoted by Δx) will bring about a corresponding change in the value of y (denoted by Δy). This can be denoted by using the difference quotient $\frac{\Delta y}{\Delta x}$ such that-

$$\Delta y = (\frac{\Delta y}{\Delta x})\Delta x \qquad .....(1)$$

This means the amount of change in y due to a small change in x, Δy can be found if $\frac{\Delta y}{\Delta x}$ and Δx are known.

But, when the value of x is very small, corresponding change in the value of y is small, then the difference quotient will become the derivative $\frac{dy}{dx}$ and the small change in x and y are denoted by dx and dy respectively. Thus, equation (1) can be re-written as-

$$dy = (\frac{dy}{dx})dx$$

$$\Rightarrow dy = f'(X).dX$$

The symbols dx and dy are called differentials of x and y respectively.

To know how to find out dy, let us take an example-

Suppose, we have $y= 10x^3$

Then the differentials of y is-

$$dy=(\frac{dy}{dx})dx$$

$$\Rightarrow dy=30X^2.dX$$

If the value of x and dx are given then we can easily evaluate the dy expression above.

Total differentials:

The concept of total differentials is applied to functions of more than one independent variables. Suppose y= f(x_1, x_2), if the x_1 and x_2 change simultaneously for a smaller amount then the total change in y can be shown as-

$$dy = \frac{\delta y}{\delta x_1} dx_1 + (\frac{\delta y}{\delta x_2}) dX_2$$

$$\therefore dy = f_1 dx_1 + f_2 dx_2 \quad \text{..... ....(2)}$$

The equation (2) is called total differential of the y function.

The process of finding total differentials is known as total differentiation.

Note :
The total differentiation technique can be extended to function having more than two independent variables.

Let us take some examples -

Example 6 :

Find differential dy of the function-

$$y= 10 +2x+ 3x^2$$

Solution:

$$dy=\frac{d}{dx}(10+2x+3x^2)dx$$

$$\Rightarrow dy=(2+6x)dx$$

Example 7 :

Find differential dy of the function

$$y=2x_1x_2+x_1^2-2x_2^2$$

Solution:

$$dy = \frac{\delta y}{\delta x_1}.dx_1 + \frac{\delta y}{\delta x_2}.dx_2$$

$$\Rightarrow dy = \frac{\delta}{\delta x_1}(2x_1x_2 + x_1^2 - 2x_2^2).dx_1 + \frac{\delta}{\delta x_2}(2x_1x_2 + x_1^2 - 2x_2^2).dx_2$$

$$\Rightarrow dy = (2x_2 + 2x_1)dx_1 + (2x_1 - 4x_2)dx_2$$

Example 8 :

Find the total differential of the function-

$$y = 2\log x_1^2 + 10 - x_1x_2^2$$

Solution:

$$dy = \frac{\delta y}{\delta x_1}.dx_1 + \frac{\delta y}{\delta x_2}.dx_2$$

$$= \frac{d}{\delta x_1}(2\log\ x_1^2 + 10 - x_1x_2^2)dx_1 + \frac{d}{\delta x_2}(2\log\ x_1^2 + 10 - x_1x_2^2)dx_2$$

$$= (\frac{2}{x_1^2}.2x_1 - x_2^2)dx_1 + (-2x_1x_2)dx_1$$

$$= (\frac{4}{x_1} - x_2^2)dx_1 - (2x_1x_2)dx_2$$



CHECK YOUR PROGRESS

Q3. Find the differential dy for each of the following functions-

(a) $y= 5x^2 + 10x - 8$

(b) $y= 110x+\sqrt{x}-3x^2$

(c) $y= \frac{x^2+10}{(3x-2)^2}$

Q4. Find the total differential dy for each of the following functions-

(a) $y= x_1^2x_2-2x_1^3+x_2^2$

(b) $y= 2x_1^2x_2^3+x_1x_2+10x_1-3x_2$

5.7 TOTAL DERIVATIVES

In the previous section, we discussed about the technique, where we considered a function with two independent variables such that $y= f(x_1, x_2)$.

But in case of total derivative technique the variables x1 and x2 are not independent. So we have the function-

$y= f(x_1, x_2)$ where $x_2= g(x_1)$

Here, x_1 and x_2 are related via function g. So a change in x_1 will affect y directly (via function f) and also affects indirectly (via function g).

The total derivative is obtained from the total differential $dy= f_1dx_1+ f_2dx_2$ by dividing both sides by dx_1.

$$\therefore \frac{dy}{dx_1} = f_1 \frac{dx_1}{dx_1} + f_2 \frac{dx_2}{dx_1}$$

$$\Rightarrow \frac{dy}{dx_1} = f_1 + f_2 \frac{dx_2}{dx_1}$$

$$\Rightarrow \frac{dy}{dx_1} = \frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2} . \frac{dx_2}{dx_1} \qquad \text{.....(1)}$$

In the above equation (1), the first term on the right hand side of equation (i.e $\frac{dy}{dx_1}$) indicates the direct effect of change in x_1 on y. But the second term $\frac{\partial y}{\partial x_2} . \frac{dx_2}{dx_1}$ measures the change in y resulting from change in x_2 with the change in x_2 resulting from initial change in x_1. It means that the second term measures the indirect effect of x_1 on y.

Let us take an example-

Example 9 :

Given a function y= $x_1^2 + 2x_2$, Where $x_2 = x_1^3 + 5$

Solution:

By applying the total derivatives equation-

$$\Rightarrow \frac{dy}{dx_1} = \frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2} . \frac{dx_2}{dx_1}$$

$$= \frac{\partial}{\partial x_1}(x_1^2 + 2x_2) + \frac{\partial}{\partial x_2}(x_1^2 + 2x_2) . \frac{d}{dx_1}(x_1^3 + 5)$$

$$= 2x_1 + 2.3x_1^2$$

$$\therefore \frac{dy}{dx_1} = 2x_1 + 6x_1^2$$

Example 10 :

Given, y= $3x_1^2 + x_2^2$, Where $x_2 = x_1^2 - 4$

Solution:

$$\frac{dy}{dx_1}=\frac{\partial y}{\partial x_1}+\frac{\partial y}{\partial x_2}.\frac{dx_2}{dx_1}$$
$$=\frac{\partial}{\partial x_1}(3x_1^2+x_2^2)+\frac{\partial}{\partial x_2}(3x_1^2+x_2^2).\frac{d}{dx_1}(x_1^2-4)$$
$$=6X_1+2X_2\times 2X_1$$
$$=6X_1+4X_1X_2$$
$$=6X_1+4X_1(x_1^2-4)$$

Note:

Let us consider a function y= $f(x_1, x_2)$;

where x_1= g(t)

x_2= h(t)

To derive the total derivative, we have to divide the total differentials by dt.

$$\therefore\frac{dy}{dt}=f_1\frac{dx_1}{dt}+f_2\frac{dx_2}{dt}$$

$$\frac{dy}{dt}=\frac{\partial y}{\partial x_1}.\frac{dx_1}{dt}+\frac{\partial y}{\partial x_2}.\frac{dx_2}{dt}$$

5.8 DERIVATIVES OF IMPLICIT FUNCTION

Suppose, we have a function

$y= f(x) = x^2 + 10$ …….. (1)

The above function is known as explicit function. However, the same function can be rewritten as-

$y- x^2 - 10= 0$ …… (2)

The equation (1) is implicitly defined by equation (2). The form of equation (2) is called implicit function. The implicit function can be expressed in general form as F(x, y)=0. Capital letter F is used here to distinguish it from the function f.

The function F represents the left hand side equation (2) whereas f represents the right hand side of equation (1).

Suppose, we have m explanatory variable, in the form-

$y= f(x_1, x_2, x_3,\ldots\ldots\ldots\ldots,x_m)$

then we can write it in the implicit form as-

$F(y, x_1, x_2,\ldots\ldots\ldots\ldots\ldots,x_m) = 0$

We can find out the derivative $(\frac{dy}{dx})$ of implicit function by using implicit function rule-

Implicit function rules:

Let, we have an implicit function in the form-

$$F(x, y) = 0$$

$$\Rightarrow \frac{\partial F}{\partial x}.dx + \frac{\partial F}{\partial y}.dy = 0 \qquad \text{(taking differenti al of both sides)}$$

$$\Rightarrow \frac{\partial F}{\partial y}.dy = -\frac{\partial F}{\partial x}.dx$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \qquad \text{(taking cross multiplica tion)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} \qquad \text{(where } F_x = \frac{\partial F}{\partial x},\ F_y = \frac{\partial F}{\partial y}\text{)}$$

Thus it is seen that the first order derivative $(\frac{dy}{dx})$ of the implicit function F(x, y) = 0 is the negative of the ratio of the two partial derivative $F_x\left(=\frac{\partial F}{\partial x}\right)$ and $F_y\left(=\frac{\partial F}{\partial y}\right)$.

Let us take some examples-

Example 11 :

Find the derivative - $F(x, y)= x^2-xy-2x + 3= 0$

Solution :

Given, $F(x, y)= x^2- xy-2x + 3= 0$

We know that,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}; \qquad \text{Where } F_x = \frac{\partial F}{\partial x}$$

$$F_y = \frac{\partial F}{\partial y}$$

To find out $\frac{dy}{dx}$, we have to require the $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ value.

$$\therefore F_x = \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(x^2 - xy - 2x + 3)$$

$$= 2x - y - 2$$

$$F_y = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(x^2 - xy - 2x + 3)$$

$$= -x$$

$$\therefore \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= \frac{2x - y - 2}{x}$$

Example 12 :

Given $F(x, y) = xy - y - 2x^2= 0$, find $\frac{dy}{dx}$.

Solution :

Here, $F(x, y) = xy - y - 2x^2= 0$

We know that,

$$\frac{dy}{dx} = \frac{F_x}{F_y}$$

$$\therefore F_x = \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(xy - y - 2x^2)$$
$$= y - 4x$$

and

$$F_y = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(xy - y - 2x^2)$$

$$= x - 1$$

We have,

$$xy - y - 2x^2 = 0$$
$$\Rightarrow \quad xy - y = 2x^2$$
$$\Rightarrow \quad y(x-1) = 2x^2$$
$$\Rightarrow \quad y = \frac{2x^2}{x-1}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{y - 4x}{x - 1}$$

$$= -\frac{\frac{2x^2}{x-1} - 4x}{x-1} \qquad \left(\text{substitut ing } y = \frac{2x^2}{x-1}\right)$$

$$= -\frac{\frac{2x^2 - 4x(x-1)}{(x-1)}}{(x-1)}$$

$$= -\frac{\frac{2x^2 - 4x^2 + 4x}{(x-1)}}{(x-1)}$$

$$= -\frac{\frac{-2x^2 + 4x}{(x-1)}}{(x-1)}$$

$$= -\frac{(-2x^2 + 4x)}{(x-1)}.(x-1)$$

$$= -2x^2 + 4x$$



5.9 LET US SUM UP

- Partial derivatives are assigned distinctive symbols. Instead of using the letter 'd' as used in the ordinary derivative, the symbol ∂ (delta) is employed to indicate partial derivative such as $\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2} \ldots \ldots \frac{\partial y}{\partial x_n}$ etc
- In case of differentiation, when we differentiate Y= f(x), then we get f'(x) or $\frac{dy}{dx}$ which is also a function of x.
- Suppose, we have a function-

  $y= f(x_1, x_2)$

- In case of partial differentiation, f_1 measures the rate of change of y with respect to a small change in the value of x_1 assuming x_2 to be constant. Similarly, f_2 represents the rate of change in y with respect to a small change in the value of x_2 holding x_1 to be constant.
- The concept of total differentials is applied to functions of more than one independent variables.
- The total derivative is obtained from the total differential $dy= f_1dx_1 + f_2dx_2$ by dividing both sides by dx_1

$$\frac{dy}{dx_1} = f_1 \frac{dx_1}{dx_1} + f_2 \frac{dx_2}{dx_1}$$

5.10 FURTHER READING

(1) A.C. Chiang. *Fundamental Methods of Mathematical Economics*. Mc Graw Hill.

(2) Mehta & Madnani. *Mathematics for Economist*. Sultan Chand & Sons.

5.11 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q. No. 1 : $f_1 = 3 + \frac{1}{x_2}$ $f_2 = -\frac{x_1}{x_2^2} + 20x_2$

Ans to Q. No. 2 : $f_1 = 12x_1^3 + 12x_1x_2^2$ $f_2 = 12x_1^2x_2 - 2x_1^2 - 12x_2^2$

Ans to Q. No. 3 : a) 10(x+1) dx

b) $\left(110 + \frac{1}{2\sqrt{x}} + 6x\right)dx$

c) $\frac{(3x-2)^2 2x - 6(x^2+10)(3x-2)}{(3x-2)^4}$

Ans to Q. No. 4 : a) $(2x_1x_2 - 6_1^2)dx_1 + (x_1^2 + 2x_2)dx_2$

b) $(4x_1x_2^3 + x_2 + 10)dx_1 + (6x_1^2x_2^2 + x_1 - 3)dx_2$

5.12 MODEL QUESTIONS

Q1. Given y= $4x_1x_2 + x_1^2$ where x_1= $3x_2$ + 5, find out total derivative $\frac{dy}{dx_2}$.

Q2. Find the total derivative $\frac{dy}{dt}$ given y= $2x_1^2 - 5x_1x_2 - 6x_2^2$ where x_1= $3t^2$, x_2= 5 - 2t

Q3. Find $\frac{dy}{dx}$ using the implicit function rule-

a) $2x^2 - x^2y + 10 = 0$

b) $y \log x - \sqrt{x} + 3 = 0$

*** ***** ***

UNIT 6: ECONOMIC APPLICATION OF PARTIAL AND TOTAL DIFFERENTIATION

UNIT STRUCTURE:

6.1 Learning Objectives

6.2 Introduction

6.3 Comparative Static Analysis of Market Model

6.4 Comparative Static Analysis in a National Income Model

6.5 Determination of Partial Elasticities

6.6 Application of Euler's Theorem

6.7 Derivation of Shape of Indifference curves and Isoquants

6.8 Let Us Sum Up

6.9 Further Reading

6.10 Answer to Check Your Progress

6.11 Model Questions

6.1 OBJECTIVES

After going through this unit, you will be able to –

- apply partial differentiation in the analysis of economic models
- apply partial differentiation in deriving the shapes of indifference curves and isoquants
- prove Euler's Theorem.

6.2 INTRODUCTION

Static models are generally related with the static or stationary process. When the static method is used to compare the equilibrium positions of different sets of data, then it is called comparative statics. With the help of comparative static analysis, we can find how the equilibrium value of an endogenous variable with change (both direction and magnitude) as a result of a change in the value of any one of the exogeneous variables or parameters in the model.

Let us discuss, how comparative static analysis are used in economics -

6.3 COMPARATIVE STATIC ANALYSIS OF MARKET MODEL

Let us consider a simple one commodity market model with two equations - Demand and Supply - and an equilibrium equation such that -

$Qd = a - bp$ ($a, b > o$) $\rightarrow$ Demand Function

$Qs = - c + dp$ ($c, d > o$) $\rightarrow$ Supply Function

$Qd = Qs$ ..... (1)

where Qd = quantity demanded

Qs = quantity supplied

P = Price of the Product

Both demand and supply are taken to be functions of price. In the comparative static analysis, we will try to know what happen to equilibrium price or equilibrium quantity when the value of the parameters a,b,c and d changes.

In the above model, a and c denotes the intercept of demand and supply curves respectively. The parameter 'b' and 'd' denotes the slope of the demand and supply curves respectively.

Now, from the model -

$Qd = Qs$

$\Rightarrow a - bP = - c + dP$ [Putting the value of Qd and Qs]

$\Rightarrow bP + dP = a + c$

$\Rightarrow P (b+d) = a + c$

$\Rightarrow \bar{P} = \frac{a+c}{b+d}$ .....(2)

Here, $\bar{P}$ is equilibrium price.

For getting the equilibrium quantity ($\bar{Q}$), we have to substitute ($\bar{P}$) in either demand equation or supply equation -

$$\begin{aligned} & Q_d = a - bP \\ \Rightarrow\ & \bar{Q} = a - b\bar{P} \\ & = a - b\left(\frac{a+c}{b+d}\right) \\ & = \frac{ab + ad - ab - bc}{b+d} \\ \therefore \bar{Q} & = \frac{ad - bc}{b+d} \qquad \text{.....(3)} \end{aligned}$$

Now, to get the information about the effect of parameters (a, b, c and d) on equilibrium price ($\bar{P}$), we take the partial derivatives of $\bar{P}$ in equation (2),

$$\frac{\partial \bar{P}}{\partial a} = \frac{\partial}{\partial a}\left(\frac{a+c}{b+d}\right)$$

$$= \frac{1}{b+d} \qquad (\text{assuming } b, c, d \text{ as constant})$$

$$\frac{\partial \bar{P}}{\partial b} = \frac{\partial}{\partial b}\left(\frac{a+c}{b+d}\right)$$

$$= \frac{(b+d).0 - (a+c).1}{(b+d)^2} \qquad (\text{taking quotient rule})$$

$$= -\frac{(a+c)}{(b+d)^2}$$

$$\frac{\partial \bar{P}}{\partial c} = \frac{\partial}{c}\left(\frac{a+c}{b+d}\right)$$

$$= \frac{1}{b+d}$$

$$\frac{\partial \bar{P}}{\partial d} = \frac{\partial}{\partial d}\left(\frac{a+c}{b+d}\right)$$

$$= \frac{(b+d).0-(a+c).1}{(b+d)^2}$$

$$= -\frac{(a+c)}{(b+d)^2}$$

From the above it is seen that

$$\frac{\partial \bar{P}}{\partial a} = \frac{\partial \bar{P}}{\partial c} > 0 \qquad \text{.....4(a)}$$

$$\frac{\partial \bar{P}}{\partial b} = \frac{\partial \bar{P}}{\partial d} < 0 \qquad \text{.....4(b)}$$

Thus when the intercept of demand and supply increases (i.e. a and c), the equilibrium price ($\bar{P}$) also increases. Again when the slope coefficient of demand and supply (i.e. b and d) increased, the equilibrium price ($\bar{P}$) decreases.

Similarly, taking partial derivative of $\bar{Q}$ in equation (3) to get the information about the effect of parameters (a, b, c, d) on equilibrium quantity-

$$\frac{\partial \bar{Q}}{\partial a} = \frac{a}{\partial a}\left(\frac{ad-bc}{b+d}\right)$$

$$= \frac{d}{b+d}$$

$$\frac{\partial \bar{Q}}{\partial b} = \frac{a}{\partial b}\left(\frac{ad-bc}{b+d}\right)$$
$$= \frac{(b+d).(-c)-(ad-bc).1}{(b+d)^2}$$
$$= \frac{-cb-cd-ad+bc}{(b+d)^2}$$
$$= -\frac{d(a+c)}{(b+d)^2}$$

$$\frac{\partial \bar{Q}}{\partial c} = \frac{\partial}{\partial c}\left(\frac{ad-bc}{b+d}\right)$$
$$= -\frac{b}{b+d}$$

$$\frac{\partial \bar{Q}}{\partial d} = \frac{\partial}{\partial d}\left(\frac{ad-bc}{b+d}\right)$$
$$= \frac{(b+d).(a)-(ad-bc).1}{(b+d)^2}$$
$$= \frac{ab+ad-ad+bc}{(b+d)^2}$$
$$= \frac{ab+bc}{(b+d)^2}$$
$$= \frac{b(a+c)}{(b+d)^2}$$

Thus,

$$\frac{\partial \bar{Q}}{\partial a} > 0,\ \frac{\partial \bar{Q}}{\partial b} < 0,\ \frac{\partial \bar{Q}}{\partial c} < 0,\ \text{and}\ \frac{\partial \bar{Q}}{\partial d} > 0 \quad \text{.....(5)}$$

From the above it is seen that an increase in the intercept of demand function (i.e. a) raise the equilibrium quantity because it follows the second order condition of maximization. While an increase in the intercept of supply curve (i.e. c) reduces the equilibrium quantity because it fulfill the second order condition of minimization.

On the other hand, if the slope coefficient of demand curve (i.e. b) increase then it reduces the equilibrium output (Q) as it follows the second order condition of minimization and if the the slope coefficient of supply curve increase (i.e. d) then it follows the second order condition of maximization and increase the equilibrium output (Q).

6.4. COMPARATIVE STATIC ANALYSIS IN A NATIONAL INCOME MODEL

We know that the growth of national income is affected by several factors like government expenditure, autonomous investment, marginal propensity to consume (mpc), marginal propensity to save(mps), rate of income tax etc. With the help of partial differentiation technique we can determine the direction and magnitude of change in national income as a result of changes in the government expenditure, autonomous investment, m.p.c, m.p.s, etc.

Let us consider a simple national income model with three equations such that -

$$Y = C + I + G. \qquad (1)$$

$$C = \alpha + \beta (Y - T) \quad (\alpha > 0;\ 0 < \beta < 1) \qquad (2)$$

$$T = r + \delta Y \qquad (r > 0;\ 0 < \delta < 1) \qquad (3)$$

where Y is national income, C is consumption expenditure, T is taxes, I is autonomous investment, G is government expenditure and α, β, δ, r are the parameters having appropriate economic interpretations.

In the above model, Y,C and T are the dependent variables and I and G are independent variables.

Note: Here, α = Autonomous Consumption, β = Marginal propensity to consume, r = Non- income tax revenue, δ =Income tax rate

In the above model, we have to find out the equilibrium value of national income, consumption expenditure and tax revenue which is indicated as $\bar{Y}$, $\bar{C}$, and $\bar{T}$ respectively.

We have

$$Y = C+I+G$$

$$\Rightarrow Y = \alpha + \beta (Y - T) + I + G \quad \text{(substituting C value the model)}$$

$$\Rightarrow Y = \alpha + \beta Y - \beta T + I + G$$

$$\Rightarrow Y = \alpha + \beta Y - \beta (r + \delta Y) + I + G \quad \text{(substituting T value from the model equations)}$$

$$\Rightarrow Y = \alpha + \beta Y - \beta r - \beta \delta Y + I + G$$

$$\Rightarrow Y-\beta Y + \beta \delta Y = \alpha - \beta r + I + G$$

$$\Rightarrow (1 - \beta + \beta \delta) Y = \alpha - \beta r + I + G$$

$$\therefore \bar{Y} = \frac{\alpha - \beta r + I + G}{(1-\beta+\beta\delta)} \qquad (4)$$

Now substituting $\bar{y}$ value in the third equation of the model-

$$T = r + \delta\left(\frac{\alpha - \beta r + I + G}{(1-\beta+\beta\delta)}\right)$$

$$T = \frac{r(1-\beta+\beta\delta) + \delta(\alpha - \beta r + I + G)}{(1-\beta+\beta\delta)}$$

$$T = \frac{r - \beta r + r\beta\delta + \alpha\delta - r\beta\delta + \delta I + \delta G}{(1-\beta+\beta\delta)}$$

$$T = \frac{r - \beta r + \alpha\delta + \delta I + \delta G}{(1-\beta+\beta\delta)}$$

$$\therefore \bar{T} == \frac{r(1-\beta) + \delta(\alpha + I + G)}{(1-\beta+\beta\delta)}$$

Similarly, by substituting the $\bar{Y}$ and $\bar{T}$ in the second equation of the model, we found that -

$$C = \alpha + \beta\{\frac{\alpha - \beta r + I + G}{(1-\beta+\beta\delta)} - \frac{r(1-\beta)+\delta(\alpha+I+G)}{(1-\beta+\beta\delta)}\}$$

$$= \frac{\alpha(1-\beta+\beta\delta)+\beta(\alpha-\beta r+I+G)-\beta\{r(1-\beta)+\delta(\alpha+I+G)\}}{(1-\beta+\beta\delta)}$$

$$= \frac{\alpha-\alpha\beta+\alpha\beta\delta+\alpha\beta-\beta^2 r+\beta I+\beta G-\beta r+\beta^2 r-\alpha\beta\delta-\beta\delta I-\beta\delta G}{(1-\beta+\beta\delta)}$$

$$= \frac{\alpha+\beta I+\beta G-\beta r-\beta\delta I-\beta\delta G}{(1-\beta+\beta\delta)}$$

$$= \frac{\alpha+\beta(I+G)-\beta\delta(I+G)-\beta r}{(1-\beta+\beta\delta)}$$

$$= \frac{\alpha+(I+G)(\beta-\beta\delta)-\beta r}{(1-\beta+\beta\delta)}$$

$$\therefore \bar{C} = \frac{\alpha+\beta(I-\delta)(I+G)-\beta r}{(1-\beta+\beta\delta)}$$

Now in order to investigate the effect of small change in I, G, α, β, r and δ on the $\bar{Y}$ (or $\bar{C}$ or $\bar{T}$) we are to find out the partial differentiation of with respect to $\bar{Y}$ (or $\bar{C}$ or $\bar{T}$) with respect to I, G, α, β, r and δ.

Let us see what happens to $\bar{Y}$ if I, G, α, β, r and δ change -

$$\frac{\partial \bar{Y}}{\partial I} = \frac{\partial}{\partial x}(\frac{\alpha-\beta r+I+G}{1-\beta+\beta\delta})$$

$$= \frac{1}{(1-\beta+\beta\delta)} > 0 \qquad (5)$$

$$\frac{\partial \bar{Y}}{\partial G} = \frac{1}{(1-\beta+\beta\delta)} > 0 \qquad (6)$$

$$\frac{\partial \bar{Y}}{\partial \alpha} = \frac{1}{(1-\beta+\beta\delta)} > 0 \qquad (7)$$

$$\frac{\partial \bar{Y}}{\partial \beta} = \frac{(1-\beta+\beta\delta)(-r) - (\alpha - \beta r + I + G)(-1+\delta)}{(1-\beta+\beta\delta)^2}$$

$$= \frac{-r+\beta r-r\delta\beta+\alpha-\beta r+I+G-\alpha\delta+\beta\delta r-\delta I-\delta G}{(1-\beta+\beta\delta)^2}$$

$$= \frac{-r + (1-\delta)(\alpha + I + G)}{(1-\beta+\beta\delta)^2} > 0 \qquad (8)$$

$$\frac{\partial \bar{Y}}{\partial \delta} = \frac{(1-\beta+\delta\beta).0 - (\alpha - \beta r + I + G).\beta}{(1-\beta+\beta\delta)^2}$$

$$= \frac{-\beta(\alpha - \beta r + I + G)}{(1-\beta+\beta\delta)^2}$$

$$= \frac{-\beta \bar{Y}}{(1-\beta+\beta\delta)} < 0 \qquad (9)$$

$$\frac{\partial \bar{Y}}{\partial r} = \frac{(1-\beta+\delta\beta).(-\beta) - (\alpha - \beta r + I + G).0}{(1-\beta+\beta\delta)^2}$$

$$= \frac{-\beta}{(1-\beta+\beta\delta)} < 0 \qquad (10)$$

Similarly, the effects of change in I, G, α, β, r and δ on $\bar{C}$ and $\bar{T}$ can also be investigated by taking partial derivatives of C ? and T ? with respect to I, G, α, β, r and δ.



Check your Progress;

Q1. If a market model is given as :

Q_d=50-2P

Q_s=-10+3P

Q_d=Qs

Then find $\bar{P}$ & $\bar{Q}$

.....

.....

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6.5. DETERMINATION OF PARTIAL ELASTICITIES

In our earlier chapter we discussed that the elasticity of demand is a function of price alone. But when the elasticity of demand is a function of more than one independent variables then we use the partial derivatives to determine the partial elasticities.

Let us assume that quantity demanded is a function of three variables-

$Q_1 = f(P_1, P_2, Y)$;

where Q_1, P_1, P_2 and Y indicates quantity demanded, price of the commodity, price of the substitute good and income of the consumer respectively.

Now, by using the partial derivative in the demand function, we can define the three partial elasticities -

- the own price elasticity of demand,
- the cross price elasticity of demand, and
- the income elasticity of demand.

Own price elasticity of demand:-

The own price elasticity of demand is defined as the ratio of proportionate change in quantity demanded of good 1 to proportionate change in the price of good 1, assumption price of the substitutes or second goods and income remain constant.

$$E_{11} = -\frac{\frac{\partial Q_1}{Q_1}}{\frac{\partial P_1}{P_1}}$$

$$= -\frac{\partial Q_1}{\partial P_1} \cdot \frac{P_1}{Q_1}$$

where $\frac{\partial Q_1}{\partial P_1}$ is the partial differentiation of Q, with respect to P1 .

Cross price elasticity of demand :-

The cross price elasticity of demand is defined as the ratio of the proportionate change in quantity demanded of good-1 to proportionate change in price of good -2, assuming the price of good -1 and income remain constant.

$$E_{12} = \frac{\frac{\partial Q_1}{Q_1}}{\frac{\partial P_2}{P_2}}$$

$$= \frac{\partial Q_1}{\partial P_2} \cdot \frac{P_2}{Q_1}$$

If $E_{12}>0$, then goods -1 and 2 are substitutes of each other, if $E_{12}<0$, they are complementary of each other and if E_{12} is approximately equal to zero, then it implies that there is hardly any relationship between good -1 and good-2.

Income elasticity of demand :-

The income elasticity of demand is defined as the ratio of proportionate change in demand for good -1 to proportionate change in the income of the consumer.

$$E_{1y} = \frac{\frac{\partial Q_1}{Q_1}}{\frac{\partial Y}{Y}}$$

$$= \frac{\partial Q_1}{\partial Y} \cdot \frac{y}{Q_1}$$

Here, $\frac{\partial Q_1}{\partial Y}$ represents the partial derivative of Q_1 with respect to Y.

Example 1:

The demand function for Q_1 commodity is given as $Q_1=50-2P_1+3P_2+0.2Y$. Find

(1) Own price elasticity of demand

(2) Cross price elasticity of demand

(3) Income price elasticity of demand ; when Y= 500, and P_1= 2 and P_2= 5.

Solution:

Given, $Q_1=50-2P_1+3P_2+0.2Y$

$=50-2(2)+3(5)+0.2(500)$

$= 161$

(1) Own price elasticity of demand (E_{11})

$$E_{11} = -\frac{\partial Q_1}{\partial P_1}.\frac{P_1}{Q_1}$$

$$= -\frac{\partial}{\partial P_1}(50 - 2P_1 + 3P_2 + 0.2Y).\frac{P_1}{Q_1}$$

$$= -(-2)\frac{P_1}{Q_1}$$

$$= 2\frac{2}{161}$$

$$= \frac{4}{161}$$

(2) Cross price elasticity of demand (E_{12})

$$E_{12} = \frac{\partial Q_1}{\partial P_2} . \frac{P_2}{Q_1}$$

$$= \frac{\partial}{\partial P_2}(50 - 2P_1 + 3P_2 + 0.2Y). \frac{P_2}{Q_1}$$

$$= 3. \frac{P_2}{Q_1}$$

$$= 3. \frac{5}{161}$$

$$= \frac{15}{161}$$

(3) Income price elasticity of demand (E_{1Y})

$$E_{1y} = \frac{\partial Q_1}{\partial Y} . \frac{Y}{Q_1}$$

$$= \frac{\partial}{\partial Y}(50 - 2P_1 + 3P_2 + 0.2Y). \frac{Y}{Q_1}$$

$$= 0.2 \frac{500}{161}$$

$$= \frac{100}{161}$$

$$\therefore E_{11} = \frac{4}{161}$$

$$E_{12} = \frac{15}{161}$$

$$E_{1Y} = \frac{100}{161}$$



Check Your Progress:

Q2. Find the Own price elasticity, Cross price elasticity of demand and Income price elasticity of demand of the following demand function-

$$Q_1 = 16 - 5P_1 - \frac{1}{4}P_2 + \frac{1}{3}Y$$

where Y= 120, P_1 = 2 and P_2 = 4 (Answer in around 70 words)

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6.6. APPLICATION OF EULER'S THEOREM

Euler's theorem states that if factors of production are paid according to their marginal productivity, the total product will be exhausted.

In a production function -

$$Q = Q(K, L)$$

where K is capital, L is labour and Q is quantity produced. If $\frac{\partial Q}{\partial K}$ and $\frac{\partial Q}{\partial L}$ are the factor prices for capital and labour respectively then

$$Q = \frac{\partial Q}{\partial K}K + \frac{\partial Q}{\partial L}L \qquad \text{(i)}$$

The condition of product exhaustion shown in equation (1) is the core of Euler's theorem. The Euler's theorem is applicable to both Cobb-Douglas (C.D) and constant Elasticity of substitution (CES) production functions when both of them are linearly homogeneous or when both of them assume the operation of constant returns to scale.

The partial derivatives are used in the application Euler's theorem to check whether CD and CES production function satisfies the Eulers theorem or not.

(A) Let us take a Cobb- Douglas(CD) production function-

$$Q = AK^{\alpha}L^{\beta} \qquad (2)$$

where A, α, β are positive parameters.

The equation (2) is a homogeneous production function of degree $(\alpha+\beta)$. If $\alpha+\beta = 1$ then it is linearly homogeneous we can rewriting the equation (2) as-

$$Q = AK^{\alpha}L^{1-\alpha} \qquad (\because \alpha+\beta = 1)$$

Now, taking partial derivatives of Q with respect to K and L, we have

$$\frac{\partial Q}{\partial K} = \frac{\partial}{\partial K}[AK^{\alpha}L^{1-\alpha}]$$
$$= \alpha AK^{\alpha-1}L^{1-\alpha}$$
$$= \frac{\alpha AK^{\alpha}L^{1-\alpha}}{K}$$
$$= \alpha\frac{Q}{K}$$

Similarly,

$$\frac{\partial Q}{\partial L} = \frac{\partial}{\partial L}[AK^{\alpha}L^{1-\alpha}]$$
$$= (1-\alpha)AK^{\alpha}L^{1-\alpha-1}$$
$$= (1-\alpha)\frac{AK^{\alpha}L^{1-\alpha}}{L}$$
$$= (1-\alpha)\frac{Q}{L}$$

Now,

$$\frac{\partial Q}{\partial K}.K+\frac{\partial Q}{\partial L}.L=\alpha.\frac{Q}{K}.K+(1-\alpha)\frac{Q}{L}.L$$
$$=\alpha Q+(1-\alpha)Q$$
$$=Q(\alpha+1-\alpha)$$
$$=Q$$

Hence, CD production function satisfies the Eulers theorem.

(B) Let us consider a linearly homogeneous CES production function-

$$Q=A\left\{\delta K^{-\rho}+(1-\delta)L^{-\rho}\right\}^{-\frac{1}{\rho}};\ A>0;\ 1>\delta>0;\ \rho>-1;$$

where Q, K and L represent output, capital and labour respectively and A, δ, and ρ the parameters.

Taking partial derivatives of Q with respect to K and L, we have-

$$\frac{\partial Q}{\partial K}=\frac{\partial}{\partial K}[A\left\{\delta K^{-\rho}+(1-\delta)L^{-\rho}\right\}^{-\frac{1}{\rho}}]$$
$$=(-\frac{1}{\rho})A\{\delta A^{-\rho}+(1-\delta)L^{-\rho}\}^{-\frac{1}{\rho}-1}\times(-\rho)\delta K^{-\rho-1}$$
$$=\delta A[\delta K^{-\rho}+(1-\delta)L^{-\rho}]^{-\frac{(1+\rho)}{\rho}}\times K^{-(1+\rho)}$$
$$=\delta\frac{A^{1+\rho}}{A^{\rho}}[\delta K^{-\rho}+(1-\delta)L^{-\rho}]^{-\frac{(1+\rho)}{\rho}}\times K^{-(1+\rho)}$$

$$=\frac{\delta}{A^{\rho}}Q^{1+\rho}K^{-(1+\rho)}\qquad\qquad\therefore Q^{1+\rho}=A^{1+\rho}[\delta\delta^{-\rho}+(1-\delta)L^{-\rho}]^{-\frac{(1+\rho)}{\rho}}$$
$$\therefore\frac{\partial Q}{\partial K}=\frac{\delta}{A^{\rho}}\left(\frac{Q}{K}\right)^{1+\rho}$$

Similarly,

$$\frac{\partial Q}{\partial L}=\frac{\partial}{\partial L}[A\{\delta K^{-\rho}+(1-\delta)L^{-\rho}\}^{-\frac{1}{\rho}}]$$

$$= (-\frac{1}{\rho})A\left\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\right\}^{-\frac{1}{\rho}-1} \times (-\rho)(1-\delta)L^{-\rho-1}$$

$$= A\left\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\right\}^{\frac{(1+\rho)}{\rho}} \times (1-\delta)L^{-\rho-1}$$

$$\frac{\partial Q}{\partial L} = (1-\delta)\frac{A^{1+\rho}}{A^{\rho}}[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\frac{(1+\rho)}{\rho}} \times L^{-(1+\rho)}$$

$$= \frac{(1-\delta)}{A\rho}Q^{(1+\rho)}L^{-(1-\rho)} \qquad \because Q^{1+\rho} = A^{(1+\rho)}[\,\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\frac{(1+\rho)}{\rho}}$$

$$\therefore \frac{\partial Q}{\partial K} = \frac{1-\delta}{A\rho}\left(\frac{Q}{L}\right)^{1+\rho}$$

$$\frac{\partial Q}{\partial K}.K + \frac{\partial Q}{\partial L}.L$$

$$= \frac{\delta}{A^{\rho}}\left(\frac{Q}{K}\right)^{1+\rho}.K + \frac{1-\delta}{A^{\rho}}\left(\frac{Q}{L}\right)^{1+\rho}.L$$

$$= \frac{\delta}{A^{\rho}}\left(\frac{Q}{K}\right)^{\rho}(\frac{Q}{K}).K + \frac{1-\delta}{A^{\rho}}\left(\frac{Q}{L}\right)^{\rho}(\frac{Q}{L}).L$$

$$= Q[\frac{\delta}{A^{\rho}}.\frac{Q^{\rho}}{K^{\rho}} + \frac{1-\delta}{A^{\rho}}.\frac{Q^{\rho}}{L^{\rho}}]$$

$$= Q[\frac{\delta}{A^{\rho}}.\frac{A^{\rho}\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}^{-\frac{1}{\rho}\rho}}{K^{\rho}}] + \frac{1-\delta}{A^{\rho}} \times \frac{A^{\rho}\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}^{-\frac{1}{\rho}\rho}}{L^{\rho}}]$$

$$= Q[\frac{\delta}{K^{\rho}\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}} + \frac{(1-\delta)}{L^{\rho}\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}}]$$

$$= Q[\frac{\delta L^{\rho} + (1-\delta)K^{\rho}}{K^{\rho}L^{\rho}\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}}]$$

$$= Q[\frac{\delta L^{\rho} + (1-\delta)K^{\rho}}{\delta K^{-\rho}K^{\rho}L^{\rho} + (1-\delta)L^{-\rho}K^{\rho}L^{\rho}}]$$

$$= Q[\frac{\delta L^{\rho} + (1-\delta)K^{\rho}}{\delta L^{\rho} + (1-\delta)K^{\rho}}]$$

$$\text{or } \frac{\partial Q}{\partial K}.K + \frac{\partial Q}{\partial L}.L = Q$$

Hence, CES production function being linearly homogeneous satisfies the Euler's theorem.

6.7. DERIVATION OF SHAPE OF INDIFFERENCE CURVES AND ISOQUANTS

The concepts of partial derivative and total derivative can be used to find out the shape of an indifference curve and isoquants. The indifference curve is related with the theory of consumers behavior and the isoquant is related with the production function. Both the indifference curve and the isoquant are always convex to the origin. The convexity of these two curves are shown with the help of the partial derivative and total derivative technique.

For getting convexity of a curve, the first order derivative should be negative and the second order derivative should be positive.

Let us consider a utility function -

U= u(x, y) ..... (1)

where x and y are two commodities consumed by the consumer.

The convexity of indifference curve requires-

$$\frac{dy}{dx} < 0 \quad \text{and} \quad \frac{d^2y}{dx} > 0$$

Now, by taking total differential of equation (1), we get-

$$du = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy$$

$$\Rightarrow 0 = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy \qquad [\because \text{Total utility does not change, so } du = 0]$$

$$\Rightarrow \frac{\partial u}{\partial y}.dy = -\frac{\partial u}{\partial x}.dx$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial u / \partial x}{\partial u / \partial y} < 0 \qquad [\text{first order condition fulfilled}]$$

$$= \frac{u_x}{u_y} < 0 \qquad \text{.....(2)}$$

Here, u_x and u_y represents marginal utilities of x and y respectively.

However, marginal utility of x is also a function of x and y, and marginal utility of y is also a function of x and y. So, we can write-

$$u_x = u_x(x, y) \qquad \text{.....(3A)}$$

$$u_y = u_y(x, y) \qquad \text{..... (3B)}$$

Again, the second order condition requires-

$$\frac{d^2y}{dx^2} > 0$$

Now, $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\frac{u_x}{u_y})$$

$$\frac{d^2y}{dx^2} = [\frac{u_y.\frac{d}{dx}(u_x) - u_x.\frac{d}{dx}(u_y)}{u_y^2}] \qquad \text{.....(4)}$$

From the equation 3(A) we found that-

$$du_x = u_{xx}.dx + u_{xy}.dy$$

$$\Rightarrow \frac{d}{dx}(u_x) = u_{xx}\frac{dx}{dx} + u_{xy}\frac{dy}{dx}$$

$$\Rightarrow \frac{d}{dx}(u_x) = u_{xx} + u_{xy}(-\frac{u_x}{u_y}) \qquad [\because \frac{dx}{dx} = 1 \text{and} \frac{dy}{dx} = (-\frac{u_x}{u_y})]$$

$$\Rightarrow \frac{d}{dx}(u_x) = \frac{u_y u_{xx} - u_x u_{xy}}{u_y} \qquad \text{.....(5)}$$

Similarly from 3(B)-

$$du_y = u_{yx}.dx + u_{yy}.dy$$

$$\Rightarrow \frac{d}{dx}(u_y) = u_{yx}\frac{dx}{dx} + u_{yy}\frac{dy}{dx}$$

$$\Rightarrow \frac{d}{dx}(u_y) = u_{yx} + u_{yy}(-\frac{u_x}{u_y})$$

$$\Rightarrow \frac{d}{dx}(u_y) = \frac{u_y u_{yx} - u_x u_{yy}}{u_y} \qquad \text{.....(6)}$$

Now, substituting equation (5) and (6) in (4), we get-

$$= -[\frac{u_y\{\frac{u_y u_{xx} - u_x u_{xy}}{u_y}\} - u_x\{\frac{u_y u_{yx} - u_x u_{yy}}{u_y}\}}{u_y^2}]$$

$$= -[\frac{\frac{u_y^2 u_{xx} - u_x u_y u_{xy} - u_x u_y u_{yx} + u_x^2 u_{yy}}{u_y}}{u_y^2}]$$

$$= \frac{-u_y^2 u_{xx} + u_x u_y u_{xy} + u_x u_y u_{yx} - u_x^2 u_{yy}}{u_y^3}$$

$$\frac{d^2y}{dx^2} = \frac{2u_x u_y u_{xy} - u_x^2 u_{yy} - u_x^2 u_{xx}}{u_y^3} > 0 \quad [\because u_{xy} = u_{yx}]$$

The value of the expression in the numerator of $\frac{d^2y}{dx^2}$ is positive.

Since $u_y > 0$ and so u_y^3 is also positive and $\frac{d^2y}{dx^2} > 0$.

Therefore, it can be concluded that the indifference curve slope downward and convex to the origin.

In similar fashion, the shape of the isoquant can also be defined.

If a production function is given Q = Q (K, L) where K and L are the inputs, then the required two condition will be-

$$\frac{dK}{dL} < 0 \text{ and } \frac{d^2K}{dL^2} > 0$$



6.8 LET US SUM UP

- Static models are generally related with the static or stationary process. When the static method is used to compare the equilibrium positions of different sets of data, then it is called comparative statics.
- With the help of partial differentiation technique we can determine the direction and magnitude of change in national income as a result of changes in the government expenditure, autonomous investment, m.p.c, m.p.s, etc.
- The own price elasticity of demand is defined as the ratio of proportionate change in quantity demanded of good 1 to proportionate change in the price of good 1, assumption price of the substitutes or second goods and income remain constant.
- The cross price elasticity of demand is defined as the ratio of the proportionate change in quantity demanded of good-1 to proportionate change in price of good -2, assuming the price of good -1 and income remain constant.

- Euler's theorem states that if factors of production are paid according to their marginal productivity, the total product will be exhausted.



6.9. FURTHER READING

1) Chiang A.C., Wainwrite K (2013). *Fundamental Methods of Mathematical Economics.* McGraw Hill Education (India) Pvt. Ltd.

2) Mehta, Madnani (1996). *Mathematics for Economics*. S. Chand & Sons.



6.10 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q. No. 1 : $\bar{P} = 12, \bar{Q} = 26$

Ans to Q. No. 2 : $E_{11} = \frac{2}{9}, \quad E_{12} = -\frac{1}{45}, \quad E_{1y} = \frac{8}{9}$



6.11 MODEL QUESTIONS

Q1. Distinguish between static and comparative static analysis.

Q1. Prove that linearly homogeneous production function satisfy the Euler's Theorem.

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UNIT 7: OPTIMIZATION TECHNIQUE – I: UNCONSTRAINED MAXIMA AND MINIMA WITH SINGLE EXPLANATORY VARIABLE

UNIT STRUCTURE

7.1 Learning Objectives

7.2 Introduction

7.3 Relative and Absolute Extrema

7.4 Criteria for Relative Extrema

7.5 Let us Sum Up

7.6. Further Reading

7.7 Answers to Check Your Progress

7.8 Model Questions

7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to –

- discuss relative extrema of a function
- discuss absolute extrema of a function
- compare relative and absolute extrema
- discuss criteria for finding out relative extrema of a function
- find out relative extrema of a function.

7.2 INTRODUCTION

In the earlier units, you have learned the concept of partial and total differentiation and its application in economic. Partial derivatives give the effect of individual independent variable on the dependent variable keeping all other variables as constant. On the other hand, total differentiation gives us the total net effect on the dependent variable when all the independent variables changes simultaneously.

Economics is by and large a science of choice. Optimizing decision behaviour is at the core of most economic analysis. Whether we ask how a consumer should behave, how firms should compete in a market or how

government should decide on their monetary policy, the ability to solve optimization problems is key to finding an answer.

Most economic problems revolve around the problem of choice and we have to choose among alternatives in order to optimize certain objectives. The most common criteria of choice among alternatives in economics is the goal of maximizing or of minimizing. Such maximization or minimization problems can be categorized under the broad heading of 'optimization', meaning "the quest for the best."

7.3 RELATIVE AND ABSOLUTE EXTREMA

'Extrema' meaning simply an extreme value is a common term used to indicate the maximum and minimum values in mathematical languages.

It is possible to have either of the two extreme values – maximum and minimum – or both of them, when we plot a function on graph paper. The high and low points on a graph are called the extrema of the function. An extremum that is higher or lower than any other points nearby is called a relative extremum. A relative extremum (the plural of extremum is called extrema) that is higher than points nearby is called relative maximum. A relative extremum that is lower than any points nearby is called a relative minimum.

The point {c, f (c)} is a relative maximum of a function f if there exists an open interval (a,b) in the domain of f containing c such that $f(x) \leq f(c)$ for all x in (a, b).

The point {c, f (c)} is a relative minimum of a function f if there exists an open interval (a, b) in the domain f of containing c such that $f(x) \leq f(c)$ for all x in (a, b).

With the help of following examples, the relative maximum and minimum has explained. The function f (x) is defined on the interval [0, 7] in the figure 1.

The relative extrema are the highest and lowest points on the graph compared to the points nearby. We can check this by drawing a small circle around each potential relative extrema to illustrate the open interval.

For x values in the small circles, the points (1, -14) and (6,7) are lower than any other points in the immediate neighborhood and are relative mimima. The point (4, 39) is higher than any other points in the neighborhood of points indicated by the circle. This makes (4, 39) a relative maximum. On the other hand, the function f(x) is defined on the interval $(\infty, 8)$ in the figure 2.

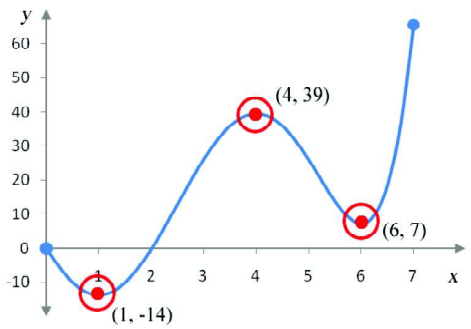


Figure 1

Since the points (5, 10) is higher than any other points in an open interval centered on (5, 10), the relative maximum occurs at that point. There are no troughs in the function so there are no relative minima on the function over this interval.

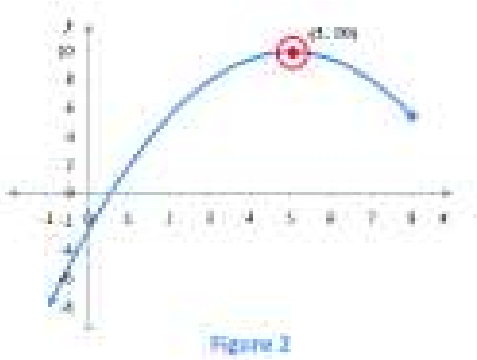


Figure 2

A function's absolute extremum occurs at the highest or lowest point of a function. The highest point on a function is called the absolute maximum and the lowest point on a function is called the absolute minimum.

Let f be a function defined on some interval and c be a number in that interval.

An absolute maximum of the function f occurs at {c, f(c)} if f(x) $\geq$ f(c) for every x value in the interval.

The function f(x) is defined on the interval [0, 7] in the figure 3.

The absolute extrema of f(x) are the highest and lowest points on the graph over the interval [0, 7]

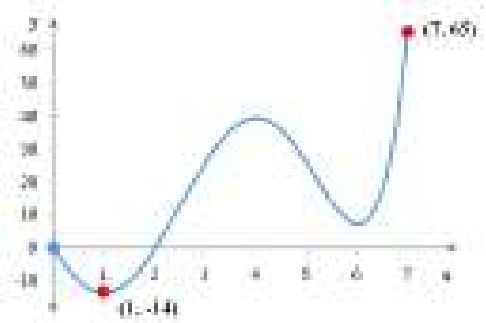


Figure 3

The absolute maximum occurs at the point (7, 65) since it is higher than any other point on the function over the interval [0, 7]. The absolute minimum occurs at the point (1, -14) since it is lower than any other point on the function over the interval [0, 7]

The function f(x) is defined on the interval $(-\infty, 8)$ in the figure 4.

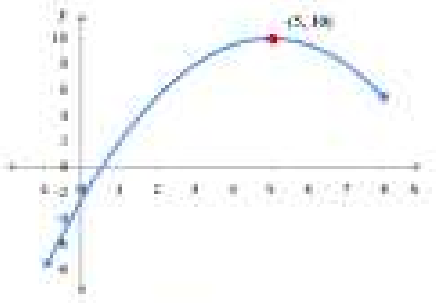


Figure 4

This function is not defined on a closed interval. From the figure 4, it is apparent that the highest point on the graph is (5, 10) so the absolute maximum occurs at that point. The graph continues downward on the left side, so there is no lowest point. Therefore, there is no absolute minimum. Sometimes we may have a function gives a horizontal line. All the values of x in such a function will give the same value of y. In such case all the points of the curve can be considered as maximum value or minimum value of the function- or indeed neither.

7.4 CRITERIA FOR RELATIVE EXTREMA:

Given a function y=f(x) the first derivative f'(x) plays an important role in the determination of its extreme values.

A function y=f(x) is to satisfy two criteria to have relative extreme. The first criterion is known as „necessary condition..

a) The slope of the function y=f(x) should be equal to zero. As the slope of a graph is represented by the first derivatives of the function, hence,

$$\frac{dy}{dx} = f'(x) = 0$$

The second criterion for a function to have a relative extreme is also known as „sufficient condition'.

b) Given the function y=f(x) whose first derivatives f'(x)=0 at the point $x = x_1$, then the value of the function will be a relative maximum if the derivative f'(x) changes its sign from ***positive to negative*** as x increases in the neighborhood of $x = x_1$. Since the direction of change of first derivative can be stated in terms of the sign of the second derivative $f''(x)$, hence the sufficient condition for maximization is that $f''(x)<0$.

Similarly, the function y=f(x), whose first derivatives f'(x)=0 at the point $x = x_2$, then the value of the function will be a relative minimum if the derivative f'(x) changes its sign from ***negative to positive*** as x increases in the neighborhood of $x = x_2$. Since the direction of change of first derivative can be stated in terms of the sign of the second derivative $f''(x)$, hence the sufficient condition for minimization is that $f''(x)>0$.

We can summarize the conditions for the relative extrema (relative maximum and minimum) as follows:

1) A function $y = f(x)$ will attain relative maximum if

$$f'(x) = \frac{dy}{dx} = 0$$

and $$f''(x) = \frac{d^2y}{dx^2} < 0$$

2) A function $y = f(x)$ will attain relative minimum if

$$f'(x) = \frac{dy}{dx} = 0$$

$$and \quad f''(x) = \frac{d^2y}{dx^2} > 0$$

Example 1 : Find the relative extrema of the function $Y=20x-2x^2$

Solution: First, we will find out the first order condition to find out the relative extrema of the function $=20x-2x^2$; i.e.,

$$f'(x) = 0$$

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}(20x - 2x^2) = 0$$

$$\Rightarrow 20 - 4x = 0$$

$$\Rightarrow x = 5$$

Now, we have to check whether the function has maximum or minimum value when $x = 5$.

Hence, the second order condition whether $f''(x) > 0 \, or \, f''(x) < 0 \, for \, x = 5$

Now,

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(20 - 4x) = -4 < 0$$

Therefore, $x = 5$ will give the maximum value of the function $Y = 20x - 2x^2$

So, the maximum value of the function will be $y = 20 \times 5 - 2 \times 5^2 = 100 - 50 = 50$

Example 2 :

Find the relative extrema of the function $y = 2x^3 - 21x^2 + 36x - 40$

Given,

$$y = 2x^3 - 21x^2 + 36x - 40$$

1st order condition for relative extrema is

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{d}{dx}(2x^3 - 21x^2 + 36x - 40) = 0$$

$$\Rightarrow 6x^2 - 42x + 36 = 0$$

$$\Rightarrow x^2 - 7x + 6 = 0$$

$$\Rightarrow x^2 - 6x - x + 6 = 0$$

$$\Rightarrow x(x-6) - (x-6) = 0$$

$$\Rightarrow (x-1)(x-6) = 0$$

$$\Rightarrow (x-1) = 0 \, or \, (x-6) = 0$$

$$\therefore x = 1 \quad 0r \quad x = 6$$

2nd order condition for relative extrema is

$$\frac{d^2y}{dx^2} < 0 \quad \text{(for maximization)}$$

$$\frac{d^2y}{dx^2} > 0 \quad \text{(for maximization)}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(6x^2 - 42x + 36)$$

$$= 12x - 42$$

$$\text{When} x = 6; \text{the} \frac{d^2y}{dx^2} = 12 \times 6 - 42 = 30 > 0$$

$$\text{When} x = 1; \text{the} \frac{d^2y}{dx^2} = 12 \times 1 - 42 = -30 < 0$$

Hence, $x = 6$ at the given function is minimum and $x = 1$ the function is maximum.

When $x = 6$; Minimum value = $2 \times 6^3 - 21 \times 6^2 + 36 \times 6 - 40 = -148$

When $x = 1$; Maximum= $2 \times 1^3 - 21 \times 1^2 + 36 \times 1 - 40 = -23$

Example 3 : Show that $y = x^3 - 6x^2 + 12x + 10$ function has neither a maximum nor a minimum value at $x = 2$.

Solution:

Given,

$$y = x^3 - 6x^2 + 12x + 10$$

The 1st order condition for relative extrema:

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{d}{dx}(x^3 - 6x^2 + 12x + 10) = 0$$

$$\Rightarrow 3x^2 - 12x + 12 = 0$$

$$\Rightarrow x^2 - 4x + 4 = 0$$

$$\Rightarrow (x-2)^2 = 0$$

$$\Rightarrow (x-2) = 0$$

$$\therefore x = 2$$

2nd Order Condition for relative extrema:

$$\frac{d^2y}{dx^2} < 0 \qquad \text{(for maximization)}$$

$$\frac{d^2y}{dx^2} > 0 \qquad \text{(for maximization)}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2 - 12x + 12)$$

$$= 6x - 12$$

When $x = 2$, then $\frac{d^2y}{dx^2} = 6 \times 2 - 12 = 0$

Although, the 1st order condition for relative extrema has satisfied, but the 2nd order condition has not satisfied at $x = 2$; i.e. the function has neither maximum nor minimum value at $x - 2$

Example 4 : Find the extreme values of the following function

$$y = x^3 - 9x^2 + 15x + 20$$

Solution:

Given,

$$y = x^3 - 9x^2 + 15x + 20$$

1st Order Condition for extreme value:

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{d}{dx}(x^3 - 9x^2 + 15x + 20) = 0$$

$$\Rightarrow 3x^2 - 18x + 15 = 0$$

$$\Rightarrow x^2 - 6x + 5 = 0$$

$$\Rightarrow x^2 - 5x - x + 5 = 0$$

$$\Rightarrow x(x-5) - (x-5) = 0$$

$$\Rightarrow (x-1)(x-5) = 0$$

$$\Rightarrow (x-1) = 0 \; or \; (x-5) = 0$$

$$\therefore \; x = 1 \; or \; x = 5$$

2nd Order condition for extreme value:

$$\frac{d^2y}{dx^2} < 0 \qquad \text{(for maximization)}$$

$$\frac{d^2y}{dx^2} > 0 \qquad \text{(for maximization)}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2 - 18x + 15)$$

$$= 6x - 18$$

Now,

When $x = 1;\ \frac{d^2y}{dx^2} = 6 \times 1 - 18 = -12 < 0$ and $x = 1;$ will maximize the function. The maximum value of the function,

$$Y = 1^3 - 9(1)^2 + 15 \times 1 + 20$$
$$= 27$$

When $x = 5$; $\frac{d^2y}{dx^2} = 6 \times 5 - 18 = 12 > 0$ and $x = 5$ will minimize the function and the minimum value of the function

$$x = 5 \qquad Y = 5^3 - 9(5)^2 + 15 \times 5 + 20$$
$$= -5$$

Example 5 : Show that the maximum value of $x + \frac{1}{x}$ is less than the minimum value.

Solution:

Given,

$$x + \frac{1}{x}$$

1st order condition for extreme value:

$$\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{d}{dx}(x + \frac{1}{x}) = 0$$
$$\Rightarrow 1 - x^{-2} = 0$$
$$\Rightarrow 1 - \frac{1}{x^2} = 0$$
$$\Rightarrow \frac{1}{x^2} = 1$$
$$\Rightarrow x^2 = 1$$
$$\Rightarrow x = \pm 1$$
$$\therefore \; x = 1 \; or \; x = -1$$

2nd Order Condition form extreme value:

$$\frac{d^2y}{dx^2} < 0 \qquad \text{(for maximization)}$$

$$\frac{d^2y}{dx^2} > 0 \qquad \text{(for maximization)}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(1 - x^{-2}) = 2x^{-3} = \frac{2}{x^3}$$

When $x = 1,\ \frac{d^2y}{dx^2} = \frac{2}{1^3} = 2 > 0$ hence, the function has minimum value at $x = 1$ and the minimum value is $Y = 1 + \frac{1}{1^3} = 2$

When $x = -1, \frac{d^2y}{dx^2} = \frac{2}{(-1)^3} = -2 < 0$ hence, the function has maximum value at $x = -1$ and the maximum value is $Y = -1 + \frac{1}{(-1)^3} = -2$

Hence, the maximum value of the function (-2) is less than the minimum value (2).



CHECK YOUR PROGRESS

Q1: Find out the extreme values of the following function

$$y = \frac{1}{3}x^2 + 4x + 10$$

Q2: Find out the extreme values of the following function

$$y = x^3 - 9x^2 + 15x + 20$$



7.5 LET US SUM UP

- Economics is by and large a science of choice. Optimizing decision behaviour is at the core of most economic analysis.

- Maximization or minimization problems can be categorized under the broad heading of 'optimization', meaning "the quest for the best."

- A function $y = f(x)$ is to satisfy two criteria to have relative extreme. The first criterion is known as .necessary condition :

  The slope of the function $y = f(x)$ should be equal to zero. As the slope of a graph is represented by the first derivatives of the function, hence,

  $$\frac{dy}{dx} = f'(x) = 0$$

  The second criterion for a function to have a relative extreme is also known as 'sufficient condition':

  Given the function $y = f(x)$ whose first derivatives $f'(x) = 0$ at the point $x = x_1$, then the value of the function will be a relative maximum if the derivative $f'(x)$ changes its sign from ***positive to negative*** as x increases in the neighborhood of $x = x_1$. Since the direction of change of first derivative can be stated in terms of the sign of the second derivative $f''(x)$, hence the sufficient condition for maximization is that $f''(x) < 0$.

- We can summarize the conditions for the relative extrema (relative maximum and minimum) as follows:

  A function $y = f(x)$ will attain relative maximum if

  $$f'(x) = \frac{dy}{dx} = 0$$

  $$and\ f''(x) = \frac{d^2y}{dx^2} < 0$$

A function $y = f(x)$ will attain relative minimum if

$$f'(x) = \frac{dy}{dx} = 0$$

$$and\ f''(x) = \frac{d^2y}{dx^2} > 0$$

7.6 FURTHER READING

1) Baruah, S (2011). *Basic Mathematics and Its Applications in Economics.* Macmillan.

2) Chiang, A.C. (1986). *Fundamental Methods of Mathematical Economics*. NewYork: McGraw Hill.

3) Allen, R.G.D. (1974). *Mathematical Analysis for Economists*. London : Macmillan Press and ELBS.

7.7 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No. 1 Maximum at $x = 1$ and minimum at $x = 4$

Ans to Q No. 2 Maximum at $x = 1$ and minimum at $x = 4$

7.8 MODEL QUESTIONS

Q 1: Find the extreme values of the following function

a) $y = -2x^2 + 20x + 8$

b) $y = x^3 - 24x^2 + 20$

c) $y = x^2 + 15x - 20$

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UNIT 8: ECONOMIC APPLICATIONS OF UNCONSTRAINED MAXIMA AND MINIMA WITH SINGLE EXPLANATORY VARIABLE

UNIT STRUCTURE

8.1 Learning objectives

8.2 Introduction

8.3 Cost Minimization

8.4 Profit Maximization

8.5 Maximization of Tax revenue

8.6 Taxation and Monopoly Output

8.7 Let us Sum Up

8.8 Further Reading

8.9 Answers to Check Your Progress

8.10 Model Questions

8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to–

- find out the output level of a firm which minimizes its cost
- find out the output level which maximizes the firm's profit
- find out the rate of tax which will maximize government tax revenue
- find out monopoly output under different tax rate.

8.2 INTRODUCTION

In the earlier unit, we have discussed, how to find out absolute and relative extrema without any constraints. Also, we discussed the technique for finding out maxima and minima of a function. The use of unconstrained maxima and minima with single explanatory variable is extensive in economics. Some of the common economic problems and their solution with the help of unconstrained maxima and minima with single explanatory variable are discussed in the unit.

8.3 COST MINIMIZATION

Each producer or a firm has a basic objective to minimise cost of production in order to maximize profit. The conditions for minimization (as explained in the earlier unit) can apply to find out the level of output at which the cost of the producer will be minimum.

Let us a total cost function of a firm as given below.

$$TC = aQ^2 + bQ + C$$

Where Q is output and C is fixed cost and a, b and c are positive parameter. The average cost, i.e., the per unit cost is given by

$$AC = \frac{TC}{Q} = aQ + b + \frac{C}{Q}$$

In order to find out the output level at which average cost (AC) will be minimum; we have to satisfy the following two conditions as shown below.

$$\frac{d(AC)}{dQ} = 0 \; and \; \frac{d^2(AC)}{dQ^2} > 0$$

$$Now, \frac{d(AC)}{dQ} = \frac{d}{dQ}\left(aQ + b + \frac{C}{Q}\right) = 0$$

$$\Rightarrow a + 0 - \frac{C}{Q^2} = 0$$

$$\Rightarrow \frac{C}{Q^2} = a$$

$$\Rightarrow Q^2 = \frac{C}{a}$$

$$\Rightarrow Q = \pm\sqrt{\frac{C}{a}}$$

$$\therefore Q = +\sqrt{\frac{C}{a}} \; or \; -\sqrt{\frac{C}{a}}$$



Total Cost (TC) is the expenses incurred in producing a particular amount of output. Total cost of a firm can further be defined as the summation of Total Fixed Cost (TFC) and Total Variable Cost (TVC).

The Average Cost (AC) is the total cost dived by the number of units of output produced. It is cost per unit of output. Thus,

$$Ac = \frac{TC}{Q}$$

$$Now, \quad \frac{d^2(AC)}{dQ^2} = \frac{d}{dQ}\left\{\frac{d(AC)}{dQ}\right\} = \frac{d}{dQ}(a + 0 - \frac{C}{Q^2})$$

$$= 0 - (-2)cQ^{-3} = \frac{2C}{Q^3}$$

$$When\ Q = \sqrt{\frac{C}{a}}, = \frac{d^2(AC)}{dQ^2} = \frac{2C}{\left(\sqrt{\frac{c}{a}}\right)^3} > 0;\ \text{since a} > 0\ and\ C > 0$$

$$When\ \ Q = -\sqrt{\frac{C}{a}}, \quad \frac{d^2(AC)}{dQ^2} = \frac{2C}{\left(\sqrt{\frac{c}{a}}\right)^3} < 0;$$

$$\text{since}\ \ a > 0\ and\ C > 0$$

Thus, the average cost will be minimum at $Q = \sqrt{\frac{c}{a}}$ Similarly, we can also establish the relationship between average variable cost and marginal cost. The marginal cost passes through the minimum point of average variable cost from below.

In figure 7.1, the MC cuts the AVC at P where AVC is minimum.

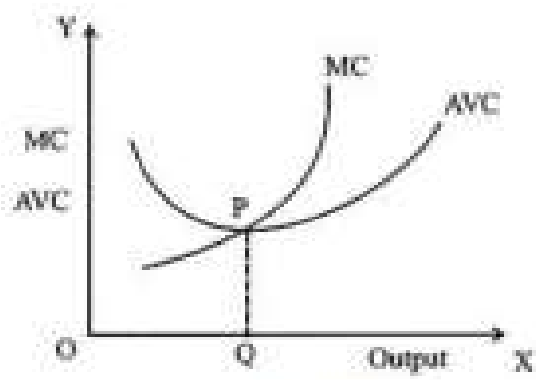


Figure 7.1

We can derive the following relation

a) When AVC is declining, MC is below AVC.

$$\frac{d}{dQ}(AVC) < 0; \quad MC < AVC \qquad (1)$$

The Average Variable Cost (AVC) is variable cost per unit of output. The AVC is the total variable cost divided by the number of units of output produced. Therefore,

$$AVC = \frac{TVC}{Q}$$

The Marginal Cost (MC) is the addition to the total cost caused by producing one more unit of output. In other words, MC is the addition to the total cost of producing units of output instead of $(n-1)$. In symbols,

Since MC is a change in total cost as a result of a unit change in output, it can also be written as:

$$MC = \frac{\Delta TC}{\Delta Q}$$

b) When AVC is increasing, MC is above AVC.

$$\frac{d}{dQ}(AVC) > 0; \quad MC > AVC \qquad (2)$$

c) When AVC is minimum, MC equals AVC.

$$\frac{d}{dQ}(AVC) = 0; \quad MC = AVC \qquad (3)$$

We know that,

$$\mathrm{TC} = \mathrm{TVC} + \mathrm{TFC}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{dQ}}(\mathrm{TC}) = \frac{\mathrm{d}}{\mathrm{dQ}}(\mathrm{TVC}) + \frac{\mathrm{d}}{\mathrm{dQ}}(\mathrm{TFC})$$

$$\Rightarrow \mathrm{MC} = \frac{\mathrm{d}}{\mathrm{dQ}}(\mathrm{TVC}) \quad (\mathrm{Since} \frac{\mathrm{d}}{\mathrm{dQ}}(\mathrm{TFC}) \mathrm{is\,zero, as\,TFC\,is\,constant})$$

$$\Rightarrow \mathrm{MC} = \frac{\mathrm{d}}{\mathrm{dQ}}(\mathrm{AVC}.\mathrm{Q}) \quad (\mathrm{Since} \mathrm{AVC} = \frac{\mathrm{TVC}}{\mathrm{Q}})$$

$$\Rightarrow \mathrm{MC} = \mathrm{Q}.\frac{\mathrm{d}}{\mathrm{dQ}}(\mathrm{AVC}) + \mathrm{AVC}.\frac{\mathrm{d}}{\mathrm{dQ}}(\mathrm{Q}) \quad (\mathrm{by\ using product method})$$

$$\Rightarrow \mathrm{MC} = \mathrm{Q}.\frac{\mathrm{d}}{\mathrm{dQ}}(\mathrm{AVC}) + \mathrm{AVC}$$

$$\Rightarrow \mathrm{Q}.\frac{\mathrm{d}}{\mathrm{dQ}}(\mathrm{AVC}) = \mathrm{MC} - \mathrm{AVC}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{dQ}}(\mathrm{AVC}) = \frac{\mathrm{MC} - \mathrm{AVC}}{\mathrm{Q}}$$

Now, from equation (1)

$$\Rightarrow \frac{d}{dQ}(AVC) < 0$$

$$\Rightarrow \frac{MC - AVC}{Q} < 0$$

$$\Rightarrow MC - AVC < 0$$

$$\Rightarrow MC < AVC$$

Thus, when AVC is declining, MC is below AVC.

From eq. (2)

$$\Rightarrow \frac{d}{dQ}(AVC) > 0$$
$$\Rightarrow \frac{MC - AVC}{Q} > 0$$
$$\Rightarrow MC - AVC > 0$$
$$\Rightarrow MC > AVC$$

Thus, when AVC is increasing, MC is above AVC.

From eq. (3)

$$\Rightarrow \frac{d}{dQ}(AVC) = 0$$
$$\Rightarrow \frac{MC - AVC}{Q} = 0$$
$$\Rightarrow MC - AVC = 0$$
$$\Rightarrow MC = AVC$$

Thus, when AVC is minimum, MC is equal to AVC.

Example 1 : Find the output level at which the average variable cost is minimum from the following total cost function.

$$TC = 2Q^3 - 20Q^2 + 18$$

Solution: The total variable cost is given by

$$TVC = 2Q^3 - 2Q^2 \qquad (TVC \text{ changes with output})$$
$$\Rightarrow AVC = \frac{TVC}{Q}$$
$$\Rightarrow AVC = \frac{2Q^3 - 2Q^2}{Q}$$
$$\Rightarrow AVC = 2Q^2 - 2Q$$

Minimization of AVC requires the following two conditions:

$$1)\frac{d}{dQ}(AVC)=0 \qquad \& \qquad ii)\frac{d^2(AVC)}{dQ^2}>0$$

Now,

$$\frac{d}{dQ}(AVC)=0$$

$$\Rightarrow \frac{d}{dQ}(2Q^2-2Q)=0$$

$$\Rightarrow 4Q-2=0$$

$$\Rightarrow Q=0.5$$

And $$\frac{d^2(AVC)}{dQ^2}=\frac{d}{dQ}(4Q-2)=4>0$$

Hence, $Q=0.5$ is the level of output which will minimize AVC.

Example 2 : Derive AVC function and show that when AVC is minimum, AVC=MC from the following function

$$TC=Q^3-7Q^2+20Q+16$$

Also, derive the AC function and check that when $Q=4$ the AC is minimum and at that level of output $MC=AC$

Solution: Given,

$$TC=Q^3-7Q^2+20Q+16$$

The Total Variable Cost (TVC) is given as

$$TVC=Q^3-7Q^2+20Q$$

$$AVC=\frac{TVC}{Q}$$

$$AVC=\frac{Q^3-7Q^2+20Q}{Q}$$

$$AVC=Q^2-7Q+20$$

Minimisation of AVC requires the fulfillment of the following two conditions:

$$i)\frac{d}{dQ}(AVC)=0 \quad \& \quad ii)\frac{d^2(AVC)}{dQ^2}>0$$

Now,

$$\frac{d}{dQ}(AVC)=0$$
$$\Rightarrow \frac{d}{dQ}(Q^2-7Q+20)=0$$
$$\Rightarrow 2Q-7=0$$
$$\Rightarrow Q=3.5$$

And

$$\frac{d^2AVC}{dQ^2}$$
$$\Rightarrow \frac{d^2(AVC)}{dQ^2}=\frac{d}{dQ}(2Q-7)=2>0$$

Thus, When $Q=3.5$ the AVC will be minimum.

Now, $$MC=\frac{d(TC)}{dQ}=\frac{d}{dQ}(Q^3-7Q^2+20Q+16)$$
$$=3Q^2-14Q+20$$

When $Q=3.5$

$$MC=3\times(3.5)^2-14(3.5)+20=7.75$$
$$AVC=(3.5)^2-7\times(3.5)+20=7.75$$

And

Thus, $\text{MC} = \text{AVC, at the minimum point of AVC}$

2nd part of the problem:

Given,

$$TC = Q^3 - 7Q^2 + 20Q + 16$$

The Total Cost (TC) is given as

$$AC = \frac{TC}{Q}$$

$$AC = \frac{Q^3 - 7Q^2 + 20Q + 16}{Q}$$

$$AC = Q^2 - 7Q + 20 + \frac{16}{Q}$$

Minimisation of AC requires the fulfillment of the following two conditions:

$$i)\ \frac{d}{dQ}(AC) = 0 \quad \& \quad ii)\ \frac{d^2(AC)}{dQ^2} > 0$$

$$\frac{d}{dQ}(AC) = 0$$

$$\Rightarrow \frac{d}{dQ}\left(Q^2 - 7Q + 20 + \frac{16}{Q}\right) = 0$$

$$\Rightarrow 2Q - 7 - \frac{16}{Q^2} = 0$$

$$\frac{d^2(AC)}{dQ^2}$$

$$\Rightarrow \frac{d^2(AC)}{dQ^2} = \frac{d}{dQ}\left(2Q - 7 - \frac{16}{Q^2}\right)$$

$$\Rightarrow 2 + \frac{32}{Q^3} > 0$$

When $Q = 4$; $\frac{d}{dQ}(AC) = 2Q - 7 - \frac{16}{Q^2} = 2 \times 4 - 7 - \frac{16}{(4)^2} = 0$

And $\frac{d^2(AC)}{dQ^2} = 2 + \frac{32}{Q^3} = 2 + \frac{32}{4^3} > 0$

Now at $Q = 4; \quad MC = 3Q^2 - 14Q + 20 = 3 \times 4^2 - 14 \times 4 + 20 = 12$

$$AC = Q^2 - 7Q + 20 + \frac{16}{Q} = 4^2 - 7 \times 4 + 20 + \frac{16}{4} = 12$$

Thus, $MC = AC$ when $Q = 4$



CHECK YOUR PROGRESS

Q1: Given the total cost function of a firm,

$TC = 4Q^2 + 7Q + 81$

a) Find the level of output where AC is minimum.

b) Chech whether MC= AC when AC is minimum..

Q2: Find the level of output at which the average variable cost (AVC) is minimum from the following function

$TC = Q^3 - 6Q^2 + 2Q + 50$

Also show that AVC=MC at that level of output.

Q3: The total cost function of a firm is given as follows

$TC = \propto Q^2 + \beta Q + \gamma$

a) Find AC

b) Find level of output where AC will be minimum

8.4 PROFIT MAXIMIZATION

The attempt to maximize profit by a producer is regarded as a rational behaviour. A producer or an entrepreneur has to choose the combination of price and quantity to maximize profit. In order to maximize profit, a firm must equate marginal cost and marginal revenue. Let us show the mathematical derivation of this condition

Profit, denoted by π is the difference between total revenue (TR) and total cost (TC). On the other hand, both TR and TC are function of Output (Q), as TR and TC changes with the change in output. Therefore, profit can be expressed as

$$\pi = TR - TC$$
$$\pi = R(Q) - C(Q)$$

To find out the profit maximizing output level, the following two conditions must be satisfied-

$$i) \frac{d\pi}{dQ} = 0$$
$$ii) \frac{d^2\pi}{dQ^2} < 0$$

Now,

$$\frac{d\pi}{dQ} = 0$$
$$\frac{d}{dQ}\{R(Q) - C(Q)\} = 0$$
$$\Rightarrow R'(Q) - C'(Q) = 0$$
$$\Rightarrow R'(Q) = C'(Q)$$

Thus the optimum output (equilibrium output) $\overline{Q}$ must satisfy the equation $R'(Q) = C'(Q)$ or MR=MC.

The first order condition explained above may lead to maximum or minimum. Therefore, we have to check the second order condition next.

$$\frac{d^2\pi}{dQ^2} = \pi''(Q)$$
$$\Rightarrow R''(Q) - C''(Q) < 0$$

If and only if $R''(Q) < C''(Q)$ i.e. the slope of MR { $R''(Q)$ } should be less than slope of MC { $C''(Q)$ }

Example 3 : The total revenue (TR) and total cost (TC) functions are given below

$$R(Q) = 1200Q - 2Q^2$$
$$C(Q) = Q^3 - 61.25Q^2 + 1528.5Q = 2000$$

Find the profit maximizing output.

Solution: The Profit function is

$$\pi(Q) = R(Q) - C(Q)$$
$$= (1200Q - 2Q^2) - (Q^3 - 61.25Q^2 + 1528.5Q + 2000)$$
$$= -Q^3 + 59.25Q^2 - 328.5Q - 2000$$

Maximisation of the profit function requires the fulfillment of following two conditions

$$i)\frac{d\pi}{dQ} = 0 \qquad and \quad ii)\frac{d^2\pi}{dQ^2} < 0$$

Now,

$$=> \frac{d}{dQ}(-Q^3 + 59.25Q^2 - 328.5Q - 2000) = 0$$
$$=> -3Q^2 + 118.5Q - 328.5 = 0$$
$$\therefore Q = \frac{-118.5 \pm \sqrt{(-118.5)^2 - 4.(-3).(-328.5)}}{2(-3)}$$
$$= \frac{-118.5 \pm \sqrt{1404225 - 3942}}{-6}$$
$$= \frac{-118.5 \pm \sqrt{1010025}}{-6}$$
$$= \frac{-118.5 \pm 100.5}{-6}$$
$$\therefore Q = 36.5 \quad or \quad 3$$

Total Revenue (TR) is the revenue earned by selling a particular amount of output. It changes with changes in output (Q). Thus,

$$TR = f(Q)$$

It can be calculated by multiplying total quantity sales by its price. Thus,

$$TR = P \times Q$$

Where, P is price.

Total Cost (TC) is the cost of producing a particular amount of output. It also changes with output. Thus,

$$TC = f(Q)$$

And

$$\frac{d^2\pi}{dQ^2}$$
$$= \frac{d}{dQ}\left(\frac{d\pi}{dQ}\right)$$
$$= \frac{d}{dQ}(-3Q^2 + 118.5Q - 328.5)$$
$$= -6Q + 118.5$$

Now, when

$$Q = 36.5 \quad \frac{d^2\pi}{dQ^2} = -6 \times (36.5) + 118.5 = -219 + 118.5 = -100.5 < 0$$

Now, when

$$Q = 3 \quad \frac{d^2\pi}{dQ^2} = -6 \times (3) + 118.5 = -18 + 118.5 = 100.5 > 0$$

Hence, the profit maximizing level of output is 36.5 units and the maximum profit will be

$$= -(36.5)^3 + 59.25(36.5)^2 - 328.5 \times 36.5 - 2000$$
$$= 16{,}318.44$$

Example 4 : A monopolist has the following total revenue (TR) and total cost (TC) functions

$$TR = 30Q - Q^2$$
$$TC = Q^3 - 15Q^2 + 10Q + 100$$

Find

a) Profit Maximising output
b) Maximum Profit
c) Point elasticity of demand at equilibrium level of output.

Solution:

a) The Profit function is

$$\pi = TR - TC$$
$$\pi = 30Q - Q^2 - (Q^3 - 15Q^2 + 10Q + 100)$$
$$\pi = -Q^3 + 14Q^2 + 20Q - 100$$

Maximisation of the profit function requires the fulfillment of following two conditions

$$i)\frac{d\pi}{dQ} = 0 \qquad and \quad ii)\frac{d^2\pi}{dQ^2} < 0$$

Now,

$$\frac{d\pi}{dQ} = 0$$
$$\frac{d}{dQ}(-Q^3 + 14Q^2 + 20Q - 100) = 0$$
$$\Rightarrow -3Q^2 + 28Q + 20 = 0$$
$$\Rightarrow 3Q^2 - 28Q - 20 = 0$$
$$\Rightarrow Q = \frac{28 \pm \sqrt{(-28)^2 - 4 \times 3 \times (-20)}}{2 \times 3}$$
$$\Rightarrow \frac{28 \pm \sqrt{1024}}{6}$$
$$\Rightarrow Q = 10 \quad or -\frac{2}{3}$$

And

$$\frac{d^2\pi}{dQ^2}$$
$$= \frac{d}{dQ}\left(\frac{d\pi}{dQ}\right)$$
$$= \frac{d}{dQ}(-3Q^2 + 28Q + 20)$$
$$= -6Q + 28$$

When

$$Q = 10 \qquad \frac{d^2\pi}{dQ^2} = -6\times 10 + 28 = -32 < 0$$

When

$$Q = -\frac{2}{3}, \qquad \frac{d^2\pi}{dQ^2} = -6\times\left(-\frac{2}{3}\right) + 28 = 32 > 0$$

Thus, the profit maximizing output $Q = 10$

b) The maximum profit

$$\pi = -(10)^3 + 14(10)^2 + 20\times 10 - 100 = -1000 + 1400 + 200 - 100 = 500$$

c) Point elasticity of demand is defined as

$$e = \frac{AR}{MR - MR}$$

Thus, $$AR = \text{Price} = \frac{TR}{Q} = \frac{30Q - Q^2}{Q} = 30 - Q$$

$$MR = \frac{d}{dQ}(TR) = \frac{d}{dQ}(30Q - Q^2) = 30 - 2Q$$

At profit maximizing output $Q = 10$, $$AR = 30 - 10 = 20$$

$$MR = 30 - 2\times 10 = 10$$

Hence, $$e = \frac{20}{20-10} = 2$$

Example 5 : In a perfectly competitive market, the market price of a commodity is Rs. 3 per unit of output (Q). The total cost function of a firm is given by

$$TC = \frac{1}{3}Q^3 - 4Q^2 + 10Q + 10$$

Find a) Profit maximizing output.

b) Maximum profit.

Solution: a) As the price remains fixed under perfect competition, the total

revenue (TR) is given by the multiplication of price with output. Thus,

$$TR = P \times Q = 3Q$$

The profit function is

$$\pi = TR - TC$$
$$\pi = 3Q - \frac{1}{3}Q^3 + 4Q^2 - 10Q - 10$$

Maximization of the profit function requires the fulfillment of following two conditions

$$i)\frac{d\pi}{dQ} = 0 \qquad and \quad ii)\frac{d^2\pi}{dQ^2} < 0$$

Now, $\frac{d\pi}{dQ} = 0$

$$\Rightarrow \frac{d}{dQ}\left(3Q - \frac{1}{3}Q^3 + 4Q^2 - 10Q - 10\right) = 0$$
$$\Rightarrow 3 - Q^2 + 8Q - 10 = 0$$
$$\Rightarrow Q^2 - 8Q + 7 = 0$$
$$\Rightarrow Q^2 - 7Q - Q + 7 = 0$$
$$\Rightarrow Q(Q-7) - (Q-7) = 0$$
$$\Rightarrow (Q-1)(Q-7) = 0$$
$$\Rightarrow Q = 1 \ \ or \ \ 7$$

Now,

$$\frac{d^2\pi}{dQ^2}$$
$$= \frac{d}{dQ}\left(\frac{d\pi}{dQ}\right)$$
$$= \frac{d}{dQ}(3 - Q^2 + 8Q - 10)$$
$$= -2Q + 8$$

When $Q = 1$;

$$\frac{d^2\pi}{dQ^2} = -2 \times 1 + 8 = 6 > 0$$

When $Q = 7$;

$$\frac{d^2\pi}{dQ^2} = -2 \times 7 + 8 = -6 < 0$$

Thus, the profit maximizing output is 7 units.

b) The maximum profit will be

$$\pi = 3 \times 7 - \frac{1}{3}(7)^3 + 4(7)^2 - 10 \times 7 - 10$$

$$\pi = 22.67$$



CHECK YOUR PROGRESS

Q4: The average revenue (AR) and total cost (TC) functions are

$$AR = 294 - Q$$

$$TC = 200 - 6Q + 14Q^2$$

a) Profit maximizing output.

b) Maximum profit

c) Point elasticity of demand at profit maximizing output.

Q5: A monopolist average revenue (AR) function and total cost (TC) functions are given below

$$AR = 16 - 2Q$$

$$TC = 40 - 2Q + 6Q + \frac{1}{3}Q^3$$

Find a) Profit maximizing output and price.

b) Maximum profit

c) Point elasticity of demand at equilibrium output level.

Q6: The total cost function of a firm is given by

$TC = PQ = (\beta - aq^2)Q$

Where q is the quantity and the demand function is given by

$p = \beta - aq^2$; where P is the price

Find out the profit maximizing output.

8.5 MAXIMIZATION OF TAX REVENUE

When a government imposes taxes (excise duty or sales tax) on producer, one purpose of the government is to maximize the tax revenue. Under such a situation, government has to decide the rate of tax (t) so that the tax revenue (T) is maximized.

We will discuss the maximization of tax revenue by the government under two different cases:

1) Maximization of tax revenue in a competitive market.
2) Maximization of tax revenue under imperfect competition.

Maximization of tax revenue in a competitive market:

When a government decides to maximize tax revenue in a competitive market, it has to find the appropriate rate of tax (t) which will maximize its total tax revenue (T).

Let us explain find the solution with the help of simplest form of competitive market as follows

$$Q_d = a - bP \quad (a, b > 0) \qquad (1)$$
$$Q_s = -C + dP \quad (c, d > 0) \qquad (2)$$
$$Q_d = Q_s \qquad (3)$$

Where $Q_d, Q_s \text{ and } p$ are quantity demanded, quantity supplied and price respectively.

Let us suppose, government imposes a tax rate 't' per unit of output. Under such circumstances, the supply function will depend on pre-tax price. Thus, the new supply function will be

$$Q_s = -c + d(p - t) \quad (c, d > 0,\ 0 < t < 1$$

Now, from the equilibrium function (3), we have

$$\begin{aligned} & a - bP = -c + d(p - t) \\ \Rightarrow & a - bP = -c + dP - dt \\ \Rightarrow & a + c + dt = bP + dP \\ \Rightarrow & a + c + dt = P(b + d) \\ \Rightarrow & \overline{P} = \frac{a + c + dt}{b + d} \end{aligned}$$

Substituting the value of $\overline{P}$ in equation (1), we have the equilibrium output.

$$\begin{aligned} & \overline{Q} = a - b\overline{P} \\ \Rightarrow & \overline{Q} = a - b\left(\frac{a + c + dt}{b + d}\right) \\ \Rightarrow & \overline{Q} = \frac{a(b + d) - b(a + c + dt)}{(b + d)} \\ & = \frac{ab + ad - ab - bc - bdt}{(b + d)} \\ & = \frac{ad - bc}{(b + d)} - \frac{bdt}{(b + d)} \end{aligned}$$

Now, $T = t.\overline{Q}$ *(Total tax revenue is obtained by multiplying tax rate 't' and equilibrium output)*

$$T = t.\left\{\frac{ad - bc}{b + d} - \frac{bdt}{(b + d)}\right\}$$

$$T = \frac{ad - bc}{(b + d)}t - \frac{bd}{(b + d)}t^2$$

Maximization of tax revenue requires the fulfillment of the following two condtions:

$$i)\frac{dT}{dt}=0 \qquad and \quad ii)\frac{d^2T}{dt^2}<0$$

Now,

$$\frac{dT}{dt}=0$$

$$\Rightarrow \frac{d}{dt}\{\frac{ad-bc}{b+d}t-\frac{bd}{(b+d)}t^2\}=0$$

$$\Rightarrow \frac{ab-bc}{(b+d)}-2.\frac{bd}{(b+d)}t=0$$

$$\Rightarrow \frac{ad-bc-2bdt}{(b+d)}=0$$

$$\Rightarrow ad-bc-2bdt=0$$

$$\Rightarrow 2bdt=ad-bc$$

$$\Rightarrow t=\frac{ad-bc}{2bd}$$

And

$$\frac{d^2T}{dt^2}=\frac{d}{dt}\left(\frac{dT}{dt}\right)=\frac{d}{dt}\left\{\frac{ad-bc}{(b+d)}-2.\frac{bd}{(b+d)}t\right\}=-\frac{2bd}{(b+d)}<0 \quad (\text{since b, d}>0)$$

Hence, the tax rate which maximize tax revenue is $\frac{(ad-bc)}{2bd}$

Example 6 : Given the competitive market of the form

$$Q_d=20-3P$$
$$Q_s=-10+2P$$
$$Q_d=Q_s;$$

where P is price.

If the government imposes a tax of t per unit of output, find the value for which the tax revenue will be maximum. Also find out equilibrium price after the imposition of tax.

Solution: With the imposition of a tax of per unit of output, the supply function becomes

$$Q_s = -10 + 2(P - t) \qquad (1)$$

Now from equilibrium condition, we have,

$$20 - 3P = -10 + 2(P - t)$$
$$\Rightarrow 20 - 3P = -10 + 2P - 2t$$
$$\Rightarrow 5P = 30 + 2t$$
$$= \overline{P} = \frac{30 + 2t}{5} \qquad (2)$$

Substituting $\overline{P}$ in the demand function, we get the equilibrium output ($\overline{Q}$)

$$Q_d = 20 - 3(\frac{30 + 2t}{5})$$
$$Q_d = 20 - \frac{90 + 6t}{5}$$
$$Q_d = \frac{100 - 90 - 6t}{5}$$
$$Q_d = \frac{10 - 6t}{5} \qquad (3)$$

The maximization of T, tax revenue, requires the fulfillment of the following two conditions

$$i)\ \frac{dT}{dt} = 0 \quad and \quad ii)\ \frac{d^2T}{dt^2} < 0$$

Now,

$$\frac{dT}{dt} = 0$$
$$\Rightarrow \frac{d}{dt}\left(\frac{10t - 6t^2}{5}\right) = 0$$
$$\Rightarrow \frac{1}{5}(10 - 12t) = 0$$
$$\Rightarrow t = \frac{10}{12} = 0.83$$



Tax Rate:

A tax rate is the percentage at which an individual or corporation is taxed. The tax rate is the tax imposed by the federal government and some States based on an individual's taxable income or corporation's earnings. In our country, a progressive tax rate system is followed, where the percentage of tax increases as taxable income increase.

Tax Revenue:

Tax revenue is the income gained by the government through taxation.

Now, to get the equilibrium price, we substitute the value of *t* in equation (2), we have

$$\overline{P} = \frac{30 + 2(0.83)}{5} = 6.33$$



CHECK YOUR PROGRESS

Q7: Given the competitive market of the following form

$$Q_d = 16 - 2P$$
$$Q_s = -4 + 2P$$
$$Q_d = Q_s$$

If the government imposes tax at the rate of per unit of output, find the value for which the tax revenue will be maximum.

Q8: In a competitive market with the following demand, supply and equilibrium equations,

$$Q_d = 18 - 2P$$
$$Q_s = -7 + 3P$$
$$Q_d = Q_s$$

If the government imposes taxes, find out the rate of taxes which will maximize tax revenue of the government.

Maximization of tax revenue under imperfect competition:

When a specific rate of tax is imposed on an imperfectly competitive firm, then the firm's total cost of production increases. An imposition of tax is added to total cost function and as result of it the profit of the firm reduces. Here in this section, we will discuss the implication of imposition of tax on profit under imperfect competition.

The total tax revenue obtained by the government depends on the equilibrium output (Q) after imposition of tax at the rate *'t'* such that *T=tQ*

Let us the assume the total revenue (TR) and total cost (TC) are given as follows-

$$TR = R(Q) = aQ - bQ^2 \qquad (a, b > 0)$$
$$TC = C(Q) = cQ^2 + dQ + e \qquad (c, d, e > 0)$$

After imposition of tax *'t'*, the profit function will be

$$\pi = TR - (TC + tax)$$
$$\pi = aQ - bQ^2 - cQ^2 - dQ - e - tQ \qquad (1)$$

Maximization of the profit function requires the fulfillment of following two conditions

$$i)\frac{d\pi}{dQ} = 0 \qquad and \qquad ii)\frac{d^2\pi}{dQ^2} < 0$$

Now,

$$\frac{d\pi}{dQ} = 0$$
$$\Rightarrow \frac{d}{dQ}(aQ - bQ^2 - cQ^2 - dQ - e - tQ) = 0$$
$$\Rightarrow a - 2bQ - 2cQ - d - t = 0$$
$$\Rightarrow a - 2(b + c)Q - (d + t) = 0$$
$$\Rightarrow \overline{Q} = \frac{a - (d + t)}{2(b + c)} = \frac{a - d - t}{2(b + c)} \qquad (2)$$

And

$$\frac{d^2\pi}{dQ^2} = \frac{d}{dQ}\left(\frac{d\pi}{dQ}\right) = \frac{d}{dQ}(a - 2bQ - 2cQ - d - t) = -2b - 2c = -2(b + c) < 0$$
$$(Since\ b, c > 0)$$

Thus, the value of $\overline{Q}$ given by equation (2) maximize profit of the firm under imperfect completion.

$$\therefore Total\ tax\ revenue\ T = t\overline{Q} = t\left\{\frac{a-d-t}{2(b+c)}\right\} = \frac{(a-b)t-t^2}{2(b+c)}$$

Now, the maximization of T, tax revenue, requires the fulfillment of the following two conditions

$$i)\frac{dT}{dt} = 0 \qquad and \qquad ii)\frac{d^2T}{dt^2} < 0$$

Now,

$$\frac{dT}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}\left\{\frac{(a-b)t-t^2}{2(b+c)}\right\} = 0$$

$$\Rightarrow \frac{a-b-2t}{2(b+c)} = 0$$

$$\Rightarrow a-b-2t = 0$$

$$\Rightarrow t = \frac{(a-b)}{2}$$

And

$$\frac{d^2T}{dt^2} = \frac{d}{dt}\left(\frac{dT}{dt}\right) = \frac{d}{dt}\left\{\frac{a-b-2t}{2(b+c)}\right\} = -\frac{2}{2(b+c)} < 0$$

$$(since\ b, c > 0)$$

Example 7 : A monopolist has the total revenue (AR) and total cost (TC) function

$$AR = 21 - 4Q$$

$$TC = 3Q^2 - 12Q + 14$$

Where Q is quantity of output

A tax at the rate t is imposed on the output. Find

i) The rate of tax which will maximize tax revenue of the government.

ii) The profit maximizing output and price after payment of tax.

Solution:

Given,

$$AR = 21 - 4Q \qquad (1)$$

Total revenue (TR) is

$$TR = AR.Q = (21 - 4Q).Q = 21Q - 4Q^2 \qquad (2)$$

i) When rate of tax *'t'* is imposed, profit function becomes

$$\pi = (TR - tax) - TC$$
$$\pi = 21Q - 4Q^2 - tQ - 3Q^2 + 12Q - 14$$
$$\pi = 33Q - 7Q^2 - tQ - 14$$

Maximization of the profit function requires the fulfillment of following two conditions

$$i)\frac{d\pi}{dQ} = 0 \qquad and \qquad ii)\frac{d^2\pi}{dQ^2} < 0$$

Now,

$$\frac{d\pi}{dQ} = 0$$
$$\Rightarrow \frac{d}{dQ}(33Q - 7Q^2 - tQ - 14) = 0$$
$$\Rightarrow 33 - 14Q - t = 0$$
$$\Rightarrow 14Q = 33 - t$$
$$\Rightarrow Q = \frac{33 - t}{14}$$

And

$$\frac{d^2\pi}{dQ^2} = \frac{d}{dQ}\left(\frac{d\pi}{dQ}\right) = \frac{d}{dQ}(33 - 14Q - t) = -14 < 0$$

Hence, the profit maximizing output after tax $\overline{Q} = \frac{33-t}{14}$ (3)

Now, the total tax revenue, $T = t\overline{Q} = t\left(\frac{33-t}{14}\right) = \frac{33t - t^2}{14}$

Now, the maximization of T, tax revenue, requires the fulfillment of the following two conditions

$$i)\frac{dT}{dt} = 0 \qquad and \qquad ii)\frac{d^2T}{dt^2} < 0$$

Now,

$$\frac{dT}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{33t - t^2}{14}\right) = 0$$

$$\Rightarrow \frac{1}{14}(33 - 2t) = 0$$

$$\Rightarrow t = \frac{33}{2} = 16.5$$

And

$$\frac{d^2T}{dt^2} = \frac{d}{dt}\left(\frac{dT}{dt}\right) = \frac{d}{dt}\left\{\frac{1}{14}(33 - 2t)\right\} = -\frac{2}{14} < 0$$

Hence, the tax rate at which total revenue will be maximum is 16.5 per unit.

ii) Substituting the rate of tax in equation (3), $\overline{Q} = \frac{33 - 16.5}{14} = 1.18$

Now, substituting the value of profit maximizing output Q in equation (1), we have

$$AR\ or\ \text{Price} = 21 - 4 \times 1.18 = 16.28$$



CHECK YOUR PROGRESS

Q9: The total revenue (TR) and total cost (TC) function of a monopolist are given below:

$TR = 46Q - 3Q^2$

$TC = 2Q^2 - 4Q + 10$ Where Q is quantity of output

A tax at the rate *'t'* is imposed on the output.

Find a) the tax rate which will maximize total tax revenue of the government.

b) the monopolist's maximum profit after payment of tax and profit maximizing output and the price at which the product is sold.

Q10: A monopolist has the following demand and total cost function

$P = 100 - Q$

$TC = 50 - 2Q + 4Q^2$

Where P is price and Q is quantity of output. The government imposes a tax at the rate of Rs.2 per unit of output. Find

a) Profit maximizing output.

b) What will the effect of tax on equilibrium output, price and profit?

8.6 TAXATION AND MONOPOLY OUTPUT

Government can imposes different types of indirect taxes on firm. Depending on the nature of the tax, both price and output of a firm may change. It may remain unaltered in some cases. A government can imposes lump-sum tax, profit tax, ad valorem tax or sales tax. Here, in this section of the unit, we will discuss the impact of different taxes on a monopolist's price and output. In case of lump-sum or a profit tax, a monopolist profit will reduce but it will not alter his optimum price quantity combination. Whereas, in case of ad-valerom or sales tax- it will reduce both profit and output of the monopolist and increases the price.

Let the profit function of a monopolist with the total revenue (TR) and total cost (TC) function as follows:

$$\pi = TR - TC$$
$$\pi = R(Q) - C(Q)$$

$$Where, TR = R(Q); TC = C(Q)$$

Now, Profit maximization requires:

$$\frac{d\pi}{dQ} = 0 \quad or \quad R'(Q) - C'(Q) = 0 \qquad (1)$$
$$or \; R'(Q) = C'(Q) \qquad (2)$$
$$\frac{d^2\pi}{dQ^2} = R''(Q) - c''(Q) < 0$$

Lump-sum tax:

Let us assume a lump-sum tax (T) is imposed on the monopolist. So, the monopolist profit after tax will be

$$\pi = R(Q) - C(Q) - T \qquad (3)$$

Now, the profit maximization represented by equation (3) requires:

$$\frac{d\pi}{dQ} = 0 \quad or \quad R'(Q) - C'(Q) = 0 \qquad (4)$$

$$or \; R'(Q) = C'(Q) \qquad (5)$$

$$\frac{d^2\pi}{dQ^2} = R''(Q) - C''(Q) < 0$$

Thus, both the conditions of monopoly equilibrium remain same even after imposition of lump-sum tax. Thus, imposition of lump-sum tax will not alter equilibrium price-output combination.

Profit tax:

Now, let us suppose government imposes a profit tax at the rate of *t* on the profit of the monopolist. Thus the profit function of the monopolist after imposition of a profit tax will become

$$\pi = R(Q) - C(Q) - t\{R(Q) - C(Q)\}$$

$$\pi = (1-t)\{R(Q) - C(Q)\}$$

Now, profit maximization requires:

$$\frac{d\pi}{dQ} = 0$$

$$\Rightarrow \frac{d}{dQ}[(1-t)\{R(Q) - C(Q)\}] = 0$$

$$\Rightarrow (1-t)[R'(Q) - C'(Q)] = 0 \quad (\text{since } t \text{ is positive fraction})$$

$$\Rightarrow R'(Q) = C'(Q) \qquad (6)$$

And $$\frac{d^2\pi}{dQ^2} = (1-t)[R''(Q) - C''(Q)] < 0 \quad (\text{since } t \text{ is a positive fraction})$$

$$= R''(Q) - C''(Q) < 0$$

Here, profit maximizing conditions are not disturbed too even after imposition of a profit tax.

Specific sales tax:

Now, let us discuss the impact of specific sales tax on monopolist output and price. Specific sales tax means sales tax on quantity.

Let the government imposes a specific sales tax of Rs. β per unit. Thus, the total sales tax will be βQ . Thus, the profit function after payment of specific sales tax will be

$$\pi = R(Q) - C(Q) - \beta Q \qquad (7)$$

Maximization of (7) requires:

$$\begin{aligned} \frac{d\pi}{dQ} &= \frac{d}{dQ}\{R(Q) - C(Q) - \beta Q\} = 0 \\ &\Rightarrow R'(Q) - C'(Q) - \beta = 0 \\ &\Rightarrow R'(Q) = C'(Q) + \beta \qquad (8) \\ &\Rightarrow MR = MC + \beta \end{aligned}$$

The second order condition is that

$$\frac{d^2\pi}{dQ^2} = \frac{d}{dQ}\left(\frac{d\pi}{dQ}\right) = \frac{d}{dQ}\{R'(Q) - C'(Q) - \beta\} = R''(Q) - C''(Q) < 0$$

Thus, the monopolist maximizes profit after tax by equating MR with MC plus tax rate per unit of output.

In order to investigate the impact on output when tax rate increases, let us take the total differentials of equation (9).

$$\begin{aligned} &R''(Q).dQ = C''(Q).dQ + d\beta \\ &\Rightarrow [R''(Q) - C''(Q)]dQ = d\beta \\ &\Rightarrow \frac{dQ}{d\beta} = \frac{1}{[R''(Q) - C''(Q)]} < 0 \end{aligned}$$

$$Since [R''(Q) - C''()Q\} < 0$$

Thus, imposition of specific sales tax results in a smaller output and higher price. The same is also true in case of **ad velorem sales tax.**

Example 8 : The total revenue and total cost function of a monopoly firm is as follows

$$R(Q) = 270Q - 5Q^2$$
$$C(Q) = 4Q^2 - 8Q + 200, \qquad where\ Q\ is\ output$$

If the government imposes a sale tax of Rs. 8 per unit of output, find

a) Profit maximizing price, output and maximum profit after tax.
b) The impact of sale tax on equilibrium price, output and profit of the firm.

Solution:

Given,

$$R(Q) = 270Q - 5Q^2$$
$$C(Q) = 4Q^2 - 8Q + 200, \text{Where Q is output}$$
$$\text{Total sales tax} = 8Q$$

Profit function before tax:

The profit function will be

$$\pi = R(Q) - C(Q)$$
$$\pi = 270Q - 5Q^2 - 4Q^2 + 8Q - 200$$
$$\pi = 278Q - 9Q^2 - 200$$

Conditions for profit maximization-

$$\frac{d\pi}{dQ} = 0 \qquad and \frac{d^2Q}{d\pi^2} < 0$$

Now,

$$\frac{d\pi}{dQ} = 0$$

$$\frac{d}{dQ}(278Q - 9Q^2 - 200) = 0$$

$$\Rightarrow 278 - 18Q = 0$$

$$\Rightarrow Q = 15.4$$

And $$\frac{d^2Q}{d\pi^2} = \frac{d}{dQ}\left(\frac{d\pi}{dQ}\right) = \frac{d}{dQ}(278 - 18Q) - 18 < 0$$

Thus, *Q=15.4* is the profit maximizing output.

Hence, the maximum profit,

$$\pi = 278 \times 15.4 - 9(15.4)^2 - 200 = 1946.76$$

Also, $$AR = \frac{TR}{Q} = \frac{270Q - 5Q^2}{Q} = 270 - 5Q$$

Substituting the value of *Q=15.4* in *AR* function, we will get the price before tax

$$AR = P = 270 - 5 \times 15.4 = 193$$

Profit function after tax:

The profit function after tax will be

$$\pi = R(Q) - C(Q) - tax$$
$$\pi = 270Q - 5Q^2 - 4Q^2 + 8Q - 200 - 8Q$$
$$\pi = 270Q - 9Q^2 - 200$$

Conditions for profit maximization-

$$\frac{d\pi}{dQ} = 0 \qquad and \quad \frac{d^2Q}{d\pi^2} < 0$$

Now,

$$\frac{d\pi}{dQ} = 0$$

$$\Rightarrow \frac{d}{dQ}(270Q - 9Q^2 - 200) = 0$$

$$\Rightarrow 270 - 18Q = 0$$

$$\Rightarrow Q = 15$$

And $\frac{d^2Q}{d\pi^2} = \frac{d}{dQ}\left(\frac{d\pi}{dQ}\right) = \frac{d}{dQ}(270 - 18Q) = -18 < 0$

Hence, the profit maximizing output after tax will be 15.

Thus, the maximum profit after tax will be $\pi = 270 \times 15 - 9(15)^2 - 200 = 1825$

Also, $AR = \frac{TR}{Q} = \frac{270Q - 5Q^2}{Q} = 270 - 5Q$

Substituting the value of *Q=15* in *AR* function, we will get the price after tax

$$AR = P = 270 - 5 \times 15 = 195$$

Thus, imposition of sale tax results increase in price from Rs. 193 to Rs.195, reduce output from 15.4 units to 15 units and reduce profit from 1946.76 to 1825.



CHECK YOUR PROGRESS

Q 11: The average revenue (AR) and total costs (TC) functions of a monopolist are given by

$$AR = 100 - 2Q$$
$$TC = 50 - 4Q + 2Q^2$$

The government also imposes a specific sale tax of Rs.8 per unit of output. Examine the effect of tax on equilibrium output, equilibrium price and profit.

Q 12: A monopolist has the following demand and total cost functions

$$P = 100 - Q$$
$$TC = 50 - 2Q + 4Q^2$$

The government also imposes a specific sale tax of Rs.2 per unit of output. Examine the effect of tax on equilibrium output, equilibrium price and profit.



8.7 LET US SUM UP

- When AVC is declining, MC is below AVC.

$$\frac{d}{dQ}(AVC) < 0; \qquad MC < AVC$$

When AVC is increasing, MC is above AVC.

$$\frac{d}{dQ}(AVC) > 0; \qquad MC > AVC$$

When AVC is minimum, MC equals AVC.

$$\frac{d}{dQ}(AVC) = 0; \qquad MC = AVC$$

- Maximization of tax revenue requires the fulfillment of the following two condtions:

$$i)\frac{dT}{dt}=0 \quad \text{and} \quad ii)\frac{d^2T}{dt^2}<0$$

- A government can imposes lump-sum tax, profit tax, ad valorem tax or sales tax.

- Imposition of specific sales tax results in a smaller output and higher price. The same is also true in case of ad velorem sales tax.

8.8 FURTHER READING

1) Baruah, S. (2011). *Basic Mathematics and Its Applications in Economics*. Macmillan.

2) Chiang, A.C. (1986). *Fundamental Methods of Mathematical Economics.* NewYork : McGraw Hill.

3) Allen, R.G.D. (1974). *Mathematical Analysis for Economists.* London: Macmillan Press and ELBS.

8.9 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No. 1: *Q=3, AC is minimum*

Ans to Q No. 2: Q=3

Ans to Q No. 3: $Q=\sqrt{\frac{\gamma}{\alpha}}$; AC is minimum.

Ans to Q No. 4: Q=10, $\pi(\max.)=1300$, ed= 28.4

Ans to Q No. 5: $\pi(\max.)=203$, Q=9, , ed=$\frac{2}{27}$

Ans to Q No. 6: $Q=\frac{-2a+\sqrt{4a^2-12a(b-\beta)}}{6a}$

Ans to Q No. 7: *t=3*

Ans to Q No. 8: $t = \frac{10}{3}$

Ans to Q No. 9: $a) t = 25,\ b) \pi = 21.25$

Ans to Q No. 10: $a) t = 12,\ \ b) \overline{Q} = 1.5 \,\mathrm{Pr}\, ice = 17, \mathrm{Pr}\, ofit = 15$

Ans to Q No. 11: Output lowers from 13 to 12, price increase from 74 to 76 and reduces profit from 626 to 526.

Ans to Q no 12: Output lowers by 0.2, price increase by 0.2 and profit reduce by 20.2.

8.10 MODEL QUESTIONS

Q1: Given the total cost function of a firm, $\mathrm{TC} = 4\mathrm{Q}^2 + 7\mathrm{Q} + 81, \mathrm{find\ the\ level\ of}$ output at where average cost is minimum. Also test whether at that output marginal cost equals to average cost.

Q 2: The total cost function of a firm under perfect competition is

$$TC = 2Q^2 + 4Q + 30$$

If the price of output is fixed at 40 per unit, find

a) Profit maximizing output level;

b) Maximum Profit

Q 3: In a competitive market with the following demand, supply and equilibrium equations,

$$Q_d = 18 - 2P$$
$$Q_s = -7 + 3P$$
$$Q_d = Q_s$$

If the government imposes a tax, find out the rate of tax which will maximize tax revenue of the government.

Q 4: A monopolist has the total revenue and total cost functions are:

$TR = 44Q - 5Q^2$

$TC = 4Q^2 - 6Q - 20$ where Q is output.

If the government imposes an excise duty, find-

a) the rate of tax which will maximize total excise revenue of the government.

b) the amount of maximum profit, price and output after imposition of excise duty.

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