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# MATHEMATICAL METHODS FOR ECONOMIC ANALYSIS

## SEMESTER - I

# ECONOMICS

**BLOCK - 2**

**KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY**

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## BLOCK INTRODUCTION

The second block of the course on mathematical methods of economic analysis has 7 units. Condition for optimisation in case of three explanatory variables, unconstrained maxima and minima (with more than one explanatory variable) are explained in unit 9. Unit 10 deals with equilibrium of discriminating monopoly. This unit also deals with equilibrium of multi-product firm and equilibrium of multi-plant firm. Unit 11 deals with the concept of constrained extrema. The lagrange multiplier method is also explained in this unit. Unit 12 explains the economic applications of constrained optimisation.This unit discuss the utility maximisation and consumer behaviour, comparative static analysis of utility maximisation and the least cost combination of inputs. Unit 13 deals with the concepts of integration and its basic rules. Unit 14 explains elaborately the economic applications of integrals. Unit 15 is on first order linear differential equation with constant coefficient and constant term. Economic applications of first order differential equation and first order difference equation are explained in this unit.

While going through a unit, you will notice some along-side boxes, which have been included to help you to know some of the difficult, unseen terms. Again, we have included some relevant concepts in "LET US KNOW" along with the text. And, at the end of each section, you will get "CHECK YOUR PROGRESS" questions. These have been designed to self-check your progress of study. It will be better if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with "ANSWERS TO CHECK YOUR PROGRESS" given at the end of each unit.

# UNIT 9: OPTIMIZATION TECHNIQUE – II: UNCONSTRAINED MAXIMA AND MINIMA WITH MORE THAN ONE EXPLANATORY VARIABLE

**UNIT STRUCTURE**


9.1 Learning Objectives

9.2 Introduction

9.3 Unconstrained Maxima and Minima (with more than one explanatory variable)

9.4 Condition for Optimization In Case Of Three Explanatory Variables

9.5 Let us Sum Up

9.6 Further Reading

9.7 Answers to Check Your Progress

9.8 Model Questions


## 9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to–

- put forward the meaning of optimization technique
- calculate unconstrained maxima and minima with more than one explanatory variable.

## 9.2 INTRODUCTION

In the previous unit, we have discussed the technique of finding relative extrema (maxima and minima) of a function with single explanatory variable. But in reality, many economic problems of optimization contain more than one explanatory variable. For example, price and output determination of discriminating monopolist, equilibrium output of a multi-plant firm, equilibrium output of a multi-product firm.

In such cases, although the basic principle remains unchanged but the technique of optimization changed.

In the in the following section of the unit, we will discuss the optimization technique of unconstrained maxima and minima with more than one variable.

## 9.3 UNCONSTRAINED MAXIMA AND MINIMA (WITH MORE THAN ONE EXPLANATORY VARIABLE)

Let us assume a function with more than one explanatory variable as follows

$$Z = f(x, y) \qquad (9.1)$$

Equation (9.1) has to fulfill two conditions for attaining **maximization**. They are

a) Necessary Condition:

The 1st order condition or the necessary condition is given by the differential of the function and equating it with zero. i.e.

$$dZ = 0 \qquad (9.2)$$

b) Sufficient Condition:

The 2nd order condition or the sufficient condition is given by the second order differential of

$$d^2Z < 0 \text{ for maximizati on and}$$

Let us explain these two conditions,

The total differential of the function (9.1) is given by

$$dZ = \frac{\partial z}{\partial x}.dx + \frac{\partial z}{\partial y}.dy = 0 \qquad (\text{according to } 1^{st} \text{ order condition})$$

$$or \;\; dZ = f_x.dx + f_y.dy = 0 \qquad (\text{where } f_x = \frac{\partial z}{\partial x} \; and \; f_y = \frac{\partial z}{\partial y})$$

$$dZ = 0 \text{ if and only if } \frac{\partial z}{\partial x} = f_x = 0 \text{ and } \frac{\partial z}{\partial y} = f_y = 0$$

The differential version of second order condition of maximization is that $d^2Z<0$

$$d^2Z = d(dZ)$$

$$=> \frac{\partial(dZ)}{\partial x}.dx + \frac{\partial(dZ)}{\partial y}.dy$$

$$=> \frac{\partial}{\partial x}\left\{\frac{\partial Z}{\partial x}.dx + \frac{\partial Z}{\partial y}.dy\right\}.dx + \frac{\partial}{\partial y}\left\{\frac{\partial Z}{\partial x}.dx + \frac{\partial Z}{\partial y}.dy\right\}.dy$$

$$=> \left\{\frac{\partial^2 Z}{\partial x^2}.dx + \frac{\partial^2 Z}{\partial y \partial x}\right\}dx + \left\{\frac{\partial^2 Z}{\partial y \partial x}dx + \frac{\partial^2 Z}{\partial y^2}dy\right\}dy$$

$$=> \frac{\partial^2 Z}{\partial x^2}dx^2 + \frac{\partial^2 Z}{\partial y \partial x}dx.dy + \frac{\partial Z}{\partial x \partial y}dx.dy + \frac{\partial^2 Z}{\partial y^2}dy^2$$

$$=> f_{xx}dx^2 + f_{yx}dx.dy + f_{xy}dx.dy + f_{yy}dy^2$$

$$=> f_{xx}dx^2 + 2f_{xy}dx.dy + f_{yy}dy^2 \qquad (\sin ce\ f_{xy} = f_{yx}) \quad (9.3)$$

$$Where, f_{xx} = \frac{\partial^2 z}{\partial x^2},\ f_{yx} = \frac{\partial^2 z}{\partial y \partial x},\ f_{xy} = \frac{\partial z}{\partial x \partial y},\ f_{yy} = \frac{\partial^2 z}{\partial y^2}$$

Since $d^2Z < 0$ for maximization, we can have $d^2Z < 0$ if and only if

$$f_{xx} < 0;\ \ f_{yy} < 0\ \ and$$

$$f_{xx}y_{yy} > f_{xy}^2$$

$$Or \qquad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} must\ be\ positive.$$

The above determinant formed by the second order partial derivatives of *Z* with respect to *x* and *y* is called a "Hessian determinant of order 2" symbolize by $|H_2|$.

Thus, the conditions (necessary and sufficient) for maximization are:

i. $f_x = 0\ and\ f_y = 0$

ii. $f_{xx} < 0\ and\ |H_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$

On the other hand, when we want to minimize the function *Z=f(x,y)*

The 1st order condition or the necessary condition for **minimization** is given by the differential of the function and equating it with zero. i.e.

$$dZ = 0$$

$$dZ = \frac{\partial z}{\partial x}.dx + \frac{\partial z}{\partial y}.dy = 0$$

$$or\ dZ = f_x.dx + f_y.dy = 0 \qquad (Where\ f_x = \frac{\partial z}{\partial x}\ and\ f_y = \frac{\partial z}{\partial y})$$

$$dZ = 0\ if\ and\ only\ if\ \frac{\partial z}{\partial x} = f_x = 0\ and\ \frac{\partial z}{\partial y} = f_y = 0$$

The differential version of second order condition of minimization is that $d^2Z>0$

$$d^2Z = d(dZ)$$

$$=> \frac{\partial (dZ)}{\partial x}.dx + \frac{d(dZ)}{\partial y}.dy$$

$$=> \frac{\partial}{\partial x}\left\{\frac{\partial Z}{\partial x}.dx + \frac{\partial Z}{\partial y}.dy\right\}.dx + \frac{\partial}{\partial y}\left\{\frac{\partial Z}{\partial x}.dx + \frac{\partial Z}{\partial y}.dy\right\}.dy$$

$$=> \left\{\frac{\partial^2 Z}{\partial x^2}dx + \frac{\partial^2 Z}{\partial y \partial x}dy\right\}dx + \left\{\frac{\partial^2 Z}{\partial x \partial y}dx + \frac{\partial^2 Z}{\partial y^2}dy\right\}dy$$

$$=> \frac{\partial^2 Z}{\partial x^2}dx^2 + \frac{\partial^2 Z}{\partial y \partial x}dx.dy + \frac{\partial Z}{\partial x \partial y}.dx.dy + \frac{\partial^2 Z}{\partial y^2}dy^2$$

$$=> f_{xx}dx^2 + 2f_{xy}dx.dy + f_{yy}dy^2 \qquad (\sin ce\ f_{xx} = f_{yx}) \qquad (9.4)$$

$$Where, f_{xx} = \frac{\partial^2 Z}{\partial x^2}, f_{yx} = \frac{\partial^2 Z}{\partial y \partial x}, \quad f_{xy} = \frac{\partial z}{\partial x \partial y}, \ f_{yy}\frac{\partial^2 z}{\partial y^2}$$

Thus, $d^2Z$ will be greater than zero if and only if

$$f_{xx} > 0; \quad f_{yy} > 0\ and$$

$$f_{xx}f_{yy} > f_{xy}^2$$

or $$|H_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}\ must\ be\ positive$$

Hence, for minimization of a function with two explanatory variables is

i. $f_x = 0 \, and \, f_y = 0$

ii. $f_{xx} > 0 \, and \, |H_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$

## 9.4 CONDITION FOR OPTIMIZATION IN CASE OF THREE EXPLANATORY VARIABLES

Let us assume

$$Z = f(x_1, x_2, x_3) \qquad (9.5)$$

be a function with three explanatory variables.

Equation (9.5) has to fulfill two conditions for attaining **maximization.** They are

a) Necessary Condition:

The 1st order condition or the necessary condition is given by the differential of the function and equating it with zero. i.e.

$$dZ = 0$$

b) Sufficient Condition:

The 2nd order condition or the sufficient condition is given by the second order differential of

$$d^2Z < 0 \text{ for maximizati on}$$

Let us explain these two conditions,

$$dZ = \frac{\partial z}{\partial x_1}.dx_1 + \frac{\partial z}{\partial x_2}.dx_3 + \frac{\partial z}{\partial x_3}.dx_3 = 0$$

$$or dZ = f_1.dx_1 + f_2.dx_2 + f_3.dx_3 = 0$$

$$(Where f_1 = \frac{\partial z}{\partial x_1}, f_2 = \frac{\partial z}{\partial x_2} and f_3 = \frac{\partial z}{\partial x_3})$$

$$dZ = 0 if and only if f_1 = f_2 = f_3 = 0$$

The differential version of second order condition of minimization is that

$d^2Z>0$

$$d^2Z = d(dZ)$$

$$\Rightarrow \frac{\partial(dZ)}{\partial x_1}.dx_1 + \frac{\partial(dZ)}{\partial x_2}.dx_2 + \frac{\partial z}{\partial x_3}.dx_3$$

$$\Rightarrow \frac{\partial}{\partial x_1}\left\{\frac{\partial Z}{\partial x_1}.dx_1 + \frac{\partial Z}{\partial x_2}.dx_2 + \frac{\partial Z}{\partial x_3}.dx_3\right\}.dx_1$$

$$+ \frac{\partial}{\partial x_2}\left\{\frac{\partial Z}{\partial x_1}.dx_1 + \frac{\partial Z}{\partial x_2}.dx_2 + \frac{\partial Z}{\partial x_3}.dx_3\right\}.dx_2$$

$$+ \frac{\partial}{\partial x_3}\{\frac{\partial Z}{\partial x_1}.dx_1 + \frac{\partial Z}{\partial x_2}.dx_2 + \frac{\partial Z}{\partial x_3}.dx_3\}dx_3$$

$$\Rightarrow \left\{\frac{\partial^2 Z}{\partial x_1^2}.dx_1 + \frac{\partial^2 Z}{\partial x_1 \partial x_2}.dx_2 + \frac{\partial^2 Z}{\partial x_1 \partial x_3}.dx_3\right\}dx_1$$

$$+ \left\{\frac{\partial^2 Z}{\partial x_2 \partial x_1}dx_1 + \frac{\partial^2 Z}{\partial x_2^2}.dx_2 + \frac{\partial^2 Z}{\partial x_2 \partial x_3}.dx_3\right\}dx_2$$

$$+ \left\{\frac{\partial Z}{\partial x_3 \partial x_1}.dx_1 + \frac{\partial Z}{\partial x_3 \partial x_2}.dx_2 + \frac{\partial^2 Z}{\partial x_3^2}.dx_3\right\}dx_3$$

$$\Rightarrow \left\{\frac{\partial^2 Z}{\partial x_1^2}.dx_1^2 + \frac{\partial^2 Z}{\partial x_1 \partial x_2}.dx_1dx_2 + \frac{\partial^2 Z}{\partial x_1 \partial x_3}.dx_1dx_3\right\}$$

$$+ \left\{\frac{\partial^2 Z}{\partial x_2 \partial x_1}.dx_1dx_2 + \frac{\partial^2 Z}{\partial x_2^2}.dx_2^2 + \frac{\partial^2 Z}{\partial x_2 \partial x_3}.dx_2dx_3\right\}$$

$$+ \left\{\frac{\partial^2 Z}{\partial x_3 \partial x_1}.dx_1dx_3 + \frac{\partial^2 Z}{\partial x_3 \partial x_2}.dx_2dx_3 + \frac{\partial^2 Z}{\partial x_3^2}.dx_3^2\right\}$$

$$\Rightarrow \{f_{11}.dx_1^2 + f_{12}.dx_1dx_2 + f_{13}.dx_1dx_3\} + \{f_{21}dx_1dx_2 + f_{22}dx_2^2$$
$$+ f_{23}.dx_2dx_3\} + \{f_{31}.dx_1dx_3 + f_{32}.dx_2dx_3 + f_{33}.dx_3^2\} \qquad (9.6)$$

$$Where, f_{11} = \frac{\partial^2 z}{\partial x_1^2}, f_{12} = \frac{\partial^2 z}{\partial x_1 \partial x_2}, f_{12} = \frac{\partial^2 z}{\partial x_1 \partial x_3}$$

$$f_{21} = \frac{\partial^2 z}{\partial x_2 \partial x_1}, f_{22} = \frac{\partial^2 z}{\partial x^2}, f_{23} = \frac{\partial^2 z}{\partial x_2 \partial x_3}$$

$$f_{31} = \frac{\partial^2 z}{\partial x_3 \partial x_1}, f_{32} = \frac{\partial^2 z}{\partial x_3 \partial x_2}, f_{33} = \frac{\partial^2 z}{\partial x_3^2}$$

The sign of $d^2Z$ will depend on the restrictions of the derivatives $f_{ij}$'s in equation (9.6). The coefficient of $f_{ij}$'s give rise to a symmetric Hessian determinant

$$|H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

Whose principal minors are

$$|H_1| = f_{11} \quad and \quad |H_2| \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

As $d^2Z<0$ for minimization, it implies that

$$|H_1| < 0, \quad |H_2| > 0 \quad and \quad |H_3| < 0 \qquad (9.7)$$

Similarly, the condition for **minimization**, are shown below:

i) $f_1 = f_2 = f_3 = 0$

ii) $|H_1| > 0, \quad |H_2| > 0 \quad and \quad |H_3| > 0$

**Condition for Optimization In Case Of *m*- Explanatory Variables:**

Let us assume

$$Z = f(x_1, x_2; .....x_m) \qquad (9.8)$$

be a function with m explanatory variables.

Equation (9.8) has to fulfill two conditions for attaining maximization. They are

a) Necessary Condition:

The 1st order condition or the necessary condition is given by the differential of the function and equating it with zero.

$$dZ = 0$$
$$\Rightarrow dZ = f_1.dx_1 + f_2.dx_2 + ..... + f_m.dx_m = 0$$
$$Now, dZ = 0 \; if \; and \; only \; if \; f_1 = f_2 = ..... = f_m = 0 \qquad (9.9)$$

b) Sufficient Condition:

The 2nd order condition or the sufficient condition is given by the second order differential of

$$d^2Z = d(dZ)$$

It will be similar to equation (9.6) and the coefficients $f_{ij}$'s will give us symmetric Hessian determinant of order $m$.

$$|H_m| = \begin{vmatrix} f_{11} & f_{12} ..... f_{1m} \\ f_{21} & f_{22} ..... f_{2m} \\ ..... \\ ..... \\ f_{m1} & f_{m2} & f_{mm} \end{vmatrix}$$

The above symmetric Hessian determinant of order $m$ will give us the following principal minors

$$|H_1| = f_{11}, \; |H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}, |H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} .....$$

$d^2Z$ will be negative if

$$|H_1| < 0; \; |H_2| > 0; \; |H_3| < 0, .....(-1)^m |H_m| > 0 \qquad (9.10)$$

And this will maximize the function (9.8)

On the other hand, the minimization of the function (9.8) $d^2Z>0$ and therefore we must have

$$|H_1| > 0; \ |H_2| > 0; \ |H_3| > 0, \ldots\ldots\ldots\ldots\ldots (-1)^m |H_m| > 0 \qquad (9.11)$$

**Example 1 :** Find the extreme values of the following function and check whether they are maximum or minimum.

$$Y = 2x_1^2 + x_1x_2 + 2x_2^2 - 10x_1 - 5x_2 + 30$$

**Solution:** The first order condition for of relative extreme requires that

$$f_1 = \frac{\partial Y}{\partial x_1} = 0$$

$$\Rightarrow f_1 = \frac{\partial}{\partial x_1}(2x_1^2 + x_1x_2 + 2_2^2 - 10x_1 - 5x_2 + 30) = 0$$

$$\Rightarrow 4x_1 + x_2 - 10 = 0 \qquad (i)$$

$$f_2 = \frac{\partial Y}{\partial x_2} = 0$$

$$\Rightarrow f_2 = \frac{\partial}{\partial x_2}(2x_1^2 + x_1x_2 + 2x_2^2 - 10x_1 - 5x_2 + 30) = 0$$

$$\Rightarrow x_1 + 4x_2 - 5 = 0 \qquad (ii)$$

Multiplying equation (ii) by 4, we have

$$4x_1 + 16x_2 - 20 = 0 \qquad (iii)$$

Now, equation (i) – (ii),

$$\Rightarrow 4x_1 + x_2 - 10 - 4x_1 - 16x_2 + 20$$

$$\Rightarrow -15x_2 + 10 = 0$$

$$\Rightarrow x_2 = \frac{10}{15} = 0.67$$

Substituting the value of $x_2=0.67$ in equation (i), we have

$$\Rightarrow 4x_1 + 0.67 - 10 = 0$$
$$\Rightarrow 4x_1 = 9.33$$
$$\Rightarrow x_1 = \frac{9.33}{4} = 2.33$$

In order to ascertain whether $x_1$=2.33 and $x_2$=0.67 will give us maximum or minimum value, we have to verify the following

| For Maximization: | For Minimization: |
|---|---|
| $\lvert H_1\rvert = \lvert y_{11}\rvert < 0$ | $\lvert H_1\rvert = \lvert y_{11}\rvert > 0$ |
| $\lvert H_2\rvert = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} > 0$ | $\lvert H_2\rvert = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} > 0$ |

$$y_{11} = \frac{\partial}{dx_1}\left(\frac{\partial Y}{\partial x_1}\right) = \frac{\partial}{dx_1}(4x_1 + x_2 - 10) = 4$$
$$y_{12} = \frac{\partial}{dx_2}\left(\frac{\partial Y}{\partial x_1}\right) = \frac{\partial}{dx_2}(4x_1 + x_2 - 10) = 1$$
$$y_{21} = \frac{\partial}{dx_1}\left(\frac{\partial Y}{\partial x_2}\right) = \frac{\partial}{dx_1}(x_1 + 4x_2 - 5) = 1$$
$$y_{22} = \frac{\partial}{dx_2}\left(\frac{\partial Y}{\partial x_2}\right) = \frac{\partial}{dx_1}(x_1 + 4x_2 - 5) = 4$$

Now,

$$\lvert H_1\rvert = \lvert y_{11}\rvert = 4 > 0$$
$$\lvert H_2\rvert = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = 15 > 0$$

Hence, at $x_1 = 2.33$ and $x_2 = 0.67$, the function has a minimum value. And the minimum value is

$$y = 2(2.33)^2 + 2.33 \times 0.67 + 2 \times (0.67)^2 - 10 \times 2.33 - 5 \times 0.67 + 30$$
$$= 10.86 + 1.56 + 0.90 - 23.3 - 3.35 + 30$$
$$= 16.67$$

**Example 2 :** Find the extreme values of the following function and check whether they are maximum or minimum.

$$y = 4x_1^3 + 8x_1x_2 - 4x_1^2 - x_2^2 + 10$$

**Solution:** The first order condition for of relative extreme requires that

$$f = \frac{\partial Y}{\partial x_1} = 0$$

$$\Rightarrow f_1 = \frac{\partial}{\partial x_1}(4x_1^3 + 8x_1x_2 - 4x_1^2 - x_2^2 + 10) = 0$$

$$\Rightarrow 12x_1^2 + 8x_2 - 8x_1 = 0$$

$$\Rightarrow 3x_1^2 + 2x_2 - 2x_1 = 0 \qquad (i)$$

And

$$f_2 = \frac{\partial Y}{\partial x_2} = 0$$

$$\Rightarrow f_2 = \frac{\partial}{\partial x_2}(4x_1^2 + 8x_1x_2 - 4x_1^2 - x_2^2 + 10) = 0$$

$$\Rightarrow 8x_1 - 2x_2 = 0 \qquad (ii)$$

From equation (ii),

$$\Rightarrow 8x_1 - 2x_2 = 0$$

$$\Rightarrow x_2 = 4x_1 \qquad (iii)$$

Substituting $x_2 = 4x_1$ in equation (i), we have,

$$3x_1^2 + 2(4x_1) - 2x_1 = 0$$

$$\Rightarrow 3x_1^2 + 8x_1 - 2x_1 = 0$$

$$\Rightarrow 3x_1^2 + 6x_1 = 0$$

$$\Rightarrow x_1(x_1 + 2) = 0$$

$$\Rightarrow x_1 = 0 \qquad or \qquad x_1 = -2$$

In order to ascertain whether $x_1 = 0$ and $x_2 = -2$ will give us maximum or minimum value, we have to verify the following

| For Maximization: | For Minimization: |
|---|---|
| $\|H_1\| = \|y_{11}\| < 0$ | $\|H_1\| = \|y_{11}\| > 0$ |
| $\|H_2\| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} > 0$ | $\|H_2\| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} > 0$ |

$$y_{11} = \frac{\partial}{dx_1}\left(\frac{\partial Y}{\partial x_1}\right) = \frac{\partial}{dx_1}(12x_1^2 + 8x_2 - 8x_1) = 24x_1 - 8 \qquad (iv)$$

$$y_{12} = \frac{\partial}{dx_2}\left(\frac{\partial Y}{\partial x_1}\right) = \frac{\partial}{dx_2}(12x_1^2 + 8x_2 - 8x_1) = 8$$

$$y_{21} = \frac{\partial}{dx_1}\left(\frac{\partial Y}{\partial x_2}\right) = \frac{\partial}{dx_1}(8x_1 - 2x_2) = 8$$

$$y_{22} = \frac{\partial}{dx_2}\left(\frac{\partial Y}{\partial x_2}\right) = \frac{\partial}{dx_1}(8x_1 - 2x_2) = -2 \qquad (iv)$$

Case 1: When $x_1=0$

$$y_{11} = 24 \times 0 - 8 = -8$$

Hence,

$$|H_1| = Y_{11} = -8 < 0$$

$$|H_2| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} = \begin{vmatrix} -8 & 8 \\ 8 & -2 \end{vmatrix} = -48 < 0$$

Therefore, at $x_1 = 0$, the function has neither maximum or minimum as $|H_2| < 0$

Case 2: When $x_1 = -2; \quad x_2 = 4X_1 = 4 \times (-2) = -8$

$$y_{11} = 24 \times (-2) - 8 = -56 < 0$$

Hence,

$$|H_1| = Y_{11} = -56 < 0$$

$$|H_2| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} = \begin{vmatrix} -56 & 8 \\ 8 & -2 \end{vmatrix} = 48 > 0$$

Thus, the function in equation (i) has a relative maximum value when $x_1 = -2; \quad x_2 = -8$

**Example 3 :** Find the extreme values of the following function and check whether they are maximum or minimum (by using Crammer's rule).

$$y = 64x_1 - 10x_1^2 + 26x_2 - \frac{5}{2}x_2^2 - 2x_1x_2 - 55$$

**Solution:** The first order condition for of relative extreme requires that

$$f_1 = \frac{\partial y}{\partial x_1} = 0$$

$$\Rightarrow f_1 = \frac{\partial}{\partial x_1}(64x_1 - 10x_1^2 + 26x_2 - \frac{5}{2}x_2^2 - 2x_1x_2 - 55) = 0$$

$$\Rightarrow 64 - 20x_1 - 2x_2 = 0$$

$$\Rightarrow 20x_1 + 2x_2 = 64 \qquad (i)$$

And

$$f_2 = \frac{\partial y}{\partial x_2} = 0$$

$$\Rightarrow f_2 = \frac{\partial}{\partial x_2}(64x_1 - 10x_1^2 + 26x_2 - \frac{5}{2}x_2^2 - 2x_1x_2 - 55) = 0$$

$$\Rightarrow 26 - 5x_2 - 2x_1 = 0$$

$$\Rightarrow 2x_1 + 5x_2 = 26 \qquad (ii)$$

Arranging equation (i) and (ii) in matrix form, we have

$$\begin{bmatrix} 20 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 64 \\ 26 \end{bmatrix}$$

$$\Rightarrow AX = C:$$

$$Where, A = \begin{bmatrix} 20 & 2 \\ 2 & 5 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} and\ C = \begin{bmatrix} 64 \\ 26 \end{bmatrix}$$

Following Crammer's rule,

$$x_1 = \frac{|A_1|}{|A|} \quad and \quad x_2 = \frac{|A_1|}{|A|}$$

$$|A_1| = \begin{vmatrix} 64 & 2 \\ 26 & 5 \end{vmatrix} = 320 - 52 = 268$$

$$|A_2| = \begin{vmatrix} 20 & 64 \\ 2 & 26 \end{vmatrix} = 520 - 128 = 392$$

$$|A| = \begin{vmatrix} 20 & 2 \\ 2 & 5 \end{vmatrix} = 100 - 4 = 96$$

$$\therefore \quad x_1 = \frac{268}{96} = 2.79$$

and

$$x_2 = \frac{392}{96} = 4.08$$

In order to ascertain whether $x_1 = 2.79$ and $x_2 = 4.08$ will give us maximum or minimum value, we have to verify the following

| For Maximization: | For Minimization: |
|---|---|
| $\|H_1\| = y_{11} < 0$ | $\|H_1\| = \|y_{11}\| > 0$ |
| $\|H_2\| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} > 0$ | $\|H_2\| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} > 0$ |

$$y_{11} = \frac{\partial}{dx_1}\left(\frac{\partial Y}{\partial x_1}\right) = \frac{\partial}{dx_1}(64 - 20x_1 - 2x_2) = -20$$

$$y_{12} = \frac{\partial}{dx_2}\left(\frac{\partial Y}{\partial x_1}\right) = \frac{\partial}{dx_2}(64 - 20x_1 - 2x_2) = -2$$

$$y_{21} = \frac{\partial}{dx_1}\left(\frac{\partial Y}{\partial x_2}\right) = \frac{\partial}{dx_1}(26 - 5x_2 - 2x_1) = -2$$

$$y_{22} = \frac{\partial}{dx_2}\left(\frac{\partial Y}{\partial x_2}\right) = \frac{\partial}{dx_1}(26 - 5x_2 - 2x_1) = -5$$

Hence,

$$|H_1| = y_{11} = -20 < 0$$

$$|H_2| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} = \begin{vmatrix} -20 & -2 \\ -2 & -5 \end{vmatrix} = 96 > 0$$

Thus, the function has a relative maximum value when $x_1 = 2.79;\quad x_2 = 4.08$ and the maximum value of the function

$$y = 64 \times 2.79 - 10(2.79)^2 + 26 \times 4.08 - \frac{5}{2}(4.08)^2$$
$$- 2 \times 2.79 \times 4.08 - 55 = 87.41$$



**CHECK YOUR PROGRESS**

**Q 1:** Find the extreme value (s) of the following function

a) $y = 32x_1 - 5x_1^2 + 13x_2 - \frac{5}{2}x_2^2 - x_1x_2 - 25$

b) $y = x_1^3 + 8x_1x_2 - 2x_1^2 - 2x_2^1$

c) $y = x + 2ey - e^x - e^{2y}$



## 9.5 LET US SUM UP

- Many economic problems of optimization contain more than one explanatory variable. For example, price and output determination of discriminating monopolist, equilibrium output of a multi-plant firm, equilibrium output of a multi-product firm.
- In such cases, although the basic principle remains unchanged but the technique of optimization changed.
- A function with more than one explanatory variable as follows

  $Z = f(x, y)$

  It has to fulfil two conditions for attaining maximization. They are

The 1st order condition or the necessary condition is given by the differential of the function and equating it with zero. i.e.

$$dZ = 0$$

The 2nd order condition or the sufficient condition is given by the second order differential of

$$d^2Z < 0 \text{ for maximizati on and}$$
$$d^2Z > 0 \text{ for minimizati on}$$

## 9.6 FURTHER READING


1) Baruah, S. (2011). *Basic Mathematics and Its Applications in Economics.* Macmillan.

2) Chiang, A.C. (1986). *Fundamental Methods of Mathematical Economics.* NewYork : McGraw Hill.

3) Allen, R.G.D. (1974). *Mathematical Analysis for Economists.* London: Macmillan Press and ELBS.


## 9.7 ANSWERS TO CHECK YOUR PROGRESS

**Ans to Q No. 1 (a):** The function has maximum value at $x_1 = 3, x_2 = 2$

**(b):** The function has maximum value at $x_1 = 4, x_2 = -8$

**(c):** The function has maximum value at $x = 0, y = \frac{1}{2}$

## 9.8 MODEL QUESTIONS

**Q 1:** Find the extreme value of the function and check whether they have maxima or minima.

$$z = x^2 + xy + 2y^2 + 3$$

**Q 2:** Find the extreme value of the function and check whether they have maxima or minima.

$$z = -x^2 + xy - y^2 + 2x + y$$

**Q 3:** Find the extreme value of the function and check whether they have maxima or minima.

$z = -x^2 + xy - y^2 + 2x + y$

**Q 4:** Find the extreme value of the function and check whether they have maxima or minima.

$y = x_1^3 - 8x_1x_2 - x_1^2 + 2x_2^2 + 15x_1 + 50$

*** ***** ***

# UNIT 10: ECONOMIC APPLICATIONS OF UNCONSTRAINED MAXIMA AND MINIMA WITH MORE THAN ONE EXPLANATORY VARIABLE

**UNIT STRUCTURE**


10. 1 Learning Objectives

10. 2 Introduction

10. 3 Equilibrium of Discriminating Monopoly

10. 4 Equilibrium of Multi-Product Firm

10. 5 Equilibrium of Multi-Plant Firm

10. 6 Let us Sum Up

10. 7 Further Reading

10. 8 Answers to Check Your Progress

10. 9 Model Questions


## 10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to–

- find out the equilibrium level of output and prices of a price discriminating monopolist
- find out maximum profit of a price discriminating monopolist
- find out equilibrium level of output and maximum profit of a multi-product firm
- find out equilibrium level of output and maximum profit of a multi-plant firm.

## 10.2 INTRODUCTION

In the previous unit 9, we have discussed the optimization technique of unconstrained maxima and minima with more than one variable. As discussed in the previous unit, the conditions for maximization with m-explanatory variables for a function,

$$Z = f(x_1, x_2 .....x_m) \qquad is$$

$$i)\ f_1 = f_2 = ..... = f_m = 0$$

$$ii)\ |H_1| < 0;\ |H_1| > 0;\ |H_3| < 0, .....(-1)^m |H_m| > 0$$

Similarly, the conditions for minimization with m-explanatory variables for the same function are

$$i)\ f_1 = f_2 = ..... = f_m = 0$$

$$ii)\ |H_1| > 0;\ |H_1| > 0;\ |H_3| > 0, .....(-1)^m |H_m| > 0$$

In the following sections of the unit, we will discuss various economic applications of maxima and minima for function with more than one explanatory variables.

## 10.3 EQUILIBRIUM OF DISCRIMINATING MONOPOLY

Under simple monopoly, a single price is charged for the whole output; but under price discrimination the monopolist will charge different prices in different sub-markets. First of all, therefore, the monopolist has to divide his total market into various sub-markets on the basis of differences in elasticity of demand in them. The monopolist can divide his total market into several sub-markets accordingly as there are differences in demand elasticity, but for the sake of making our analysis, we shall explain the case when total market is divided into sub-markets.

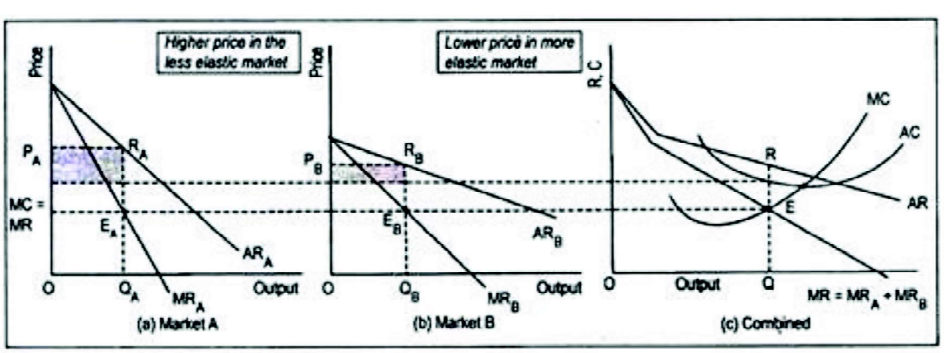



Figure 10.1

Figure 10.1 shows the diagrammatic presentation of determination of equilibrium output and price in a discriminating monopolist.

Let us assume that the total revenue (TR) function in sub-market 1 and sub-market 2 are as follows

$$TR_1 = R_1(Q_1)$$
$$TR_2 = R_2(Q_2)$$

Where, $TR_1$ and $TR_2$ are total revenue in sub-market 1 and sub-market 2 where respective output are $Q_1$ and $Q_2$.

The total cost (TC) function is given as follows:

*TC=C(Q)*

Now, the profit function is defined as

$$\pi = TR - TC$$
$$\pi = (TR_1 + TR_2) - TC \qquad Where\, TR = (TR_1 + TR_2)$$
$$\pi = R_1(Q_1) + R_2(Q_2) - C(Q)$$

The 1st order condition for maximization of profit is

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = 0 \quad and \quad \pi_2 = \frac{\partial \pi}{\partial Q_2} = 0$$

And $|H_1| < 0 \qquad and\, |H_2| > 0$

Now

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = 0$$
$$\Rightarrow \frac{\partial}{\partial Q_1}\{R_1(Q_1) + R_2(Q_2) - C(Q)\} = 0$$
$$\Rightarrow R'_1(Q_1) - C'(Q) = 0 \qquad (i)$$

Also,

$$\pi_2 = \frac{\partial \pi}{\partial Q_2} = 0$$

$$\Rightarrow \frac{\partial}{\partial Q_2}\{R_1(Q_1) + R_2(Q_2) - C(Q)\} = 0$$

$$\Rightarrow R'_2(Q_2) - C'(Q) = 0 \qquad (ii)$$

From equation (i) and (ii), we have

$$R'_1(Q_1) = R'_2(Q_2) = C'(Q)$$

OR $$MR_1 = MR_2 = MC \qquad (iii)$$

The second order condition requires:

$$|H_1| = \pi_{11} = \frac{\partial^2 \pi}{\partial Q_1^2} = R''_1(Q_1) - C''(Q) < 0$$

And $$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} \frac{\partial^2 \pi}{\partial Q_1^2} & \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} \\ \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} & \frac{\partial^2 \pi}{\partial Q_2^2} \end{vmatrix} > 0 \qquad (iv)$$

Now,
$$\frac{\partial^2 \pi}{\partial Q_1^2} = \frac{\partial}{\partial Q_1}\left(\frac{\partial \pi}{\partial Q_1}\right) = \frac{\partial}{\partial Q_1}\{R'_1(Q_1) - C'(Q)\} = R''_1(Q_1) - C''(Q)$$

$$\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = \frac{\partial}{\partial Q_2}\left(\frac{\partial \pi}{\partial Q_1}\right) = \frac{\partial}{\partial Q_2}\{R'_1(Q_1) - C'(Q)\} = -C''(Q)$$

$$\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = \frac{\partial}{\partial Q_1}\left(\frac{\partial \pi}{\partial Q_2}\right) = \frac{\partial}{\partial Q_1}\{R'_2(Q_2) - C'(Q)\} = -C''(Q)$$

$$\frac{\partial^2 \pi}{\partial Q_2^2} = \frac{\partial}{\partial Q_2}\left(\frac{\partial \pi}{\partial Q_2}\right) = \frac{\partial}{\partial Q_2}\{R'_2(Q_2) - C'(Q)\} = R''_2(Q_2) - C''(Q)$$

Thus, from equation (iv),

$$|H_2| == \begin{vmatrix} R''_1(Q_1) - C''(Q) & -C''(Q) \\ -C''(Q) & R''_2(Q_2) - C''(Q) \end{vmatrix} > 0$$

**Impact of Equilibrium Conditions on Prices of Discriminating Monopolist:**

The equilibrium condition of a discriminating monopolist as given by equation (iii) has an implication on prices of each sub-market of a discriminating monopolist.

We know that the relation among marginal revenue (MR), average revenue (AR) and elasticity of demand is as follows,

$$MR = AR\left(1-\frac{1}{|e|}\right) = P\left(1-\frac{1}{|e|}\right)$$

Where $P(price) = AR$ and $|e|$ is absolute value of elasticity of demand.

In case of discriminating monopoly

$$MR_1 = P_1\left(1-\frac{1}{|e_1|}\right) and\ MR_2 = P_2\left(1-\frac{1}{|e_2|}\right)$$

From equation (iii), we have,

$$MR_1 = MR_2$$

$$P_1\left(1-\frac{1}{|e_1|}\right) = P_2\left(1-\frac{1}{|e_2|}\right)$$

$$\Rightarrow \frac{p_1}{P_2} = \frac{\left(1-\frac{1}{|e_2|}\right)}{\left(1-\frac{1}{|e_1|}\right)} \qquad (v)$$

Relation (v) indicates that when $|e_1| > |e_2|, \left(1-\frac{1}{|e_1|}\right) > \left(1-\frac{1}{|e_2|}\right)$ and therefore $P_1 < P_2$. Thus, higher the elasticity, lower will be the price.

**Example 1 :** The demand functions of a monopoly in two different markets are given by

$$P_1 = 180 - 15Q_1$$
$$P_2 = 280 - 20Q_2$$

And the total cost function is *C=150+120Q, Where $Q_1$ and $Q_2$ are output in Market- 1 and market- 2 respectively so that $Q=Q_1+Q_2$*

Find

(a) Profit maximizing output to be sold in market-1 and market-2.
(b) Equilibrium prices in both the markets
(c) maximum profit.
(d) Show that the market with higher elasticity of demand has lower price and vice versa

**Solution:**

(a) Given

$$P_1 = 180 - 15Q_1 \qquad (i)$$
$$P_2 = 280 - 20Q_2 \qquad (ii)$$

The total revenue function of market -1 and market -2 are

$$TR_1 = P_1 \times Q_1 = (180 - 15Q_1) \times Q_1 = 180Q_1 - 15Q_1^2$$
$$TR_2 = P_2 \times Q_2 = (280 - 20Q_2) \times Q_2 = 280Q_2 - 20Q_2^2$$

And $C = 150 + 120Q$

$$C = 150 + 120(Q_1 + Q_2)$$
$$C = 150 + 120Q_1 + 120Q_2$$

Given the total cost function, the profit function of the monopolist will be

$$\pi = TR_1 + TR_2 - C(Q)$$
$$\pi = 180Q_1 - 15Q_1^2 + 280Q_2 - 20Q_2^2 - 150 - 120Q_1 - 120Q_2$$
$$\pi = 60Q_1 - 15Q_1^2 + 160Q_2 - 20Q_2^2 - 150 \qquad (iii)$$

The 1st order condition for maximization of profit is

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = 0 \qquad and \qquad \pi_2 = \frac{\partial \pi}{\partial Q_2} = 0$$

and $\left|H_1\right| < 0 \qquad and \quad \left|H_2\right| > 0$

Now

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = 0$$

$$\Rightarrow \frac{\partial}{\partial Q_1}(60Q_1 - 15Q_1^2 + 160Q_2 - 20Q_2^2 - 150) = 0$$

$$\Rightarrow 60 - 30Q_1 = 0$$

$$\Rightarrow Q_1 = \frac{60}{30} = 2 \qquad (iv)$$

Also,

$$\pi_2 = \frac{\partial \pi}{\partial Q_2} = 0$$

$$\Rightarrow \frac{\partial}{\partial Q_2}(60Q_1 - 15Q_1^2 + 160Q_2 - 20Q_2^2 - 150) = 0$$

$$\Rightarrow 160 - 40Q_2 = 0$$

$$\Rightarrow Q_2 = \frac{160}{40} = 4 \qquad (v)$$

The second order condition requires:

$$\left|H_1\right| = \pi_{11} < 0$$

and $$\left|H_2\right| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} \frac{\partial^2 \pi}{\partial Q_1^2} & \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} \\ \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} & \frac{\partial^2 \pi}{\partial Q_2^2} \end{vmatrix} > 0$$

Now,

$$\frac{\partial^2 \pi}{\partial Q_1^2} = \frac{\partial}{\partial Q_1}\left(\frac{\partial \pi}{\partial Q_1}\right) = \frac{\partial}{\partial Q_1}(60 - 30Q_1) = -30$$
$$\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = \frac{\partial}{\partial Q_2}\left(\frac{\partial \pi}{\partial Q_1}\right) = \frac{\partial}{\partial Q_2}(60 - 30Q_1) = 0$$
$$\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = \frac{\partial}{\partial Q_1}\left(\frac{\partial \pi}{\partial Q_2}\right) = \frac{\partial}{\partial Q_1}(160 - 40Q_2) = 0$$
$$\frac{\partial^2 \pi}{\partial Q_2^2} = \frac{\partial}{\partial Q_2}\left(\frac{\partial \pi}{\partial Q_2}\right) = \frac{\partial}{\partial Q_2}(160 - 40Q_2) = -40$$

Now $|H_1| = \pi_{11} = -30 < 0$

And $$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} -30 & 0 \\ 0 & -40 \end{vmatrix} = 1200 > 0$$

As both the conditions of maximization are satisfied, therefore the profit maximizing output attain at output level $Q_1 = 2;\ Q_1 = 4$

(b) Substituting the value of $Q_1 = 2;\ Q_1 = 4$ in equation (i) & (ii), we have equilibrium prices in market-1 and market-2 are as follows

$$P_1 = 180 - 15Q_1 = 180 - 15 \times 2 = 180 - 30 = 150$$
$$P_2 = 280 - 20Q_2 = 280 - 20 \times 4 = 280 - 80 = 200$$

(c) Substituting the value of $Q_1 = 2;\ Q_1 = 4$ in equation (iii) , we have,

$$\pi = 60 \times 2 - 15 \times 2^2 + 160 \times 4 - 20 \times 4^2 - 150$$
$$\pi = 120 - 60 + 640 - 320 - 150$$
$$\pi = 230$$

(d) We know that,

$$e_d = \frac{AR}{AR - MR}$$

Therefore, elasticity in market-1

$$e_{d_1} = \frac{AR_1}{AR_1 - MR_1}$$

Now, $MR_1 = \frac{d(TR_1)}{dQ_1} = \frac{d}{dQ_1}(180Q_1 - 15Q_1^2) = 180 - 30Q_1$

When $Q_1$=2;

$$MR_1 = 180 - 30 \times 2 = 180 - 60 = 120$$
$$P_1 = AR_1 = 180 - 15 \times 2 = 180 - 30 = 150$$

Therefore,

$$e_{d_1} = \frac{150}{150 - 120} = \frac{150}{30} = 5$$

And elasticity in market-2

$$e_{d_2} = \frac{AR_2}{AR_2 - MR_2}$$

Now, $MR_2 = \frac{d(TR_2)}{dQ_2} = \frac{d}{dQ_2}(280Q_2 - 20Q_2^2) = 280 - 40Q_2$

When $Q_2$=4

$$MR_2 = 280 - 40 \times 4 = 280 - 160 = 120$$
$$P_2 = AR_2 = 280 - 20 \times 4 = 280 - 80 = 200$$

Therefore,

$$e_{d_2} = \frac{200}{200 - 120} = \frac{200}{80} = 2.5$$

Thus, when $e_{d_1} = 5 > e_{d_2} = 2.5$ then $P_1 = 150 < p_2 = AR_2 = 200$. This imply that the market where elasticity of demand is high have lower price and vice versa.

**Example 2 :** A monopolist discriminates in prices in three different markets of its products and the price equations of the three markets are given by

$$P_1 = 88 - 5Q_1$$
$$P_2 = 62 - 4Q_2$$
$$P_3 = 56 - 3Q_3$$

Where $Q_1$, $Q_2$ *and* $Q_3$ the output levels of are first, second and third markets and the total cost function is given by

$$C = 50 - 8Q + 2Q^2$$

Find Profit maximizing output

**Solution:**

(a) The total revenue functions in each of the three market is given by

$$TR_1 = P_1 \times Q_1 = (88 - 5Q_1) \times Q_1 = 88Q_1 - 5Q_1^2$$
$$TR_2 = P_2 \times Q_2 = (62 - 4Q_2) \times Q_2 = 62Q_2 - 4Q_2^2$$
$$TR_3 = P_3 \times Q_3 = (56 - 3Q_3) \times Q_3 = 56Q_3 - 3Q_3^2$$

With total cost function, $C = 50 - 8Q + 2Q^2$ , the profit function is given by

$$\pi = (TR_1 + TR_2 + TR_3) - C$$
$$\pi = 88Q_1 - 5Q_1^2 + 62Q_2 - 4Q_2^2 + 56Q_3 - 3Q_3^2 - 50 + 8Q - 2Q^2 \quad (i)$$

Now,

The 1st order condition for maximization of profit is

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = 0 \qquad and \qquad \pi_2 = \frac{\partial \pi}{\partial Q_2} = 0$$

Now

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = 0$$

$$\Rightarrow \frac{\partial}{\partial Q_1}(88Q_1 - 5Q_1^2 + 62Q_2 - 4Q_2^2 + 56Q_3 - 3Q_3^2 - 50 + 8Q - 2Q^2) = 0$$

$$\Rightarrow 88 - 10Q_1 + 8 - 4Q = 0$$

$$\Rightarrow 88 - 10Q_1 + 8 - 4(Q_1 + Q_2 + Q_3) = 0$$

$$\Rightarrow 14Q_1 + 4Q_2 + 4Q_3 = 96 \qquad (ii)$$

Also,

$$\pi_2 = \frac{\partial \pi}{\partial Q_2} = 0$$

$$\Rightarrow \frac{\partial}{\partial Q_2}(88Q_1 - 5Q_1^2 + 62Q_2 - 4Q_2^2 + 56Q_3 - 3Q_3^2 - 50 + 8Q - 2Q^2) = 0$$

$$\Rightarrow 62 - 8Q_2 + 8 - 4Q = 0$$

$$\Rightarrow 62 - 8Q_2 + 8 - 4(Q_1 + Q_2 + Q_3) = 0$$

$$\Rightarrow 4Q_1 + 12Q_2 + 4Q_3 = 70 \qquad (iii)$$

$$\pi_3 = \frac{\partial \pi}{\partial Q_3} = 0$$

$$\Rightarrow \frac{\partial}{\partial Q_3}(88Q_1 - 5Q_1^2 + 62Q_2 - 4Q_2^2 + 56Q_3 - 3Q_3^2 - 50 + 8Q - 2Q^2) = 0$$

$$\Rightarrow 56 - 6Q_3 + 8 - 4Q = 0$$

$$\Rightarrow 56 - 6Q_3 + 8 - 4(Q_1 + Q_2 + Q_3) = 0$$

$$\Rightarrow 4Q_1 + 4Q_2 + 10Q_3 = 64 \qquad (iv)$$

From equation (ii), (iii) & (iv) and using Crammer's rule:

$$14Q_1 + 4Q_2 + 4Q_3 = 96$$
$$4Q_1 + 12Q_2 + 4Q_3 = 70$$
$$4Q_1 + 4Q_2 + 10Q_3 = 64$$

$$Q_1 = \frac{\begin{vmatrix} 96 & 4 & 4 \\ 70 & 12 & 4 \\ 64 & 4 & 10 \end{vmatrix}}{\begin{vmatrix} 14 & 4 & 4 \\ 4 & 12 & 4 \\ 4 & 4 & 10 \end{vmatrix}} = \frac{6256}{1232} = 5.07$$

$$Q_2 = \frac{\begin{vmatrix} 14 & 96 & 4 \\ 4 & 70 & 4 \\ 4 & 64 & 10 \end{vmatrix}}{\begin{vmatrix} 14 & 4 & 4 \\ 70 & 12 & 4 \\ 4 & 4 & 10 \end{vmatrix}} = \frac{3816}{1232} = 3.04$$

$$Q_3 = \frac{\begin{vmatrix} 14 & 4 & 96 \\ 4 & 12 & 70 \\ 4 & 4 & 64 \end{vmatrix}}{\begin{vmatrix} 14 & 4 & 4 \\ 4 & 12 & 4 \\ 4 & 4 & 10 \end{vmatrix}} = \frac{3664}{1232} = 2.97$$

The second order condition requires:

$$|H_1| = |\pi_{11}| < 0$$

$$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} \frac{\partial^2 \pi}{\partial Q_1^2} & \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} \\ \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} & \frac{\partial^2 \pi}{\partial Q_2^2} \end{vmatrix} > 0$$

$$|H_3| = \begin{vmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{vmatrix} = \begin{vmatrix} \frac{\partial^2 \pi}{\partial Q_1^2} & \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} & \frac{\partial^2 \pi}{\partial Q_1 \partial Q_3} \\ \frac{\partial^2 \pi}{\partial Q_2 \partial Q_1} & \frac{\partial^2 \pi}{\partial Q_2^2} & \frac{\partial^2 \pi}{\partial Q_2 \partial Q_3} \\ \frac{\partial^2 \pi}{\partial Q_3 \partial Q_1} & \frac{\partial^2 \pi}{\partial Q_3 \partial Q_2} & \frac{\partial^2 \pi}{\partial Q_3^2} \end{vmatrix}$$

$$\frac{\partial^2\pi}{\partial Q_1^2}=\frac{\partial}{\partial Q_1}\left(\frac{\partial\pi}{\partial Q_1}\right)=\frac{\partial}{\partial Q_1}(88-10Q_1+8-4Q)=-10-4=-14$$

$$\frac{\partial^2\pi}{\partial Q_2\partial Q_1}=\frac{\partial}{\partial Q_2}\left(\frac{\partial\pi}{\partial Q_1}\right)=\frac{\partial}{\partial Q_2}(88-10Q_1+8-4Q)=-4$$

$$\frac{\partial^2\pi}{\partial Q_3\partial Q_1}=\frac{\partial}{\partial Q_3}\left(\frac{\partial\pi}{\partial Q_1}\right)=\frac{\partial}{\partial Q_3}(88-10Q_1+8-4Q)=-4$$

$$\frac{\partial^2\pi}{\partial Q_2^2}=\frac{\partial}{\partial Q_2}\left(\frac{\partial\pi}{\partial Q_2}\right)=\frac{\partial}{\partial Q_2}(62-8Q_2+8-4Q)=-8-4=-12$$

$$\frac{\partial^2\pi}{\partial Q_1\partial Q_2}=\frac{\partial}{\partial Q_1}\left(\frac{\partial\pi}{\partial Q_2}\right)=\frac{\partial}{\partial Q_1}(62-8Q_2+8-4Q)=-4$$

$$\frac{\partial^2\pi}{\partial Q_3\partial Q_2}=\frac{\partial}{\partial Q_3}\left(\frac{\partial\pi}{\partial Q_2}\right)=\frac{\partial}{\partial Q_3}(62-8Q_2+8-4Q)=-4$$

$$\frac{\partial^2\pi}{\partial Q_3^2}=\frac{\partial}{\partial Q_3}\left(\frac{\partial\pi}{\partial Q_3}\right)=\frac{\partial}{\partial Q_3}(56-6Q_3+8-4Q)=-6-4=-10$$

$$\frac{\partial^2\pi}{\partial Q_1\partial Q_3}=\frac{\partial}{\partial Q_1}\left(\frac{\partial\pi}{\partial Q_3}\right)=\frac{\partial}{\partial Q_1}(56-6Q_3+8-4Q)=-4$$

$$\frac{\partial^2\pi}{\partial Q_2\partial Q_3}=\frac{\partial}{\partial Q_2}\left(\frac{\partial\pi}{\partial Q_3}\right)=\frac{\partial}{\partial Q_2}(56-6Q_3+8-4Q)=-4$$

Now

$$|H_1|=\pi_{11}=-14<0$$

$$|H_2|=\begin{vmatrix}\pi_{11} & \pi_{12}\\ \pi_{21} & \pi_{22}\end{vmatrix}=\begin{vmatrix}-14 & -4\\ -4 & -12\end{vmatrix}=152>0$$

$$|H_3|=\begin{vmatrix}\pi_{11} & \pi_{12} & \pi_{13}\\ \pi_{21} & \pi_{22} & \pi_{23}\\ \pi_{31} & \pi_{32} & \pi_{33}\end{vmatrix}=\begin{vmatrix}-14 & -4 & -4\\ -4 & -12 & -4\\ -4 & -4 & -10\end{vmatrix}=-1232<0$$

Hence, $Q_1=5.07;\quad Q_2=3.04\,\&\,Q_3=2.97$ will maximize the profit of the discriminating monopolist and equilibrium output is $Q_1+Q_2+Q_3=5.07+3.04+2.97=11.08$

b) Maximum profit =

$$\pi = 88\times 5.44 - 5Q_1^2 + 62Q_2 - 4Q_2^2 + 56Q_3 - 3Q_3^2 - 50 + 8Q - 2Q^2$$
$$\pi = 88\times 5.44 - 5\times(5.44)^2 + 62\times 2.55 - 4\times(2.55)^2 + 56\times 2.24 - 3$$
$$\times(2.24)^2 - 50 + 8\times 10.23 - 2\times(10.23)^2 = 395.28$$



## CHECK YOUR PROGRESS

**Q 1:** The average revenue and total cost functions of a discriminating monopolist are given by

$$AR_1 = 80 - 3Q_1$$
$$AR_2 = 104 - 4Q_2$$
$$TC = 50 - 10Q + 2Q^2 \qquad Where\ Q_1 + Q_2 = Q$$
$$Q_1\ and\ Q_2\ are\ the\ outputs\ be\ sold\ in\ two\ markets.$$

Find

(a) Profit maximizing output

(b) Equilibrium prices of the two markets

(c) Elasticity of demand of the markets

(d) Maximum profit

**Q 2:** The price equations and total cost functions of a discriminating monopolist, who discriminates prices in three markets, are given by

$$P_1 = 88 - 5Q_1$$
$$P_2 = 62 - 4Q_2$$
$$P_3 = 56 - 3Q_3$$

$$TC = 50 - 8Q + 2Q^2 \qquad Where\ Q_1 + Q_2 = Q$$
$$Q_1\ and\ Q_2\ are\ the\ outputs\ to\ be\ sold\ in\ two\ markets.$$

Find

(a) Profit maximizing output

(b) Equilibrium prices of the three markets

(c) Maximum profit

## 10.4 EQUILIBRIUM OF MULTI-PRODUCT FIRM

In real world, the firms are generally multi-product firm, that is, they produce more than one product. For example, in our country the firm Hindustan Unilever produces Dalda ghee as well as various verities of soaps and also some other goods. Besides, there are joint products in which case a single production process yielding joint products. Sheep raising is the most familiar example of single production process yielding joint products. Two products, wool and mutton, can be jointly produced in varying proportions by a single production process. The case of joint products exists when the production of two or more products are technically dependent upon one another. In such cases, total revenue of a firm will depend on more than one product.

In this section, we will discuss the equilibrium of a firm producing two commodities, $Q_1 \; and \; Q_2$, jointly under the condition of

a) Perfect competition
b) Imperfect competition

The total cost *(TC)* function of a firm jointly producing two products $Q_1 \; and \; Q_2$ has defined as follows:

$$TC = C(Q_1, Q_2)$$

Similarly, the marginal cost *(MC)* functions is derived as follows, which are also function of $Q_1 \; and \; Q_2$

$$MC_1 = \frac{\partial C}{\partial Q_1} = C_1(Q_1 Q_2) \; and$$
$$MC_2 = \frac{\partial C}{\partial Q_2} = C_2(Q_1 Q_2)$$

a) Equilibrium of multi-product firm under perfect competition

The price of the product is given and fixed in a perfectly competitive

market. If the prices of $Q_1 \; and \; Q_2$ are denoted by $P_1 \; and \; P_2$ respectively, the total revenue (TR) function is defined as follows

$$TR = \overline{P_1}Q_1 + \overline{P_2}Q_2 \qquad (i)$$

And the marginal revenue of the first and second products are

$$MR_1 = \frac{\partial(TR)}{\partial Q_1} = \overline{P_1} \qquad (ii)$$

and

$$MR_2 = \frac{\partial(TR)}{\partial Q_2} = \overline{P_2} \qquad (iii)$$

Now, the profit function of the multi-product firm can be defined as follows

$$\pi = TR - TC = \overline{P_1}Q_1 + \overline{P_2}Q_2 - C(Q_1, Q_2) \qquad (iv)$$

Maximization of profit requires the fulfillment of the following conditions -

The first order condition for maximization of profit is

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = 0 \qquad and \qquad \pi_2 = \frac{\partial \pi}{\partial Q_2} = 0$$

The second order condition requires:

$$|H_1| = \pi_{11} = \frac{\partial^2 \pi}{\partial Q_1^2} < 0$$

And $$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial^2 \pi}{\partial Q_1^2} & \dfrac{\partial^2 \pi}{\partial Q_1 \partial Q_2} \\ \dfrac{\partial^2 \pi}{\partial Q_1 \partial Q_2} & \dfrac{\partial^2 \pi}{\partial Q_2^2} \end{vmatrix} > 0$$

Now

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = 0$$

$$\Rightarrow \frac{\partial}{\partial Q_1}\{\overline{P_1}Q_1 + \overline{P_2}Q_2 - C(Q_1, Q_2)\} = 0$$

$$\Rightarrow \overline{P_1} - \frac{\partial C}{\partial Q_1} = 0$$

$$\Rightarrow \overline{P_1} = \frac{\partial C}{\partial Q_1}$$

$$\Rightarrow MR_1 = MC_1$$

Also,

$$\pi_2 = \frac{\partial \pi}{\partial Q_2} = 0$$

$$\Rightarrow \frac{\partial}{\partial Q_2}\{\overline{P_1}Q_1 + \overline{P_2}Q_2 - C(Q_1, Q_2)\} = 0$$

$$\Rightarrow \overline{P_2} - \frac{\partial C}{\partial Q_2} = 0$$

$$\Rightarrow \overline{P_2} = \frac{\partial C}{\partial Q_2}$$

$$\Rightarrow MR_2 = MC_2$$

Now,

$$\pi_{11} = \frac{\partial^2 \pi}{\partial Q_1^2} = \frac{\partial}{\partial Q_1}\left(\frac{\partial \pi}{\partial Q_1}\right) = \frac{\partial}{\partial Q_1}\left(\overline{P_1} - \frac{\partial C}{\partial Q_1}\right) = -\frac{\partial^2 C}{\partial Q_1^2}$$

$$(\overline{P_1} \textit{ is cons}\tan \textit{t in perfect competion})$$

$$\pi_{12} = \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = \frac{\partial}{\partial Q_2}\left(\frac{\partial \pi}{\partial Q_1}\right) = \frac{\partial}{\partial Q_2}\left(\overline{P_1} - \frac{\partial C}{\partial Q_1}\right) = -\frac{\partial^2 C}{\partial Q_1 \partial Q_2}$$

$$\pi_{21} = \frac{\partial^2 \pi}{\partial Q_2 \partial Q_1} = \frac{\partial}{\partial Q_1}\left(\frac{\partial \pi}{\partial Q_2}\right) = \frac{\partial}{\partial Q_1}\left(\overline{P_2} - \frac{\partial C}{\partial Q_2}\right) = -\frac{\partial^2 C}{\partial Q_1 \partial Q_2}$$

$$\pi_{22} = \frac{\partial^2 \pi}{\partial Q_2^2} = \frac{\partial}{\partial Q_2}\left(\frac{\partial \pi}{\partial Q_2}\right) = \frac{\partial}{\partial Q_2}\left(\overline{P_2} - \frac{\partial C}{\partial Q_2}\right) = -\frac{\partial^2 C}{\partial Q_2^2}$$

$$(\overline{P_2} \textit{ is cons}\tan \textit{t in perfect competion})$$

The second order condition requires:

$$|H_1| = \pi_{11} = \frac{\partial^2 \pi}{\partial Q_1^2} = -\frac{\partial^2 C}{\partial Q_1^2} < 0$$

And $$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} -\frac{\partial^2 C}{\partial Q_1^2} & -\frac{\partial^2 C}{\partial Q_1 \partial Q_2} \\ -\frac{\partial^2 C}{\partial Q_1 \partial Q_2} & -\frac{\partial^2 C}{\partial Q_2^2} \end{vmatrix} > 0$$

b) Equilibrium of multi-product firm under imperfect competition

Since, in case of imperfect condition $\overline{P_1}$ *and* $\overline{P_2}$, are not given. So the second order condition will be as follows

Now,

$$\pi_{11} = \frac{\partial^2 \pi}{\partial Q_1^2} = \frac{\partial}{\partial Q_1}\left(\frac{\partial \pi}{\partial Q_1}\right) = \frac{\partial^2 (TR)}{\partial Q_1^2} - \frac{\partial^2 C}{\partial Q_1^2}$$

$$\pi_{12} = \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = \frac{\partial}{\partial Q_2}\left(\frac{\partial \pi}{\partial Q_1}\right) = \frac{\partial^2 (TR)}{\partial Q_1 \partial Q_2} - \frac{\partial^2 C}{\partial Q_1 \partial Q_2}$$

$$\pi_{21} = \frac{\partial^2 \pi}{\partial Q_2 \partial Q_1} = \frac{\partial}{\partial Q_1}\left(\frac{\partial \pi}{\partial Q_2}\right) = \frac{\partial^2 (TR)}{\partial Q_2 \partial Q_1} - \frac{\partial^2 C}{\partial Q_2 \partial Q_1}$$

$$\pi_{22} = \frac{\partial^2 \pi}{\partial Q_2^2} = \frac{\partial}{\partial Q_2}\left(\frac{\partial \pi}{\partial Q_2}\right) = \frac{\partial^2 (TR)}{\partial Q_2^2} - \frac{\partial^2 C}{\partial Q_2^2}$$

**Example 3 :** A monopolist firm produces two products $Q_1$ *and* $Q_2$ jointly. The total cost function is given by.

$$C = Q_1^2 + \frac{1}{2}Q_2^2 + Q_1 Q_2 + 10$$

His average revenue functions are given by

$$P_1 = 40 - 2Q_1$$
$$P_2 = 34 - 3Q_2$$

Find

(a) the profit maximizing output levels $Q_1$ *and* $_2$

(b) Equilibrium prices

(c) the maximum profit

**Solution:**

(a) The total revenue function of the monopolist is given by

$$TR = (P_1 \times Q_1) + (P_2 \times Q_2)$$
$$TR = (40 - 2Q_1) \times Q_1 + (34 - 3Q_2) \times Q_2$$
$$TR = (40Q_1 - 2Q_1^2) + (34Q_2 - 3Q_2^2)$$

Thus, the profit function is

$$\pi = TR - TC$$
$$\pi = (40Q_1 - 2Q_1^2) + (34Q_2 - 3Q_2^2) - (Q_1^2 + \frac{1}{2}Q_2^2 + Q_1Q_2 + 10)$$
$$\pi = 40Q_1 - 2Q_1^2 + 34Q_2 - 3Q_2^2 - Q_1^2 - \frac{1}{2}Q_2^2 - Q_1Q_2 - 10$$

Maximization of profit requires the fulfillment of the following conditions -

The first order condition for maximization of profit is

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = \frac{\partial}{\partial Q_1}\left(40Q_1 - 2Q_1^2 + 34Q_2 - 3Q_2^2 - Q_1^2 - \frac{1}{2}Q_2^2 - Q_1Q_2 - 10\right)$$
$$\Rightarrow 40 - 4Q_1 - 2Q_1 - Q_2 = 0$$
$$\Rightarrow 40 - 6Q_1 - Q_2 = 0$$
$$\Rightarrow 6Q_1 + Q_2 = 40 \qquad (ii)$$

$$\pi_2 = \frac{\partial \pi}{\partial Q_2} = \frac{\partial}{\partial Q_2}\left(40Q_1 - 2Q_1^2 + 34Q_2 - 3Q_2^2 - Q_1^2 - \frac{1}{2}Q_2^2 - Q_1Q_1 - 10\right)$$
$$\Rightarrow 34 - 6Q_2 - Q_2 - Q_1 = 0$$
$$\Rightarrow 34 - 7Q_2 - Q_1 = 0$$
$$\Rightarrow Q_1 + 7Q_2 = 34 \qquad (iii)$$

We arrange the equation (ii) and (iii) in matrix form as follows

$$\begin{bmatrix} 6 & 1 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 40 \\ 34 \end{bmatrix}$$
$$\Rightarrow AX = C$$

Where,

$$A = \begin{bmatrix} 6 & 1 \\ 1 & 7 \end{bmatrix};\ Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} and\ C\begin{bmatrix} 40 \\ 34 \end{bmatrix}$$

According to Crammer's rule

$$Q_1 = \frac{|A_1|}{|A|} \quad and \quad Q_2 = \frac{|A_2|}{|A|}$$

Now,

$$|A| = \begin{vmatrix} 6 & 1 \\ 1 & 7 \end{vmatrix} = 42 - 1 = 41$$
$$|A_1| = \begin{vmatrix} 40 & 1 \\ 34 & 7 \end{vmatrix} = 280 - 34 = 246$$
$$|A_2| = \begin{vmatrix} 6 & 40 \\ 1 & 34 \end{vmatrix} = 204 - 40 = 164$$
$$\therefore Q_1 = \frac{246}{41} = 6 \quad and \quad Q_2 = \frac{164}{41} = 4$$

The second order condition requires:

$$|H_1| = \pi_{11} = \frac{\partial^2 \pi}{\partial Q_1^2} < 0$$

And $$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{11} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial^2 \pi}{\partial Q_1^2} & \dfrac{\partial^2 \pi}{\partial Q_1 Q_2} \\ \dfrac{\partial^2 \pi}{\partial Q_1 Q_2} & \dfrac{\partial^2 \pi}{\partial Q_2^2} \end{vmatrix} > 0$$

$$\pi_{11} = \frac{\partial^2 \pi}{\partial Q_1^2} = \frac{\partial}{\partial Q_1}\left(\frac{\partial \pi}{\partial Q_1}\right) = \frac{\partial}{\partial Q_1}(40 - 6Q_1 - Q_2) = -6$$

$$\pi_{12} = \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = \frac{\partial}{\partial Q_2}\left(\frac{\partial \pi}{\partial Q_1}\right) = \frac{\partial}{\partial Q_2}(40 - 6Q_1 - Q_2) = -1$$

$$\pi_{21} = \frac{\partial^2 \pi}{\partial Q_2 \partial Q_1} = \frac{\partial}{\partial Q_1}\left(\frac{\partial \pi}{\partial Q_2}\right) = \frac{\partial}{\partial Q_1}(34 - 7Q_2 - Q_1) = -1$$

$$\pi_{22} = \frac{\partial^2 \pi}{\partial Q_2^2} = \frac{\partial}{\partial Q_2}\left(\frac{\partial \pi}{\partial Q_1}\right) = \frac{\partial}{\partial Q_2}(34 - 7Q_2 - Q_1) = -7$$

$$|H_1| = \pi_{11} = -6 < 0$$

And $$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} -6 & -1 \\ -1 & -7 \end{vmatrix} = 42 - 1 = 41 > 0$$

Thus, equilibrium output $Q_1 = 6 \quad and \quad Q_2 = 4$.

(b) The equilibrium prices are given by

$$P_1 = 40 - 2 \times 6 = 40 - 12 = 28$$

$$P_2 = 34 - 3 \times 6 = 34 - 12 = 22$$

(c) The maximum profit

$$\pi = 40Q_1 - 2Q_1^2 + 34Q_2 - 3Q_2^2 - Q_1^2 - \frac{1}{2}Q_2^2 - Q_1Q_2 - 10$$

$$\pi = 40 \times 6 - 2(6)^2 + 34 \times 4 - 3 \times (4)^2 - (6)^2 - \frac{1}{2} \times (4)^2 - 6 \times 4 - 10$$

$$\pi = 178$$



**CHECK YOUR PROGRESS**

**Q 3:** The total revenue (R) and total cost (C) functions of a firm under perfect competition are given by

$$R = 10Q_1 + 4Q_2$$

$$C = 2Q_1^2 + \frac{1}{4}Q_2^2 + Q_1Q_2 + 5$$

Find the profit maximing output and maximum profit.

**Q 4:** A monopolist has the following average revenue function in two market

$$AR_1 = 60 - 2Q_1$$
$$AR_2 = 45 - 3Q_2$$

The total cost function of the monopolist is given by

$$TC = 3Q^2 - 6Q + 40; \quad where\ Q = Q_1 + Q_2$$

Find profit maximizing output level in each market.

## 10.5 EQUILIBRIUM OF MULTI-PLANT FIRM

When a firm produces a product in more than one plant under different cost condition is known as multi-plant firm. Under such a situation, the profit maximizing condition of the firm is different from the normal condition.

Let us assume that a firm has two production plant- plant-1 and plant-2. The output produce in plant-1 is $Q_1$ and in plant-2 is $Q_2$. The cost condition of plant-1 will depend on $Q_1$ and the cost condition of plant-2 will depend on $Q_2$Thus,

$$TC_1 = C_1(Q_1) \qquad and \qquad TC_2 = C_2(Q_2)$$

The total output produced by the multi-plant firm is assumed to be sold in a single market. Therefore, the total revenue (TR) function is given by

$$TR = R(Q) = R(Q_1 + Q_2)$$

Having defined the total cost and total revenue function, the profit function can be defined as follows:

$$\pi = TR - (TC_1 + TC_2)$$
$$\pi = R(Q) - C_1(Q_1) - C_2(Q_2)$$

Maximization of profit requires the fulfillment of the following conditions -

The first order condition for maximization of profit is

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = 0 \qquad and \qquad \pi_2 = \frac{\partial \pi}{\partial Q_2} = 0$$

The second order condition requires:

$$|H_1| = \pi_{11} = \frac{\partial^2 \pi}{\partial Q_1^2} < 0$$

And $$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} \frac{\partial^2 \pi}{\partial Q_1^2} & \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} \\ \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} & \frac{\partial^2 \pi}{\partial Q_2^2} \end{vmatrix} > 0$$

Now

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = \frac{\partial}{\partial Q_1}\{R(Q) - C_1(Q_1) - C_2(Q_2)\} = 0$$
$$\Rightarrow R'(Q)\frac{\partial Q}{\partial Q_1} - C'_1(Q_1) = 0$$
$$\Rightarrow R'(Q) - C'_1(Q_1) = 0 \qquad here, \frac{\partial Q}{\partial Q_1} = 1$$
$$\Rightarrow R'(Q) = C'_1(Q_1) \qquad (i)$$

Similarly,

$$\pi_2 = \frac{\partial \pi}{\partial Q_2} = \frac{\partial}{\partial Q_2}\{R(Q) - C_1(Q_1) - C_2(Q_2)\} = 0$$
$$\Rightarrow R'(Q)\frac{\partial Q}{\partial Q_2} - C'_2(Q_2) = 0$$
$$\Rightarrow R'(Q) - C'_2(Q_2) = 0 \qquad here, \frac{\partial Q}{\partial Q_1} = 1$$
$$\Rightarrow R'(Q) = C'_2(Q_2) \qquad (ii)$$

From equation (i) & (ii), we have,

$$R'(Q) = C'_1(Q_1) = C'_2(Q_2)$$
$$MR = MC_1 = MC_2$$

For the second order condition, we obtain the following,

$$\pi_{11} = \frac{\partial^2 \pi}{\partial Q_1^2} = \frac{\partial}{\partial Q_1}\left(\frac{\partial \pi}{\partial Q_1}\right) = R''(Q) - C''_1(Q_1)$$
$$\pi_{12} = \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = \frac{\partial}{\partial Q_2}\left(\frac{\partial \pi}{\partial Q_1}\right) = R''(Q)$$
$$\pi_{21} = \frac{\partial^2 \pi}{\partial Q_2 \partial Q_1} = \frac{\partial}{\partial Q_1}\left(\frac{\partial \pi}{\partial Q_2}\right) = R''(Q)$$
$$\pi_{22} = \frac{\partial^2 \pi}{\partial Q_2^2} = \frac{\partial}{\partial Q_2}\left(\frac{\partial \pi}{\partial Q_2}\right) = R''(Q) - C''_2(Q_2)$$

Now,

$$|H_1| = \pi_{11} = R''(Q) - C''_1(Q_1) < 0$$

And $$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} R''(Q) - C''_1(Q_1) & R''(Q) \\ R''(Q) & R''(Q) - C''_2(Q_2) \end{vmatrix} > 0$$

In case of a monopolist producing its output in three different plants, the conditions for profit maximization are-

$$MR = MC_1 = MC_2 = MC_3$$

And

$$|H_1| < 0; \quad |H_2| > 0 \quad and \quad |H_3| < 0$$

**Example : 4** A monopolist produces his product in two different plants and his total cost functions are given by

$$TC_1 = 20 - 4Q_1 + \frac{1}{2}Q_1^2$$
$$TC_2 = 40 - 8Q_2 + Q_2^2$$

If the average revenue function is given by

$$AR = 40 - 0 \quad where \quad Q = Q_1 + Q_2$$

Find (a) Profit maximizing output

(b) Maximum profit

**Solution:**

(a) Given

$$AR = 40 - 0 \quad where \quad Q = Q_1 + Q_2$$

So, $$TR = AR \times Q = (40 - Q) \times Q = 40Q - Q^2$$
$$= 40(Q_1 + Q_2) - (Q_1 + Q_2)^2$$
$$= 40Q_1 + 40Q_2 - Q_1^2 - 2Q_1Q_2 - Q_2^2$$

Given total revenue and total cost of the firm, we can define the profit function as follows-

$$\pi = 40Q_1 + 40Q_2 - Q_1^2 - 2Q_1Q_2 - Q_2^2 - 20 + 4Q_1 - \frac{1}{2}Q_1^2 - 40 + 8Q_2 - Q_2^2 \quad (I)$$

Maximization of profit requires the fulfillment of the following conditions -

The first order condition for maximization of profit is

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = 0 \qquad and \qquad \pi_2 = \frac{\partial \pi}{\partial Q_2} = 0$$

Now $$\pi_1 = \frac{\partial \pi}{\partial Q_1} = \frac{\partial}{\partial Q_1}\left\{\begin{matrix} 40Q_1 + 40Q_2 - Q_1^2 - 2Q_1Q_2 - Q_2^2 - 20 \\ + 4Q_1 - \frac{1}{2}Q_1^2 - 40 + 8Q_2 - Q_2^2 \end{matrix}\right\} = 0$$
$$\Rightarrow 40 - 2Q_1 - 2Q_2 + 4 - Q_1 = 0$$
$$\Rightarrow 44 - 3Q_1 - 2Q_2 = 0$$
$$\Rightarrow 3Q_1 + 2Q_2 = 44 \qquad (ii)$$

Similarly,

$$\pi_2 = \frac{\partial \pi}{\partial Q_2} = \frac{\partial}{\partial Q_2}\left\{\begin{matrix} 40Q_1 + 40Q_2 - Q_1^2 - 2Q_1Q_2 - Q_2^2 - 20 \\ +4Q_1 - \frac{1}{2}Q_1^2 - 40 + 8Q_2 - Q_2^2 \end{matrix}\right\} = 0$$

$$\Rightarrow 40 - 2Q_1 - 2Q_2 + 8 - 2Q_2 = 0$$

$$\Rightarrow 48 - 2Q_1 - 4Q_2 = 0$$

$$\Rightarrow 2Q_1 + 4Q_2 = 48 \qquad (iii)$$

We arrange the equation (ii) and (iii) in matrix form as follows

$$\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 44 \\ 48 \end{bmatrix}$$

$$\Rightarrow AX = C$$

Where,

$$A = \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix},\ Q = \begin{vmatrix} Q_1 \\ Q_2 \end{vmatrix} and\ C = \begin{vmatrix} 44 \\ 48 \end{vmatrix}$$

According to Crammer's rule

$$Q_1 = \left|\frac{A_1}{A}\right| \quad and \quad Q_2 = \left|\frac{A_2}{A}\right|$$

Now,

$$|A| = \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} = 12 - 4 = 8$$

$$|A_1| = \begin{vmatrix} 44 & 2 \\ 48 & 4 \end{vmatrix} = 176 - 96 = 80$$

$$|A_2| = \begin{vmatrix} 3 & 44 \\ 2 & 48 \end{vmatrix} = 144 - 88 = 56$$

$$\therefore Q_1 = \frac{80}{8} = 10 \quad and \quad Q_2 = \frac{56}{8} = 7$$

The second order condition requires:

$$|H_1| == |\pi_{11}| = \frac{\partial^2 \pi}{\partial Q_1^2} < 0$$

and $$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial^2 \pi}{\partial Q_1^2} & \dfrac{\partial^2 \pi}{\partial Q_1 \partial Q_2} \\ \dfrac{\partial^2 \pi}{\partial Q_1 \partial Q_2} & \dfrac{\partial^2 \pi}{\partial Q_2^2} \end{vmatrix} > 0$$

For the second order condition, we obtain the following,

$$\pi_{11} = \frac{\partial^2 \pi}{\partial Q_1^2} = \frac{\partial}{\partial Q_1}\left(\frac{\partial \pi}{\partial Q_1}\right) = \frac{\partial}{\partial Q_1}\{44 - 3Q_1 - 2Q_2\} = -3$$

$$\pi_{12} = \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = \frac{\partial}{\partial Q_2}\{44 - 3Q_1 - 2Q_2\} = -2$$

$$\pi_{21} = \frac{\partial^2 \pi}{\partial Q_2 \partial Q_1} = \frac{\partial}{\partial Q_1}\{48 - 2Q_1 - 4Q_2\} = -2$$

$$\pi_{22} = \frac{\partial^2 \pi}{\partial Q_2^2} = \frac{\partial}{\partial Q_2}\{48 - 2Q_1 - 4Q_2\} = -4$$

Now,

$$|H_1| = \pi_{11} = -3 < 0$$

and $$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{11} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} -3 & -2 \\ -2 & -4 \end{vmatrix} = (12 - 4) = 8 > 0$$

Thus, the profit maximizing output levels are $Q_1 = 10 \quad and \quad Q_2 = 7$

(b) the maximum profit

$$\pi = 40 \times 10 + 40 \times 7 - (10)^2 - 2 \times 10 \times 7 - (7)^2$$
$$-20 + 4 \times 10 - \frac{1}{2} \times (10)^2 - 40 + 8 \times 7 - (7)^2$$
$$\pi = 328$$



## CHECK YOUR PROGRESS

**Q 5 :** A monopolist produces his product in two different plants and his total cost functions are given by

$$TC_1 = 10 - 2Q_1 + Q_1^2$$
$$TC_2 = 15 - 6Q_2 + 2Q_2^2$$

If the average revenue function is given by

$$AR = 50 - 2Q \qquad where\, Q = Q_1 + Q_2$$

Find (a) Profit maximizing output

(b) Maximum profit

**Q 6:** A monopolist produces his product in three different plants and his total cost functions are given by

$$TC_1 = 20 - 4Q_1 + Q_1^2$$
$$TC_2 = 50 - 6Q_1 + Q_2^2$$
$$TC_3 = 25 - 3Q_3 + Q_3^2$$

If the average revenue function is given by

$$AR = 50 - 2Q \qquad where\, Q = Q_1 + Q_2$$

Find (a) Profit maximizing output

(b) Maximum profit



## 10.6 LET US SUM UP

- The conditions for maximization of a monopolist profit are

$$\pi_1 = \frac{\partial \pi}{\partial Q_1} = 0 \qquad and \qquad \pi_2 = \frac{\partial \pi}{\partial Q_2} = 0$$

And $$|H_1| = |\pi_{11}| = \frac{\partial^2 \pi}{\partial Q_1^2} = R''_1(Q_1) - C''(Q) < 0$$

and $$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} \frac{\partial^2 \pi}{\partial Q_1^2} & \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} \\ \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} & \frac{\partial^2 \pi}{\partial Q_2^2} \end{vmatrix} > 0$$

- When $|e_1| > |e_2|$, $\left(1 - \frac{1}{|e_1|}\right) > \left(1 - \frac{1}{|e_2|}\right)$ and therefore $P_1 < P_2$. Thus, higher the elasticity, lower will be the price.

- In real world most firms are joint products or multi-product firm. Multi-product firm exists when the production of two or more products are technically dependent upon one another. In such cases, total revenue of a firm will depend on more than one product.

- When a firm produces a product in more than one plant under different cost condition is known as multi-plant firm. Under such a situation, the profit maximizing condition of the firm is different from the normal condition.

## 10.7 FURTHER READING

1) Baruah, S. (2011). *Basic Mathematics and Its Applications in Economics.* Macmillan.

2) Chiang, A.C. (1986). *Fundamental Methods of Mathematical Economics.* NewYork : McGraw Hill.

3) Allen, R.G.D. (1974). *Mathematical Analysis for Economists*. London: Macmillan Press and ELBS.

4) Mehta and Madnani (2007). *Mathematics for Economists*. Sultan Chand & Sons.

## 10.8 ANSWER TO CHECK YOUR PROGRESS

**Ans to Q 1:**

(a) $Q_1 = 6;\ Q_2 = 7.5;$

(b) $P_1 = 62;\ P_1 = 74;$

(c) $e_1 = 31/9;\ \ e_1 = 37/15;$

(d) $\pi = 647.5$

**Ans to Q 2:**

(a) $Q_1 = 5.07;\ Q_2 = 3.09;\ Q_3 = 3.14;$

(b) $P_1 = 62.65;\ P_2 = 49.64;\ P_3 = 46.58;$

(c) $\pi = 402.30$

**Ans to Q No. 3:** $Q_1 = 1;\ Q_2 = 6;\ \text{Pr}\,ofit = 12$

**Ans to Q No. 4:** $Q_1 = 5.79;\ Q_2 = 1.36;$

**Ans to Q No. 5:** $Q_1 = 6;\ Q_2 = 4;\ \text{Pr}\,ofit = 243$

**Ans to Q No. 6**: $Q_1 = 3;\ Q_2 = 4;\ Q_3 = 5\ \ \text{Pr}\,ofit = 230.5$

## 10.9 MODEL QUESTIONS

Q 1: The average revenues and average cost functions are

$$AR_1 = 80 - 3Q_1$$
$$AR_2 = 104 - 4Q_2$$
$$AC = 2Q - 10 \qquad where\ \ Q = Q_1 + Q_2$$

$Q_1\ and\ Q_2$ are the outputs to be sold in two markets.

Find profit maximizing outputs and equilibrium prices.

Q 2: A monopolist discriminates prices in three markets on the basis of demand elasticities where average revenue (AR) functions are such that

$$AR_1 = 4 - 2Q_1$$
$$AR_2 = 8 - 2Q_2$$
$$AR_3 = 10 - 3Q_3$$

If the total cost function is $TC = -20 - 2Q + Q^2 \qquad where\, Q = Q_1 + Q_2$

Find

(a) Profit maximizing output
(b) Prices in each market
(c) Maximum profit

Q 3: A monopolist produces his product in three different plants and his total cost functions are given by

$$TC_1 = 40 - 8Q_1 + 2Q_1^2$$
$$TC_2 = 100 - 12Q_2 + 2Q_2^2$$
$$TC_3 = 50 - 6Q_3 + 2Q_3^2$$

If the average revenue function is given by

$$AR = 100 - 4Q \qquad where\, Q = Q_1 + Q_2$$

Find

(a) Profit maximizing output
(b) Maximum profit

*** ***** ***

# UNIT-11: OPTIMIZATION TECHNIQUE-III : OPTIMIZATION WITH EQUALITY CONSTRAINT

**UNIT STRUCTURE**


11. 1 Learning Objectives

11. 2 Introduction

11. 3 Concept of Constrained Extrema

11. 4 Determination of Optimal Value

11. 5 Lagrange Multiplier Method

11. 6 Constrained Extremum with more than one Constraint

11. 7 Let us Sum Up

11. 8 Further Reading

11. 9 Answers to Check Your Progress

11. 10 Model Questions


## 11.1. LEARNING OBJECTIVES

After going through this unit, you will be able to–

- state the meaning of constrained extrema
- explain structure of optimization with equality constraint
- understand the Lagrange multiplier method.

## 11.2 INTRODUCTION

The case of unconstrained extrema, the optimal value of a function with more than one explanatory variable is derived without having any constraint on the optimal choice of the values of the explanatory variables. In other words, all the variables are independent of one another. The decision made regarding one variable does not have an impact of the choices of the remaining variables. But in real economic world, there are many economic problems which involve finding an consumer behaviour, a consumer maximized his utility subject to his budget constraint. Similarly in the theory of production, a firm's output is maximized subject to a cost constraint. These types of problems are called constrained optimization.

In the present chapter, we try to disuess the technique of finding the optimal value of a function with equality constraint(s).

## 11.3 CONCEPT OF CONSTRAINED EXTREMA

Let us consider a function with two explanatory variables '$x_1$' & '$x_2$' such that

$$y= f(x_1, x_2) \text{ ———— } (11.1)$$

subject to an equality constraint

$$g(x_1, x_2)= C \text{ ———— } (11.2)$$

Where C is a constant.

Now, in order to get an optimal solution for the functional equation (11.1), it must satisfy the equation (11.2). Equation (11.1) is called the "objective function" & the equation (11.2) is called as equality constraint.

The difference between a constrained extremum and an unconstrained extremum may be illustrated in a three dimensional graph as shown in figure (11.1).

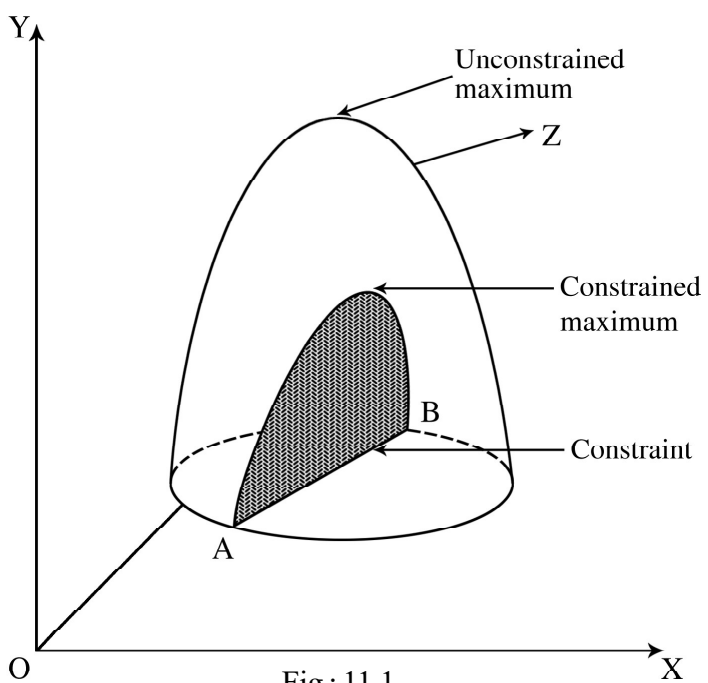



Fig : 11.1

The unconstrained (free) extremum in this particular graph is the peak point of the entire dome, but the constrained extremum is at the peak of the inverse U–shaped curve lying directly above the constraint line AB. In general, a constrained maximum is expected to have a lower value than the unconstrained maximum, although, by coincidence, the two maxima may happen to have the same value. But the constrained maximum can never exceed the unconstrained maximum.

## 11.4 DETERMINATION OF OPTIMAL VALUE

In case of optimization technique with an equality constraint, the optimal choice of the values of the explanatory variables are very much dependent on the constraint function. Even withut any new technique of solution, we can find out the optimal solution by the method of elimination. In such case, we substitute one of the explanatory variables in terms of the other in usual unconstrained optimization technique is followed to find out the extreme values.

Suppose the objective function to be maximized is given by

$Y=f(x_1, x_2)= x_1 x_2+2x_1$ ———— (11.3)

& the constraint $g(x_1, x_2) = C$ is in the form

$x_1+x_2 = 20$ ———— (11.4)

Now from equation (11.4)

$x_1 = 20–x_2$ ———— (11.5)

Substituting equation (11.5) in equation (11.3), we have

$Y=(20–x_2) x_2+2(20–x_2)$

$\Rightarrow Y = 20x_2–x_2^2+40 – 2x_2$

$\Rightarrow Y = –x_2^2+18x_2+40$ ———— (11.6)

Now maximization of equation (11.6) requires that

$$\frac{dy}{dx_2}=0 \; and \; \frac{d^2y}{dx_2^{\,2}}<0$$

Thus $\frac{dy}{dx_2} = 0$ gives

$-2x_2+18=0$

$\Rightarrow 2x_2=18$

$\Rightarrow \overline{x_2} = 9$

The second order condition

$$\frac{d^2y}{dx_2^2} = -2 < 0$$

Now putting $\overline{x_2} = 9$ in equation (11.5), we get

$$\overline{x_1} = 20 - \overline{x_2} = 20 - 9 = 11$$

$\therefore \overline{x_1} = 11 \text{ and } \overline{x_2} = 9$ will maximize the objective function $y=x_1x_2+2x_1$ and also satisfy the constraint function $x_1+x_2=20$

Therefore, the maximum value of Y is

$\overline{y} = x_1x_2+2x_1$

$= 11.9+2.11$

$= 99+22=121$

But the above technique is of no use when the constraint itself is a complicated function or when we will have more than one constraint. In such cases, we generally apply the Lagrange multiplier method to find out the optimal value of the variables.

## 11.5 LAGRANGE MULTIPLIER METHOD

Lagrange multiplier method seeks to convert the constrained extremum problem into a form to which the first order and second order conditions of the unconstrained extremum problem can still be applied.

In an optimization problem with two explanatory variables $x_1$ & $x_2$ with the objective function as in equation (11.1) & the equality constraint as in equation (11.2), for obtaining optimal values, we first construct the Lagrange function as follows :

$$L=f(x_1, x_2)+\lambda[c-g(x_1, x_2)]-(11.7)$$

The symbol $\lambda$ (lambda) representing some undetermined value is called a Lagrange multiplier. If we have the objective of maximizing equation (11.1) subject to equation (11.2), then the maximization of Lagrange function is equation (11.7) will provide the optimal values of $x_1$ and $x_2$. Since the Lagrange function is now a function of three variables $x_1$, $x_2$ and $\lambda$, the maximization of L in equation (11.7) requires to satisfy the first order and second order conditions of maximization.

**(I) First order conditions**

The first order condition requires

$$\frac{\delta L}{\delta x_1}=0,\ \frac{\delta L}{\delta x_2}=0 \text{ and } \frac{\delta L}{\delta \lambda}=0$$

Now

$$\left.\begin{aligned}\frac{\delta L}{\delta x_1}&=L_1=f_1-\lambda g_1=0\\ \frac{\delta L}{\delta x_2}&=L_2=f_2-\lambda g_2=0\\ \frac{\delta L}{\delta \lambda}&=L_\lambda=c-g(x_1x_2)=0\end{aligned}\right\}(11.8)$$

(Since c is constant, its derivative is zero)

From the first two equations of (11.8)

$$\lambda=\frac{f_1}{g_1}=\frac{f_2}{g_2} \qquad -(11.9)$$

Since the last equation in (11.8) is simply a restatement of the

equality constraint (11.2), Lagrange function L will automatically satisfy the constraint of the original objective function (11.1).

The same first order conditions are true for the minimization problem also if we want to minimize the objective function (equation 11.1) subject to the equality constraint (equation 11.2).

**(II) Second order conditions**

The introduction of a Lagrange multiplier as an additional variable makes it possible to apply the same first order condition of unconstrained extremum problem in constrained extremum problem also. However, we should not apply the second order condition of unconstrained extremum in the Lagrange function. This is because of the fact that atthough the optimal solution of L depends on the optimal choice of $x_1$, $x_2$ and $\lambda$, but unlike $\overline{x_1}$ and $\overline{x_2}$, the choice of $\overline{\lambda}$ is meaningless. Even if $\overline{\lambda}$ is replaced by any other value of $\lambda$, no effect will be produced on $\overline{L}$, since [c–g($x_1$, $x_2$)] is identically zero. While it is harmless to treat $\lambda$ as just another variable in the discussion of the first order condition, we must be careful not to apply blindly the second order conditions of unconstrained extremum in the present constrained case. Rather, we must derive a set of new ones. The new conditions can be restated in terms of second order total differential $d^2y$.

When we have to determine the constrained extremum of Y=f($x_1$, $x_2$) subject to g($x_1$, $x_2$)=c, the second order sufficient conditions will depend on the algebraic sign of the second order total differential $d^2y$.

(i) For maximum value of y, $d^2y$ should be negative definite subject to dg=0

(ii) For minimum value of y, $d^2y$ should be positive definite subject to dg=0

In case of unconstrained extremum, the sufficient conditions are expressed in terms of Hessian determinant |H|, but is case of sufficient

conditions in terms of bordered Hessian determinant $|\overline{H}|$.

In maximization problem, $d^2y<0$ subject to $dg=0$ requires that

$$\begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} > 0 \quad \text{——— (11.10)}$$

In minimization problem, $d^2y>0$ subject to $dg=0$ requires that

$$\begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} < 0 \quad \text{——— (11.11)}$$

The determinants in equation (11.10) & (11.11) are known as bordered Hessian determinants $|\overline{H}|$. In case of two explanatory variables in the optimization problem, it is specifically denoted by $|\overline{H}|$.

Thus in case of two explanatory variables in the optimization problem, the second order condition requires.

For maximization problem,

$$|\overline{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} > 0$$

For minimization problem,

$$|\overline{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} < 0$$

Similarly, if we have an objective function having three explanatory variables with a constraint, say,

Maximiza

$y=f(x_1, x_2, x_3)$ ——— (11.12)

Subject to

$g(x_1, x_2, x_3) = c$ ——— (11.13)

then we have the Lagrange function as

$L=f(x_1, x_2, x_3) + \lambda[c-g(x_1, x_2, x_3)]$

The first order condition requires

$$L_1 = \frac{\delta l}{\delta x_1} = 0,\ L_2 = \frac{\delta L}{\delta x_2} = 0,\ L_3 = \frac{\delta L}{\delta x_3} = 0 \text{ and } L_\lambda = \frac{\delta L}{\delta \lambda} = 0$$

The second order condition requires

$$\left|\overline{H}_2\right| \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} > 0$$

and $$\left|\overline{H}_3\right| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{vmatrix} < 0$$

For minimization of the objective function (equation 11.12) subject to the constraint (equation 11.13), the first order conditions remain the same as in case of maximization but the second order condition requires

$$\left|\overline{H}_2\right| < 0 \ and \left|\overline{H}_3\right| < 0$$

In general, when we have an objective function with 'n' explanatory variables in the form

$y=f(x_1, x_2, \text{----------}, x_n)$ ——— (11.14)

Subject to

$g(x_1, x_2, \text{---------}, x_n) = c$ ——— (11.15)

then we have the Lagrange function as

$L=f(x_1, x_2, \text{--------}, x_n) + \lambda g(x_1, x_2, \text{--------}, (x_n)$

The first order condition for both maximum and minimum requires

$L_1= L_2= L_3= \text{--------} = L_n = L_\lambda =0$

The second order condition for maximum requires

$$\left|\overline{H}_2\right|\rangle 0,\ \left|\overline{H}_3\right|\langle 0, ------, (-1)^n \left|\overline{H}_n\right|\rangle 0$$

Whereas the second order condition for minimum requires

$$\left|\overline{H}_2\right|\langle 0,\ \left|\overline{H}_3\right|\langle 0, ------, (-1)^n \left|\overline{H}_n\right|\langle 0$$

Where

$$\left|\overline{H}_n\right| = \begin{vmatrix} 0 & g_1 & g_2 & g_n \\ g_1 & L_{11} & L_{12} & L_{1n} \\ g_2 & L_{21} & L_{22} & L_{2n} \\ \vdots & & & \\ g_n & L_{n1} & L_{n2} & L_{nn} \end{vmatrix}$$

**Example 1 :**

Maximize $y = x_2x_2 + 2x_1$

Subject to $x_1+x_2 = 20$

Solution : The Lagrange function is given by

$L=x_1x_2+2x_1+\lambda(20-x_1-x_2)$ —— (11.16)

The first order condition of maximization requires

$$L_1 = \frac{\delta L}{\delta x_1} = 0 \ ; \ L_2 = \frac{\delta L}{\delta x_2} = 0 \text{ and } L_\lambda = \frac{\delta L}{\delta \lambda} = 0$$

Now

$$\left.\begin{aligned} L_1 &= x_2 + 2 - \lambda = 0 \\ L_2 &= x_1 - \lambda = 0 \\ L_\lambda &= 20 - x_1 - x_2 = 0 \end{aligned}\right\} ---(11.17)$$

From first two equations of equation (11.17),

we have $\lambda = x_2 + 2$

and $\lambda = x_1$

$\therefore x_2 + 2 = x_2$

Or $x_1 - x_2 = 2$ -----(11.18)

Now from the third equation of equation (11.17), we get

$x_1 + x_2 = 20$ ----- (11.19)

Now adding equations (11.18) and (11.19), we get

$2x_1 = 22$

$\therefore \bar{x}_1 = 11$

Now substituting $x_1 = 11$ in equation (11.18), we get

$x_2 = x_1 - 2$

or $\bar{x}_2 = 11 - 2 = 9$

To test whether $\bar{x}_1 = 11$ and $\bar{x}_2 = 9$ will maximize the objective function or not, we have to verify the value of $|\bar{H}_2|$ as second order condition.

New from equation (11.17),

$L_{11} = 0 \quad L_{12} = 1$

$L_{21} = 1 \quad L_{22} = 0$

again $g_1 = 1$ and $g_2 = 1$

$$\therefore |\bar{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2 > 0$$

Since $\therefore |\bar{H}_2| > 0, \bar{x}_1 = 11 \; and \; \bar{x}_2 = 9$ will maximize the objection function subject to the constraint.

$\therefore$ The maximum value of Y will be

$Y = x_1x_2 + 2x_1$

$= 11\times9 + 2\times11$

$= 99 + 22$

$= 121$

**Example 2 :**

Minimize $y=3x_1^2-4x_1x_2 + 2x_2$

Subject to $x_1+x_2=22$

**Solution :**

The Lagrange function is given by

$L = 3x_1^2-4x_1x_2+2x_2+\lambda[22-x_1-x_2]$ ——— (11.20)

The first order contions requires.

$$L_1 = \frac{\delta L}{\delta x_1} = 0 \; ; \; L_2 = \frac{\delta L}{\delta x_2} = 0 \text{ and } L_\lambda = \frac{\delta L}{\delta \lambda} = 0$$

Now,
$$\left.\begin{aligned} L_1 &= 6x_1 - 4x_2 - \lambda = 0 \\ L_2 &= -4x_1 + 2 - \lambda = 0 \\ L_\lambda &= 22 - x_1 - x_2 = 0 \end{aligned}\right\} ----(11.21)$$

From first two equations of equation (11.21), we have

$6x_1- 4x_2 = \lambda$

$- 4x_1+2= \lambda$

$\therefore 6x_1-4x_2= -4x_1+2$

$\Rightarrow 6x_1+4x_1-4x_2=2$

$\Rightarrow 10x_1-4x_2= 2$ ----- (11.22)

From the third equation of equation (11.21), we get

$x_1+x_2=22$ ----- (11.23)

Now multiplying both sides of equation (11.23) by 4 and adding to equation (11.22), we get

$10x_1 - 4x_2 = 2$

$4x_1 + 4x_2 = 88$

$14x_1 = 90$

$$\therefore \bar{x}_1 = \frac{90}{14} = \frac{45}{7}$$

Substituting $x_1 = \frac{45}{7}$ in equation (11.23), we get

$x_2 = 22 - x_1$

or $= 22 - \frac{45}{7}$

$$\bar{x}_2 = \frac{154 - 45}{7} = \frac{109}{7}$$

The second order condition for minimization requires that $|\bar{H}_2| < 0$ from equation (11.21)

$L_{11} = 6$ $L_{12} = -4$

$L_{21} = -4$ $L_{22} = 0$

Again from equation (11.23)

$g_1 = 1$ and $g_2 = 1$

$$\therefore |\bar{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 6 & -4 \\ 1 & -4 & 0 \end{vmatrix}$$

$= -14 < 0$

Therefore, $|\bar{x}_1| = \frac{45}{7}$ and $\bar{x}_2 = \frac{109}{7}$ will minimize the function

$y = 3x_1^2 - 4x_1x_2 + 2x_2$ and the minimum value will be

$$y = 3\left(\frac{45}{7}\right)^2 - 4\cdot\frac{45}{7}\cdot\frac{109}{7} + 2\cdot\frac{109}{7}$$

$$= 3\cdot\frac{2025}{49} - \frac{19620}{49} + \frac{218}{7}$$

$$= \frac{6075}{49} - \frac{19620}{49} + \frac{218}{7}$$

$$= \frac{6075 - 19620 + 1526}{49}$$

$$= \frac{7601 - 19620}{49}$$

$$= \frac{-12019}{49} = \frac{-1717}{7}$$

**Example 3 :**

Maximize $y=3x_1x_2$

Subject to $2x_1+x_2=10$

**Solution :**

The Lagrange function is given by

$L=3x_1x_2+\lambda(10-2x_1-x_2)$ ------- (11.24)

The first order condition requires

$$L_1 = \frac{\delta L}{\lambda x_1} = 0 \; ; \; L_2 = \frac{\delta L}{\delta x_2} = 0 \text{ and } L_\lambda = \frac{\delta L}{\delta \lambda} = 0$$

Now

$$\left.\begin{aligned} L_1 &= 3x_2 - 2\lambda = 0 \\ L_2 &= 3x_1 - \lambda = 0 \\ L_\lambda &= 10 - 2x_1 - x_2 = 0 \end{aligned}\right\} ----(11.25)$$

From first two equations of equation (11.25), we get

$3x_2=2\lambda$

or $\lambda = \frac{3}{2}x2$

and $3x_1=\lambda$

$\therefore \frac{3}{2}x_2 = 3x_1$

or $3x_2=6x_1$

or $6x_1-3x_2=0$ ----- (11.26)

From the third equation of equation (11.25), we get

$2x_1+x_2=10$ ----- (11.27)

Now multiplying both sides of equation (11.7) by 3 and adding to equation (11.26), we get

$6x_1-3x_2= 0$

$6x_1+3x_2=30$

---

$12x_1 \quad =30$

$\bar{x}_1 = \frac{30}{12} = \frac{5}{2}$

Substituting $x_1 = \frac{5}{2}$ in equation (11.27), we get

$2 \cdot \frac{5}{2} + x_2 = 10$

or $5+x_2=10$

or $\bar{x}_2 = 10-5=5$

The second order condition of maximization requires that $|\bar{H}_2| \rangle 0$

From equation (11.25),

$L_{11} = 0 \quad L_{12} = 3$

$L_{12} = 0 \quad L_{22} = 0$

Again from equation (11.27),

$g_1=2$ and $g_2=1$

$$\therefore \left|\overline{H}_2\right| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{vmatrix}$$

$$= 12 > 0$$

Therefore, $\bar{x}_1 = \frac{5}{2}$ and $\bar{x}_2 = 5$ will maximize the function $y=3x_1x_2$ and the maximum value will be

$$y = 3 \cdot \frac{5}{2} \cdot 5 = \frac{75}{2}$$

## 11.6 CONSTRAINED EXTREMUM WITH MORE THAN ONE CONSTRAINT

When we have more than one constraint in an optimizing problem, the first order and second order conditions remain the same, but the elements of the bordered Hessian determinant will change depending on the number of equality constraints.

For instance, when we have an optimizing problem of two explanatory variables $x_1$ & $x_2$ with two equality constraints, say

Objective function $y=f(x_1, x_2)$

Subject to $g^1(x_1, x_2) = c_1$,

$g^2(x_1, x_2) = c_2$

In such a case, the Lagrange function is defined as

$L=f(x_1, x_2) + \lambda_1[c_1-g^1(x_1, x_2)]+\lambda_2[c_2-g^2(x_1, x_2)]$

The first order condition for extremum (both maximum & minimum) requires

$$L_1=\frac{\delta L}{\delta x_1}=0 \; ; \; L_2=\frac{\delta L}{\delta x_2}=0 \; ; \; L_{\lambda_1}=\frac{\delta L}{\delta \lambda_1}=0 \; ; \; L_{\lambda_2}=\frac{\delta L}{19_2}=0$$

The second order condition requires

$|\overline{H}_2|>0$ for maximum

and $|\overline{H}_2|<0$ for minimum,

Where

$$|\overline{H}_2|=\begin{vmatrix} 0 & 0 & g_1^1 & g_2^1 \\ 0 & 0 & g_1^2 & g_2^2 \\ g_1^1 & g_1^2 & L_{11} & L_{12} \\ g_2^1 & g_2^2 & L_{21} & L_{22} \end{vmatrix}$$

Where $g_i^j$ is the differentiation of jth constraint with repect to the $i^{th}$variable.



**CHECK YOUR PROGRESS**

**Q 1 :** Maximize a function $y=5x_1x_2$

Subject to $x_1+2x_2=8$

**Q 2:** Find the extreme values of the function

$$y=\frac{1}{2}x_1^2-2x_1x_2+\frac{1}{2}x_2^2+20$$

Subject to $2x_1+2x_2=96$



## 11.7 LET US SUM UP

- Structure of equality constrained extrema with more than two variables '$X_1$' and '$X_2$' is

  Objective function : $Y= f(X_1, X_2)$

  Structural constraint : $g = (X_1, X_2)=C$

  Where C is a constant

- Lagrange function can be constructed as

  $L= f(x_1, x_2)+ \lambda [c-g(x_1, x_2)]$

  Where $\lambda$ = Lagrange multiplier

- First order condition for Lagrange multiplier method requires (in case of two variables $x_1$ and $x_2$)

$$\frac{\delta L}{\delta X_1}=\frac{\delta L}{\delta X_2}=\frac{\delta L}{\delta \lambda}=0$$

- Second order condition for Lagrange multiplier method requeries (in case of two variables $x_1$ and $x_2$)

$$\left|\overline{H_1}\right|>0\,(\text{For maximization})$$

$$\left|\overline{H_2}\right|<0\,(\text{For minimization})$$

$$\text{Where}\left|\overline{H_2}\right|=\begin{vmatrix} o & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}$$

- In case of n explanatory variables the Lagrange function becomes

  $L=f(x_1, x_2 .....x_w)+ \lambda [C-g(x_1, x_2 .....x_n)]$

  Then the first order condition for optimization requires

$$\frac{\delta L}{\delta x_1}=\frac{\delta L}{\delta x_2}=.....=\frac{\delta L}{\delta x_n}=\frac{\delta L}{\delta \lambda}=0$$

  And for second order condition requires

$$\left|\overline{H_2}\right| > 0,\ \left|\overline{H_3}\right| < 0 \ldots\ldots\ldots\ldots (-1)^w \left|\overline{H_n}\right| > 0 \text{ (For maximization)}$$

$$\left|\overline{H_2}\right| < 0,\ \left|\overline{H_3}\right| < 0 \ldots\ldots\ldots\ldots (-1)^n \left|\overline{H_w}\right| < 0 \text{ (For minimization)}$$

## 11.8 FURTHER READING


1) Baruah, S. (2011). *Basic Mathematics and Its Applications in Economics.* Macmillan.

2) Chiang, A.C. (1986). *Fundamental Methods of Mathematical Economics.* NewYork : McGraw Hill.

3) Allen, R.G.D. (1974). *Mathematical Analysis for Economists*. London: Macmillan Press and ELBS.

4) Mehta and Madnani. (2007). *Mathematics for Economists*. Sultan Chand & Sons.


## 11.9 ANSWER TO CHECK YOUR PROGRESS

**Ans to Q No. 1:** $\bar{x}_1 = 4,\ \bar{x}_2 = 2$ and maximum value of $\bar{y}$ = 40

**Ans to Q No. 2:** $\bar{x}_1 = 24,\ \bar{x}_2 = 24$ minimizes

## 11.10 MODEL QUESTIONS

**Q1.** Define Lagrange mutliplier.

**Q2.** Explain the structure of optimization with equality constraint.

**Q3.** Find the extreme value of the following function subject to the given constraint:

$$U = xy^{\frac{1}{2}}.\text{subject to } 4x + y = 48$$

*** ***** ***

# UNIT 12 : ECONOMIC APPLICATIONS OF CONSTRAINED OPTIMISATION

## UNIT STRUCTURE


12.1 Learning Objectives

12.2 Introduction

12.3 Utility Maximization and Theory of Consumer Bahaviour

12.4 Derivation of Conditions of Least Cost Combination of Input for a Producer

12.5 Comparative Static Analysis of Utility Maximisation

12.6 Let Us Sum Up

12.7 Further Reading

12.8 Answers to Check Your Progress

12.9 Model Questions


## 12.1 LEARNING OBJECTIVES

After going through this unit, you will be able to–

- obtain the consumer equilibrium
- obtain the minimisation of cost with a output combination
- derive that the price effect on the consumer's is the sum of income effect and substitution effect.

## 12.2 INTRODUCTION

Contrained optimization is the process of optimizing an objective function with respect to some variables in the presence of constraints on those variables. Most of the economic decisions are the results of an optimization problem subject to one or a series of constraints. For example, consumers have to choose the combination of goods in order to maximise his total utility subject to budget constraint. Similarly, firms minimise costs by choosing a combination of the factor inputs subject to the given level of output or production function constraint. In this chapter, we will discuss the important applications of constrained optimization in economic problems,

viz. utility maximisation and consumer behaviour, least cost combination of input and comparative static analysis of utility maximisation.

## 12.3 UTILITY MAXIMIZATION AND THEORY OF CONSUMER BEHAVIOUR

There are two approaches to study consumer behaviour– the first approach is known as cardinal utility approach or popularly known as Marshallian approach and the second approach is ordinal utility approach, also known as indifference curve approach. According to Marshall the consumer will be in equlibrium if the ratio of marginal utilities are equi-proportional to their price. The Hicksian indifference curve approach state that a consumer will be in equilibrium if the budget line of the consumer is become tangent to the highest possible indifference curve i.e. at the point of equilibrium the slope of indifference curve = the slope of the budget line and also the indifference curve must be convex to the origin at the point of equilibrium.

**A) Marshallian Concept of Utility Maximisation:** The rational consumer always tries to derive maximum satisfaction out of his limited income or total expenditure. Let us assume that the consumer consumes two goods, say x and y. Thus, the utility function of the consumer can be written as:

$$U = U(x, y)$$

and the budget line of the consumer will be

$$B = x.P_X + y.P_Y$$

Where, B = total income or budget

$P_X$ = price of commodity x

$P_Y$ = price of commodity y

Now, our problem is to maximise the utility function

$$U = U(x, y) \quad (1)$$

Subject to $B = xP_X + yP_Y$

$$\Rightarrow \quad B - xP_X - yP_Y = 0 \quad (2)$$

Let us first construct the Lagrange's function (L) with the help of Lagrange's multiplier ($\lambda$) as follows:

$$L = U(x, y) + \lambda[B - xP_X - yP_Y] \qquad (3)$$

Now, maximisation of utility function subject to given budget constraint requires that:

First order condition:

It requires: $L_X = \frac{\partial L}{\partial x} = 0$

$$L_Y = \frac{\partial L}{\partial y} = 0$$

$$L_\lambda = \frac{\partial L}{\partial \lambda} = 0$$

Here, $L_X = \frac{\partial L}{\partial x} = \frac{\partial U}{\partial x} - \lambda P_X$

$$L_Y = \frac{\partial L}{\partial y} = \frac{\partial U}{\partial y} - \lambda P_Y$$

$$L_\lambda = B - xP_X - yP_Y$$

According to 1st order condition:

$$L_X = \frac{\partial U}{\partial x} - \lambda P_X = 0 \qquad \Rightarrow \lambda = \frac{\partial U}{\partial X} \Big/ P_X \qquad (4)$$

$$L_Y = \frac{\partial U}{\partial y} - \lambda P_Y = 0 \qquad \Rightarrow \lambda = \frac{\partial U}{\partial Y} \Big/ P_Y \qquad (5)$$

$$L_\lambda = B - xP_X - yP_Y = 0 \quad (6)$$

Now, from eqn (4) and (5), we have

$$\lambda = \frac{\partial U}{\partial X} \Big/ P_X = \frac{\partial U}{\partial Y} \Big/ P_Y$$

$$\Rightarrow \frac{MU_X}{P_X} = \frac{MU_Y}{P_Y}$$

$$\Rightarrow \frac{MU_X}{MU_Y} = \frac{P_X}{P_Y} \quad (7)$$

$$\left[ \because \quad \frac{\partial U}{\partial x} = \frac{\partial}{\partial x} U(x, y) = MU_X \right.$$

$$\left. \text{and} \quad \frac{\partial U}{\partial y} = \frac{\partial}{\partial y} U(x, y) = MU_Y \right]$$

This eqn (7) is the 1st order necessary condition for consumer equilibrium according to Marshallian cardinal utility analysis. The 1st order condition of consumer equilibrium gives that the marginal utilities are equi-proportional to their prices.

Now, for more clearification, we must have to move for the varification

of 2nd order condition of utility maximisation.

Second order condition:

We know that, from the mathematical point of view the 2nd order condition for utility maximisation requires that the 2nd order Bordered Hessian Determinant must be positive i.e. $|\overline{H}_2| > 0$.

We have,

$$L_X = \frac{\partial U}{\partial x} - \lambda P_X \qquad L_Y = \frac{\partial U}{\partial y} - \lambda P_Y \qquad g = B - xP_X - yP_Y$$

$$\Rightarrow L_Y = U_X - \lambda P_X \qquad \Rightarrow L_Y = U_Y - \lambda P_Y \qquad g_X = -P_X$$

$$L_{XX} = U_{XX} \qquad L_{YX} = U_{YX} \qquad g_Y = -P_Y$$

$$L_{XY} = U_{XY} \qquad L_{YY} = U_{YY}$$

Now, $$|\overline{H}_2| = \begin{vmatrix} 0 & g_X & g_Y \\ g_X & L_{XX} & L_{XY} \\ g_Y & L_{YX} & L_{YY} \end{vmatrix} = \begin{vmatrix} 0 & -P_X & -P_Y \\ -P_X & U_{XX} & U_{XY} \\ -P_Y & U_{YX} & U_{YY} \end{vmatrix}$$

$$= 0 + P_X[-P_X.U_{YY} + P_Y.U_{xy}] - P_Y[-P_X.U_{YX} + P_Y.U_{XX}]$$

$$= -P_X^2.U_{YY} + P_XP_YU_{XY} + P_XP_YU_{YX} - P_Y^2.U_{XX}$$

$$\therefore |\overline{H}_2| = -P_X^2.U_{YY} + 2P_XP_YU_{XY} - P_Y^2U_{XX} \qquad [\because U_{XY} = U_{YX}]$$

Now, if the 2nd order condition for utility maximisation is to be satisfied, then we must have to get:

$$\therefore |\overline{H}_2| = -P_X^2.U_{YY} + 2P_XP_YU_{XY} - P_Y^2.U_{XX} > 0.$$

But unfortunately, Marshall didnot verify the 2nd order condition for utility maximisation. This was the drawback of Marshalian concept of utility maximization, which was latter verified by Prof. Hicks in his ordinal utility analysis.

**B) Hicksian Concept of Utility Maximisation:** Hicks states that at the point of equilibrium the consumer's budget line or price line must be tangent to the indifference curve and also indifference curve must be convex to the origin at the point of equilibrium.

Let us assume that a consumer consumes two goods x and y and so the utility function of the consumer is given by

$$U = U(x, y)$$

and the budget line of the consumer will be

$$B = xP_X + yP_Y$$

Where, B = Total income or budget

$P_X$ = Price of commodity x

$P_Y$ = Price of comkodity of y

Now, mathematically here our problem is to maximise the utility function:

$U = U(x, y)$ (1)

Subject to $B = xP_X + yP_y$ (2)

Now, Let us first find the 1st order necessary condition of consumer's eqilibrium according to Hicks as under–

To find the slope of budget line:

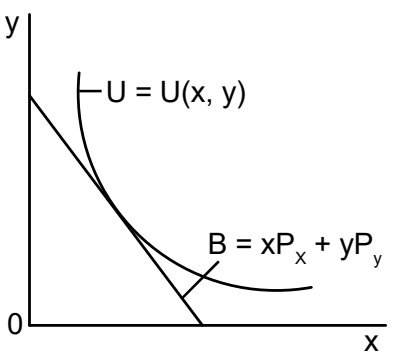
We have the budget line of the consumer given by

$B = xP_X + yP_y$

$\Rightarrow yP_Y = B - XP_X$

$\Rightarrow y = \frac{B}{P_Y} - x\frac{P_X}{P_Y}$

$\Rightarrow y = \frac{B}{P_Y} + \left(-\frac{P_X}{P_Y}\right)x$




$\therefore$ The slope of the budget line $= -\frac{P_X}{P_Y}$ (3)

To find the slope of indifference curve:

We have the utility function or indifference curve give by

$U = U(x, y)$

Now taking total differentiation of the utility function, we get

$du = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy$

$\Rightarrow 0 = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy$

$\Rightarrow \frac{\partial u}{\partial y}.dy = -\frac{\partial u}{\partial x}.dx$

$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$

[$\because$ in terms of indifference curve, there is no change in total utility along the same indifference curve
$\therefore du = 0$]

$$\Rightarrow \frac{dy}{dx} = -\frac{Mu_x}{Mu_y}$$

$\therefore$ The Slope of indifference curve $= -\frac{Mu_x}{Mu_y}$ (4)

Now, rearranging the Marshallian 1st order condition of consumer's equilibrium, we can get:

$$\frac{Mu_x}{Mu_y} = \frac{P_X}{P_Y}$$

$$\Rightarrow -\frac{Mu_x}{Mu_y} = -\frac{P_X}{P_Y}$$

$\Rightarrow$ Slope of indifference curve = Slope of the budget line. (5)

This eqn (5) is the 1st order necessary condition of consumer equilibrium according Hicks. From the first order condition of equilibrium we have seen that both the Marchallian as well as Hicksian concepts are same, even though they are terminologically different.

But to be confirm we must have to move for the verification of 2nd order condition of constraint maximisation i.e. $\left|\overline{H}_2\right| > 0$, which implies that indifference curve should be convex to the origin at the point of tengency.

Second order condition:

We have,

$L_X = \frac{\partial U}{\partial x} - \lambda P_X$ $\quad$ $L_Y = \frac{\partial U}{\partial y} - \lambda P_Y$ $\quad$ $g = B - xP_X - yP_Y$

$\Rightarrow L_x = U_X - \lambda P_X$ $\quad$ $\Rightarrow L_Y = U_Y - \lambda P_Y$ $\quad$ $g_X = -P_X$

$L_{XX} = U_{XX}$ $\quad$ $L_{YX} = U_{YX}$ $\quad$ $g_Y = -P_Y$

$L_{XY} = U_{XY}$ $\quad$ $L_{YY} = U_{YY}$

Now, $$\left|\overline{H}_2\right| = \begin{vmatrix} 0 & g_X & g_Y \\ g_X & L_{XX} & L_{XY} \\ g_Y & L_{YX} & L_{YY} \end{vmatrix} = \begin{vmatrix} 0 & -P_X & -P_Y \\ -P_X & U_{XX} & U_{XY} \\ -P_Y & U_{YX} & U_{YY} \end{vmatrix}$$

$= 0 + P_X[-P_X.U_{YY} + P_Y.U_{xy}] - P_Y[-P_X.U_{YX} + P_Y.U_{XX}]$

$= -P_X^2.U_{YY} + P_XP_YU_{XY} + P_XP_YU_{YX} - P_Y^2.U_{XX}$

$\therefore \left|\overline{H}_2\right| = -P_X^2.U_{YY} + 2P_XP_YU_{XY} - P_Y^2U_{XX}$ $\quad [\because U_{XY} = U_{YX}]$

Now, if the 2nd order condition for utility maximisation is to be satisfied, then we must have to get:

$\therefore \left|\overline{H_2}\right| = -P_X^2.U_{YY} + 2P_XP_YU_{XY} - P_Y^2.U_{XX} > 0.$ (6)

Since the indifference curve should be convex to the origin at the point of equilibrium, the necessary condition in terms of derivatives is that:

1st order condition: $\frac{dy}{dx} < 0$

2nd order condition: $\frac{d^2y}{dx^2} > 0$

Here, $\frac{dy}{dx} = -\frac{Mu_x}{Mu_y} = -\frac{U_X}{U_Y} < 0$ (7)

Now, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{dx}{dy}\right)$

$$\frac{d^2y}{dx^2} = -\left[\frac{U_y.\frac{d}{dx}(U_x) - U_X\frac{d}{dx}(U_y)}{U_Y^2}\right] \quad (8)$$

Now taking total differentiation of $U_X = U_X(x, y)$, we get:

$dU_X = \frac{\partial Ux}{\partial x}.dx + \frac{\partial Ux}{\partial y}.dy$

$\Rightarrow \quad dU_X = U_{XX}.dx + U_{XY}.dy$

$\Rightarrow \quad \frac{dUx}{dx} = U_{XX} + U_{XY}\frac{dy}{dx}$ [dividing both sides by dx] (9)

Again, taking total differentiation of $U_Y = U_Y(x, y)$, we get:

$dU_Y = \frac{\partial Uy}{\partial x}.dx + \frac{\partial Uy}{\partial y}.dy$

$\Rightarrow \quad dU_Y = U_{YX}.dx + U_{YY}.dy$

$\Rightarrow \quad \frac{dUy}{dx} = U_{YX} + U_{YY}\frac{dy}{dx}$ [dividing both sides by dx] (10)

Now, substituting equation (9) and (10) in equation (8), we get:

$$\frac{d^2y}{dx^2} = -\left[\frac{U_Y\left(U_{XX} + U_{XY}\frac{dy}{dx}\right) - U_X\left(U_{YX} + U_{YY}\frac{dy}{dx}\right)}{U_Y^2}\right]$$

$$= -\left[\frac{U_Y.U_{XX} + U_Y.U_{XY}\frac{dy}{dx} - U_x.U_{YX} - U_X.U_{YY}\frac{dy}{dx}}{U_Y^{\,2}}\right]$$

$$= -\left[\frac{U_y U_{xx} + U_y U_{xy}\left(-\frac{P_x}{P_y}\right) - U_x U_{yx} - U_x U_{yy}\left(-\frac{P_x}{P_y}\right)}{U_y^2}\right]$$

$$\because \frac{dy}{dx} = -\frac{U_x}{U_y} = -\frac{P_x}{P_y}$$

$$\therefore \frac{dy}{dx} = -\frac{P_X}{P_Y}$$

$$= -\left[\frac{P_y U_y U_{xx} - P_x U_y U_{xy} - P_y U_x U_{yx} + P_x U_x U_{yy}}{U_y^2 P_y}\right]$$

$$= -\left[\frac{P_Y U_Y U_{XY} - P_X U_Y U_{XY} - P_Y.\frac{P_X}{P_Y}U_Y U_{YX} + P_X.\frac{P_X}{P_Y}U_Y U_{YY}}{U_Y{}^2 P_Y}\right]$$

$$\because \frac{U_X}{U_Y} = \frac{P_X}{P_Y}$$

$$\therefore U_X = \frac{P_X}{P_Y}.U_Y$$

$$= -\left[\frac{P_Y{}^2 U_Y U_{XY} - P_X P_Y U_Y U_{XY} - P_Y P_X U_Y U_{YX} + P_X{}^2 U_Y U_{YY}}{U_Y{}^2.P_Y{}^2}\right]$$

$$= -\left[\frac{U_Y\left(P_Y{}^2 U_{XY} - P_X P_Y U_{XY} - P_X P_Y U_{YX} + P_X{}^2 U_{YY}\right)}{U_Y{}^2.P_Y{}^2}\right]$$

$$= \frac{-P_X{}^2 U_{YY} + 2P_X P_Y U_{XY} - P_Y{}^2 U_{XY}}{U_Y P_Y{}^2} \qquad [\because U_{XY} = U_{YX}]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\left|\bar{H}_2\right|}{U_y P_y^2} \qquad \text{(from equation 6)}$$

Since, $\left|\bar{H}_2\right| > 0$, $U_Y > 0$ and $P_Y > 0$

$$\therefore \frac{d^2y}{dx^2} > 0$$

Thus, the 2nd order condition for convexity of indifference curve is satisfied only on the basis of the fact that $\left|\bar{H}_2\right| > 0$. That is to say, only the assumption of positive $\left|\bar{H}_2\right|$ can ensure the condition of strict convexity of indifference curve.

**Example 1:** The utility function of a consumer is given by

$U = 2x + y + 2xy$

Find out the quantities of x and y which will maximise the utility function of the consumer where the budget of the consumer is fixed at Rs. 100 with the prices of x and y being Rs. 2 and Rs. 1 respectively.

**Solution:** Given, the utility function of the consumer is:

$U = 2x + y + 2xy$

Also given, Budget (B) = 100

Price of x ($P_X$) = 2

Price of y ($P_Y$) = 1

$\therefore$ The budget line of the consumer will be:

$B = xP_X + yP_Y$

$\Rightarrow$ $100 = x.2 + y.1$

$\Rightarrow$ $100 = 2x + y$

Thus, we have to maximise the objective function

$U = 2x + y + 2xy$ (1)

Subject to $100 = 2x + y$

$\Rightarrow$ $100 - 2x - y = 0$ (2)

Here, the Lagrange's function will be:

$L = 2x + y + 2xy + \lambda(100 - 2x - y)$ (3)

**Conditions for maximisation:**

First order condition:

It requires: $L_X = \frac{\partial L}{\partial x} = 0$

$L_Y = \frac{\partial L}{\partial y} = 0$

$L_\lambda = \frac{\partial L}{\partial \lambda} = 0$

Here, $L_X = \frac{\partial L}{\partial x} = \frac{\partial}{\partial x}[2x + y + 2xy + \lambda(100 - 2x - y)]$

$= 2 + 2y - 2\lambda$

$L_Y = \frac{\partial L}{\partial y} = \frac{\partial}{\partial y}[2x + y + 2xy + \lambda(100 - 2x - y)]$

$= 1 + 2x - \lambda$

$L_\lambda = \frac{\partial L}{\partial \lambda} = \frac{\partial}{\partial \lambda}[2x + y + 2xy + \lambda(100 - 2x - y)]$

$= 100 - 2x - y$

According to 1st order condition:

$L_X = 2 + 2y - 2\lambda = 0$ (4)

$L_Y = 1 + 2x - \lambda = 0$ (5)

$L_\lambda = 100 - 2x - y = 0$ (6)

From eqn (4) and (5), we can get:

$2\lambda = 2 + 2y$

$\Rightarrow \lambda = 1 + y$ (7)

$\lambda = 1 + 2x$ (8)

Now, from equation (7) and (8), we can get:

$1 + y = 1 + 2x$

$\Rightarrow y = 2x$ (9)

Substituting equation (9) in equation (6), we can have

$100 - 2x - 2x = 0$

$\Rightarrow 4x = 100$

$\Rightarrow x = 25$ (10)

Again, substituting equation (10) in equation (9), we get:

$y = 2x = 2.25 = 50$

$\therefore x = 25$

$y = 50$

2nd order condition:

We have,

$L_X = 2 + 2y - 2\lambda$ $L_Y = 1 + 2x - \lambda$ $g = 100 - 2x - y$

$L_{XX} = 0$ $L_{YX} = 2$ $g_X = -2$

$L_{XY} = 2$ $L_{YY} = 0$ $g_Y = -1$

Here, $\left|\overline{H_1}\right| = \begin{vmatrix} 0 & g_X \\ g_X & L_{XX} \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} = -4 < 0$

$$\left|\overline{H_2}\right| = \begin{vmatrix} 0 & g_X & g_Y \\ g_X & L_{XX} & L_{XY} \\ g_Y & L_{YX} & L_{YY} \end{vmatrix} = \begin{vmatrix} 0 & -2 & -1 \\ -2 & 0 & 2 \\ -1 & 2 & 0 \end{vmatrix}$$

$= 0 - (-2)(0 + 2) + (-1)(-4 - 0)$

$= 4 + 4$

$= 8 > 0$

Since, $\left|\overline{H_1}\right| < 0$ and $\left|\overline{H_2}\right| > 0$

$\therefore$ 2nd order condition for utility maximisation is satisfied

when $x = 25$ and $y = 50$.

Here, $x = 25$ and $y = 50$ will maximise the utility function of the consumer subject to the given budget constraint.

**Maximum utility:** The maximum utility of the consumer will be:

$U = 2x + y + 2xy$

$= 2.25 + 50 + 2.25.50$

$= 50 + 50 + 2500$

$= 2600$, is the required maximum utility of the consumer.

**Example 2:** Given the utility function:

$U = 18xy + 9y$

and the budget constraint:

$6x + 3y = 15$

Find out optimum purchase of x and y which will maximise the utility of the consumer subject to the satisfaction of the budget constraint.

**Solution:** Given, the utility function:

$U = 18xy + 9y$

to be maximised subject to the budget constraint

$6x + 3y = 15$

or, $15 - 6x - 3y = 0$

Here, the Lagrange's function will be:

$L = 18xy + 9y + \lambda(15 - 6x - 3y)$

**Conditions for maximisation:**

First order condition:

If requires: $L_X = \frac{\partial L}{\partial x} = 0$

$L_Y = \frac{\partial L}{\partial y} = 0$

$L_\lambda = \frac{\partial L}{\partial \lambda} = 0$

Here, $L_X = \frac{\partial L}{\partial x} = \frac{\partial}{\partial x}[18xy + 9y + \lambda(15 - 6x - 3y)]$

$= 18y - 6\lambda$

$L_Y = \frac{\partial L}{\partial y} = \frac{\partial}{\partial y}[18xy + 9y + \lambda(15 - 6x - 3y)]$

$= 18x + 9 - 3\lambda$

$L_\lambda = \frac{\partial L}{\partial \lambda} = \frac{\partial}{\partial \lambda}[18xy + 9y + \lambda(15 - 6x - 3y)]$

$= 15 - 6x - 3y$

According to 1st order condition:

$L_X = 18y - 6\lambda = 0$ (1)

$L_Y = 18x + 9 - 3\lambda = 0$ (2)

$L_\lambda = 15 - 6x - 3y = 0$ (3)

From eqn (1) and (2), we get:

$3y = \lambda$ (4)

$6x + 3 = \lambda$ (5)

From eqn (4) and (5), we have

$3y = 6x + 3$

$y = 2x + 1$ (6)

Substituting eqn (6) in eqn (3), we have

$15 - 6x - 3(2x + 1) = 0$

$\Rightarrow 15 - 6x - 6x - 3 = 0$

$\Rightarrow 12x = 12$

$\Rightarrow x = 1 \quad (7)$

Substituting eq$^n$ (7) in eq$^n$ (6), we get

$y = 2x + 1 = 2.1 + 1 = 3$

$\therefore \quad x = 1$

$y = 3$

2$^{nd}$ order condition:

We have,

$L_X = 18y - 6\lambda$ $\quad L_Y = 18x + 9 - 3\lambda$ $\quad g = 15 - 6x - 3y$

$L_{XX} = 0$ $\quad L_{YX} = 18$ $\quad g_X = -6$

$L_{XY} = 18$ $\quad L_{YY} = 0$ $\quad g_Y = -3$

Here, $\left|\overline{H_1}\right| = \begin{vmatrix} 0 & g_x \\ g_x & L_{xx} \end{vmatrix} = \begin{vmatrix} 0 & -6 \\ -6 & 0 \end{vmatrix} = -36 < 0$

$$\left|\overline{H_2}\right| = \begin{vmatrix} 0 & g_X & g_Y \\ g_X & L_{XX} & L_{XY} \\ g_Y & L_{YX} & L_{YY} \end{vmatrix} = \begin{vmatrix} 0 & -6 & -3 \\ -6 & 0 & 18 \\ -3 & 18 & 0 \end{vmatrix}$$

$= 0 - (-6)(0 + 54) - 3(-108 - 0)$

$= 324 + 324$

$= 648 > 0$

Since, $\left|\overline{H_1}\right| < 0$ and $\left|\overline{H_2}\right| > 0$

$\therefore$ 2$^{nd}$ order condition for utility maximisation is satisfied when

$x = 1$ and $y = 3$.

Here, $x = 1$ and $y = 3$ will maximise the utility function of the consumer subject to the given budget constraint.

**Maximum utility:** The maximum utility of the consumer will be:

$U = 18xy + 9y$

$= 18.1.3 + 9.3$

$= 54 + 27$

$= 61$

**Example 3 :** Given the utility function $U = x\sqrt{y}$ and the budget constraint 4x + y = 48, find the equilibrium purchase of x and y in order to maximise total utility and maximum utility.

Solution :

Given,

$U = x\sqrt{y}$ or $xy^{1/2}$ subject to budget constraint

4x + y = 48 or 48 – 4x – y = 0

The Lagrange's function is given by

$$L = xy^{1/2} + \lambda(48 - 4x - y)$$

To maximise total utility :

First order condition :

It requires : $L_1 = \frac{\partial L}{\partial x} = 0$

$$L_2 = \frac{\partial L}{\partial y} = 0$$

$$L_\lambda = \frac{\partial L}{\partial \lambda} = 0$$

Here $L_1 = \frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left[xy^{1/2} + \lambda(48 - 4x - y)\right]$

$$= y^{\frac{1}{2}} - 4\lambda = \sqrt{y} - 4\lambda$$

$$L_2 = \frac{\partial L}{\partial y} = \frac{\partial}{\partial y}\left[xy^{1/2} + \lambda(48 - 4x - y)\right]$$

$$= x\frac{1}{2}y^{1/2-1} - \lambda = \frac{x}{2}y^{-1/2} - \lambda$$

$$= \frac{x}{2\sqrt{y}} - \lambda$$

$$L_\lambda = \frac{\partial L}{\partial \lambda} = \frac{\partial}{\partial y}\left[xy^{1/2} + \lambda(48 - 4x - y)\right]$$

$$= 48 - 4x - y$$

According to 1st order condition

$$L_1 = \sqrt{y} - 4\lambda = 0 \qquad \Rightarrow \lambda = \frac{\sqrt{y}}{4}$$

$$L_2 = \frac{x}{2\sqrt{y}} - \lambda = 0 \qquad \Rightarrow \lambda = \frac{x}{2\sqrt{y}}$$

$$L_\lambda = 48 - 4x - y = 0$$

$$\Rightarrow 4x + y = 48 \qquad \text{—— (3)}$$

From eq$^n$ (1) and (2), we have,

$$\lambda = \frac{\sqrt{y}}{4} = \frac{x}{2\sqrt{y}}$$

$$\Rightarrow 4x = 2\left(\sqrt{y}\right)^2$$

$$\Rightarrow 4x = 2y$$

$$\Rightarrow 4x - 2y = 0 \qquad \text{—— (4)}$$

Now, from eq$^n$ (3) and (4). We get

$4x + y = 48$

$4x - 2y = 0$

Solving these two equations, we get,

$3y = 48$

$$y = \frac{48}{3} = 16$$

Putting y = 16 in eq$^2$ (3), we get

$4x + y = 48$

$\Rightarrow 4x + 16 = 48$

$\Rightarrow 4x = 48 - 16$

$\Rightarrow 4x = 32$

$$\Rightarrow x = \frac{32}{4} = 8$$

$\therefore x = 8$

and $y = 16$

Second order condition :

We have

$$L_1 = y^{1/2} - 4\lambda \qquad L_2 = \frac{x}{2}y^{1/2} - \lambda \qquad L_\lambda = 48 - 4x - y$$

$L_{11} = 0$

$L_{12} = \frac{1}{2}y^{\frac{1}{2}-1}$

$= \frac{1}{2}y^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{y}}$

$= \frac{1}{2\sqrt{16}}$

$= \frac{1}{8}$

$L_{21} = \frac{1}{2}y^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{y}}$

$= \frac{1}{2\sqrt{16}}$

$= \frac{1}{8}$

$L_{22} = \frac{x}{2}\left(-\frac{1}{2}\right)y^{-\frac{1}{2}-1}$

$= -\frac{x}{4}y^{-\frac{3}{2}}$

$= -\frac{x}{4\sqrt{y^3}}$

$= -\frac{8}{4\sqrt{16^3}}$

$= -\frac{8}{4.16.\sqrt{16}}$

$= -\frac{8}{4.16.4}$

$= -\frac{1}{32}$

$g_1 = -4$

$g_2 = -1$

Now $|H_1| = \begin{vmatrix} 0 & g_1 \\ g_1 & L_{11} \end{vmatrix} = \begin{vmatrix} 0 & -4 \\ -1 & 0 \end{vmatrix} = 0 - 4 = -4 < 0$

$$|H_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -4 & -1 \\ -4 & 0 & \frac{1}{8} \\ -1 & \frac{1}{8} & -\frac{1}{32} \end{vmatrix}$$

$$= 0 - (-4)\left(\frac{1}{8} + \frac{1}{8}\right) - 1\left(-\frac{1}{2} - 0\right)$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2} > 0$$

$\because$ $|H_1| < 0$ and $|H_2| > 0$

$\therefore$ 2nd order condition for maximisation of utility function is satisfied, when x = 8 and y = 16

Hence, x = 8 and y = 16 will mximise the utility function, subject to the given budget constraint.

Maximum utility will be

$$U = x\sqrt{y}$$

$$= 8\sqrt{16}$$

$$= 8.4$$

$$= 32$$

## 12.4 DERIVATION OF CONDITIONS OF LEAST-COST COMBINATION OF INPUTS FOR A PRODUCER

We know that, with a given production function, a producer tries to minimise the total cost of production by choosing the optimal size of the combination of factors of production. i.e., the producer tries to minimise the cost function subject to the given level of output or production function constriant (isoquant).

Therefore, here our problem is to minimise the cost function:

$$C = LP_L + KP_K$$

Subject to $Q^0 = f(L, K)$

$$\Rightarrow Q^0 - f(L, K) = 0$$

Let, us first construct the Lagrange's function (Z) with the help of Lagrange's multiplier ($\lambda$) as follows:

$$Z = LP_L + KP_K + \lambda[Q^0 - f(L, K)]$$

Now minimisation of the cost function subject to the given production

function requires that:

First order condition:

If requires: $Z_L = \frac{\partial Z}{\partial L} = 0$

$Z_K = \frac{\partial Z}{\partial K} = 0$

$Z_\lambda = \frac{\partial Z}{\partial \lambda} = 0$

Here, $Z_L = \frac{\partial Z}{\partial L} = \frac{\partial}{\partial L}[LP_L + KP_K + \lambda\{Q^0 - f(L, K)\}] = P_L - \lambda f_L$

$(\because f(L, K) = f_L = MP_L)$

$Z_K = \frac{\partial Z}{\partial K} = P_K - \lambda f_K$

$Z_\lambda = Q^0 - f(L, K)$

According to 1st order condition:

$$Z_L = P_L - \lambda f_L = 0 \quad \Rightarrow \lambda = \frac{P_L}{f_L} \qquad (1)$$

$$Z_K = P_K - \lambda f_K = 0 \quad \Rightarrow \lambda = \frac{P_K}{f_K} \qquad (2)$$

$$Z_\lambda = Q^0 - f(L, K) = 0 \quad \Rightarrow Q^0 = f(L, K) \qquad (3)$$

Now, from (1) and (2), we get:

$$\lambda = \frac{P_L}{f_L} = \frac{P_K}{f_K}$$

$$\Rightarrow \frac{f_L}{P_L} = \frac{f_K}{P_K}$$

$$\Rightarrow \frac{MP_L}{P_L} = \frac{MP_K}{P_K} \qquad (4)$$

This eqn (4) is first order necessary condition for a producer to be in equilibrium in the factor market which implies that the marginal products of the factor inputs equi-proportional to their price ratio.

We can represent the first order condition of producer equilibrium in terms of iso-cost line and isoquant as follows:

We know that a producer will be in equilibrium in factor market if his

cost line become tangent to the isoquant. i.e., at the point of equilibrium,

the slope of iso-cost line = slope of isoquent.

We have the iso-cost line given by

$C = LP_L + KP_K$

$\Rightarrow \quad LP_L = C - KP_K$

$\Rightarrow \quad L = \frac{C}{P_L} - \frac{P_K}{P_L}K$

$\Rightarrow \quad L = \frac{C}{P_L} + \left(-\frac{P_K}{P_L}\right)K$

The slope of iso-cost line $= -\frac{P_K}{P_L}$ (5)

We have the iso-quant or production function given by

$Q^0 = f(L, K)$

Taking total differentiation of the production function we get:

$dQ^0 = \frac{\partial f}{\partial L}.dL + \frac{\partial f}{\partial K}.dK$

$\Rightarrow \quad 0 = \frac{\partial f}{\partial L}.dL + \frac{\partial f}{\partial K}.dK \quad (\because\ dQ^0=0)$

$\Rightarrow \quad \frac{\partial f}{\partial L}.dL = -\frac{\partial f}{\partial K}.dK$

$\Rightarrow \quad \frac{dL}{dK} = -\frac{\frac{\partial f}{\partial K}}{\frac{\partial f}{\partial L}}$

$\Rightarrow \quad \frac{dL}{dK} = -\frac{f_K}{f_L} = -\frac{MP_K}{MP_L}$

$\therefore$ The slope of isoquant $= -\frac{MP_K}{MP_L} = -\frac{f_K}{f_L}$ (6)

Now, from eq$^n$ (4):

$$\frac{MP_L}{P_L} = \frac{MP_K}{P_K}$$

$$\Rightarrow \quad \frac{MP_K}{MP_L} = \frac{P_K}{P_L}$$

$$\Rightarrow \quad -\frac{MP_K}{MP_L} = -\frac{P_K}{P_L}$$

$\Rightarrow$ Slope of isoquant = Slope of iso-cost line (7)

This eq$^n$ (7) is the 1$^{st}$ order necessary condition for a producer to be in equilibrium in the factor market in terms of iso-cost line and iso-quant.

Now, we move to the process of varification of 2$^{nd}$ order condition of constraint minimisation, i.e., $\left|\overline{H}_2\right| < 0$, which implies that iso-quant should be convex to the origin at the point of tangency.

Second order condition:

We have,

$$Z_L = P_L - \lambda f_L \quad Z_K = P_K - \lambda f_K \quad g = Q^0 - f(L, K)$$
$$Z_{LL} = -\lambda f_{LL} \quad Z_{KL} = -\lambda f_{KL} \quad g_L = -f_L$$
$$Z_{LK} = -\lambda f_{LK} \quad Z_{KK} = -\lambda f_{KK} \quad g_K = -f_K$$

Now, $$\left|\overline{H}_2\right| = \begin{vmatrix} 0 & g_L & g_K \\ g_L & L_{LL} & L_{LK} \\ g_K & L_{KL} & L_{KK} \end{vmatrix} = \begin{vmatrix} 0 & -f_L & -f_K \\ -f_L & -\lambda f_{LL} & -\lambda f_{LK} \\ -f_K & -\lambda f_{KL} & -\lambda f_{KK} \end{vmatrix}$$

$$= 0 + f_L(f_L\lambda f_{KK} - f_K\lambda f_{LK}) - f_K(f_L\lambda f_{KL} - f_K\lambda f_{LL})$$
$$= f_L{}^2\lambda f_{KK} - f_Lf_K\lambda f_{LK} - f_Lf_K\lambda f_{KL} + f_K{}^2\lambda f_{LL}$$
$$\Rightarrow \left|\overline{H}_2\right| = f_L{}^2\lambda f_{KK} - 2f_Lf_K\lambda f_{LK} + f_K{}^2\lambda f_{LL} \quad [\because f_{LK} = f_{KL}]$$

Now, if the 2$^{nd}$ order condition for minimisation of cost function is to be satisfied, then we must have to get:

$$\left|\overline{H}_2\right| = \lambda(f_L{}^2f_{KK} - 2f_Lf_Kf_{LK} + f_K{}^2f_{LL}) < 0 \quad (8)$$

Since, $\lambda > 0$

$$\therefore f_L{}^2f_{KK} - 2f_Lf_Kf_{LK} + f_K{}^2f_{LL} < 0 \quad (9)$$

Since the iso-quant should be convex to the origin at the point of equilibrium, the conditions for convexity of iso-quant requires that:

First order condition: $\frac{dL}{dK} < 0$

Second order condition: $\frac{d^2L}{dK^2} > 0$

Here, $\frac{dL}{dK} = -\frac{f_K}{f_L} < 0$

Now, $\frac{d^2L}{dK^2} = \frac{d}{dK}\left(\frac{dL}{dK}\right) = \frac{d}{dK}\left(-\frac{f_K}{f_L}\right)$

$$\frac{d^2L}{dK^2} = -\left[\frac{f_L \frac{d}{dK} f_K - f_K \frac{d}{dK} f_L}{f_L^{\,2}}\right] \qquad (10)$$

Now taking total differentiation of $f_K = f_K(L, K)$, we get:

$$df_K = \frac{\partial f_K}{\partial L}.dL + \frac{\partial f_K}{\partial K}.dK$$

$$\Rightarrow df_K = f_{KL}dL + f_{KK}dK$$

$$\Rightarrow \frac{df_K}{dK} = f_{KL}.\frac{dL}{dK} + f_{KK} \quad \text{[dividing both sides by dK]} \qquad (11)$$

Again, taking total differentian of $f_L = f_L(L, K)$, we get:

$$df_L = \frac{\partial f_L}{\partial L}.dL + \frac{\partial f_L}{\partial K}.dK$$

$$\Rightarrow df_L = f_{LL}dL + f_{LK}dK$$

$$\Rightarrow \frac{df_L}{dK} = f_{LL}.\frac{dL}{dK} + f_{LK} \quad \text{[dividing both sides by dK]} \qquad (12)$$

Now, substituting eq$^n$ (11) and (12) in eq$^n$ (10), we get:

$$\frac{d^2L}{dK^2} = -\left[\frac{f_L\left(f_{KL}\frac{dL}{dK} + f_{KK}\right) - f_K\left(f_{LL}\frac{dL}{dK} + f_{LK}\right)}{f_L^{\,2}}\right]$$

$$= -\frac{1}{f_L^{\,2}}\left[f_L f_{KL}\frac{dL}{dK} + f_L f_{KK} - f_K f_{LL}\frac{dL}{dK} - f_K f_{LK}\right]$$

$$= -\frac{1}{f_L^{\,2}}\left[f_L f_{KL}\left(-\frac{f_K}{f_L}\right) + f_L f_{KK} - f_K f_{LL}\left(-\frac{f_K}{f_L}\right) - f_K f_{LK}\right]$$

$$= -\frac{1}{f_L^3}\left[-f_L f_K f_{KL} + f_L^2 f_{KK} + f_K^2 f_{LL} - f_L f_K f_{LK}\right]$$

$$\Rightarrow \frac{d^2L}{dK^2} = -\frac{1}{f_L^3}\left[f_L^2 f_{KK} - 2f_L f_K f_{LK} + f_K^2 f_{LL}\right] \qquad [\because f_{LK} = f_{KL}]$$

Since, $f_L^3$ is positive by its nature and $(f_L^2 f_{KK} - 2f_L f_K f_{LK} + f_K^2 f_{LL}) < 0$ from eq$^n$ (9):

$\therefore \frac{d^2L}{dK^2} > 0$, i.e. 2$^{nd}$ order sufficient condition for convexity of isoquant is satisfied.

Thus, the 2$^{nd}$ order condition for convexity of isoquant is satisfied only on the basis of the fact that $|\overline{H}_2| < 0$. That is to say, only the assumption of negative $|\overline{H}_2|$ can ensure the condition of strict convexity of isoquant.

**Example 3:** A producer desires to minimise the cost of production given the cost function as,

$C = 4L + K$

Where K and L are capital and Labour respectively subject to the given production function:

$Q = L^{1/2}K^{1/2}$

Find out the equilibrium of inputs K and L in order to minimise the cost of production when output Q = 16.

**Solution:** Given, the cost function:

$C = 4L + K$, to be minimise subject to the output constant,

$Q = L^{1/2}K^{1/2} = 16$

Here, the Lagrange's function will be:

$Z = 4L + K + \lambda(16 - L^{1/2}K^{1/2})$

Conditions for minimisation:

First order condition:

It requires: $Z_L = \frac{\partial Z}{\partial L} = 0$

$$Z_K = \frac{\partial Z}{\partial K} = 0$$

$$Z_\lambda = \frac{\partial Z}{\partial \lambda} = 0$$

We have, $Z_L = \frac{\partial Z}{\partial L} = \frac{\partial}{\partial L}[4L + K + \lambda(16 - L^{1/2}K^{1/2})]$

$$= 4 - \frac{1}{2}\lambda \cdot L^{\frac{1}{2}-1}K^{\frac{1}{2}}$$

$$= 4 - \frac{1}{2}\lambda L^{1/2}K^{1/2}L^{-1}$$

$$= 4 - \frac{1}{2}\lambda\frac{Q}{L}$$

$$Z_K = \frac{\partial Z}{\partial K} = \frac{\partial}{\partial K}[4L + K + \lambda(16 - L^{1/2}K^{1/2})]$$

$$= 1 - \lambda\frac{1}{2}L^{\frac{1}{2}}K^{\frac{1}{2}-1}$$

$$= 1 - \frac{1}{2}\lambda L^{1/2}K^{1/2}K^{-1}$$

$$= 1 - \frac{1}{2}\lambda\frac{Q}{K}$$

$$Z_\lambda = \frac{\partial Z}{\partial \lambda} = \frac{\partial}{\partial \lambda}[4L + K + \lambda(16 - L^{1/2}K^{1/2})]$$

$$= 16 - L^{1/2}K^{1/2}$$

According to 1st order condition:

$$Z_L = 4 - \frac{1}{2}\lambda\frac{Q}{L} = 0 \qquad (1)$$

$$Z_K = 1 - \frac{1}{2}\lambda\frac{Q}{K} = 0 \qquad (2)$$

$$Z_\lambda = 16 - L^{1/2}K^{1/2} = 0 \qquad (3)$$

From eqn (1) and (2), we get:

$$4 - \frac{1}{2}\lambda\frac{Q}{L} = 0$$

$$\Rightarrow \quad 4 = \frac{\lambda Q}{2L}$$

$$\Rightarrow \quad \lambda Q = 8L$$

$$\Rightarrow \quad \lambda = \frac{8L}{Q} \qquad (5)$$

and, $1 - \frac{1}{2}\lambda\frac{Q}{K} = 0$

$$\Rightarrow \quad 1 = \frac{\lambda Q}{2K}$$

$$\Rightarrow \quad \lambda Q = 2K$$

$$\Rightarrow \quad \lambda = \frac{2K}{Q} \qquad (6)$$

From eq$^n$ (5) and (6), we have

$$\lambda = \frac{8L}{Q} = \frac{2K}{Q} \qquad (7)$$

or, $8L = 2K$

$$\therefore \quad K = 4L \qquad (8)$$

Now, substituting eq$^n$ (8) in eq$^n$ (3), we get:

$16 - L^{1/2}K^{1/2} = 0$

$$\Rightarrow \quad 16 = L^{1/2}.(4L)^{1/2}$$

$$\Rightarrow \quad 16 = L^{1/2}.2L^{1/2}$$

$$\Rightarrow \quad 8 = L^{1/2}.L^{1/2}$$

$$\Rightarrow \quad L = 8 \qquad (9)$$

Substituting L = 8, in eq$^n$ (8), we have

$K = 4L = 4.8 = 32$

$\therefore \quad L = 8$

$K = 32$

Again, from eq$^n$ (7), we get:

$$\lambda = \frac{8L}{Q} = \frac{8\times 8}{16} = 4 \ (\because \ Q=16)$$

Second order condition:

We have, $Z_L = 4 - \frac{1}{2}\lambda L^{\frac{1}{2}-1}K^{\frac{1}{2}}$

$$\therefore Z_{LL} = -\frac{1}{2}\lambda\left(\frac{1}{2}-1\right)L^{\frac{1}{2}-1-1}K^{\frac{1}{2}}$$

$$= \frac{1}{4}\lambda L^{\frac{1}{2}}K^{\frac{1}{2}}.L^{-2}$$

$$= \frac{1}{4}\lambda\frac{Q}{L^2}$$

$$= \frac{1}{4}.4.\frac{16}{8^2}$$

$$= \frac{1}{4}$$

$$Z_{LK} = -\frac{1}{2}\lambda L^{\frac{1}{2}-1}\left(\frac{1}{2}\right)K^{\frac{1}{2}-1}$$

$$= -\frac{1}{4}\lambda L^{1/2}K^{1/2}L^{-1}K^{-1}$$

$$= -\frac{1}{4}\lambda\frac{Q}{LK}$$

$$= -\frac{1}{4}.4.\frac{16}{8\times 32}$$

$$= -\frac{1}{16}$$

And, $Z_K = 1 - \frac{1}{2}\lambda L^{\frac{1}{2}}K^{\frac{1}{2}-1}$

$$\therefore Z_{KL} = -\frac{1}{2}\lambda\left(\frac{1}{2}\right)L^{\frac{1}{2}-1}K^{\frac{1}{2}-1}$$

$$= -\frac{1}{4}\lambda L^{1/2}K^{1/2}L^{-1}K^{-1}$$

$$= -\frac{1}{4}\lambda\frac{Q}{LK}$$

$$= -\frac{1}{4}.4.\frac{16}{8\times 32}$$

$$= -\frac{1}{16}$$

$$Z_{KK} = -\frac{1}{2}\lambda L^{\frac{1}{2}}\left(\frac{1}{2}-1\right)K^{\frac{1}{2}-1-1}$$

$$= \frac{1}{4}\lambda L^{1/2}K^{1/2}K^{-2}$$

$$= \frac{1}{4}\lambda\frac{Q}{K^2}$$

$$= \frac{1}{4}.4.\frac{16}{32^2}$$

$$= \frac{16}{1024}$$

$$= \frac{1}{64}$$

Again, $g = 16 - L^{1/2}K^{1/2}$

$$\therefore g_L = -\frac{1}{2}L^{\frac{1}{2}-1}K^{\frac{1}{2}}$$

$$= -\frac{1}{2}L^{\frac{1}{2}}K^{\frac{1}{2}}L^{-1}$$

$$= -\frac{1}{2}.\frac{Q}{L}$$

$$= -\frac{1}{2}.\frac{16}{8} = -1$$

$$g_K = -\frac{1}{2}L^{\frac{1}{2}}K^{\frac{1}{2}-1}$$

$$= -\frac{1}{2}L^{\frac{1}{2}}K^{\frac{1}{2}}K^{-1}$$

$$= -\frac{1}{2}\frac{Q}{K}$$

$$= -\frac{1}{2}.\frac{16}{32} = -\frac{1}{4}$$

Now, $\left|\overline{H}_1\right| = \begin{vmatrix} 0 & g_L \\ g_L & Z_{LL} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ -1 & \frac{1}{4} \end{vmatrix} = 0 - 1 = -1 < 0$

$$\left|\overline{H}_2\right| = \begin{vmatrix} 0 & g_L & g_K \\ g_L & Z_{LL} & Z_{LK} \\ g_K & Z_{KL} & Z_{KK} \end{vmatrix} = \begin{vmatrix} 0 & -1 & -\frac{1}{4} \\ -1 & \frac{1}{4} & -\frac{1}{16} \\ -\frac{1}{4} & -\frac{1}{16} & \frac{1}{64} \end{vmatrix}$$

$$= 0 + 1\left(-\frac{1}{64} - \frac{1}{64}\right) - \frac{1}{4}\left(\frac{1}{16} + \frac{1}{16}\right)$$

$$= -\frac{2}{64} - \frac{2}{64}$$

$$= -\frac{4}{64} < 0$$

Since, $\left|\overline{H}_1\right| < 0$ and $\left|\overline{H}_2\right| < 0$

$\therefore$ 2nd order condition for cost minimisation is satisfied when

L = 8 and K = 32

Here, L = 8 and K = 32 will minimise the cost function of the producer subject to the given production function.

**Example 4 :** A producer desires to minimise his cost of production C = 5L + 2K, where L and K are the inputs, subject to satisfaction of the production function Q = LK. Find the optimum combination of L and K in order to minimise cost of production when output is 90.

**Solution :**

Given the cost function:

C = 5L + 2K, to be minimise subject to the output constraint

Q = LK = 90

Here, the Lagrange's function will be:

$Z = 5L + 2K + \lambda(90 - LK)$

Conditions for minimisation:

First order condition:

It requires: $Z_L = \frac{\partial Z}{\partial L} = 0$

$Z_K = \frac{\partial Z}{\partial K} = 0$

$$Z_\lambda = \frac{\partial Z}{\partial \lambda} = 0$$

Here, $Z_L = 5 - \lambda K$

$Z_K = 2 - \lambda L$

$Z_\lambda = 90 - LK$

According to 1st order condition:

$$Z_L = 5 - \lambda K = 0 \qquad \Rightarrow \lambda = \frac{5}{K} \qquad (1)$$

$$Z_K = 2 - \lambda L = 0 \qquad \Rightarrow \lambda = \frac{2}{L} \qquad (2)$$

$$Z_\lambda = 90 - LK = 0 \qquad (3)$$

From eqn (1) and (2), we get:

$$\lambda = \frac{5}{K} = \frac{2}{L}$$

$$\Rightarrow \quad 5L = 2K$$

$$\Rightarrow \quad L = \frac{2K}{5} \qquad (4)$$

Substituting $L = \frac{2K}{5}$ in eqn (3), we get:

$90 - LK = 0$

$$\Rightarrow \quad 90 = \frac{2K}{5}.K$$

$$\Rightarrow \quad 90 = \frac{2K^2}{5}$$

$$\Rightarrow \quad 2K^2 = 450$$

$$\Rightarrow \quad K^2 = 225$$

$\therefore K = 15$

Substituting K = 15 in eqn (4), we get:

$$L = \frac{2K}{5} = \frac{2.15}{5} = 6$$

$\therefore \quad L = 6$

$K = 15$

Again, $\lambda = \frac{5}{K} = \frac{5}{15} = \frac{1}{3}$

Second order condition:

We have,

$Z_L = 5 - \lambda K$ $Z_K = 2 - \lambda L$ $g = 90 - LK$

$Z_{LL} = 0$ $Z_{KL} = -\lambda = -\frac{1}{3}$ $g_L = -K = -15$

$Z_{LK} = -\lambda = -\frac{1}{3}$ $Z_{KK} = 0$ $g_K = -L = -6$

Now, $\left|\overline{H_1}\right| = \begin{vmatrix} 0 & g_L \\ g_L & Z_{LL} \end{vmatrix} = \begin{vmatrix} 0 & -15 \\ -15 & 0 \end{vmatrix} = -225 < 0$

$$\left|\overline{H_2}\right| = \begin{vmatrix} 0 & g_L & g_K \\ g_L & Z_{LL} & Z_{LK} \\ g_K & Z_{KL} & Z_{KK} \end{vmatrix} = \begin{vmatrix} 0 & -15 & -6 \\ -15 & 0 & -\frac{1}{3} \\ -6 & -\frac{1}{3} & 0 \end{vmatrix}$$

$= 0 - (-15)(0 - 6.\frac{1}{3}) - 6(15.\frac{1}{3} - 0)$

$= -30 - 30$

$= -60 < 0$

Since, $\left|\overline{H_1}\right| < 0$ and $\left|\overline{H_2}\right| < 0$

$\therefore$ 2nd order condition for cost minimisation is satisfied when

L = 6 and K = 15

Hence, L = 6 and K = 15 will minimise the cost of function of the producer subject to the given production function.

## 12.5 COMPARATIVE STATIC ANALYSIS OF UTILITY MAXIMIZATION

The utility maximization problem can be studied further in comparative static framemork assuming changes in the income of the consumer and the price of the goods. According to the theory of consumer's equilibrium, we know

that in case of normal goods the price effect on the consumer is the sum of income effect and substitution effect.

Let us prove it with the help of mathemetical analysis–

Let the utility function be,

$U = U(x, y)$

and the budget constraint is

$\beta = xP_x + yP_y$

For maximising the utility function, let us first construct the Lagrange function as–

$$L = U(x,y) + \lambda(\beta - xP_x - yP_y) \quad \text{—— (1)}$$

1st order condition requires :

$$\left.\begin{aligned} L_x &= \frac{\partial L}{\partial X} = U_x - \lambda P_x = 0 \\ L_y &= \frac{\partial L}{\partial Y} = U_y - \lambda P_y = 0 \\ \frac{\partial L}{\partial \lambda} &= \beta - xP_x - yP_y = 0 \end{aligned}\right\} \quad \text{—— (2)}$$

Since, $U_x$ (marginal utility of x) and $U_y$ (marginal utility of y) are the functions of both x and y, so equation (2) can be re-written and re-arranged as equilibrium identities, as follows –

$$\left.\begin{aligned} \beta - \bar{x}p_x - \bar{y}p_y &= 0 \\ U_x(\bar{x}, \bar{y}) - \bar{\lambda}p_x &= 0 \\ U_y(\bar{x}, \bar{y}) - \bar{\lambda}p_y &= 0 \end{aligned}\right\} \quad \text{—— (3)}$$

Where, $\bar{x}$, $\bar{y}$, $\bar{\lambda}$ equilibrium values of x, y and $\lambda$. Taking total differential of each of the above identities, we get -

$$d\beta - p_x d\bar{x} - \bar{x}dp_x - p_y d\bar{y} - \bar{y}dp_y = 0$$

$$U_{xx}d\bar{x} + U_{xy}d\bar{y} - p_x d\bar{\lambda} - \bar{\lambda}dp_x = 0$$

$$U_{yx}d\bar{x} + U_{yy}d\bar{y} - p_y d\bar{\lambda} - \bar{\lambda}dp_y = 0$$

or,

$$\left.\begin{aligned} -P_x d\bar{x} - P_y d\bar{y} &= \bar{x}dp_x + ydp_y - d\beta \\ -P_x d\bar{\lambda} + U_{xx}d\bar{x} + U_{xy}d\bar{y} &= \bar{\lambda}dp_x \\ -P_y d\bar{\lambda} + U_{yx}d\bar{x} + U_{yy}d\bar{y} &= \bar{\lambda}dp_y \end{aligned}\right\} \quad \text{—— (4)}$$

**Effect of change in income**

Let us assume that, price do not change i.e, $dp_x = 0 = dp_y$ and income changes i.e, $d\beta \neq 0$

Equilibrium values $\bar{x}$, $\bar{y}$, $\bar{\lambda}$ are functions of $p_x$, $p_y$ and $\beta$ in each case, any change in equilibrium value of $\bar{x}$ or $\bar{y}$ or $\bar{\lambda}$ with respect to change in $p_x$ or $p_y$ or $\beta$ should be shown as partial derivative. Now dividing both sides of the set of three equations by $d\beta$ in (4) and interpreting each ratio of differentials as partial detivative, we have–

$$-P_x \frac{\partial \bar{x}}{\partial \beta} - P_y \frac{\partial \bar{x}}{\partial \beta} = 0 + 0 - 1$$

$$-P_x \frac{\partial \bar{\lambda}}{\partial \beta} + U_{xx} \frac{\partial \bar{x}}{\partial \beta} + U_{xy} \frac{\partial \bar{y}}{\partial \beta} = 0$$

$$-P_y \frac{\partial \bar{\lambda}}{\partial \beta} + U_{yx} \frac{\partial \bar{x}}{\partial \beta} + U_{yy} \frac{\partial \bar{y}}{\partial \beta} = 0$$

or, in matrix notation –

$$\begin{bmatrix} 0 & -P_x & -P_y \\ -P_x & U_{xx} & U_{xy} \\ -P_y & U_{yx} & U_{yy} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{\lambda}}{\partial \beta} \\ \frac{\partial \bar{x}}{\partial \beta} \\ \frac{\partial \bar{y}}{\partial \beta} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{—— (5)}$$

The 1st part of the equation (5) is exactly the $|\bar{H}_2|$

Now, using Cramer's rule

$$\frac{\partial \bar{x}}{\partial \beta} = \frac{1}{|\bar{H}_2|} \begin{bmatrix} 0 & -1 & -P_y \\ -P_x & 0 & U_{xy} \\ -P_y & 0 & U_{yy} \end{bmatrix}$$

$$= \frac{1}{|\bar{H}_2|} \left\{ -(-1) \begin{vmatrix} -P_x & U_{xy} \\ -P_y & U_{yy} \end{vmatrix} - P_y \begin{vmatrix} -P_x & 0 \\ -P_y & 0 \end{vmatrix} \right\}$$

$$= \frac{1}{|\bar{H}_2|} \begin{vmatrix} -P_x & U_{xy} \\ -P_y & U_{yy} \end{vmatrix} \qquad \text{—— (6)}$$

Similarly,

$$\frac{\partial \bar{y}}{\partial \beta} = \frac{1}{|\bar{H}_2|}\begin{bmatrix} 0 & -P_x & -1 \\ -P_x & U_{xx} & 0 \\ -P_y & U_{yx} & 0 \end{bmatrix}$$

$$= \frac{1}{|\bar{H}_2|}\left\{-P_x\begin{vmatrix} -P_x & 0 \\ -P_y & 0 \end{vmatrix} - 1\begin{vmatrix} -P_x & U_{xx} \\ -P_y & U_{yx} \end{vmatrix}\right\}$$

$$= \frac{-1}{|\bar{H}_2|}\begin{vmatrix} -P_x & U_{xx} \\ -P_y & U_{yx} \end{vmatrix} \quad \text{—— (7)}$$

$\because |\bar{H}_2| > 0$ the signs of $\frac{\partial \bar{x}}{\partial \beta}$ and $\frac{\partial \bar{y}}{\partial \beta}$ depend on relative values of $P_x$, $P_y$, $U_{xx}$, $U_{yy}$ and $U_{xy}$.

$\frac{\partial \bar{x}}{\partial \beta} > 0$ if and only if $P_x Y_{yy} < P_y U_{xy}$

$\frac{\partial \bar{y}}{\partial \beta} > 0$ if and only if $P_x Y_{yx} > P_y U_{xx}$

In case of normal goods, both $\frac{\partial \bar{x}}{\partial \beta}$ and $\frac{\partial \bar{y}}{\partial \beta}$ should be greater then zero. In case of inferrior goods, both $\frac{\partial \bar{x}}{\partial \beta}$ and $\frac{\partial \bar{y}}{\partial \beta}$ should be negative.

**Effect of change in price (price of good x i.e. $p_x$)**

Assuming $\partial p_y = 0$, $\partial \beta = 0$ and $\partial p_x \neq 0$ and dividing both sides of equation(4) by $\partial p_x$, we get,

$$-P_x \frac{\partial \bar{x}}{\partial p_x} - P_y \frac{\partial \bar{y}}{\partial p_x} = \bar{x}$$

$$-P_x \frac{\partial \lambda}{\partial p_x} + U_{xx} \frac{\partial \bar{x}}{\partial p_x} + U_{xy} \frac{\partial \bar{y}}{\partial p_x} = \lambda$$

$$-P_y \frac{\partial \bar{\lambda}}{\partial p_x} + U_{yx} \frac{\partial \bar{x}}{\partial p_x} + U_{yy} \frac{\partial \bar{y}}{\partial p_x} = 0$$

in matrix notation,

$$\begin{bmatrix} 0 & -P_x & -P_y \\ -P_x & U_{xx} & U_{xy} \\ -P_y & U_{yx} & U_{yy} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{\lambda}}{\partial p_x} \\ \frac{\partial \bar{x}}{\partial p_x} \\ \frac{\partial \bar{y}}{\partial p_x} \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{\lambda} \\ 0 \end{bmatrix}$$

$$\frac{\partial \bar{x}}{\partial P_x} = \frac{1}{|\bar{H}_2|} \begin{bmatrix} 0 & \bar{x} & -P_y \\ -P_x & \bar{\lambda} & U_{xy} \\ -P_y & 0 & U_{yy} \end{bmatrix}$$

$$= \frac{-\bar{x}}{|\bar{H}_2|} \begin{vmatrix} -P_x & U_{xy} \\ -P_y & U_{yy} \end{vmatrix} + \frac{-P_y}{|\bar{H}_2|} \begin{vmatrix} -P_x & \bar{\lambda} \\ -P_y & 0 \end{vmatrix}$$

$$= -\bar{x}\left(\frac{\partial \bar{x}}{\partial \beta}\right) + \frac{-P_y}{|\bar{H}_2|}\left(0 + \bar{\lambda} P_y\right) \text{ [ From equ}^{n} \text{ (6) ]}$$

$$\Rightarrow \frac{\partial \bar{x}}{\partial P_x} = -\bar{x}\left(\frac{\partial \bar{x}}{\partial \beta}\right) - \frac{\bar{\lambda} P^2{}_y}{|\bar{H}_2|}$$

$$\Rightarrow \frac{\partial \bar{x}}{\partial P_x} = E_1 + E_2 \qquad \text{—— (8)}$$

Similarly,

$$\frac{\partial \bar{y}}{\partial P_x} = \frac{1}{|\bar{H}_2|} \begin{bmatrix} 0 & -P_x & \bar{x} \\ -P_x & U_{xx} & \bar{\lambda} \\ -P_y & U_{yx} & 0 \end{bmatrix}$$

$$= \frac{P_x}{|\bar{H}_2|} \begin{vmatrix} -P_x & \bar{\lambda} \\ -P_y & 0 \end{vmatrix} + \frac{\bar{x}}{|\bar{H}_2|} \begin{vmatrix} -P_x & U_{xx} \\ -P_y & U_{yx} \end{vmatrix}$$

or, $\frac{\partial \bar{y}}{\partial P_x} = -\bar{x}\frac{\partial \bar{y}}{\partial \beta} + \frac{\bar{\lambda} P_x P_y}{|\bar{H}_2|}$ [ From equ$^{n}$ (7) ] —— (9)

or, $\frac{\partial \bar{y}}{\partial P_x} = E_3 + E_4$ [ From equ$^{n}$ (8) ]

$\frac{\partial \bar{x}}{\partial P_x}$ indicates how a change in the price of commodity x($p_x$) brings about a change in the optimal purchase of x i.e. nothing but the price effect. There are two companents of price effect viz. $E_1$ and $E_2$. The first term

$E_1 = -\bar{x}\left(\frac{\partial \bar{x}}{\partial \beta}\right)$ represents the effect of change in $\beta$ i.e., income of consumer on optimal purchase of $\bar{x}$ i.e. nothing but the income effect of price change and it indicates that as $p_x$ increase, real income of consumer dacreases and optimal purchase of x will fall by $\bar{x}\left(\frac{\partial \bar{x}}{\partial \beta}\right)$.

Now, the decrease in the consumption of x due to fall in real income of the consumer is compensated by substitution of y for x which is substitition effect of price change and is denotied by $E_2$.

Similarly, in equation (9), $E_3$ represents the income effect and $E_4$ represents the substitution effect.



## CHECK YOUR PROGRESS

**Q 1:** The consumer's utility function is given as $u = xy^{\frac{1}{2}}$ and the budget constraint is given as 4x+y=48. Find out optimum purchase of x and y which will maximize the utility of the consumer at the same time satisfying the budget constraint.

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**Q 2:** The production function of a firm is given by

Q= $Q = 10L^{\frac{5}{9}}K^{\frac{4}{9}}$

where L and k are labour and capital inputs. The cost function is given by C=4L+5k. Find least cost capital-labour ratio.

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## 12.6 LET US SUM UP

- In Manshallian concept of consumer behaviour, a consumer will attain equilibrium when the marginal utilities are equi-proportional to their prices.
- In Hicksian analysis a consumer achieves equilibrium when the indifference curve becomes tangent to the budget line.
- At the point of equilibrium, the slope of indifference curve must be convex to the origin.
- Cost minimisation requires that the cost curve should be tangent to the isoquant as well as the isoquant must be convex to the origin.
- According to the theory of consumer's equilibrium, the price effect on the consumer's is the sum of income effect and substitution effect.



## 12.7 FURTHER READING

1) Baruah, S. (2012). *Basic Mathematics and its Applications in Economics.* Trinity Press Pvt. Ltd.

2) Chiang, A. C. (1986). *Fundamental Methods of Mathematical Economics.* McGraw Hill Book Company.

3) Rosser, M. *Basic Mathematics for Economists*. Routledge.



## 12.8 ANSWERS TO CHECK YOUR PROGRESS

**Ans to Q No 1:** x=8 and y=16

**Ans to Q No 2:** $\frac{K}{L} = 0.64$



## 12.9 MODEL QUESTIONS

**Q.1:** The utility function of a consumer is given by U = xy and the budget of the consumer is 24 = 6x + 3y. Find the combination of x and y which will maximise the utility of the consumer. Also calculate maximum utility derive by the consumer.

**Q.2:** The utility function of a consumer is given by U = 2logx + logy and $P_X$ = 2, $P_Y$ = 4 and total income (B) = 36. Find out–

a) the utility maximising consumption of x and y

b) maximum utility derive by the consumer

**Q.3:** A producer cost function and the production function are given as:

C = 8L + 2K, subject to $Q = 2L^{1/3}K^{2/3} = 64$

Find the optimum combination of the inputs K and L in order to minimise the cost, satisfying the given production function.

**Q.4:** The total cost function of a firm is given by c = 2L + 3k when L and K are Labour and capital employed respectively. If the production function is $Q = L^{1/2}K^{1/2}$. Find optimum combination of L and K in order to minimise the cost of production when Q = 8L.

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# UNIT 13: INTEGRATION

**UNIT STRUCTURE**


13.1 Learning objectives

13.2 Introduction

13.3 Concept of integration

13.4 Basic rules of integration

- 13.4.1 Indefinite integral
- 13.4.2 Definite integral
- 13.4.3 Improper integral

13.5 Let us sum up

13.6 Further Reading

13.7 Answers to check your progress

13.8 Model Questions


## 13.1 LEARNING OBJECTIVES

After going through this unit, you will be able to–

- find the integral of a function
- apply the definite integral
- distinguish between definite integral and indefinite integral
- know the use of improper integral.

## 13.2 INTRODUCTION

In the earlier units we have discussed the concept of derivatives, differentiation and optimisation under equality constraints. We discussed marginal analysis in terms of marginal revenue and marginal cost, utility maximisation under budget constraint, profit maximisation and cost minimisation by firms. We have concentrated on finding the rate of change of a function as one or more of its arguments changed. In this unit we will discuss about the integration, which is the reverse process of differentiation. Also we will learn the distinction between definite and indefinite integral.

## 13.3 CONCEPT OF INTEGRATION

Integral calculus has an important role in dynamic analysis. It provides the basic mathematics required to do the dynamic analysis. Dynamic analysis seeks to trace and study the time paths of the variables or to determine whether these variables will tend to converge to certain values, given sufficient time. This fills a gap between statics and comparative statics. In the latter we compare the equilibriums assuming that these are attainable. In dynamic analysis the question of attainability is not assumed away- rather it is considered seriously.

The concept of an integral of a function has two aspects. It refers to an area. It measures the area enclosed by the graph of a function f(X) over some range of X values. This is done through definite integral of the function. Secondly the integral is obtained by reversing the process of differentiation.

$$\frac{d[f(x)]}{dx} \equiv f'(x) \equiv \varphi(X)$$

Starting with $\varphi(X)$, we can discover the function f(X). For example through differentiation you can derive a marginal cost function MC from a total cost function C(q). MC= $\frac{d\,C(q)}{dq}$. If you are given the marginal cost function , can you derive the total cost function. For this we have to find out the indefinite integral of the function in question.

Dating of variables is an important feature of dynamic analysis as Dynamic analysis introduces the explicit consideration of time into the picture. Time can be considered as a continuous variable or as a discrete variable. Some thing is happening to a continuous variable at each point of time. A discrete variable undergoes a change only once within a period of time . The mathematical techniques of integral calculus and differential equations are appropriate for the continuous time case. Integral calculus deals with the process of tracing a given derivative function back to its

primitive function. A differential equation is defined as any equation containing differential or derivative expressions. An example of such an equation is

$$\frac{dH}{dx} = t^{-1/2}$$

In a static model we try to find the values of the endogenous variables that satisfy some specified equilibrium conditions. In the context of optimisation models we have to find those values of the choice variables that maximise or minimise a specific objective function. In a dynamic model the focus is on tracing the time path of some variables on the basis of a known pattern of change. Differential equations will be covered in the next unit.

## 13.4 BASIC RULES OF INTEGRATION

Integration is the reverse of differentiation. If differentiation of a given primitive function F(x) yields the derivative f(x) , then f(x) can be integrated to find F(x), provided it is possible to definitize the arbitrary constant that arises in the process of integration. The function F(x) is known as the integral of f(x).

The notation for integration of f(x) with respect to x is

$$\int f(x)dx$$

$$\frac{d}{dx}F(x) = f(x) \Rightarrow \quad \int f(x)dx = F(x) + C$$

Where C is the arbitrary constant of integration. f( X) is called the integrand. In general since the derivative of a constant is zero ,if the function f(X) has a derivative function $f'$(X) then any function g(X) =f(X) +C has derivative $f'$(X). The knowledge of a derivative allows one to determine the function upto some unknown constant term

### 13.4.1 Indefinite integral

$\int f(x)dx$ is known as the indefinite integral of f(x) because it has no definite numerical value. An indefinite integral is itself a function of the variable x. f(x)dx is differential of F(x) where f(x) is called integrand and the symbol dx indicates that x is the variable of integration.

Rule I (the power rule)

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c \qquad (n \neq -1)$$

Rule II (the exponential rule)

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + c \ \ where\, a\, (a \neq 0)$$

When a=1 we can write

$$\int e^x dx = e^x + c$$

Rule III (the logarithmic rule)

$$\int \frac{1}{x} \mathrm{dx} = \ln \mathrm{x} + \mathrm{c} \qquad \mathrm{x} > 0$$

$$\int \frac{\mathrm{f}'(\mathrm{x})}{\mathrm{f}(\mathrm{x})} \mathrm{dx} = \ln \mathrm{f}(\mathrm{x}) + \mathrm{c} \qquad \mathrm{f}(\mathrm{x}) > 0$$

$$\ln \left| \mathrm{f}(\mathrm{x}) \right| + \mathrm{c} \qquad \mathrm{f}(\mathrm{x}) \neq 0$$

Rule IV (the integral of a sum)

$$\int [\mathrm{f}(\mathrm{x}) + \mathrm{g}(\mathrm{x})] \mathrm{dx} = \int \mathrm{f}(\mathrm{x}) \mathrm{dx} + \int \mathrm{g}(\mathrm{x}) \mathrm{dx}$$

Rule V (the integral of a multiple)

$$\int kg(x)dx = k \int g(x)dx$$

Rule VI (the substitution rule)

$$\int f(u)\frac{du}{dx}dx = \int f(u)du = F(u) + C$$

Rule VII (integration by parts)

$$\int vdu = uv - \int udv$$

Rule VIII integration of $b^x$

$$\int b^x dx = b^x / \log b + C$$

**Examples : 1** Evaluate i) $\int x^9 dx$ ii) $\int \frac{1}{x^7} dx$

iii) $\int \sqrt[3]{x^5} dx$

Solutions : i) $\int x^9 dx = \frac{x^{9+1}}{9+1} + C = \frac{1}{10}x^{10} + C$

ii) $\int \frac{1}{x^7} dx = \int x^{-7} dx = \frac{x^{-7+1}}{-7+1} + C = \frac{1}{-6}x^{-6} + C$

iii) $\int \sqrt[3]{x^5}\ dx = \int (x^5)^{1/3} = \int x^{\frac{5}{3}} dx = \frac{3}{8}x^{\frac{8}{3}} + C$

Example : 2 Find $\int (2x^2 + \sqrt{x} + 4/x) dx$

Solution: $\int (2x^2 + \sqrt{x} + 4/x) dx$

$= \int 2x^2 dx + \int \sqrt{x} dx + \int 4/x\, dy$

$= \frac{2}{3}x^3 + \frac{x^{1/2+1}}{1/2+1} + 4\log x + C$

$= \frac{2}{3}x^3 + \frac{2}{3}x^{\frac{3}{2}} + 4\log x + C$

**Example : 3** Evaluate $\int \frac{1-x^3}{1-x}dx = \int \frac{(1-x)(1+x+x^2)}{(1-x)}dx$

$$= \int (1+x+x^2)dx = x + \frac{x^2}{2} + \frac{x^3}{3} + C$$

**Example : 4** Find $\int x(x+1)^9 dx$

**Solution :** Let u=x and v= $\frac{1}{10}(x+1)^{10}$

du = dx and dv=$(x+1)^9$dx

Now $\int x(x+1)^9 dx = \int udv$

$$= uv - \int vdu$$

$$= x\frac{1}{10}(x+1)^{10} - \int \frac{1}{10}(x+1)^{10} dx$$

$$= \frac{x}{10}(x+1)^{10} - \frac{1}{10} \times \frac{1}{11}(x+1)^{11} + C$$

$$= \frac{x}{10}(x+1)^{10} - \frac{1}{110}(x+1)^{11} + C$$

**Example : 5** Find $\int (e^x + \frac{1}{x^2})dx$

**Solution :** $\int (e^x + \frac{1}{x^2})dx$

$$= \int e^x dx + \int \frac{1}{x^2} dx$$

$$= e^x + \frac{x^{-2+1}}{-2+1} + C$$

$$= e^x - \frac{1}{x} + C$$

**Example : 6** Find $\int x \log x \, dx$

**Solution :** Let u = logx, $v = \frac{1}{2}x^2$

$$du = \frac{1}{x}dx, dv = \frac{1}{2}2x\,dx = xdx$$

Now, $\int x\log x dx = \int u dv =$ uv – $\int$ v du

$$= \frac{1}{2}x^2\log x - \int \frac{1}{2}x^2 \frac{1}{x}dx$$

$$= \frac{1}{2}x^2\log x - \frac{1}{2}\frac{x^2}{2} + C$$

$$= \frac{1}{2}x^2\left(\log x - \frac{1}{2}\right) + C$$

**Example : 7**

$$\int x\sqrt{x+a}\ dx$$

**Solution :** $\int x\sqrt{x+a}\ dx = \int [(x+a)-a]\ \sqrt{x+a}\, dx$

$$= \int (x+a)\sqrt{x+a}\ dx - \int a\sqrt{x+a}\ dx$$

$$= \int (x+a)^{3/2} dx - a\int (x+a)^{1/2} dx$$

$$= \frac{2}{5}(x+a)^{5/2} - a.\frac{2}{3}(x+a)^{3/2} + C$$

$$= \frac{2}{5}(x+a)^{5/2} - \frac{2}{3}a\,(x+a)^{3/2} + C$$

**Example : 8** $\int \frac{(a^x + b^x)^2}{a^x b^x} dx$

**Solution :** $\int \frac{(a^x + b^x)^2}{a^x b^x} dx = \int \frac{a^{2x} + b^{2x} + 2a^x b^x}{a^x b^x} dx$

$$= \int \left( \frac{a^x}{b^x} + \frac{b^x}{a^x} + 2 \right) dx$$

$$= \int \left( \frac{a}{b} \right)^x dx + \int \left( \frac{b}{a} \right)^x dx + 2\int 1 dx$$

$$= \frac{\left(a/b\right)^x}{\log\left(a/b\right)} + \frac{\left(b/a\right)^x}{\log\left(\frac{b}{a}\right)} + 2x + k$$

$$= \frac{1}{\log\left(a/b\right)} \left[ \left( \frac{a}{b} \right)^x - \left( \frac{b}{a} \right)^x \right] + 2x + k$$

$$\left[ \because \log\left(\frac{b}{a}\right) = -\log\left(\frac{a}{b}\right) \right]$$

**Example : 9** Evaluate $\int \frac{x+1}{x-1} dx$ $= \int \frac{x-1+2}{x-1} dx$

**Solution :** $\int \frac{x+1}{x-1} dx = \int \frac{x-1+2}{x-1} dx$

$$= \int \left( 1 + \frac{2}{x-1} \right) dx$$

$$= x + 2\log(x-1) + C$$

**Example : 10** $\int \frac{1}{a+bx} dx$

**Solution :** $\int \frac{1}{a+bx} dx$

$$= \log(a+bx).\frac{1}{b} + K$$

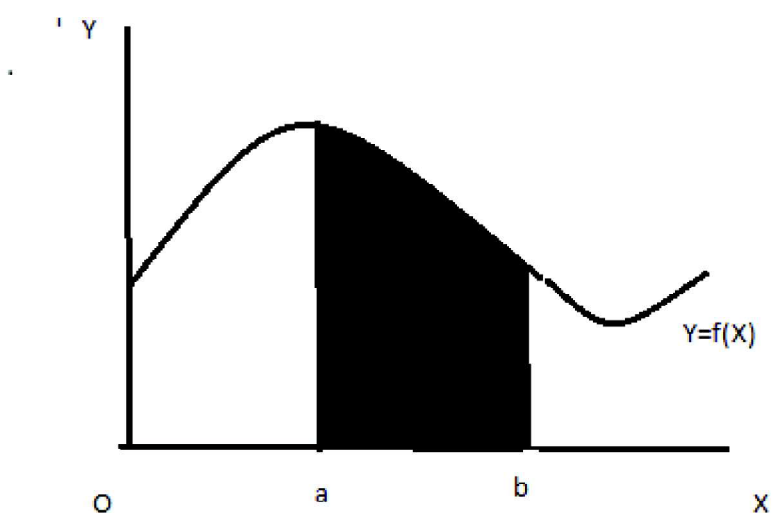
$$= \frac{1}{b}\log(a+bx) + K$$

## 13.4.2 Definite integral

Meaning of definite integral (Riemann integral)

Unlike the indefinite integral a definite integral has a definite value. It can be interpreted as a particular area under a given curve. A function is integrable in [a,b] if it is continuous in that interval.

**Figure 11.1 Definite Integral**



Let us evaluate $\int_1^5 2XdX$ . The indefinite integral is $X^2 + C$. The definite integral is given by

$$\int_1^5 2XdX = \left[X^2\right]_1^5 = 5^2 - 1^2 = 25 - 1 = 24$$

Properties of definite integrals

- The interchange of the limits of integration changes the sign of the definite integral $\int_a^b f(x)dx = -\int_a^a f(x)dx$
- When the two limits of integration are identical, the value of the integral is zero. $\int_b^a f(X)dx = 0$
- A definite integral can be represented by a sum of a finite number of definite subintegrals $\int_a^d f(X)dx = \int_a^b f(x)dx + \int_c^d f(x)dx$ where a<b<c<d
- $\int_a^b -f(x)dx = -\int_a^b f(x)dx$
- $\int_a^b Kf(x)dx = K\int_a^b f(x)dx$
- $\int_b^a [f(x) + g(X)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

**Example : 11** Solve $\int_0^1 x^2 dx$

**Solution :** $\int x^2 dx = \frac{x^3}{3}$

$$\therefore \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_1^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

**Example : 12** $\int_a^b e^x dx$

**Solution :** $\int e^x dx = e^x$

$$\therefore \int_a^b e^x dx = \left[e^x\right]_a^b = e^b - e^a$$

**Example : 13** Solve $\int_0^1 xe^x dx.$

**Solution :** $\int xe^x dx$

$$= x\int e^x dx - \int\left\{\frac{d}{dx}x\int e^x dx\right\}dx$$

$$= xe^x - \int 1.e^x.dx$$

$$= xe^x - e^x$$

$$\therefore \int_0^1 xe^x dx = \left[xe^x - e^x\right]_0^1 = (1.e^1 - 0) - (e^1 - e^0)$$

$$= e - e + 1$$

$$= 1$$

**Example : 14** Solve $\int_1^e \log x \ dx$

**Solution :** $\int \log x \ dx = \log x\int dx - \int\left\{\frac{d}{dx}\log x\int dx\right\}dx$

$$= \log x.x - \int \frac{1}{x}.x \ dx$$

$$= x \ \log x - x$$

$$\therefore \int_1^e \log x \ dx = \left[x \log x - x\right]_1^e$$

$$= (e \log e - 1.\log 1) - (e - 1)$$

$$= e \times 1 - 0 - e + 1$$

$$= 1$$

**Example : 15** $\int_0^1 x^2 e^x dx$

**Solution :** $\int x^2 e^x dx = x^2 \int e^x dx - \int \left\{ \frac{d}{dx} x^2 \int e^x dx \right\} dx$

$= x^2 e^x - 2\int x e^x dx$

$= x^2 e^x - 2\left[ x\int e^x dx - \int \left\{ \frac{d}{dx} x \int e^x d \right\} dx \right]$

$= x^2 e^x - 2\left[ x e^x - \int 1.e^x dx \right]$

$= x^2e^x-2(xe^x-e^x)$

$= x^2e^x-2xe^x+2e^x$

$\therefore \int_0^1 x^2 e^x dx = \left[ x^2 e^x - 2xe^x + 2e^x \right]_0^1$

$= (e-2e+2e-2)$

$= e-2$

**Example : 16** $\int_0^1 \frac{x^7}{1+x^8} dx$

**Solution :** $\int_0^1 \frac{x^7}{1+x^8} dx$

Put $1+x^8=z$

$\therefore 8x^7 dx = dz$

or $x^7 dx = \frac{dz}{8}$

$= \frac{1}{8}\int_1^2 \frac{dz}{z}$

$\therefore \frac{1}{8}\left[\log z\right]_1^2 = \frac{1}{8}\left[\log 2 - \log 1\right] = \frac{1}{8}\log 2$

When x=0, z=1+0=1

When x=1, z=1+1=2

**Example : 17** $\int_0^1 \frac{e^x}{e^x+2} dx$

**Solution :** $\int \frac{e^x}{e^x+2}dx$

$= \left[\log(e^x+2)\right]$

$\therefore \int_0^1 \frac{e^x}{e^x+2}dx = \left[\log(e^x+2)\right]_0^1 = \log(e^1+2)-\log 3$

$= \log(e+2)-\log 3$

$= \log\frac{e+2}{3}$

**Example : 18** $\int_0^1 (2ax+b)dx$

**Solution :** $\int (2ax+b)dx$

$= 2a\frac{x^2}{2}+bx$

$= ax^2+bx$

$\therefore \int_0^1 (2ax+b)dx = \left[ax^2+bx\right]_0^1 = (a+b)$

**Example : 19**: $\int_0^2 \frac{2x+5}{3x+4}dx$

**Solution :** $\int \frac{2x+5}{3x+4}dx$

$= \frac{1}{3}\int \frac{6x+15}{3x+4}dx$ (multiplying the numerator by 3)

$= \frac{1}{3}\int\left(2+\frac{7}{3x+4}\right)dx$

$= \frac{1}{3}\int 2dx+\frac{7}{3}\int\frac{1}{3x+4}dx$

$$= \frac{2}{3}x + \frac{7}{9}\int \frac{3}{3x+4}dx$$

$$= \frac{2}{3}x + \frac{7}{9}\log(3x+4)$$

$$\therefore \int_0^2 \frac{2x+5}{3x+4}dx = \left[\frac{2}{3}x + \frac{7}{9}\log(3x+4)\right]_0^2$$

$$= \frac{4}{3} + \frac{7}{9}\log(10) - \frac{7}{9}\log 4$$

$$= \frac{4}{3} + \frac{7}{9}\left\{\log\left(\frac{10}{4}\right)\right\}$$

$$= \frac{4}{3} + \frac{7}{9}\left\{\log\left(\frac{5}{2}\right)\right\}$$

**Example : 20** Solve $\int_2^4 \frac{x^2-4}{x}dx$

**Solution :** $\int \frac{x^2-4}{x}dx = \int\left(x - \frac{4}{x}\right)dx$

$$= \int x\,dx - 4\int \frac{1}{x}dx = \frac{x^2}{2} - 4\log x$$

$$\therefore \int_2^4 \frac{x^2-4}{x}dx = \left[\frac{x^2}{2} - 4\log x\right]_2^4 = \frac{1}{2}\left\{4^2 - 2^2\right\} - 4\left\{\log 4 - \log 2\right\}$$

$$= \frac{1}{2}.12 - 4\log\frac{4}{2} = 6 - 4\log 2$$

### 13.4.3 Improper integral

An integral is said to be improper integral when we have infinite bounds or a discontinuity in the interval under consideration (or both).

$\int_1^{\alpha} \frac{1}{x} dx$ is improper because the upper bound is infinite.

$\int_0^1 \frac{1}{x} dx$ is improper because the function is discontinuous at x=0 even though the bounds of integration are finite.

$\int_{-\alpha}^{1} \frac{1}{x+5} dx$ is improper because the lower bound is infinite and the function $\frac{1}{x+5}$ is discontinuous at X =-5

$\int_a^{\alpha} f(X)dx$ *and* $\int_{-\alpha}^{a} f(X)dx$ are examples of improper integrals as one of the limits of integration is infinite. Such integrals are computed over an infinite interval of integration. The lower limit of integration may approach $-\infty$. The upper limit may approach $\infty$.

$$\int_1^{\alpha} \frac{1}{x} dx$$

Such integrals are not evaluated directly . It is done using the concept of limits. For example,

$$\int_a^{\alpha} f(x)dx = \lim \quad \int_a^b f(x)$$
$$\int_{-\alpha}^{a} f(x)dx = \lim \quad \int_b^a f(x)$$

If the limit exists , the limiting process will yield the value of the improper integral. Such an improper integral is said to be convergent. If the limit doesnot exist then the integral cannot be evaluated.To evaluate such convergent improper integrals we have to use the concept of limit.

You may be wondering why we have to study improper integrals. We do it because there are many instances in economics where evaluation of improper integrals becomes necessary. Curves asymptotic to either X or Y axis or both are common in economics . Our indifference curves never touch any of the axis. When we estimate consumer surplus associated with a demand function with constant elasticity of demand it has to be done through integration of an improper integral.

Take the demand function

$q = aP^{-b}, \qquad a, b > 0$

or the inverse demand function $P=(q/a)^{-1/b}$

where b is the own price elasticity of demand. The graph of this demand curve never cuts the p or q axis.

$$Consumer \quad surplus \quad = \int_{p_0}^{\alpha} (ap^{-b})dp$$

$$= \lim \quad \int_{p_0}^{p} (a\,P^{-b})dp$$

$$= \frac{a}{1-b}[\lim \quad P^{1-b} - P_0^{1-b}]$$

The limit exists iff b > 1

The consumer surplus for such a demand function has a finite value only if the elasticity of demand exceeds one.



**Check your progress**

**Q 1:** Find the integral of

i. $f(x) = X^4$

ii. $f(x)=X^3 +15$

**Q2:** Evaluate the following integrals

i. $\int_2^5 x\,dx$

ii. $\int_0^3 \frac{1}{\sqrt{2-x}}dx$



## 13.5 LET US SUM UP

- Integration is the reverse of differentiation.
- The notation for integration of f(x) with respect to x is

$$\int f(x)dx$$

$$\frac{d}{dx}F(x) = f(x) \Rightarrow \quad \int f(x)dx = F(x) + C$$

Where C is the arbitrary constant of integration.

- $\int f(x)dx$ is known as the indefinite integral of f(x) because it has no definite numerical value.
- Unlike the indefinite integral a definite integral has a definite value. It can be interpreted as a particular area under a given curve.
- An integral is said to be improper integral when we have infinite bounds or a discontinuity in the interval under consideration (or both).

## 13.6 FURTHER READING


- Allen, R.G.D. (1974). *Mathematical Analysis for Economists.* London: Macmillan Press and ELBS.
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## 13.7 ANSWERS TO CHECK YOUR PROGRESS

**Ans to Q No. 1.**

i. $\frac{x^5}{5}+C$

ii. $\frac{x^4}{4}+15x+c$

**Ans to Q No. 2.**

i. $\frac{21}{2}$

ii. $2\sqrt{2}$

## 13.8 MODEL QUESTIONS

**Q 1:** state whether the following statements are true (T) or false(F)

i. An improper integral cannot be evaluated.(T/F)

ii. $\int_0^1 \frac{1}{x} dx$ is a definite integral (T/F)

**Q 2:**

i. What is the difference between an indefinite integral and a definite integral ?

ii. What is the difference between an improper integral and an indefinite integral ?

**Q 3:**

Find the integral of

i. $f(x) = e^{12X}$

ii. $f(x) = X^a + X^b$

**Q4:** Evaluate the following integrals

i. $\int_2^3 (3x + 2) dx$

ii. $\int_1^3 x^2 \log x \ dx$

*** ***** ***

# UNIT 14: INTEGRATION AND ECONOMIC APPLICATIONS

**UNIT STRUCTURE**


14.1 Learning objectives

14.2 Introduction

14.3 Some Applications of Integration in Economics

- 14.3.1 Derivation of total function from marginal function
- 14.3.2 Demand function from elasticity of demand
- 14.3.3 Consumer's surplus
- 14.3.4 Producer's surplus
- 14.3.5 Foreign exchange reserves of a country
- 14.3.6 Investment and capital formation
- 14.3.7 Present discounted value of a continuous future income stream
- 14.3.8 Present value of a perpetual flow
- 14.3.9 Domar growth model

14.4 Let us sum up

14.5 Further Reading

14.6 Answers to check your progress

14.7 Model Questions


## 14.1 LEARNING OBJECTIVES

After going through this unit, you will be able to–

- to apply definite integral to estimate consumer's surplus
- to apply definite integral to estimate producer's surplus
- to find the present value of cash flow
- to find present value of a perpetual flow.

## 14.2 INTRODUCTION

In the earlier unit we have discussed about the integration and its basic rules. The list of applications of integration in Economics clearly shows

the importance of this technique in economic analysis. In this unit we will discuss about such applications of integration in Economics. Here, we will discuss how marginal functions will be derived from total functions. We will estimate consumer's surplus from demand function and producer's surplus from supply function both for definite and improper integrals. Apart from it we will discuss about the relation between investment and capital formation. We will also find the present value of income flow of a financial asset.

## 14.3 SOME APPLICATIONS OF INTEGRATION IN ECONOMICS

The following gives you an idea of the type of questions we are going to handle through the technique of integration :

Given marginal revenue, can we derive total revenue? If the marginal utility function is given, can you derive the total utility function? Can you derive a demand function from elasticity of demand? Can you derive total cost from marginal cost? Given a demand function can you find the consumers surplus when the consumer purchases 25 units of the commodity ?

### 14.3.1 Deriving a total function from a marginal function

Marginal revenue is the derivative of total revenue. Total revenue function can be obtained by integrating the marginal revenue function.

TR= $\int MR\,dx + C$ where C is a constant of integration and its value is determined from some initial value.

Given MR = 240- 4X, the total revenue TR is given by

$\int (240 - 4X)dx$ =240X -2X$^2$ + C

Since TR=0 when X =0 $\Rightarrow$ C=0

Thus the total revenue function is

$TR=240X-2X^2$

This principle can be followed when total cost is derived from marginal cost.

Given the marginal cost function $f'(X)=2+X+X^2$ and a fixed cost of 50 units, find the total cost function, X being output produced.

$f'(X)=2+X+X^2$

Total cost $= f(X)=\int f'(X)dX=\int (2+X+X^2)\,dX = 2X+X^2/2+X^3/3+C$

$2X+X^2/2+X^3/3$ represents variable cost which varies with level of output X.

C represents fixed cost. In this exercise fixed cost is given by 50 units.

Therefore total cost function is given by

$$f(X)=2X+X^2/2+X^3/3+50$$

## 14.3.2 Demand function from elasticity of demand

A demand function may be obtained by integrating the elasticity of demand.

The elasticity of demand for a commodity is given by $\eta=\alpha-\beta P$ where $\alpha$ and $\beta$ are constants and p is the price per unit of the commodity. Through integration we can derive the demand function associated with this elasticity.

The general formula of elasticity of demand is $\eta=-\frac{dX}{dP}.\frac{P}{X}.$

Equating this with the given elasticity we get $-\frac{dX}{dP}.\frac{P}{X}.=\alpha-\beta P$

$$Or\ -\frac{dX}{X}=\frac{dP}{P}(\alpha-\beta P)=(\frac{\alpha}{P}-\beta).dP$$

Integrating both sides

$$-\quad\int\frac{dX}{X}=\int(\frac{\alpha}{P}-\beta).dP$$

$$Or\quad-\quad\log X=\alpha\log P-\beta P+C$$

$$Or\quad-\quad\log X=\log P^{\alpha}+\log e^{(\beta P+C)}$$

$$Or\log\frac{1}{X}=\log[P^{\alpha}+e^{(C-\beta P)}]$$

Taking anti-log we get

$$\frac{1}{X}=P^{\alpha}.e^{(C-\beta P)}$$

The demand function is $X=P-^{\alpha}.e^{(\beta P-C)}$

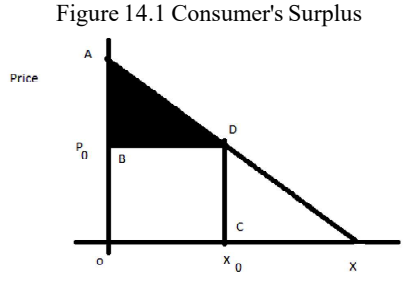
## 14.3.3 Consumer's surplus

Consumer's surplus is the difference between the amount of money a consumer is willing to pay and the amount that he actually pays for purchasing a given quantity of the commodity. Given a demand function P=f(X) , $P_0$ is the price at which $X_0$ units of the commodity are traded, the consumer's surplus is given by the shaded portion. This area can be determined by the process of integration. It depends on the nature of market- pure competition or monopoly

Consumer's surplus = Area ABD = Area AOCD-Area BOCD

$$\int_0^{X_0}f(X)dX-P_0X_0$$

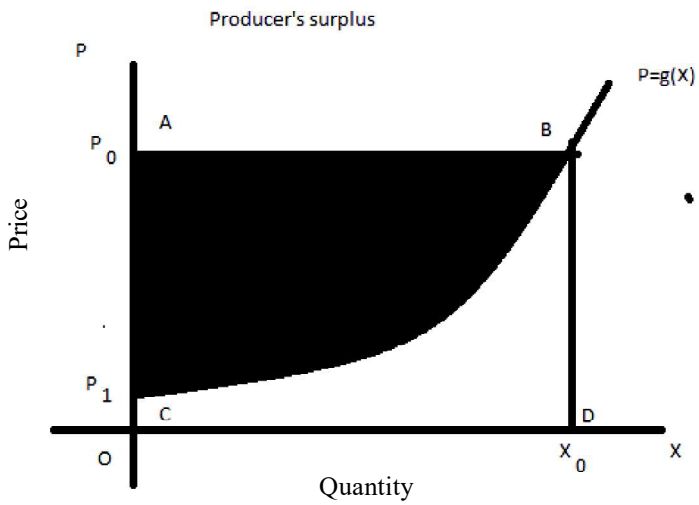
Figure 14.1 Consumer's Surplus



### 14.3.4 Producer's surplus

It is the difference between actual amount of money that a producer gets and the minimum amount at which he is willing to sell a given quantity of output.

Figure 14.2 Producer's Surplus




Given the supply function P=g(X) , $P_0$ is the price at which $X_0$ units of the product are sold. The shaded portion gives the producer's surplus.

Producers surplus = Area AODB -Area CODB

$$= P_0X_0 - \int_0^{x_0} g(x)$$

$$= P_0X_0 - \int_0^{x_0} MCdx$$

### 14.3.5 Foreign exchange reserves of a country

Let F(t) denote a country's foreign exchange reserves at time t. Assuming that F(t) is differentiable, the rate of change in the foreign exchange reserves per unit of time will be

$$f(t) = F'(t) = \frac{dF(t)}{dt}$$

there is net flow of foreign exchange into the country at time t if f(t) > 0. If f(t) < 0 foreign exchange is flowing out of the country.

A measure of the change in foreign exchange reserves over the time interval $[t_0, t_1]$ is given by

$$F(t_1) - F(t_0) = \int_{t_0}^{t_1} f(t)dt$$

### 14.3.6 Investment and capital formation

Capital formation is a very important issue in economics. It refers to the process of adding to a given stock of capital. Representing capital stock by K(t) and the rate of capital formation by $\frac{dK}{dt}$ and the rate of net investment flow at time t by I(t) we can establish a relationship between capital stock K and net investment by the following equations

$$\frac{dK}{dt} = I(t)$$

$$K(t) = \int I(t)dt = \int \frac{dK}{dt} dt = \int dK$$

### 14.3.7 Present discounted value of a continuous future income stream

Financial assets provide a monetary flow over time. Present value refers to the value as of now of future incomes which is calculated by using the technique of discounting. Discounting is a method used to value at the same date economic flows or stocks associated with different dates. The expected future incomes from an asset are converted to present values using a discount rate.A firm has to know before deciding to purchase machines - how much

are future profits worth today ?

Suppose that income is to be received continuously from time t= 0 to time t=T at the rate of f(t) at time t. Interest is compounded continuously at rate r. Let P(t) denote the present discounted value of all payments made over the time interval [0,t]. It represents the amount of money we have to deposit at time t=0 in order to match the outcome of depositing the income stream over the time interval [0,t]. The following definite integral provides the total present value over the period under consideration

$$P(t) = \int_0^t f(t)e^{-rt}dt$$

### 14.3.8 Present value of a perpetual flow

If the cash flow were to persist forever as in perpetuity, a fixed interest bond with no redemption date , the present value of the flow would be given by the following improper integral

$$\int_0^y R(t)e^{rt}dt$$

The value of the improper integral is given by the limit of a proper integral.

If a perpetuity yields an income stream of Rs D per year and the continuous rate of discount is r, the present value is

$$\int_0^y De^{-rt}dt = \lim_{y\to\infty}\int_0^y De^{-rt}dt = \lim_{y\to\infty}\frac{D}{r}(1-e^{rt})$$

$$= \frac{D}{R} \qquad \text{since} \quad \lim_{y\to\infty} e^{-ry} = 0$$

Such financial instruments also known as consol, were introduced in the UK in 1751 by replacing a variety of government bonds of different maturities and interest rates. The consols currently traded in the UK carry a nominal interest rates of 2.5 percent and 4 percent.

### 14.3.9 Domar growth model

Domar growth model answers the question-what will be the type of time path required to prevail if a certain equilibrium condition of the economy is to be satisfied. The equilibrium is a situation in which productive capacity is fully utilized. Aggregate demand should be exactly equal to the potential output producible in a year. Changes in aggregate demand should be equal to changes in capacity.

$$\frac{dy}{dt} = \frac{dK}{dt}$$

We have to find the time path of investment I(t) which will satisfy this equilibrium condition at all times. What will happen if the actual rate of growth of investment differs from the required rate?

The demand effect of a change in investment I (t) is given by

$$\frac{dy}{dt} = \frac{dI}{dt}.\frac{1}{s} \quad .....14.1$$

The capacity effect of change in investment is given by

$$\frac{k}{K} \equiv \rho$$

Where k stands for capacity or potential output flow per year and $\rho$, the given capacity-capital ratio.

It follows that k= $K\rho$

$dk = \rho\, dK$ *and*

$$\frac{dK}{dt} = \rho\, dK = \rho I .....14.2$$

As we have mentioned already, in Domar's model the equilibrium is given by the equality between aggregate demand and potential output producible in a year

y =k

If we start from an equilibrium situation, in order to sustain the equilibrium, changes in capacity should be equal to changes in aggregate demand.

$$\frac{dy}{dt} = \frac{dK}{dt} \text{.....14.3}$$

Substituting 14.1 and 14.2 into 14.3 we get

$$\frac{dI}{dt}.\frac{1}{s} = \rho I$$

$$Or \frac{1}{I}\frac{dI}{dt} = \rho s \text{.....14.4}$$

The required investment path can be obtained by integrating both sides of 14.4 with respect to t.

$$\int \frac{1}{I}\frac{dI}{dt}dt = \int \rho s\, dt$$

Using the substitution rule (rule VI) and the log rule for integration (rule III), the left hand side of 14.5 can be written

$$\int \frac{dI}{I} = In|I| + \quad C_1 \quad (I \neq 0)$$

The integral of the right hand side is

$$\int \rho s\, dt = \rho st + C_2$$

Equation 14.4 yields

$$In|I| = \rho st + C \qquad where\, C = C_1 + C_2$$

Taking the antilog of $In|I|$

$$e^{In|I|} = e^{\rho st + C}$$

$$Or\,|I| = e^{\rho st}.e^{C} = A.\ \ e^{\rho st}\ where\ A \equiv e^{C} \text{.....14.5}$$

As investment is positive, we can write |I| =I

Equation 14.5 can be re written as

I(t) = A . $e^{\rho st}$ where A is arbitrary constant.....14.6

The arbitrary constant can be eliminated by setting t=0 in equation 14.6

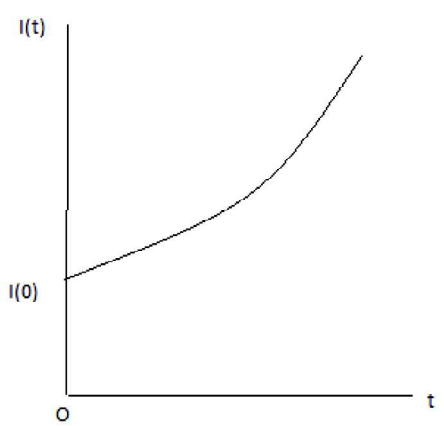
I(0) = A . $e^{0}$

Or I(0) = A where I(0) is the initial rate of investment.

Now equation 14.6 is rewritten as

I(t) = I(0) . $e^{\rho st}$ .....14.7

Figure 14.4 Domar's path




It means that the rate of investment must grow at the exponential rate of $\rho s$ along the path given in fig. 14. 4.



**Check your progress**

**Q 1:** Find the value of Q at which the average variable cost is minimum when $MC=6Q^2-24Q+5$

**Q 2:** Find consumer's surplus from the following demand function

$Q=\sqrt{42-\frac{3}{4}p}$ when price = 8

**Q 3:** Find producer's surplus from the following supply function

$Q=-5+\frac{4}{5}p$ when price = 15



## 14.4 LET US SUM UP

- The list of applications of integration in Economics clearly shows the importance of this technique in economic analysis.
- Using integration we can derive total cost, total revenue from their marginal counterparts.
- We can estimate consumer's surplus from demand function and producer's surplus from supply function both for definite and improper integrals.
- We can derive a demand function given the price elasticity of demand.
- We can also find the present value of income flow of a financial asset. We can work it out even for a perpetuity.



## 14.5 FURTHER READING

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2) Baruah, S (2011). *Basic Mathematics and Its Applications in Economics.* Macmillan.

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## 14.6 ANSWERS TO CHECK YOUR PROGRESS

**Ans to Q No. 1.** 3

**Ans to Q No. 2.** 192

**Ans to Q No. 3.** 30.625

## 14.7 MODEL QUESTIONS

**Q1.**The marginal revenue function of a monopolist is given by

$$\frac{dR}{dx} = 100 - 0.004X^2$$

Find the demand function.

**Q2.**Find the consumer's surplus and the producer's surplus under pure competition for demand function $P = \frac{8}{X+1} - 2$ and the supply function $P = \frac{X+3}{2}$ where P is price and X is quantity.

**Q3.**For a profit maximising, perfectly competitive firm with marginal cost

function $MC(q)=q^2+3$, find the following and illustrate your answer

(a) Producer's surplus at price $P_0=7$

(b) Producer's surplus at price $P_1=12$

(c) Change in producer's surplus from the price change from $P_0=7$ to $P_1=12$

**Q4.** For a consumer with demand function $q = 10 - 2p^{1/2}$, find the following and illustrate your results graphically.

(a) Consumer surplus at price $P_0=1$

(B) Consumer surplus at price $P_1= 4$

(c) Change in consumer surplus when price rises from 1 to 4.

**Q5.** The function $S(t) =100\ e^{-0.5t}$ indicates the sales of a product where t is the number of years since the introduction of the product in the market. Find the total sales in the first three years.

**Q6.** Given a continuous income stream at the constant rate of Rs 1000 per year, find the present value if the income stream lasts for 2 years and the continuous discount rate is 0.05 per year .

**Q7.** What is the present value of a perpetual cash flow of Rs 1500 discounted at r=5% ?

**Q8.** Given the marginal propensity to import $M'(Y) =0.1$ and that $M=20$ when $Y =0$, find the import function $M(Y)$.

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# UNIT 15: DIFFERENTIAL EQUATIONS AND ECONOMIC APPLICATIONS

**UNIT STRUCTURE**


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- 15.3.2 Types of differential equation

15.4 First order liner differential equations with constant coefficient and constant term

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15.5 First order liner differential equations with variable coefficient and variable term

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15.10 Further Reading

15.11 Answers to check your progress

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## 15.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

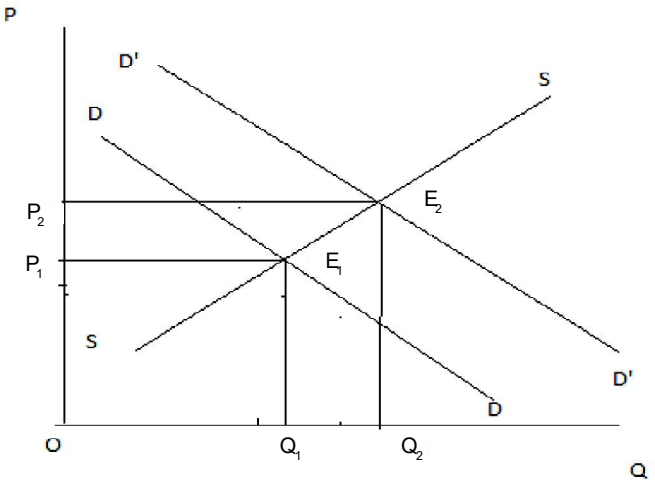
- learn some basic concepts related to differential equations
- know the importance of differential equations in dynamic analysis
- understand general and particular solutions of first order linear differential equations
- solve applications of first order linear differential equations in Economics.

## 15.2 INTRODUCTION

The essence of dynamics is the explicit role of time. It is no longer confined to dating of variables. It also matters whether time enters as continuous variable or a discrete variable. In the former we consider infinitesimal change in time and in the latter the variable t takes only discrete values. When t is continuous, the derivative dy/dt or y'(t) shows the pattern of change. When t is discrete the pattern of change is no longer described by differentials but by differences ($y_t - y_{t-1}$). The values of time refer to periods, not points of time. In dynamics analysis using difference equations, also known as period analysis, time is discrete and in differential equation time is continuous. This chapter will discuss the nature of such time paths of variables when time is considered continuous through differential equations. Differential equations are increasingly used to model the time path of variables.

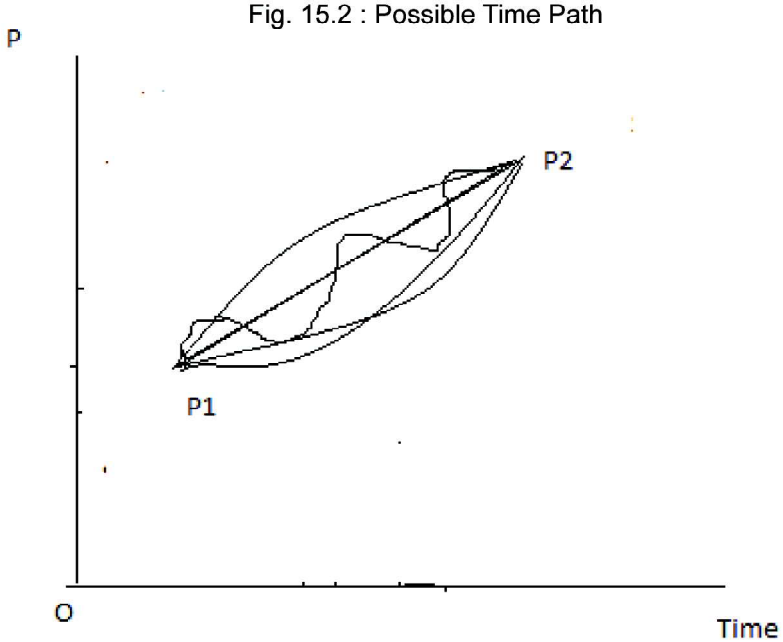
Comparative statics involves the comparison of different (static) equilibrium conditions without mentioning how the variables involved reach the new equilibriums. Fig 15.1 is a familiar diagram in microeconomics. $E_1$ is the initial market equilibrium. Demand curve shifts from DD to D'D' with the equilibrium shifting to $E_2$. The price changes from $OP_1$ to $OP_2$.

Fig. 15.1 : Comparative statics




However we do not know exactly how it has changed. The time path P(t) may be one of the following alternative paths between $P_1$ and $P_2$ given by fig.15.2. we could have reached $P_2$ by following any of these paths. In addition we have taken for granted the convergence to the new equilibrium. There may be situations where the equilibrium is unstable by which we mean the possibility of moving away from the equilibrium with out any prospect for approaching a new equilibrium.Thus our interest lies in understanding the time path of the variable.

Fig. 15.2 : Possible Time Path



## 15.3 BASIC CONCEPTS OF A DIFFERENTIAL EQUATION

A differential equation is a mathematical relation that relates some function with its derivative. A differential equation expresses the rate of change of a variable as a function of the variable. The solution to a differential equation on an interval a< t < b is any function Y(t) which satisfies the differential equation in question on the interval a< t < b. There is an infinite number of solutions. The initial condition allows us to determine which solution we are after. How many initial conditions we need depends on the order of the differential equation.

Consider a variable such as GDP. The rate of change of GDP is proportional to the current GDP.

$$\frac{d\,GDP}{dt} = g\,GDP(t)$$

Differential equations describe dynamic models where variables change with time. Derivatives represent rates of change. Differential equations have been widely used in modelling change and motion. Dynamics is concerned with how variables change over time. How a variable changes over time is shown with the help of models.

In dynamic analysis time usually indicated by t is always an argument of the function. Differential equations are widely used in economics, Physics, biology and chemistry. The difference between an ordinary algebraic equation with which we are so familiar and the differential equation that is being introduced in this chapter is that it has one or more of the derivatives of the function and the unknown solution is a function, not a number .

For example, the following is a differential equation

$$\frac{dy}{dt} = 6y + 27$$

We have to find the function of y in terms of t. What are we doing when we solve a differential equation? We are trying to find all functions

satisfying that equation. When we solve an algebraic equation we find values of the variable that makes the equation true. In the equation 2x-10 =0 the solution for x is 5 as it satisfies the equation. No other value of x satisfies this equation. In quadratic equations denoted by $aX^2+ bX +c =0$, there are two values of x that satisfy this equation.

Consider the differential equation

$$\frac{dy}{dt} = 2x$$

To solve this equation means to find the unknown which will turn the equation into an identity. The solutions of differential equations are functions, not numbers. The solution to a differential equation is often referred to as the integral of that function. It is a relation among the variables which is free of derivatives or differentials. It should be in the form of an integral of the function given in the differential equation.

$$dy = 2x \,.\, dx$$
$$\int dy = \int 2x \; dx$$
$$y = x^2 + c$$

The general solution of the differential equation $\frac{dy}{dx} = 2x \; is \; y = x^2 + C$ where C is any arbitrary constant which can be definitized using the initial value of the function.

There are some fundamental questions: do differential equations always have a solution? Are the solutions always unique? Not all differential equations may have a solution. It is also possible to have more than one solution. It is also not necessary that we can find a solution despite its existence.

In this section we will study about the order and degree of differential equation and its types.

### 15.3.1 Order and degree of a differential equation

The word 'order' refers to the order of the highest derivatives (or differentials) appearing in the differential equation. A first order differential equation can contain only the first derivative $\frac{dy}{dt}$. While the term 'order' of the differential equation refers to the highest derivative, degree of the differential equation, if it is defined, refers to the highest power attained by the derivative of the highest order in the equation. The degree of a differential equation is defined if it is a polynomial equation in its derivatives. In first order differential equations only the first order derivatives of the unknown function of one variable are included..

$d^2Y/dt^2 = kY$ this is a second order differential equation because of the second order derivative.

$d^nY/dt^n + a_1 d^{n-1}Y/dt^{n-1} + .....+a_{n-1}dy/dt + a_n y = b$

It is an n th order differential equation because the order of the highest derivative is n.

Let us examine the following differential equations and determine their order and degree.

| | | |
|---|---|---|
| i. | $dy/dt = 2t + 3$ | first order, first degree |
| ii. | $(dy/dt)^3 - t^5 = 0$ | first order, third degree |
| iii. | $d^2y/dt^2 + (dy/dt)^3 + t^2 = 0$ | second order , first degree |
| iv. | $(d^2y/dt^2)^7 + (d^3y/dt^3)^5 = 75\ y$ | Third order, fifth degree. |

The highest order of the derivative in equation (iv) given above is three and it is of fifth degree as it has been raised to the power 5.

### 15.3.2 Types of differential equation

There are many forms of differential equations.

- Ordinary vs partial differential equation : A differential equation contains derivatives, either ordinary derivatives or partial derivatives. Differential equations are called partial differential equations or ordinary differential equations depending on whether or not they contain partial derivatives.

  Ordinary differential equations are differential equations whose solutions are functions of one independent variable usually denoted by t which stands for time. The term ordinary indicates that we are dealing with a function of a single variable. we are looking for X(t), a quantity which changes with time.

- Autonomous vs non autonomous or time dependent: Differential equations where time variable does not appear explicitly are known as autonomous equations. In non autonomous or time dependent equation time variable appears explicitly. The general form of the linear,autonomous, first order differential equation is

  $dy/dt + ay = b$

  An example of nonautonomous differential equation is $dy/dt + a(t)y = b(t)$. the coefficient 'a' or the constant 'b' are functions of time

- Linear vs non linear differential equations: The differential equation is said to be linear if the derivative dy/dt and y appear only in the first degree and no product of the form y(dy/dt) occurs. The general form of the linear, first order differential equation is $dy/dt + a(t)\, y = b(t)$

  An example of non-linear, first order differential equation is $dy/dt = y - y_2$ This is a first order nonlinear differential equation. It is non linear because y has been raised to a power 2 that is larger than one. It is first order because only the first order derivative has been included.

We have a single solution technique to solve any first order linear differential equation. With non linear equations there is no single solution technique which will work in all cases. Only a few nonlinear first order differential equations can be solved at all. It is rare in economic applications to be able to find solutions to nonlinear differential equations. In such a situation we do a qualitative analysis with the use of phase diagrams.



**Phase diagram**

A phase diagram enables us to determine the qualitative properties of a solution to a differential equation without solving it. The phase diagram of a single differential equation shows dy/dt as a function of y. This technique is very useful in analysis of nonlinear differential equations

In this chapter we will discuss first order linear differential equations. We know the first order linear differential equation will take the form

$$\frac{dy}{dt} + u(t)y = w(t)$$

y, u and w are continuous functions of t. However there is no restriction on t. u and w may even be constants. The first order differential equation may contain

a. A constant (this may be zero)
b. The unknown function y
c. The first order derivative $\frac{dy}{dt}$

When the function u is a constant and w is a constant additive term we have a first order linear differential equation with constant coefficient and constant term.

$$\frac{dy}{dt} + ay = b$$



**Check your progress**

**Q. 1.** Find out the order of the following differential equations

i. $\frac{dx(t)}{dt} + 2tx = 4$

ii. $(t^2+1)\frac{dX(t)}{dt} + e^t = t\,In\,t$

iii. $\frac{d\,K}{dt} = \alpha\sigma K(t) + H_0.e^{\mu t}$

iv. $(d^2y/dx^2)4 - 6(dy/dx)^3 + xy = 20$

v. $dy/dt + y^2 = 2$

vi. $(dy/dx)^3 + d^4y/dx^4 + y = 2\,Sin(x) + \cos^3(x)$

vii. $y.\frac{dy}{dx} = 2$

viii. $x^2dy/dx = x^3 - 2xy$

**Q. 2.** Classify the following equations

i. $d^2y/dt^2 - 4y = 0$

ii. $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \frac{y-x}{y+x}$

## 15.4 FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENT AND CONSTANT TERM

### 15.4.1 The Homogenous case

If u is a constant and w is zero we have

$$\frac{dy}{dt} + a\,y = 0 \ldots\ldots\ldots\ldots\ldots\ldots (15.1)$$

Such a differential equation is said to be homogenous. It is homogenous because every term is uniformly in the first degree in

terms of y and dy/dt.

This can be rewritten as $\frac{1}{y}\frac{dy}{dt} = -a$ .....(15.2)

The solution is $y(t) = Ae^{-at}$ General solution .....(15.3)

For y(0), substituting the value of t in $y(0) = Ae^0 \Rightarrow A = y(0)$

Or

$y(t) = y(0)\, e^{-at}$ Definite solution .....(15.4)

In equation (15.3) there is an arbitrary constant A. To convert this general solution into a particular solution, we should substitute a value for A. Of the infinite set of particular solutions, y(0) the initial value of y definitizes the solution of the differential equation.

Let us solve the following initial value problem

$$\frac{dy}{dt} - 2y = 0; y(0) = 9$$

Here a = -2

Substituting the values of a and y(0) in eqn. (15.4)

$$y(t) = 9e^{2t}$$

## 15.4.2 The non-homogenous case

The form of the non homogenous linear differential equation is as follows

$$\frac{dy}{dt} + ay = b \quad .....(15.5)$$

The solution of this equation will be the sum of two terms-the complementary function $y_c$ and the particular integral $y_p$. When we solve the nonhomogenous equation, the corresponding homogenous

equation will be referred to as reduced equation and the nonhomogenous equation is referred to as the complete equation.

The reduced equation is $\frac{dy}{dt} + ay = 0$

The complementary function is the general solution of the reduced equation and the particular integral is any particular solution of the complete equation.

The general solution of the reduced equation is

$$Y_c = Ae^{-at} \quad \text{.....(15.6)}$$

Since the particular integral is any solution of the nonhomogenous equation, also known as complete equation, we can start with the simplest solution.

Such a solution is y=k , a constant

If y is assumed to be a constant, $\frac{dy}{dt} = 0$

Then ay=b and $Y_p = \frac{b}{a} \qquad (a \neq 0) \qquad \text{.....(15.7)}$

The general solution of the complete equation or the nonhomogenous equation thus can be written as

$$y(t) = Y_c + Y_p$$
$$= Ae^{-at} + \frac{b}{a} \qquad (a \neq 0) \text{.....(15.8)}$$

The solution is said to be general due to the presence of the arbitrary constant A as in the homogenous case. It is in this context that the initial condition matters. The initial condition enables us to definitise this constant.

y( 0) is the initial value, the value at time zero.

$$Y(0) = A + \frac{b}{a} \qquad bacause \qquad e^0 = 1$$

$$A = y(0) - \frac{b}{a} \text{.....(15.9)}$$

Substituting the value of A from eqn. 15.9 in eqn.15.8

$$y(t) = [y(0) - \frac{b}{a}]e^{-at} + \frac{b}{a} \text{.....(15.10)}$$

This is no longer a general solution. It has become a definite solution.

let us consider as an exercise the solution of the following nonhomogenous linear differential equation

dy/dt + 4y =8 given y(0) =10

comparing this with $\frac{dy}{dt} + ay = b$ term by term we have

a=4, b=8

using eqn.15.8 the general solution is y(t) = = $Ae^{-4t} + \frac{8}{4}$ then the definite solution, using eqn.15.10, is given by the following

$$y(t) = [10 - \frac{8}{4}]e^{-4t} + 2$$

$$= 8e^{-4t} + 2$$

This holds as long as $a \neq 0$

When a =0, the differential equation is $\frac{dy}{dt} = b$

The general solution is y(t) = bt +c where c is an arbitrary constant.

Since a =0 the complementary function becomes

$y_c = Ae^{-at} = Ae^0 = A$ $\sin ce\, e^0 = 1$

To find out the particular integral let y=kt, then $\frac{dy}{dt} = K$ the complete equation dy/dt =b reduces to k=b the general solution when a=0 is

y(t) = $y_c$ +$y_p$

=A + bt

The definite solution is y(t) = y(0) + bt .....

Both show the importance of initial condition in definitising the arbitrary constant.



**Check your progress**

**Q. 3.** Find the solution of each of the following by using an appropriate formula

i. dy/dt +y =4 ; y(0) = 0

ii. 3 dy/dt + 6y =5; y(0) =0

iii. dy/dt - 5y=0 ;y(0) = 6

iv. dy/dt =23 ; y(0) =1

## 15.5 FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENT AND VARIABLE TERM

$$\frac{dy}{dt} + u(t)y = w(t)$$

u(t) and w(t) represent a variable coefficient and a variable term respectively. This is the more general first order linear differential equation.

### 15.5.1 The Homogenous case

$$\frac{dy}{dt} + u(t)y = 0$$

$$\frac{1}{y}\frac{dy}{dt} = -u(t)$$

Integrating both sides with respect to t

Left hand side = $\int \frac{1}{y}\frac{dy}{dt}dt = \int \frac{dy}{y} = iny + c$ assuming y>0

Right hand side = $\int -u(t)dt = -\int u(t)dt$

Equating both sides

lny = $-c - \int u(t)dt$

The antilog of ln y gives the time path y(t)

$$y(t) = e^{\ln y} = e^{-c}e^{\int u(t)dt}$$

$$= Ae^{-\int u(t)dt} \text{.....(15.11)} \qquad where\ A = e^{-c}$$

This is the general solution of homogenous first order linear differential equation.

The definite solution can be obtained using an appropriate initial condition as before.

### 15.5.2 The nonhomogenous case

$$\frac{dy}{dt} + u(t)y = w(t)$$

The general solution of the equation is

$$y(t) = e^{\int u(t)dt}(A + \int w(t)e^{\int u(t)dt}dt) \text{.....(15.12)}$$

where A is an arbitrary constant which can be definitized with an appropriate initial condition.

## 15.6 EXACT DIFFERENTIAL EQUATIONS

### 15.6.1 The basics

The total differential of F(y,t) is given by

$$dF(y,t) = \frac{\partial F}{\partial y}.dy + \frac{\partial F}{\partial t}dt \quad \text{.....(15.13)}$$

setting this equal to 0, we have

$$\frac{\partial F}{\partial y}.dy + \frac{\partial F}{\partial t}.dt = 0 \quad \text{.....(15.14)}$$

Eqn. 15.14 is known as exact differential equation. It is exact because its LHS is equal to the differential of F(y,t)

A differential equation

Mdy +Ndt =0 .....(15.15)

is exact if and only if there exists a function F( y,t) such that

M = $\frac{\partial F}{\partial y}$ and N = $\frac{\partial F}{\partial t}$

Eqn 15.15 is also exact if and only if

$$\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y} \quad \text{.....(15.16)}$$

Eqn.15.16 is a test for exactness of the differential equation.

When the differential equation is exact, we have

dF(y,t)=0

its general solution should be in the form

F(y,t)=c where c is an arbitrary constant. It can be definitised by introducing the initial value of y i.e. y(0)

### 15.6.2 General procedure of solving exact differential equation

What we are doing when we solve such an exact equation is to find the primitive function F(y,t) and set it equal to an arbitrary constant c.

Since M= $\frac{\partial F}{\partial y}$ the function F must contain the integral of M with respect to y. A preliminary result is

$$F(y,t) = \int M\, dy + \varphi(t) \quad \text{.....(15.17)}$$

We know that in differentiating F(y,t) partially with respect to y, any additive term containing only t and/or some constants would drop out and such terms should be reinstated in the integration process. That is why we have introduced the general term $\varphi(t)$. The exact form of $\varphi(t)$ can be obtained by using N = $N = \frac{\partial F}{\partial t}$. There is a four step sequence to solve any exact differential equation.

Let us discuss the general procedure through an exercise.

The given equation is $(t+2y)\,dy + (y+3t^2)\,dt = 0$

Set M= t +2y and N= $y+3t^2$

$\frac{\partial M}{\partial t} = 1 = \frac{\partial N}{\partial y}$ we have an exact differential equation.

Step i using eqn.15.17 we can write a preliminary result

$$F(y,t) = \int (t+2y)dy + \varphi(t) = t\int dy + 2\int y dy + \varphi(t) = ty + y^2 + \varphi(t)$$

The constant of integration has been merged into $\varphi$ (t).

Step ii differentiate partially this preliminary result with respect to t. we have

$$\frac{\partial F}{\partial t} = Y + \varphi(t)$$

Since N = $\frac{\partial F}{\partial t}$ we can write N =( y+3t$^2$) and equating this with

$$\frac{\partial F}{\partial t} = y + \varphi(t)$$

We can write $\varphi(t) = 3t^2$

Step iii. Integrating the last result i.e. $\varphi(t)$ we get

$$\varphi(t) = \int 2t^2\, dt = t^3$$

we now have a specific form for $\varphi(t)$

step iv the results of step i and iii together give

F(y,t) = ty +y$^2$ + t$^3$ = c

Any exact differential equation can be solved through these four steps.



**Check your progress**

**Q. 4.** Verify that each of the following differential equations is exact and solve by the four step procedure

i. 2yt$^3$dy + 3 y$^2$t$^2$dt = 0

ii. ( e$^{t+y}$ + 2y) dy + (e$^{t+y}$+ 3t$^2$) dt = 0

iii. x dy -y dx = x$^2$dx

## 15.7 APPLICATION OF DIFFERENTIAL EQUATIONS IN ECONOMICS

Differential equations are widely used in economics to examine several dynamic situations. Most of the models used in economics are comparative statics models. Business cycle analysis is a dynamic model.

### 15.7.1 Continuous Market adjustment model

The following market model is an illustration of nonhomogenous linear differential equation .

$Q_d = \alpha - \beta P \qquad \alpha,\ \beta > 0$ demand function

$Q_s = -y + \delta P \qquad Y,\ \ \delta > 0$ supply function

Let us assume that the initial price is not in equilibrium. The equilibrium price will be attained after a due process of adjustment. When price changes, $Q_d$ and $Q_s$ also will change. We are interested in the adjustment process i.e. the time path P(t) and want to know whether given sufficient time for adjustment it will converge to an equilibrium price. The adjustment is continuous.

Assume that the rate of price change with respect to time at any moment is directly proportional to the excess demand ($Q_d$ - $Q_s$) at that moment.

$$\frac{dP}{dt} = j(Q_d - Q_s) \qquad j > 0$$

j represents a constant adjustment coefficient.

$$\frac{dP}{dt} = j(Q_d - Q_s) = j(\alpha - \beta P - (-\gamma + \delta P)) = j(\alpha - \beta P + \gamma - \delta P)$$
$$= j(\alpha + \gamma) - j(\beta + \delta)P$$
$$\frac{dP}{dt} + j(\beta + \delta)P = j(\alpha + \gamma)$$

This is of the same form as $\frac{dy}{dt}$ + ay =b.

The solution then was y(t) = $\left[ y(0) - \frac{b}{a} \right] e^{at} + \frac{b}{a}$

Comparing with the standard form, the solution can be written as

P(t) = $[P(0) - \frac{\alpha + \gamma}{\beta + \delta}] e^{-j(\beta + \delta)t} + \frac{\alpha + \delta}{\beta + \delta}$

This is the time path of price .

Whether the path shows that as $t \to \infty$

P(t) $\to \frac{\alpha + \gamma}{\beta + \delta}$ depends on $e^{-j(\beta + \delta)}$ t . since j, $\beta$ and $\delta$ are positive

$e^{-j(\beta + \delta)}$ t tends to 0 as t $\to \infty$

In this set up the time path will lead to lead to an equilibrium t $\to \infty$

## 15.7.2 Compound interest and how to become a crorepati at retirement

The equation for the growth of an investment with continuous compounding of interest is a first order differential equation. Let s(t) be the value of the investment at time t and r be the annual interest rate compounded after everytime interval. The equation for continuous compounding of interest (r) and continuous deposit (k) is

ds/dt = rs +k; s(0) =$s_0$

This can be written as ds/dt - rs =k

Comparing this with the first order linear non- homogenous differential equation

$dy/dt + ay = b$

The definite solution will be

$y(t) = [y(0) - b/a]e^{-at} + b/a$

$a = -r$

$b = k$

The time path of investment will be

$s(t) = [s(0) + k/r]e^{rt} - k/r$

$= s(0)e^{rt} + (k/r)e^{rt}[1 - e^{-rt}]$

An interesting offshoot of this exercise is related with our cherished wish to become a crorepati at the time of retirement. What should be the investment plan? A person becomes a lecturer at the age of 25 years and he expects to retire at 65 years. The interest rate adjusted for inflation is 6 percent per annum. How much he has to invest every year to become a crorepati when he retires ?

We have

$s(0) = 0$, $r = 0.06$ $t = 40$ years $s(40) =$ Rs 10,000,000

we have to calculate $k = \frac{rs(t)}{e^{rt} - 1}$

$= \frac{0.06 * 10000000}{e^{0.06*40} - 1} = 59862$

He should save Rs 59,862 per annum. Over 40 years he would be saving Rs 2,394,451. The amount earned on the investment is Rs 7,605,549 which is three times his saving.

### 15.7.3 Population growth

Let N(t) be the population size at time t and r is the growth rate of

population. The rate at which current population N of an organism with respect to time changes is equal to the product of the constant r and the current population N. Malthusian law of population growth is dN/dt =rN

This is a first degree linear homogenous differential equation which can be rewritten as dN/dt -rN= 0

Comparing this with eqn.15.1 given by $\frac{dy}{dt}$ + a y = 0

Whose definite solution is $y(t) = y(0)\, e^{-at}$

a= -r

The time path of population variable will be given by

$N(t) = N(0)e^{rt}$ N(0) is the initial population.

We know that census is a decadal exercise. Such a formula is very useful in interpolating the population as we need estimates of population every year to work out yearly per capita income .

When the population growth is constrained by limited resources, the equation may be modified as follows

dN/dt= rN(1-N/k).

k is also known as the carrying capacity of the environment.

Population of Assam is growing continuously at an annual rate of 1.07%. If the current population is 31.2 million, what will be the population in 15 years?

Here r = 0.0107 N(0)=31.2 million t=15 years

$N(15) = N(0)e^{rt}$

$=31.2X\ e^{0.0107*15}$

=36.632 million

Using the equation $N(t) = N(0)\, e^{rt}$

We can ask many related questions such as

i. Given N(0) and r, what will be the value of N(t) for a specified period?
ii. Given N(t), N(0) and t, what will be the continuous growth rate?
iii. Given N(t), r and t, what should be the initial population?
iv. Given N(t),N(0) and r, how long it will take to reach N(t) from N(0)?

### 15.7.4 Keynesian macroeconomic model

The simple Keynesian income determination model is given below

$$C = \alpha + \beta y$$

E=C+I where C,I ,y and E indicate consumption, autonomous investment, income and total expenditure respectively

In equilibrium E=y $\Rightarrow$ y=C+I

An exogenous increase in I disturbs the equilibrium. We know the multiplier process will unfold and there will be adjustments. Assume that the rate at which the system adjusts to a new equilibrium is directly proportional to the difference between total expenditure E and current income y.

dy/dt =r(E-y) =r(C+I-y)

$= r(\alpha + \beta y + I - y) = r(\beta - 1)y + r(\alpha + I)$

dy/dt -r( $\beta - 1$)y = r( $\alpha + I$ )

Since a, b, r and I are constants, it is a first order linear non-homogenous differential equation as in eqn.15.5

$\frac{dy}{dt}$ + ay =b

The general solution is

y(t)= = $Ae^{-at} + \frac{b}{a}$ $(a \neq 0)$

comparing the equations

a= $-r(\beta - 1)$ $b = r(\alpha + I)$

the general solution will be

$$y(t) = A e^{r(\beta-1)t} + \frac{r(\alpha + I)}{-r(\beta - 1)}$$
$$= A e^{r(\beta-1)t} - \frac{(\alpha + I)}{(\beta - 1)}$$

In order to get a particular solution, by definitizing the arbitrary constant A, we have to get an initial income y(0).

If we have y(0), the particular solution, by comparison with eqn.15.10, will be

$$y(t) = \left[ y(0) - \frac{b}{a} \right] e^{-at} + \frac{b}{a}$$

our required solution is

$$y(t) = \left[ y(0) + \frac{(\alpha + I)}{(\beta - 1)} \right] e^{r(\beta-1)t} - \frac{(\alpha + I)}{(\beta - 1)} \Bigg]$$

as an illustration let us consider the following model

C= 360 +0.8y

I= 120 and y(0) = 2000 r=0.25

Then the particular solution will be

$$y(t) = [2000 + \frac{(360+120)}{(0.8-1)}]e^{0.25(0.8-1)t} - \frac{(360+120)}{(0.8-1)}$$

$$= -400e^{-0.05t} + 2400$$

## 15.8 BASIC CONCEPT OF DIFFERENCE EQUATION

**Difference Equation :**

Any equation that relates the independent variable t, the dependent variable y and its successive differences such as $\Delta y_t$, $\Delta^2 y_t$ etc, is a difference equation.

For Example, i) $y_{t+1}-y_t=10$

ii) $y_{t+2}-y_{t+1}-y_t=0$

**Degree of Difference equation:**

The highest difference of time lag contained in the equation is known as the order of the difference equation.

In the above example, (i) is of first order difference equation as higehst time lag involved in the equation is 1(t+1-t) and (ii) is of second order difference equation as the highest time lag involved in the equation is 2 (t+2-t).

**Form of difference equation :**

In the following we are discussing some important types of difference equation –

i) $y_{t+1}+ay_t=c$

Where a is a constant coefficient and C is constant term.

The above equation is of first order linear difference equation with constant coefficient and constant term.

ii) $y_{t+2}+a_1y_{t+1}+a_2\ y_t=C$

Where $a_1$ and $a_2$ are constant coefficients and c is a constant term.

The above equation is of second order linear difference equation with constant coefficient and constant term.

iii) $y_{t+2}+a_1y_{t+1}+a_2y_t=f(t)$

Where f(t) is a function of t.

The above equation is of second order linear difference equation with constant coefficiant and variable term.

**Solution of first order difference equation :**

**i) Iterative method of solution :**

In this method, we get the value of $y_t$ by substituting t=0, 1, 2, 3 ..... in the difference equation as shown below :

For example, $y_{t+1}= y_t+10$ is a difference equation

Now, putting t=0, in the above equation we get $y_1=y_0+10$

Putting t=1, we get $y_2 = y_1+10$

$= (y_0+10)+10$

$= y_0+2\times10$

Putting t=2, we get $y_3 = y_2+10$

$= (y_0+2\times10)+10$

$= y_0+3\times10$

Continuing this process for any value of t, we can write the general form of difference equation as –

$y_t=y_0+10t$

**ii) General method of solution :**

Let us consider a first order linear difference equation with constant coefficient(a) and constant term (b),

$y_{t+1}+ay_t=b$

Like the differential equation, the general solution of difference equation will also consist of two parts viz. complementary solution and particular solution.

Complementary solution implies the solution of homogeneous part i.e. $y_{t+1}+ay_t=0$

And particular solution refers to the solution of non homogeneous part i.e. $y_{t+1}+ay_t=C$

The solution of homogeneous part : $y_{t+1}+ay_t=0$ (i)

Putting, t=o, $y_1=-ay_0$

t=1, $y_2=-ay_1=-a(-ay_0)=(-a)^2y_0$

t=2, $y_3=-ay_2=-a[(-a)^2y_0]=(-a)^3y_0$

In general form we can write; $y_t=(-a)^ty_0$

Putting $A=Y_0$ and b=-a, we can write, $y_t=Ab^t$.

$\therefore$ The complementary (homogeneous part) solution : $y_t=A(-a)^t$

Solution of non-homogeneous part : $y_{t+1}+ay_t=b$ (ii)

For obtaining particular solution, let us consider;

$y_t=C$

$\therefore y_{t+1}=C$

$\therefore$ Equation (ii) becomes : $c+ca=b$

$$\Rightarrow c(1+a)=b$$

$$\therefore c=\frac{b}{1+a}$$

$\therefore$ The particular (non-homogeneous part) solution ($y_p$) is

$$c=\frac{b}{1+a}(a\neq -1)$$

If a=-1, $y_p$ becomes undefined and in such cases we consider the particular solution as

$y_t=ct$

$\therefore y_{t+1}=c(t+1)$

$\therefore$ Equation (ii) becomes : $C(t+1)+act=b$

$C(t+1+at)=b$

$$\therefore c=\frac{b}{t+1+at}$$

$\therefore$ The particular (non homogeneous part) solution ($y_p$)is

$$ct=\frac{bt}{1+t+at} \quad (a=-1)$$

$\therefore$ The complete solution is :

$y_t$= Complementary solution + Particular solution

$$y_t= A(-a)^t+\frac{b}{1+a}(a\neq -1) \qquad \text{(iii)}$$

$$y_t= A(-a)^t+\frac{bt}{1+t+at}(a=-1)$$

$$\therefore y_t=A+bt(\because a=-1) \qquad \text{(iv)}$$

Considering t=0, Equation (iii) becomes

$$y_0 = A + \frac{b}{1+a}$$

$$\therefore A = \left( y_0 - \frac{b}{1+a} \right)$$

Again considering t=0, equation (iv) becomes $y_0$=A

$\therefore$ The final solution is

$$y_t = \left( y_0 - \frac{b}{1+a} \right).(-a)^t + \frac{b}{1+a} (a \neq -1)$$

$y_t=y_0+bt$ (a=-1)

**Example : 1** Solve $y_t-2y_{t-1}=3$ with $y_0=5$

**Solution :** Comparing the given equation $y_t-2y_{t-1}=3$ with the general first order difference equation

$y_{t+1}+ay_t=b$

We get a=-2

$\therefore a \neq -1$

and b=3

So, the solution will be given by

$$y_t = \left( y_0 - \frac{b}{1+a} \right) (-a)^t + \frac{b}{1+a} (a \neq -1)$$

Now, putting the value of a, b and $y_0$ in the above equation, we get,

$$y_t = \left( 5 - \frac{3}{1-2} \right) (2)^t + \frac{3}{1-2}$$

$$= \left( 5 - \frac{3}{(-1)} \right) (2)^t + (-3)$$

$$\therefore y_t = 8(2)^t - 3$$

**Example : 2** Solve $2y_{t-1}-y_t=6$, with $y_0=7$

**Solution :** Dividing the given equation by 2 we get,

$y_{t+1}-0.5\ y_t=3$ (A)

Now; comparing the equation (A) with the general first order difference equation

$y_{t+1}+ay_t=b$

We get a=-0.5

i.e. $a \neq -1$

and b=3

So, the solution will be given by

$$y_t = \left( y_0 - \frac{b}{1+a} \right) (-a)^t + \frac{b}{1+a} (a \neq -1)$$

Now, putting the value of a,b and $y_0$ in the above equation, we get

$$y_t = \left( 7 - \frac{3}{1-0.5} \right) (-0.5)^t + \frac{3}{1-0.5}$$

$$= \left( 7 - \frac{3}{0.5} \right) (-0.5)^t + 6$$

$$= (7-6)(-0.5)^t + 6$$

$$= 1(-0.5)^t + 6$$

$y_t = (-0.5)^t+6$

**Example : 3** Solve $y_{t+2}-y_{t+1}=10$ with $y_0=10$

**Solution :** Comparing the given equation $y_{t+2}-y_{t+1}=10$

with the general first order difference equation $y_{t+1}+ay_t=b$

We get a=-1

b=10

Since, a=-1, the solution of the difference equation is given by

$y_t=y_0+bt$

Now, putting the value of b and $y_0$ in the about equation, we get,

$y_t=10+10t$



**Check your progress**

**Q.5.** Solve $y_{t+1}+3y_t=2$ with $y_0=5$

**Q.6.** Solve $y_{t+1}-y_t=10$, with $y_0=5$

**Q7.** Solve $3y_{t+1}-y_t=6$, with $y_0=1$

**Q8.** Solve $y_{t+1}-10y_t=3$ with $y_0=1$



## 15.9 LET US SUM UP

- A differential equation is a mathematical relation that relates some function with its derivative. A differential equation expresses the rate of change of a variable as a function of the variable.

- While the term 'order' of the differential equation refers to the highest derivative, degree of the differential equation refers to the highest power attained by the derivative of the highest order in the equation.

- The first order linear differential equation will take the form

$$\frac{dy}{dt}+u(t)y=w(t)$$

y, u and w are continuous functions of t. However there is no restriction on t. u and w may even be constants.

- If u is a constant and w is zero we have

$\frac{dy}{dt}$ + a y = 0 Such a differential equation is said to be homogenous.

The solution is $y(t) =Ae^{-at}$ General solution

Or $y(t) =y(0)\, e^{-at}$ Definite solution

- The form of the non homogenous linear differential equation is as follows

$\frac{dy}{dt}$ +ay =b

- The general solution of the complete equation or the nonhomogenous equation thus can be written as

  $y(t) = y_c + y_p$

  $= Ae^{-at} + b/a \qquad (a \neq 0)$

  The definite solution is

  $$y(t) = \left[ y(0) - \frac{b}{a} \right] e^{-at} + \frac{b}{a}$$

- For $\frac{dy}{dt}$ +u(t)y=0

  $$y(t) = e^{\ln y} = e^{-c} e^{-\int u(t)dt}$$
  $$= Ae^{-\int u(t)dt}$$

- For $\frac{dy}{dt} + u(t)y = w(t)$

  The general solution of the equation is

  $$y(t)e^{-\int u(t)dt}(A + \int w(t)e^{\int u(t)dt}dt)$$

- A differential equation

  Mdy +Ndt =0 is exact if and only if there exists a function F( y,t) such that

  M = $\frac{\partial F}{\partial y}$ and N = $\frac{\partial F}{\partial t}$

- When the differential equation is exact, we have dF(y,t)=0.

Its general solution should be in the form F(y,t)=c where c is an arbitrary constant.

- A general procedure of solving exact differential equation is also discussed.
- Application of differential equation in market adjustment, compound interest, population growth and Keynesian income determination has been discussed.



## 15.10 FURTHER READING


1) Allen, R.G.D. (1974). *Mathematical Analysis for Economists.* London : Macmillan Press and ELBS.

2) Baruah, S. (2011). *Basic Mathematics and Its Applications in Economics.* Macmillan.

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## 15.11 ANSWERS TO CHECK YOUR PROGRESS

**Ans to Q. No. 1.**

The word 'order' refers to the order of the highest derivatives (or

differentials) appearing in the differential equation.

i. First
ii. First
iii. First
iv. Second
v. First
vi. Fourth
vii. First
viii. First

**Ans to Q. No. 2.**

Partial differential equation has partial derivatives and ordinary differential equation has only a simple derivative. The variable is a function of one variable only.

i. Ordinary
ii. Partial

**Ans to Q. No. 3.**

First You have to identify the type of differential equation. Compare it with the standard formulae. We can obtain the solutions by substituting the initial values and known constants.

$\frac{dy}{dt}$ + ay =b

The definite solution is

$$y(t) = \left[ y(0) - \frac{b}{a} \right] e^{-at} + \frac{b}{a}$$

consider (i) $\frac{dy}{dt}$ + y =4; y(0) = 0 a= 1; b=4

substituting these values y(t) =[ 0-4/1] $e^{-t}$ + 4/1 =-4$e^{-t}$ + 4 =4(1-$e^{-t}$

(ii) can be rewritten as $\frac{dy}{dt}$ + 2y =5/3 ; a=2, b=5/3 substituting these values

$y(t) = [\ 0\ -5/3]\ e^{-2t} + 5/6 = -\frac{5}{3}e^{-2t} + 5/6$

(iii) a=-5; b=0; $y(t) = 6e^{5t}$

(iv) a= 0 ; b =23 ; y(t) = 1 +23t

**Ans to Q. No. 4.**

We examine $\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y}$ to examine exactness of the differential equations:

i. $M = 2yt^3$ $N = 3y^2t^2$

$\frac{\partial M}{\partial t} = 6yt^2$ $\frac{\partial N}{\partial y} = 6yt^2$ ; $y = (c/t^3)^{1/2}$

ii. $\frac{\partial M}{\partial t} = e^{t+y} = \frac{\partial N}{\partial y}$; $e^{t+y} + t^3 + y^2 = C$

iii. M =x N =$(-y-x^2)$

$\frac{\partial M}{\partial x} = 1$; $\frac{\partial N}{\partial y} = -1$

$\frac{\partial M}{\partial t} \neq \frac{\partial N}{\partial y}$ This is not exact.

**Ans to Q. No. 5.** $y_t = \frac{9}{2}(-3)^t + \frac{1}{2}$

**Ans to Q. No. 6.** $Y_t = 5 + 10t$

**Ans to Q. No. 7.** $y_t = -2(\frac{1}{3})^t + 3$

**Ans to Q. No. 8.** $y_t = \frac{4}{3}(10)^t - \frac{1}{3}$

## 15.12 MODEL QUESTIONS

**Q. 1.** In a Keynesian income determination model, given c =50 + 0.6 y and dy/dt =0.5(E-y) I =22 and y(0)=205, find the value of y(10).

**Q. 2.** If the demand and supply functions in a competitive market are

$Q_d$=35 - 0.5 P and $Q_s$ = -4 +0.8P

and the rate of adjustment when the market is out of equilibrium is

dp/dt= 0.25 ($Q_d$ - $Q_s$). Derive and solve the time path P(t). It is also given that p(0)=37

**Q. 3.** The annual growth rate of population of India is 1.77 percent. The population in 2011 is 1210854977. What will be the population in 2021?

**Q. 4.** If the number of rhinos in a reserve increased continually from 600 in 1991 to 1500 in 2011,what was the annual growth rate ?

**Q. 5.** Given an initial investment of Rs 1,00,000 and real interest rate at 5% which is compounded continuously, how long it will take to accumulate Rupees one crore ?

**Q. 6.** Current stock of mineral M is 250 million tonnes. If it is continuously used up at an annual rate of 5% what amount of M will remain after 15 years ?

**Q. 7.** A parent wants to ensure that his child will have a fund of Rs 5,00,000 to finance his university study. He has 10 years time and he wants to make a lumpsum investment now. How much he should invest if the investment is expected to grow continuously at an annual rate of 6%?

**Q. 8.** Solve : a) $y_{t+1} - 5y_t = 12$ with $y_0 = 10$

b) $y_t - y_{t-1} = 4$ with $y_0 = 15$

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## References (For all Units of The Course)


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