GBA S2 05-02

Business Statistics

SEMESTER 2

BBA

BLOCK 2

KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

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BLOCK INTRODUCTION

This is the second block of the course "Business Statistics". The block consists of seven units.

In Unit 7, we will focus on the concepts of probability. Here we will discuss different theorems on probability.

In Unit 8, we discuss the concept conditional probability, Multiplication theorem of probability, Law of probability and Baye's theorem.

In Unit 9, we will discuss random variables and its probability distribution. Unit 10, gives us a broad overview of theoretical probability distributions relating to discrete variables.

In Unit 11, we will discuss about index numbers, types of simple index numbers, construction of simple and weighted index numbers etc.

In Unit 12, we will discuss time series and its various components.

Unit 13 is about sampling. Here we will discuss certain concepts like, sample, population, sampling, random and non-random sampling, and errors in sampling.

While going through a unit, you will notice some along-side boxes, which have been included to hep you know some of the difficult, unseen terms. Some 'ACTIVITY' (s) has been included to help you apply your own thoughts. Again, we have included some relevant concepts in 'LET US KNOW' along with the text. At the end of sections, you will get 'CHECK YOUR PROGRESS' questions. These have been designed to self-check your progress of study. It will be better if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with 'ANSWERS TO CHECK YOUR PROGRESS' given at the end of each unit.

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UNIT 8: Conditional Probability **Page : 168-188**

Conditional probability; Multiplication theorem on probability; Total probability; Baye's theorem.

UNIT 9: Random variables and its probability distribution **Page : 189-201**

Random Variable: Definition or Random Variable, Discrete Random Variable, Continuous Random Variable; Probability Distribution: Probability Mass Function, Probability Density Function; Mathematical Expectation and Variance: Mathematical Expectation, Properties of Mathematical Expectation, Variance of Random Variable, Properties of Variances, Illustrated Examples; Moments : Raw Moments, Central Moments.

UNIT 10: Discrete Probability Distribution **Page : 202-228**

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UNIT 12: Time series **Page : 273-290**

Definition of Time Series, Importance of Time Series Analysis, Components of a Time Series, Methods of Measuring Secular Trend, Estimation of the Trend by the Method of Moving Average.

UNIT 13: Basic Concepts in Sampling **Page : 291-308**

Definition of Population, Parameter and Sampling; Random and non- random sampling; Sampling with and without replacement; Errors in Sampling.

UNIT 7: PROBABILITY

UNIT STRUCTURE

7.1 Learning Objectives

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7.3 Random Experiment

7.4 Classical Definition of Probability

7.5 Sample Space

7.6 Axiomatic Definition of Probability

7.7 Elementary Theorems on Probability

7.8 Let Us Sum Up

7.9 Further Readings

7.10 Answer to Check Your Progress

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7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to–

- explain the concept of random experiment
- discuss the concept of events
- define probability
- explain the concept of sample space
- explain limitations of classical and statistical definition and need for axiomatic definition
- calculate probabilities of events
- define some elementary theorems on probability
- explain the concept of independent events
- evaluate probabilities of independent events.

7.2 INTRODUCTION

The subject probability began in the 17th century through efforts of some mathematician (to quote Fermat and Pascal) to answer questions concerning games of chance. It was not untill the 20th century that a rigorous

mathematical theory based on axioms, definitions and theorems was developed.

All phenomena in nature and society are either deterministic or probabilistic. Till the 18th century only deterministic phenomena were amenable to scientific treatment, but during the last two centuries a great deal of research has been carried out on phenomena involving elements of chance.

As time progresses, theory of probability found its way into many applications, not only in mathematical sciences also in wider fields like agriculture, business, medical science etc. All these applications contributed to further development of the theory.

The term probability or chance we use in our daily talk to indicate the degree of personal belief. We often say "there is no probability of raining today". The statement just reflects the speaker's belief without any base on experiment or trial. But in statistics the term is used in different way which is more reliable than the former one.

The theory of probability can be discussed in two different stages. The first one is classical approach. In this approach, probability is defined as the ratio of number of favourable cases when the cases are exhaustive, equally likely and mutually exclusive. The second one is modern approach. It is based on concept of sample space, which is again based on set.

7.3 RANDOM EXPERIMENT

If we perform certain experiment under identical conditions we expect to arrive at results that are essentially same provided we can control the value of the variable that may effect the outcome of the experiment. But in many cases we are not in a position to control the value of some variables. As a result, though under identical conditions experiments are performed, the result will vary from one experiment to the others. These experiments are called random experiments.

Examples of random experiments :

1. Let the experiment be tossing of a coin. The result will be either

head (H) or tail (T). But we cannot exactly predict what the result will be. The result will depend on chance. There are various factors that will influence the result and all these factors cannot be controlled. This is an example of random experiment.

If coin is tossed twice, the result will be {(H,H), (H,T), (T,H), (T,T)}. This is also example of random experiment.

2. Let us consider the random experiment of throwing a six faced cubical die. As the die is perfect, we are sure that one of the faces will come up with one of the numbers, 1, 2, 3, 4, 5, 6.

Event

An event is a collection of possible outcome of a random experiment. An event may consist of a single outcome or a group of outcomes taken together. If it consists of single outcome then it is called elementary event. An event will be denoted by capital letter of English alphabet.

A series of events $A_1, A_2 \ldots\ldots\ldots A_K$ will be called exhaustive if at least one of them is sure to happen in any trial of random experiment. For example, when a coin is tossed, either head (H) or tail (T) must occur. Hence the event is exhaustive.

Two events A and B are said to be mutually exclusive, if the occurrence of one precludes the occurrence of the other. In other words the two events cannot occur simultaneously. For example in casting a die if 6 occurs then other numbers 1, 2, 3, 4 and 5 cannot occur. Hence it is an example of mutually exclusive events.

A series of events $A_1, A_2 \ldots\ldots\ldots A_K$ are said to be equally likely, if one of them cannot be expected to occur in preference to the others in a single trial of random experiment. For example, in tossing of a coin each face is equally likely to occur.

7.4 CLASSICAL DEFINITION OF PROBABILITY

Consistent with the conditions of a random experiment let 'n' be exhaustive, mutually exclusive and equally likely cases and 'm' of them

are favourable to an event A, then probability of A is defined by 'm/n' and is denoted by 'P (A)' or 'p' so that

$$P(A) = \frac{m}{n}$$

Since $0 \leq m \leq n$, P(A) lies between 0 and 1.

1. When there is no elementary event favourable to A, then

   P(A) = 0, since m = 0.

2. When all events are favourable to A then P (A) = 1.

Example 1 : Let a six faced die be cast. Find the probability that (i) the number shown on the die is odd (ii) number shown on the die is divisible by 2 and 3 (iii) number shown on the die is divisible by 2 and 5.

Solution: When die is cast there are six cases 1, 2, 3, 4, 5 or 6. These are exhaustive, mutually exclusive and equally likely.

(i) Of the cases the favourable cases are 1, 3, 5. Let the event be A.

$$P(A) = \frac{3}{6} = \frac{1}{2}$$

(ii) In the second case let the event be B then there is only one favourable case viz. 6.

$$P(B) = \frac{1}{6}$$

(iii) In this case let the event be C. Out of the numbers 1, 2, 3, 4, 5, 6 no one is divisible by 2 and 5 simultaneously. Hence no case is favourable to C.

P (C) = 0

Example 2 : A card is drawn at random from a well shuffled pack of cards. Find the probability that it is (i) a king (ii) a queen of spade (iii) a heart.

In a pack of cards there are 52 cards and any card can be drawn and hence total number of cases is 52.

(i) Let A denote the first event. There are 4 kings. So number of favourable cases for drawing a king is 4.

$$P(A) = \frac{4}{52} = \frac{1}{13}$$

(ii) Let B denote the second event. There are only one queen of spade.

$$P(B) = \frac{1}{52}$$

(iii) Let C denote the third event. There are 13 hearts. So favourable cases to C is 13.

$$P(C) = \frac{13}{52} = \frac{1}{4}$$

Limitations of classical definitions :

The classical definition requires that n is finite. But there are instances where it may be infinite Secondly we assume elementary events to be equally likely. This is also need not be true always. For example, in tossing of a coin, there are two events head and tail. Unless the coin is unbiased they may not be equally likely. To get a perfectly unbiased coin is a very difficult condition. Hence probability of obtaining a head with any coin cannot be obtained from classical definition.

Statistical definition :

To overcome the difficulties encountered in classical definition, a large number of trials of random experiment is considered. Let m be the number of occurrences of an event A associated with n independent trials of random experiment. The probability of the event A is defined by

$$P(A) = \underset{n \to \infty}{Lim} \frac{m}{n}$$

Example : In tossing of a coin 1000 times head comes up 537 times. The probabiity of head coming up is

$$\frac{537}{1000} = 0.537$$

This definition of probability also has drawback, because the large number involved in definition is vague.

As both the definitions suffer from certain defects another approach has been put forward to overcome all defects. This approach is known as axiomatic approach of probability.

Before considering this approach we shall introduce a concept, which is as Set. We shall briefly state the necessary results here.

7.5 SAMPLE SPACE

A set S that consists of all possible outcome of a random experiment is called a sample space. Each outcome is a sample point.

Example : Let us consider the result of casting a six-faced die. The outcome will be the appearance of one and only one of the numbers 1, 2, 3, 4, 5, 6, on the upper most face of the die. In other words the six possibilities are such that no two or more of them can occur simultaneously and at least one of them must occur. The space S is {1, 2, 3, 4, 5, 6}. If A denote the event "occurrences of even number" then A = {2, 4, 6} which is a subset of S.

Let us generalise the concept.

Let all possible outcomes of some particular experiment be e_1, e_2 ..... e_n which are such that no two or more of them can occur simultaneously and exactly one of them must occur when the experiment is performed. Any set associated with an experiment which satisfies the above mentioned two properties is called a sample space. The elements or points of the sample space associated with an experiment are called elementary events of the experiment.

If a sample space has finite number of points then it is a finite sample space. If it has many points as there are natural numbers then it is a countably infinite sample space. If it has as many points as there are in $0 \leq x \leq 1$ then it is an infinite sample space.

Examples :

1. If a coin is tossed twice, the four possible results are {HH, HT, TH, TT} which is the sample space. The event that only one head comes up will be {HT, TH} which is a subset of in sample space.
2. From an urn containing 4 balls of different colours red (R), Blue (B), White (W) and Green (G) draw two balls simultaneously. The sample space for this experiment is {RB, RW, RG, BW, BG, WG}.

Note : (1) RB and BR represent the same outcome because we draw balls simultaneously.

Instead of balls drawing simultaneously, let us now draw balls in successions with replacement. The sample space will be {RR, RB, RW,

RG, BR, BB, BW, BG, WR, WB, WW, WG, GR, GB, GW, GG}.

(2) In this case RB and BR are not same.

7.6 AXIOMATIC DEFINITION OF PROBABILITY

Let S be a sample space. To each event A in a class C of events, a real number P(A) is associated.

P (A) is called probability of the event A if the following axioms are satisfied.

Axiom 1. For any event A in the class C, $P(A) \geq 0$.

Axiom 2. For certain event S in class C, P (S)=1.

Axiom 3. For any number of mutually exclusive events A_1, A_2 ..... in the class C,

$$P(A_1 \cup A_2 \cup .....) = P(A_1) + P(A_2) + ...$$

In particular for two events, A_1 and A_2

$$P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

Let S be a sample space {e_1, e_2 ..... e_n) and P ({e_i})=p_i, i=1, 2 ..... n such that $0 \leq P_i \leq 1$ and $\sum_{i=1}^{n} P_i = 1$

Now $P(\{e_i\}) = \frac{1}{n}$

If E be any event consisting of $r(1 \leq r \leq n)$ elementary events then

$$P(E) = \frac{1}{n} + \frac{1}{n} + ...\text{r times}$$

$$= \frac{r}{n}$$

$P(E) = \frac{n(E)}{n(S)}$ where n (E) = number of events in E and n (S) = number of events in S.

7.7 ELEMENTARY THEOREMS ON PROBABILITY

Some elementary theorems on probability :

Let S = {e_1, e_2 ..... e_n} be a sample space of an experiment.

Theorem 1. P (S) = 1.

Proof : By definition,

$P(S) = P(\{e_1\}) + P(\{e_2\}) + \ldots\ldots\ldots + P(\{e_n\})$

$= p_1 + p_2 + \ldots\ldots\ldots + p_n$

$= 1$

This theorem is very trivial.

Theorem 2. If A and B are two events such that $A \subseteq B$ then $P(A) \leq P(B)$

Proof : $A \subseteq B \Rightarrow B$ contains all the elementary events of A and possibly a few more.

So, the probability of B is equal to the probability of A plus the sum of the probabilities attached to those elementary events which belongs to B but not to A i.e. to B-A. Since the probability is a non-negative number,

$$P(A) \leq P(B)$$

Theorem 3. For any event A, $0 \leq P(A) \leq 1$

By axiom 1, $P(A) \geq 0$ ..... (i)

For any event A, $A \subseteq S$

By theorem 2, $P(A) \leq P(S) = 1$ (by theorem 1) ..... (ii)

$\therefore$ From (i) and (ii)

$$0 \leq P(A) \leq 1$$

Theorem 4. Probability of impossible event is zero.

Let $\varnothing$ denote the impossible event

We know, $S = S \cup \varnothing$

$\Rightarrow P(S) = P(S \cup \varnothing) = P(S) + P(\varnothing)$ (by axiom 2)

$\Rightarrow P(\varnothing) = 0$

Let us examine the converse of theorem 1 and theorem 4.

For this let us consider the sample space.

S = {HH, HT, TH, TT}

Let us consider

$$P(TH) = \frac{1}{2} = P(HH)$$

$$P(HT) = 0 = P(TT)$$

If A = {HT, TT} then P(A) = 0, but A is not impossible event.

Hence converse of theorem 4 is not true.

Let B = {TH, HH} then P(B)=1, but B is not sure event.

Hence converse of theorem 1 is not true.

Theorem 5. $P(A_2-A_1) = P(A_2)-P(A_1)$

We know, $A_2 = A_1 \cup (A_2 - A_1)$ where A_1 and A_2 are mutually exclusive.

$$P(A_2) = P[A_1 \cup (A_2 - A_1)]$$
$$= P(A_1) + P(A_2 - A_1)$$
$$\Rightarrow P(A_2 - A_1) = P(A_2) - P(A_1)$$

Theorem 6. If A^C is the complement of A then

$P(A^C) = 1-P(A)$

We know, $A \cup A^C = S$ and $A \cap A^C = \varnothing$

$A \cap A^C = \varnothing \Rightarrow$ A and A^C are mutually exclusive

Hence

$A \cup A^C = S \Rightarrow P(A \cup A^C) = P(S)$

$$\Rightarrow P(A) + p(A^C) = 1$$
$$\Rightarrow P(A^C) = 1 - P(A)$$

Independent Events : If the probability of B occuring is not affected by the occurence or non-occurence of A then A and B are said to be independent events.

In this case $P(A \cap B)=P(A).\ P(B)$

Conversely of $P(A \cap B) = P(A).\ P(B)$ then A and B are independent.

Calculation of probabilities :

All these probability theories are used in business to evaluate financial and decision making risk. Probability is used to improve business performance.

Example 1: In a single cast with two dice find the probability of throwing (i) two aces (ii) doublets (iii) five-six

Solution: Any face of either die may turn up. So sample space consists of 6x6=36 sample points.

(i) Two aces means (6, 6) only one point in sample space.

Probability for the $aces = \frac{1}{36}$

(ii) For doublets, the point are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6), which are six in numbers.

Probability for the $doublets = \frac{6}{36} = \frac{1}{6}$

(iii) For five-six, we have two points(5,6) and (6,5).

Probability for five-six = 2/36= 1/18

Example 2. Two dice one red and other blue are thrown. Find the probability of the event E where E is

(i) Sum of the number shown by the two dice is 7.

(ii) Number on the second dice is greater than the number on the 1st.

Solution: The sample space consists of 6x6=36 elementary events.

(i) E_1={(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}

$$P(E_1) = \frac{6}{36} = 1/6$$

(ii) E_2={(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)}

$$P(E_3) = \frac{15}{36} = 5/12$$

Example 3. A coin is tossed three times in succession. Find the probability of the event that have

(i) two or more heads

(ii) number of heads equal to the number of tail

(iii) exactly one head and two tails.

Solution: The sample space is {(HHH), (HHT), (HTH), (THH), (HTT), (THT), (TTH), (TTT)}

(i) E_1 = {(HHH), (HHT), (HTH), (THH)}

$$P(E_1) = 4/8 = 1/2$$

(ii) E_2 = {(HHH), (TTT)}

$$P(E_2) = \frac{2}{8} = \frac{1}{4}$$

(iii) E_3 = {(HTT), (THT), (TTH)}

$$P(E_3) = 3/8$$

Example 4: From a pack of usual playing cards, a card is drawn at random and is noted. Calculate the probabilities of the following events :

(i) the drawn card is either a spade or a club.

(ii) the drawn card is a picture card (ace is include).

(iii) the drawn card is of denomination less than 10 and greater than 5.

Solution: The sample space consists of 52 elementary events corresponding to each of the 52 cards that can be drawn from the pack.

(i) Let, E_1 be the event that the drawn card is a spade or a club. Then E_1 consists of (13+13)=26 elementary events.

$$P(E_1) = \frac{26}{52} = \frac{1}{2}$$

(ii) E_2 be the event that drawn card is a picture card. Here aces are taken as picture card. Then E_2 contains 4x4 = 16 elementary events.

$$P(E_2) = \frac{16}{52} = \frac{4}{13}$$

(iii) E_3 be the event such that denomination lies between 5 and 10. E_3 consists of 4 x 4 = 16 elementary events.

$$P(E_3) = \frac{16}{52} = \frac{4}{13}$$

Example 5: Two drawings, each of 3 balls, are made from a bag containing 5 white and 8 black balls, the balls not being replaced before the second trial. Find the chance that the first drawing will give 3 white and the second 3 black balls.

Solution: Let E_1 be the event of drawing 3 white balls and E_2 be the event of drawing 3 black balls.

The sample space consists of 13_{C_3} elementary events for E_1.

$$P(E_1) = 5_{C_3} / 13_{C_3}$$

$$= \frac{5.4}{2.1} / \frac{13.12.11}{3.2.1}$$

$$= \frac{5.4}{2.1} / \frac{3.2.1}{13.12.11}$$

$$= \frac{5}{143}$$

For E_2 the sample space consists of 10_{C_3} elementary events.

$$P(E_2) = 8_{C_3} / 10_{C_3}$$

$$= \frac{8.7.6}{3.2.1} / \frac{10.9.8}{3.2.1}$$

$$= \frac{8.7.6}{3.2.1} x \frac{3.2.1}{10.9.8}$$

$$= \frac{7}{15}$$

Since E_1 and E_2 are independent,

$$P(E_1E_2) = P(E_1).P(E_2)$$

$$= \frac{5}{143}.\frac{7}{15}$$

$$= \frac{7}{429}$$

Example 6: Two drawings, each of 3 balls, are made from a bag containing 5 white and 8 black balls, the balls being replaced before the second trial find the chance that the first drawing will give 3 white, and the second 3 black balls.

Solution: Let, E_1 be the event of drawing 3 white balls and E_2 be the event of drawing 3 black balls.

Since balls are replaced these sample space for both the events will contained same number of elementary events.

E_1 contains 5_{C_3} elementary events and E_2 contains 8_{C_3} elementary events.

$$P(E_1) = 5_{C_3}/13_{C3}$$

$$= \frac{5.4}{2.1}\Big/\frac{13.12.11}{3.2.1}$$

$$= \frac{5}{143}$$

$$P(E_2) = 8_{c_3}/13_{c_3} = \frac{8.7.6}{3.2.1}\Big/\frac{13.12.11}{3.2.1}$$

$$= \frac{8.7.6}{13.12.11}$$

$$= \frac{28}{143}$$

Since E_1 and E_2 are independent,

$$P(E_1 \cap E_2) = P(E_1).P(E_2)$$

$$= \frac{5}{143}.\frac{28}{143} = \frac{140}{20449}$$

Example 7 : A bag contains 5 white and 7 black balls; if two balls are drawn what is the chance that one is white and the other black?

Solution: The sample space consists of 12_{C_2} =66 elementary events.

Let E_1 be the event of drawing first white and second black balls.

$$P(E_1) = \frac{5}{12} \cdot \frac{7}{11} = \frac{35}{132}$$

Let, E_2 be the event of drawing first black and second white balls.

$$P(E_2) = \frac{7}{12} \cdot \frac{5}{11} = \frac{35}{132}$$

E_1 and E_2 are mutually exclusive. Therefore

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

$$= \frac{35}{132} + \frac{35}{132}$$

$$= \frac{70}{132}$$

$$= \frac{35}{66}$$

Example 8 : Four cards are drawn from a pack of 52 cards. What is the chance that one of each suit is drawn ?

The sample space consists of 52_{C_4} elementary events.

Let E be the event of drawing one card from each suit.

$$P(E) = \frac{13_{C_4} \times 13_{C_4} \times 13_{C_4} \times 13_{C_4}}{52_{C_4}}$$

$$= \frac{13 \times 13 \times 13 \times 13}{\frac{52.51.50.49}{4.3.2.1}}$$

$$= \frac{28561}{270725}$$

Example 9 : From an ordinary pack of cards a card is drawn at random. What is the probability that it is

(i) 3 of clubs or 6 of diamonds.

(ii) any suit except heart.

(iii) a jack or a queen.

(iv) either a spade, or a heart or not a picture card.

Solution: (i) Let, A denote 3 of clubs, and B denote 6 of diamonds.

Now $A \cap B = \varnothing$

$$P(A \cup B) = P(A) + P(B) = \frac{1}{52} + \frac{1}{52} = \frac{1}{26}$$

(ii) If H denote heart that H' will denote any suit except heart.

$$P(H) = \frac{13}{52} = \frac{1}{4}$$

$$P(H') = 1 - P(H)$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4}$$

(iii) Let, A denote that the card is jack & B that is queen. Now

$$A \cap B = \varnothing$$

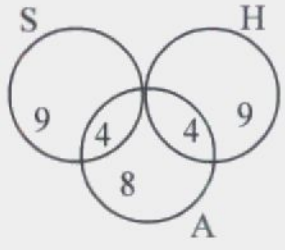
$$P(A \cup B) = P(A) + P(B) = \frac{4}{13} + \frac{4}{13} = \frac{2}{13}$$

(iv) Let, S denotes spade, H denotes heart and A denotes picture card. S and H are mutually exclusive but S and A, H and A are not mutually exclusive.

$$P(S) = \frac{13}{52}, P(H) = \frac{13}{52} and P(A) = \frac{16}{52}$$

$$P(A') = \frac{36}{52}, P(S \cap A') = \frac{9}{52}$$

$$P(H \cap A') = \frac{9}{52}$$



$$P(S \cup H \cup A') = P(S) + P(H) + P(A') - P(S \cap A')$$
$$- P(H \cap A') - P(S \cap H) + P(S \cap H \cap A')$$

$$P(S \cup H \cup A) = \frac{13}{52} + \frac{13}{52} + \frac{36}{52} - \frac{9}{52} - \frac{9}{52}$$

$$[\because P(S \cap H) = 0, P(S \cap H \cap A') = 0]$$

$$= \frac{62}{52} - \frac{18}{52} = \frac{44}{52}$$

Example 10 : Through certain election 3 persons are to be chosen out of five candidates A, B, C, D, E with equal chance being elected. Find the probability that

(i) A and B are elected or B and C are elected.

(ii) E is elected or C and D are elected.

Solution: Let E_1 be the event that A and B are elected and E_2 be the events that B and C are elected.

S = {ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE}

(i) E_1 = {ABC, ABD, ABE}

E_2 = {ABC, BCD, BCE}

$$E_1 \cap E_2 = \{ABC\}$$

$$P(E_1) = \frac{3}{10}, P(E_2) = \frac{3}{10}, P(E_1 \cap E_2) = \frac{1}{10}$$

$$\therefore P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$P(E_1 \cup E_2) = \frac{3}{10} + \frac{3}{10} - \frac{1}{10} = \frac{5}{10} = \frac{1}{2}$$

Let E_3 be the event that E is elected and E_4 be the event that B and C are elected.

(ii) E_3 = {ABE, ACE, ADE, BCE, BDE, CDE}

E_4 = {ACD, BCD, CDE}

$$E_3 \cap E_4 = \{CDE\}$$

$$P(E_3) = \frac{6}{10}, P(E_4) = \frac{3}{10}, P(E_3 \cap E_4) = \frac{1}{10}$$

$$\therefore P(E_3 \cup E_4) = P(E_3) + P(E_4) - P(E_3 \cap E_4)$$

$$P(E_3 \cup E_4) = \frac{6}{10} + \frac{3}{10} - \frac{1}{10} = \frac{8}{10} = \frac{4}{5}$$

Odds for and odds against :-

Let E be an event. Sometimes we say that the odds for E are a : b. By this we mean

$$P(E) = \frac{a}{a+b}$$

Thus given the odds for an event, the probability of an event can be calculated. Conversely given the probability of an event, the odds for the event can determined. For example

if P(E)=.7 then

$$P(E) = \frac{7}{10} = \frac{7}{7+3}$$

Hence odds for E are 7 : 3.

If odds for E is a : b then odds against E is b : a.

Composite event : When an event, comprises of more than one simple event then the event is called a composite event.

The event $A \cup B$ comprises of the sample points in the whole region bounded by A and B. $A \cup B$ is also denoted by A+B. Similarly the event $A \cap B$ is denoted by AB.

Example 11 : From a bag containing 4 white and 5 black balls a man draws 3 balls at random; what are the odds against these being all black ?

Solution: The total number of ways in which 3 balls can be drawn is ${}_9C_3$, and the number of ways of drawing 3 black balls is ${}_5C_3$; therefore the chance of drawing 3 black balls

$$= \frac{{}_5C_3}{{}_9C_3} = \frac{5.4.3}{9.8.7} = \frac{5}{42}$$

Thus the odds against the event are 37 to 5.

Example 12 : A party of ten take their seats at a round table. What are the odds against two specified persons (A, B) sitting together?

Solution: A having taken his place, B has a choice of 9 places, 2 of which are next to A, hence the odds against B sitting next to A are as 7:2.



CHECK YOUR PROGRESS

Q1. In a single throw with two dice, find the chances of throwing (i) five, (ii) six.

Q2. From a pack of 52 cards two are drawn at random; find the chance that one is a king and the other a queen.

Q3. A bag contains 5 white, 7 black and 4 red balls; find the chance that three balls drawn at random are all white.

Q9. If four coins are tossed, find the chance that, there should be two heads and two tails.

Q10. One of two events must happen : given that the chance of the one is two-thirds that of the other, find the odds in favour of the other.



ACTIVITY

You may perform some random experiments like tossing of coin, casting of a die, drawing cards or some other practical field you come across and calculate probabilities. You may verify the accuracy of the result by repeating the experiment.



7.8 LET US SUM UP

In this unit we have discussed the concept of random experiment.

- In certain experiments, value of the variables cannot be controlled. Therefore, the results of these experiments are not the same. These are known as random experiments.
- We have discussed the concept of probability.
- Ultimately we have given axiomatic definition. We have discussed the concept of probability and their computational procedure. The discussion is extended to composite events. All are discussed through rules and numerical computation.



7.9 FURTHER READINGS

1) Medhi, J. (2005) Statistical Method, New Age International Publishers.

2) Murray R. Spiegel. Jhon Schiller, R. Alu Srinivasan (2005). Probability and statistics. Schaum's Outline Series.

3) Mukherjee Kalyan Kumar (1993) Probability and Statistics and New Central Book Agency.

4) Gupta S.K. (1968) Elements of probability NCERT.



7.10 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No1: $\frac{1}{9}$

Ans to Q No.2: $\frac{5}{36}$

Ans to Q No. 3: $\frac{1}{56}$

Ans to Q No.4: $\frac{3}{8}$

Ans to Q No.5: 2 to 3



7.11 MODEL QUESTIONS

Q1: There are three events A, B, C one of which must, and only one can, happen; the odds are 8 to 3 against A, 5 to 2 against B; find the odds against C.

Q2: If two balls are drawn from a bag containing 2 white, 4 red and 5 black balls, what is the chance that

i) they are both red ?

Ii) one is red and the other black ?

Q3: Two coins are tossed simultaneously. Find the probability of getting i) exactly two heads or head is the first coin or both, ii) getting head in first and tail is the 2nd coin or both.

Q4: A card is drawn at random from a pack of playing cards. Find the probability that it is

i) a ten or a spade.

ii) neither a 4 nor a club.

iii) either a black card or an ace or both.

iv) either a spade or a picture card.

Q5: Two dice are thrown simultaneously. Find the probability that the sum is

i) divisible by either 2 or 5 but not by both.

ii) does not exceed 6.

iii) greater than 10.

iv) divisible by 2 and 5 both.

Q 6: If A and B are any events prove that

$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$

when $P(A \cup B) = P(A) + P(B)$

Q 7: If $P(A) = \frac{1}{2}, P(B) = \frac{1}{4}, P(A \cup B) = \frac{3}{8}$ find P(AB)

Q 8: There are three events A, B, C one of which must and only one can happen. The odds are 8 to 3 against A, 5 to 2 against B. Find the odds against C.

Q 9: A experiment consists of drawing 3 cards in succession from a pack of cards. A_1 is the event king on first draw, A_2 king on the 2nd and A_3 king is the 3rd. State the meaning of

i) $P(A_1 \cap A')$ (ii) $P(A_1 \cup A_2)$ (iii) $P(A_1' \cup A_2')$

iv) $P(A' \cap A_2' \cap A_3')$ v) $P[(A_1 \cap A_2) \cup (A_2' \cap A_3)]$

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UNIT 8: CONDITIONAL PROBABILITY

UNIT STRUCTURE

8.1 Learning Objectives

8.2 Introduction

8.3 Conditional probability

8.4 Multiplication theorem on probability

8.5 Total probability

8.6 Baye's theorem

8.7 Let Us Sum Up

8.8 Further Readings

8.9 Answers to check your progress

8.10 Model Questions

8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define conditional probability
- know multiplication theorem of probability
- define law of probability
- know Baye's theorem.

8.2 INTRODUCTION

In unit 7, we discussed about probbility and some important theorems related to probability. In this unit, we will discuss the concept of conditional probability. We will also discuss dependent and independent events. Multiplication theorems for dependent and independent events will also be discussed in this unit. Finally, we will disuss the law of probability and Baye's theorem.

8.3 CONDITIONAL PROBABILITY

In unit 7, we have discussed the methods of finding the probability of events. Supose, we have two events from the same sample space in a

random experiment, does the occurrence of one of the events affect the probability of the other event? Let us see this situation with the following examples

1. Consider a random experiment of throwing a die.

Let S be the sample space and A be the event of of getting an odd numbers.

$\therefore\ S=\{1,2,3,4,5,6\}$ and $A=\{1,3,5\}$

$$P(A)=\frac{3}{6}=\frac{1}{2}$$

Suppose we consider another event B such that B be the event of getting the numbers greater than 3. Then $B=\{4,5,6\}$.

When we say that B has already occurred,the event of getting ' numbers is less than or equal to 3 is not possible. Now, the number of elements of the sample space is reduced to 3 i.e $\{4,5,6\}$. Out of these three points, two points are even number $\{4,6\}$.

Thus, given that if the event B has already occurred, then $P(A)=\frac{2}{3}$. Hence, the probability of A given that B has already occurred is denoted by $P\left(\frac{A}{B}\right)$.

2. Consider the experiment of tossing three fair coins. The sample space of the experiment is $S=\{HHH,HHT,HTH,THH,HTT,THT,TTH,TTT\}$.

Let E be the event 'at least two tails appear' and F be the event 'first coin shows head'.

Now, suppose we are given that the first coin shows head, i.e. F occurs, then we have to find the probability of occurrence of E. With the information of occurrence of F, we are sure that the cases in which first coin does not result into a head should not be considered while finding the probability of E. This information reduces our sample space from the set S to its subset F for the event E. In other words, the additional information really amounts to telling us that the situation may be considered as being that of a new random experiment for which the sample space consists of all those outcomes only which are favourable to the occurrence of the event F. Now, the sample point of F which is favourable to event E is HTT.

Thus, Probability of E considering F as the sample space $=\frac{1}{4}$.

Probability of E given that the event F has occurred $=\frac{1}{4}$

This probability of the event E is called the *conditional probability of E given that F has already occurred*, and is denoted by P (E|F).

3. Consider the random experiment of drawing a card from a deck of 52 cards. Suppose we draw a queen in successive draws. In first draw the probability of drawing a queen is $\frac{4}{52}$. If the card is not replaced before the second draw, then the event of getting queen in the second draw depends on the first event. Now, there will be only 3 queens. The probability of getting a queen in second draw will also be $\frac{3}{52}$.

In the above examples,we observe that the occurence of one event depends on another event which has already happend. Hence,an event E is said to be a dependent event if the occurnce of A depends on another event F which has already happend. It is denoted by $P(E \mid F)$.

The above examples leads to the definition of conditional probability as :

Let E and F be two events defined on a sample space S. Let $P(F) > 0$,then the conditional probability of E given that F has already occurred denoted by $P(E \mid F)$ and mathematically written as :

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}, P(F) > 0$$

Similarly, $P(F \mid E) = \frac{P(E \cap F)}{P(E)}, P(E) > 0$

Note : (1) $P(E \mid F)$ and $P(F \mid E)$ can also be expressed as follows :

$$P(E \mid F) = \frac{n(E \cap F)}{n(F)} \quad \text{and} \quad P(F \mid E) = \frac{n(E \cap F)}{n(E)}.$$

(2) $P(E \mid F)$ is read as probability of event E given event F.

Laws of Conditional probability :

(a) If E_1 and E_2 be any two events of a sample space S and A be any event of S such that $P(A) \neq 0$, then

$$P(E_1 \cup E_2 \mid A) = P(E_1 \mid A) + P(E_2 \mid A) - P(E_1 \cap E_2 \mid A)$$

We have $P(E_1 \cup E_2 \mid A) = \dfrac{P((E_1 \cup E_2) \cap A)}{P(A)}$

$$= \frac{P(E_1 \cap A) \cup P(E_2 \cap A)}{P(A)}$$

$$= \frac{P(E_1 \cap A) + P(E_2 \cap A) - P(E_1 \cap E_2 \cap A)}{P(A)}$$

$$= \frac{P(E_1 \cap A)}{P(A)} + \frac{P(E_2 \cap A)}{P(A)} - \frac{P(E_1 \cap E_2 \cap A)}{P(A)}$$

$$= P(E_1 \mid A) + P(E_2 \mid A) - P(E_1 \cap E_2 \mid A)$$

(b) $P(E' \mid F) = 1 - P(E \mid F)$

We have $P(S \mid F) = 1$

$$\Rightarrow P(E' \cup E \mid F) = 1$$

$$\Rightarrow P(E' \mid F) + P(E \mid F) = 1$$

$$\Rightarrow P(E' \mid F) = 1 - P(E \mid F)$$

Independent events

Events are said to be independent if the occurrence of one does not affect the others. In the experiment of tossing a fair coin, the occurrence of the event 'head' in the first toss is independent of the occurrence of the event ' head' in the second toss and third toss.

Example 1 : A card is chosen at random from an ordinary deck of 52 cards.

Let E be the event that the cards is an ace and F the event that the card is a club. Then E and F are independent.

Dependent events:

If the occurrence of one event influences the occurrence of the other, then the second event is said to be dependent on the first. In the above example, if we do not replace the first ball drawn, this will change the composition of balls in the bag while making the second draw and therefore the event of 'drawing a red ball' in the second will depend on event (first ball is red or white) occurring in first draw. Similarly, if a person draw a card from a full pack and does not replace it, the result of the draw made afterwards will be dependent on the first draw.

Example : Two fair dice are tossed. Let E be the event that the first die is a 3, F the event that the sum is 6 and G the event that sum is 7. Then E and F are dependent, but E and G are independent.

Note: If the events A and B are independent, that is the probability of occurrence of any one of them $P(E \mid F) = P(E)$ and $P(F \mid E) = P(F)$.

8.4 MULTIPLICATION THEOREM ON PROBABILITIES

Now, we discuss multiplication theorem on probabilities for both independent and dependent events.

Multiplication theorem on probabilities for independent events :

If two events E and F are independent, the probability that both of them occur is equal to the product of their individual probabilities. i.e $P(E \cap F) = P(E)P(F)$.

Note: (1)The theorem can be extended to three or more independent events. If E, F,G be independent events, then

$$P(E \cap F \cap G) = P(E)P(F)P(G).$$

(2) If E and F are independent then the complements of E and F are also independent. i.e $P\left(\bar{E} \cap \bar{F}\right) = P\left(\bar{E}\right)P\left(\bar{F}\right)$

Multiplication theorem for dependent events: If E and F be two dependent events, i.e the occurrence of one event is affected by the occurrence of the other event, then the probability that both A and B will occur is

$$P(E \cap F) = P(E)P(F \mid E).$$

Note: In the case of three events E, F, G

$P(E \cap F \cap G) = P(E)P(F \mid E)P(G \mid E \cap F)$i.e., the probability of occurrence of E, F and G is equal to the probability of E times the probability of F given that E has occurred, times the probability of G given that both E and F have occurred.

Ilustrative Examples :

Example1: Let E and F are two events,such that $P(E) = 0.8$, $P(F) = 0.6$, $P(E \cap F) = 0.5$. Find the value of (i) $P(E \cup F)$ (ii) $P(F \mid E)$ (iii)

$P(E \mid F)$.

Solution : (i) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$$= 0.8 + 0.6 - 0.5$$

$$= 0.9$$

(ii) $P(F \mid E) = \frac{P(E \cap F)}{P(E)} = \frac{0.5}{0.8} = \frac{5}{8}$

(iii) $P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{0.5}{0.6} = \frac{5}{6}$.

Example 2 : If $P(E) = 0.5, P(F) = 0.3$ and $P(E \cap F) = 0.15$. Find $P(E \mid F)$.

Solution : By definition of conditional probability

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

$$= \frac{0.15}{0.3}$$

$$= 0.5$$

Example 3 : If $P(E) = 0.9; P(F \mid E) = 0.8$, find $P(E \cap F)$.

Solution : By definition of conditional probability

$$P(F \mid E) = \frac{P(F \cap E)}{P(E)}$$

$$\Rightarrow P(F \cap E) = P(E).P(F \mid E)$$

$$= 0.9 \times 0.8$$

$$= 0.72.$$

Example 4 : Find the chance of drawing 2 white balls in succession from a bag containing 5 red and 7 white balls, the balls drawn not being replaced.

Solution : Let E be the event that ball drawn is white in the first draw and F be the event that ball drawn is white in the second draw.

$$\therefore P(F \cap E) = P(E).P(F \mid E)$$

Here $P(E) = \frac{7}{12}, P(F \mid E) = \frac{6}{11}$

$$\therefore P(F \cap E) = \frac{7}{12} \cdot \frac{6}{11} = \frac{7}{22}.$$

Example 5 : In an examination 30% of the students have failed in Mathematics, 20% of the students have failed in chemistry and 10 % have

failed in both Mathematics and Chemistry. A student is selected at random (i) What is the probability that the students has failed in Mathematics if it is known that he has failed in Chemistry ?

Solution : Let E be the event of failing in Mathematics.

$$\therefore P(E) = 30\% = 0.3$$

Let F be the event of failing in Chemistry

$$\therefore P(F) = 20\% = 0.2$$

Let $E \cap F$ be the event of failing in both subjects.

$$P(E \cap F) = 10\% = 0.1$$

The probability that the student has failed in Mathematics given that he has failed in Chemistry is

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{0.1}{0.2}$$

$$= 0.5 .$$

Example 5 : A bag contains 5 white and 3 black balls.Two balls are drawn at random one after the other without replacement.Find the probability that both balls are black.

Solution : Probability of drawing a black ball in the first attempt is

$$P(E) = \frac{3}{5+3} = \frac{3}{8}$$

Probability of drawing the second black ball given that the first ball drawn is black

$$P(F \mid E) = \frac{2}{5+2} = \frac{2}{7}$$

$\therefore$ The probability that both balls drawn are black is given by

$$P(E \cap F) = P(A) \times P(F \mid E) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28} .$$



CHECK YOUR PROGRESS -1

Q 1: Two fair coins are tossed. Given that at least one of them is heads, what is the probability that both of them are heads?

Q 2: Suppose that a box contains 8 red balls and 4 white balls. We

randomly draw 3 balls from the box without replacement. Find the probability that all 3 are red.

Q 3: A die is rolled, If the outcome is an odd number. What is the probability that it is prime?

Q 4: A family has two children.Find the probability that both children are girls given that at least one of them is a girl.

8.5 TOTAL PROBABILITY

Theorem of total probability : If E_1 and E_2 are mutually exclusive and exhaustive events and $P(E_1) \neq 0$, $P(E_2) \neq 0$, then for any event A of S,

$$P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)$$

Proof : Since E_1 and E_2 are mutually exclusive and exhaustive events, we have $E_1 \cup E_2 = S$ and $E_1 \cap E_2 = \phi$.

$\therefore A = A \cap S$

$= A \cap (E_1 \cup E_2)$

$= (A \cap E_1) \cup (A \cap E_2)$ .... (1)

Now, $(A \cap E_1) \cap (A \cap E_2) = A \cap (E_1 \cap E_2)$

$= A \cap \phi$

$= \phi$

$\therefore$ $A \cap E_1$ and $A \cap E_2$ are mutually exclusive and exhaustive events.

$\therefore$ From (1),we have

$P(A) = P(A \cap E_1) + P(A \cap E_2)$

$= P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)$ [multiplication theorem of probability]

If E_1, E_2, E_3 are mutually exclusive and exhaustive events and $P(E_1) \neq 0, P(E_2) \neq 0, P(E_3) \neq 0$ then for any event A of S,

$$P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) + P(E_3)P(A \mid E_3)$$

8.6 BAYE'S THEOREM

The concept of conditional probability discussed earlier takes into account information about the occurrence of one event to predict the probability of another event. This concept can be extended to revise

probabilities based on new information and to determine the probability that a particular effect was due to specific cause. The procedure for revising these probabilities is known as Bayes theorem. The Principle was given by Thomas Bayes in 1763. By this principle, assuming certain prior probabilities, the posteriori probabilities are obtained.

Bayes' Theorem or Rule

Statement : Let $E_1, E_2, E_3, \cdots, E_n$ be a set of n mutually exclusive and collectively exhaustive events and $P(E_1), P(E_2), P(E_3), \cdots, P(E_n)$ are their corresponding probabilities. If A is another event such that P(A) is not zero and the priori probabilities $P(A \mid E_i)$ i =1,2..., n are also known. Then

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) + \cdots + P(E_n)P(A \mid E_n)}$$

i.e., $$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)P(A \mid E_i)}$$

Proof : Let S be the sample space of the random experiment.

The events $E_1, E_2, E_3, \cdots, E_n$ being exhaustive.

$$\therefore S = E_1 \cup E_2 \cup \cdots \cup E_n$$

$$A = A \cap S \qquad [A \subset S]$$

$$= A \cap (E_1 \cup E_2 \cup \cdots \cup E_n)$$

$$= (A \cap E_1) \cup (A \cap E_2) \cup \cdots (A \cap E_n) \quad \text{[Distributive law]}$$

$$\Rightarrow P(A) = P(A \cap E_1) + P(A \cap E_2) + \cdots + P(A \cap E_n)$$

$$= P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) + \cdots + P(E_n)P(A \mid E_n)$$

$$= \sum_{i=1}^{n} P(E_i)P(A \mid E_i) \quad \text{.... (1)}$$

Now, $P(A \cap E_i) = P(A)P(E_i \mid A)$

$$\Rightarrow P(E_i \mid A) = \frac{P(A \cap E_i)}{P(A)}$$

$$= \frac{P(E_i)P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)P(A \mid E_i)} \qquad \text{[using (1)]}$$

Note : (1) The probabilities $P(E_1), P(E_2), \cdots, P(E_n)$ are called as the 'a prori

probabilites'.

(2) The probabilities $P(E_1 \mid A), P(E_2 \mid A), \cdots, P(E_n \mid A)$ are called'posteriori probabilities'.

Example 1 : A purse contains 6 silver coins and 3 gold coins.Another purse contains 4 silver coins and 5 gold coins.A purse is selected at random and a coin is drawn from it.What is the probability that it is a silver coin.

Solution : Let Let E = Event of selecting first purse

F = Event of selecting the second purse

A= Event of getting a silver coin

$\therefore P(E) = \frac{1}{2}$ and $P(F) = \frac{1}{2}$

Also, $P(E \mid A) = \frac{6}{9} = \frac{2}{3}$ and $P(F \mid A) = \frac{4}{9}$

$\therefore$ Required probability $P(A) = P(E)P(A \mid E) + P(F)P(A \mid F)$

$$= \frac{1}{2} \times \frac{6}{9} + \frac{1}{2} \times \frac{4}{9}$$

$$= \frac{10}{18}$$

$$= \frac{5}{9}$$

Example 2 : A box contains two coins. One coin is heads on both sides and the other is heads on one side and tails on the other. One coin is selected from the box at random and the face of one side is observed. If the face is heads, what is the probability that the other side is heads ?

Solution : Let E = Event of selecting two headed coin

F = Event of selecting the other coin (one head and one tail)

A= Event of getting the second side also head.

Now, we have to find $P(E \mid A)$.

Here, $P(E) = P(F) = \frac{1}{2}$

$$P(A \mid E) = 1, P(A \mid F) = \frac{1}{2}$$

By Baye's theorem, we have

$$P(E|A)=\frac{P(E)P(A|E)}{P(E)P(A|E)+P(F)P(A|F)}$$

$$=\frac{\frac{1}{2}\times 1}{\frac{1}{2}\times 1+\frac{1}{2}\times\frac{1}{2}}$$

$$=\frac{2}{3}.$$

Example 3 : Assume that a factory has two machines. Past records show that machine I produces 20% of the items and machine II produces 80% of the items. Further, it shows that 6% of the items produced by machine I are defective and only 1 % of the items produced by machine II were defective. If a defective item is drawn at random,what is the probability that it was produced by machine I ?

Solution : Let E_1, E_2 be the events of drawing an item produced by machines I and II repectively. Let A denote the event of drawing a defective item.

We have $P(E_1)=\frac{20}{100}=\frac{1}{5}$, $P(E_2)=\frac{80}{100}=\frac{4}{5}$

Also, we have $P(A|E_1)=\frac{6}{100}$, $P(A|E_2)=\frac{1}{100}$

From Bayes' theorem, the required probability is

$$P(E_1|A)=\frac{P(E_1).P(A|E_1)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)}$$

$$=\frac{\frac{1}{5}.\frac{6}{100}}{\frac{1}{5}.\frac{6}{100}+\frac{4}{5}.\frac{1}{100}}=\frac{6}{6+4}=0.6.$$

Example4 : A bolt is manufactured by 3 machines A,B and C.A turns out twice as many items as B and machines B and C produce equal number of items. 2 % of bolts produced by A and B are defective and 4 % of bolts produced by C are defective.All bolts are put into one stock pile and 1 is chosen from this pile.What is the probability that it is defective ?

Solution : Let E_1, E_2, E_3 be the events of bolts manufactured by machine

A,B,C repectively.

Here we have to apply total theorem on probability.

B and C produce equal no. of bolts and A produces two times that of B.

$$\therefore\ P(E_1)=50\%=\frac{50}{100}=0.50$$

$$P(E_2)=25\%=\frac{25}{100}=0.25$$

$$P(E_3)=25\%=\frac{25}{100}=0.25$$

Let D be the event of getting defective bolt.

$$P(D\mid E_1)=\frac{2}{100}=0.02$$

$$P(D\mid E_2)=\frac{2}{100}=0.02$$

$$P(D\mid E_3)=\frac{4}{100}=0.04$$

By total theorem on probability.

$$\begin{aligned}P(D)&=P(E_1)P(D\mid E_1)+P(E_2)P(D\mid E_2)+P(E_3)P(D\mid E_3)\\&=0.5\times0.02+0.25\times0.02+0.25\times0.04\\&=0.0250\end{aligned}$$

Example 5 : A bag A contains 2 white and 3 red balls and bag B contains 4 white and 5 red balls.One ball is drawn at random at random from one of the bags and is found to be red,find the probability that it is drawn from bag B.

Solution : Here given that the ball drawn is red and we have to find the probability of the ball coming from bag B.

Let E_1 be the event of selecting bag A.

$$\therefore\quad P(E_1)=\frac{1}{2}$$

Let E_2 be the event of selecting bag B.

$$\therefore\ P(E_2)=\frac{1}{2}$$

Let D be the event of selecting red ball.

Probability of drawing red ball from bag A is

$$P(D \mid E_1) = \frac{3}{5}$$

Similarly, probability of drawing red ball from bag B is

$$P(D \mid E_2) = \frac{5}{9}$$

We have to find the red ball was drawn from bag B i.e., $P(E_2 \mid D)$

By Baye's Theorem

$$P(E_2 \mid D) = \frac{P(E_2)P(D \mid E_2)}{P(E_1)P(D \mid E_1) + P(E_2)P(D \mid E_2)}$$

$$= \frac{\frac{1}{2} \cdot \frac{5}{9}}{\frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{5}{9}}$$

$$= \frac{\frac{5}{18}}{\frac{3}{10} + \frac{5}{18}}$$

$$= \frac{\frac{5}{18}}{\frac{27+25}{90}}$$

$$= \frac{\frac{5}{18}}{\frac{52}{90}}$$

$$= \frac{25}{52}$$

Example 6 : In a bolt factory machines A, B, C manufacture repectively 25%, 35%, 40% of the total. Of their output 5%, 4%, 2% are defective bolts. A bolt is drawn at random and found to be defective. What is the probability that it was manufactured by machine B ?

Solution : Let E_1, E_2, E_3 be the events of bolts manufactured by machine A, B, C repectively.

$$P(E_1) = 25\% = \frac{25}{100} = 0.25$$

$$P(E_2) = 35\% = \frac{35}{100} = 0.35$$

$$P(E_3) = 40\% = \frac{40}{100} = 0.40$$

Let D be the event of drawing a defective bolt.

$$\therefore\ P(D \mid E_1) = \frac{5}{100} = 0.05$$ [defective from 'A']

$$P(D \mid E_2) = \frac{4}{100} = 0.04$$ [defective from 'B']

$$P(D \mid E_3) = \frac{2}{100} = 0.02$$ [defective from 'C']

Now, we have to find the probability of defective bolt manufactured by machine B. i.e., $P(E_2 \mid D)$.

By Baye's Theorem :

$$P(E_2 \mid D) = \frac{P(E_2)P(D \mid E_2)}{P(E_1)P(D \mid E_1) + P(E_2)P(D \mid E_2) + P(E_3)P(D \mid E_3)}$$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.4 \times 0.02}$$



CHECK YOUR PROGRESS -2

Q 1: Assume that a factory has two machines. Past records show that machine I produces 20% of the items and machine II produces 80% of the items. Further, it shows that 6% of the items produced by machine I are defective and 1% of the items produced by machine II are defective. If a defective item is drawn at random, what is the probability that it was produced by machine I ?

Q 2: Three machines A, B, C produce repectively 60%, 30%, 10% of the total number of items of a factory. The percentage of respective outputs of these machines are repectively 2%, 3% and 4%. An item is selected at random and is found to be

defective.Find the probability that the item was produced by machine C.

Q 3: A bag X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag Y.

Q 4: Urn A contains 3 red and 4 white balls and Urn B has 4 red and 5 white balls. One urn is selected at random and one ball is taken out of it. Thrown ball is red. What is the probability that it was taken from urn B ?

Q 5: A factory has three production lines I, II and III contributing 20%, 30 % and 50 %respectively, to its total output. The percentages of substandard items produced by lines I, II and III are, respectively, 15, 10 and 2. If an item chosen at random from the total output is found to be substandard, what is the probability that the item is from line I ?.



8.7 LET US SUM UP

- For two events E and F, the probability of **an** event **E** under the condition that F has already occurr is the **conditional probability P(E|F)** of **E** and is given by $P(E \mid F) = \frac{P(E \cap F)}{P(F)}$.
- Two events for which the occurrence of one has no influence on the occurrence or non-occurrence of the other are **independent** events, otherwise, they are **dependent.**
- If the events E and F are independent and $P(E) > 0, P(F) > 0$, then $P(E \cap F) = P(E)P(F)$ and $P(E \mid F) = P(E), P(F \mid E) = P(F)$.



8.8 FURTHER READINGS

1) Rao. G. Shanker, Probability and Statistics, Universitis Press.

2) Statistics, R.S.N. Pillai and Bagavathi S. Chand and Company Ltd. New Delhi-110055.



8.9 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS -1

Ans to Q No 1: Let E be the event that both of them are heads and F the event that at least one of them is heads.

Then, $P(E)=\frac{3}{4}, P(E\cap F)=\frac{1}{4}, P(F)=\frac{1}{4}$

$$\therefore P(E\mid F)=\frac{P(E\cap F)}{P(F)}$$

$$=\frac{\frac{1}{4}}{\frac{3}{4}}$$

$$=\frac{1}{3}.$$

By using the definition of conditional probability, we can easily get that

$$P(E\cap F)=P(F)P(E\mid F)=\frac{1}{4}\times\frac{1}{3}=\frac{1}{12}.$$

Ans to Q No 2: For i= 1;2;3, let E_i be the event that thei-th ball is red.

Then, $P(E_1\cap E_2\cap E_3)=P(E_1)P(E_2\mid E_1)P(E_3\mid E_1E_2)$

$$=\frac{8}{12}.\frac{7}{11}.\frac{6}{10}=\frac{14}{55}.$$

Ans to Q No 3: When a die is rolled, the sample is $S=\{1,2,3,4,5,6\}$.

Let $E=$ Event of getting an odd number

$=\{1,3,5\}$

F = Event of getting a prime number

$=\{2,3,5\}$

Then, $E\cap F=\{3,5\}$

$\therefore P(E)=\frac{3}{6}=\frac{1}{2}, P(F)=\frac{3}{6}=\frac{1}{2}$ and $P(E\cap F)=\frac{1}{3}$.

P(getting a prime already when an odd number)

= P(getting a prime number | getting an odd number)

$=P(E\mid F)$

$=P(E\mid F)$

$=\frac{P(E\cap F)}{P(F)}$

$=\frac{\frac{1}{3}}{\frac{1}{2}}=\frac{2}{3}$.

Ans to Q No 4: The sample space S is given by

$S=\{(b,b),(b,g),(g,b),(g,g)\}$

Let E : Event that both children are girls

F : Event that at least one of the child is a girl.

Then $E=\{(g,g)\}$ and $F=\{(b,g),(g,b),(g,g)\}$

$E\cap F=\{(g,g)\}$

$\therefore P(E)=\frac{3}{4}, P(E\cap F)=\frac{1}{4}$

$\therefore$ Required probability is $P(E\mid F)=\frac{P(E\cap F)}{P(F)}$

$=\frac{\frac{1}{4}}{\frac{3}{4}}$

$=\frac{1}{3}$

CHECK YOU PROGRESS-2

Ans to Q No 1: Let E_1, E_2 be the events of drawing an item produced by machines I and II repectively.Let A denote the event of drawing a defective item.

We have $P(E_1)=\frac{20}{100}=\frac{1}{5}, P(E_2)=\frac{80}{100}=\frac{4}{5}$

Also, we have $P(A \mid E_1)=\frac{6}{100}, P(A \mid E_2)=\frac{1}{100}$

From Bayes's Theorem, the required probability is

$$P(E_1 \mid A)=\frac{P(E_1)P(A \mid E_1)}{P(E_1)P(A \mid E_1)+P(E_2)P(A \mid E_2)}$$

$$=\frac{\frac{1}{5}\cdot\frac{6}{100}}{\frac{1}{5}\cdot\frac{6}{100}+\frac{4}{5}\cdot\frac{1}{100}}=\frac{6}{6+4}=0.6.$$

Ans to Q No 2: Let E, F, G denote the events in which the items are selected by machines A, B, C repectively.

Then, we have $P(E)=\frac{60}{100}=0.6, P(F)=\frac{30}{100}=0.3, P(G)=\frac{10}{100}=0.1$

Let D denote the event in which the item selected at random is a defective item.Then

$$P(D \mid E)=\frac{2}{100}=0.02, P(D \mid F)=\frac{3}{100}=0.03, P(D \mid G)=\frac{4}{100}=0.04$$

Applying Bayes's theorem, we obtain

$$P(G \mid D)=\frac{P(G)P(D \mid G)}{P(E)P(D \mid E)+P(F)P(D \mid F)+P(G)P(D \mid G)}$$

$$=\frac{(0.1)(0.04)}{(0.6)(0.02)+(0.3)(0.03)+(0.1)(0.04)}$$

$$=\frac{0.004}{0.012+0.009+0.004}$$

$$=\frac{0.004}{0.025}$$

$$=0.16.$$

Ans to Q No 3: Let E_1 : the ball is drawn from bag X

E_2: the ball is drawn from bag Y

and A ; the ball is red.

Here $P(E_1)=P(E_2)=\frac{1}{2}$

Also, $P(A \mid E_1)$=P(a red ball is drawn from bag X)

$$=\frac{3}{5}$$

$P(A \mid E_2)$=P(a red ball is drawn from bag Y)

$$=\frac{5}{9}$$

By Baye's Theorem ,we have

$$P(E_2 \mid A)=\frac{P(E_1)P(A \mid E_1)}{P(E_1)P(A \mid E_1)+P(E_2)P(A \mid E_2)}$$

$$=\frac{\frac{1}{2}\times\frac{5}{9}}{\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{5}{9}}=\frac{25}{52}.$$

Ans to Q No 4: Required probability $=\frac{28}{45}$ [By Bayes's Theorem]

Ans to Q No 5: By Bayes formula

$$P(E_1 \mid A)=\frac{P(E_1)P(A \mid E_1)}{P(E_1)P(A \mid E_1)+P(E_2)P(A \mid E_2)}$$

$$=\frac{0.2\times 0.15}{0.2\times 0.15+0.3\times 0.10+0.5\times 0.02}$$

$$=0.069.$$



8.10 MODEL QUESTIONS

Q 1: E and F are two events for which $P(E)=\frac{1}{2}$, $P(F)=\frac{1}{3}$ and $P(E\cap F)=\frac{1}{4}$.Find (i) $P(E \mid F)$,(ii) $P(E\cup F)$ and (iii) $P(E' \mid F')$.

Q 2: Three persons A, B, C supply raw mterials to a factory in the proportion $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ and $5\%, 6\%, 8\%$ of the materials supplied by them repectively are found to be defective. A material supplied is selected at random from the total supply. What are the probabilities that material was supplied by A, or by B or by C ?

Q 3: One of three identical coins is unbiased while the remaining two are biased with probabilities of getting head in a toss are $\frac{1}{3}$ and $\frac{2}{3}$. One of the coins selected randomly shows head when tossed. Find the probability of the coin to be unbiased.

Q 4: Boys from the districts of Kamrup, Nagaon and Jorhat come to a college in Assam in the proportions 50%, 30% and 20% repectively. 80%, 60% and 40% of the boys coming repectively from the districts wear Dhooti. A boy selected randomly from all the boys of the college is found to wear Dhooti. What is the probability that he comes kamrup?

Q 5: Three trains X, Y, Z carry repectively 25%, 35%, 40% of the total passengers. Of the passengers carried by the tains 5%, 4% and 3% respectively are ladies. If a pssengers chosen at random out of all the passengers of the trains found to be a lady. Prove that the probability that she was carried by X, or Y or Z are repectively $\frac{25}{69}, \frac{28}{69}, \frac{16}{69}$.

Q 6: Three are three groups of children. The first group consists of 3 girls and 1 boy, the second consists of 2 girls and 2 boys and the third consists of 1 girl and 3 boys. One child is picked up at random from each group. Find the probability of selecting 1 girl and 2 boys.

Q 7: A bag contains 5 white and 7 red balls. If a ball is drawn twice at random successively, find the probability that one ball is white and the other is red.

Q 8: In a running competition, the probabilities of the two competitiors of winning the race is $\frac{1}{3}$ and $\frac{1}{5}$. What is the probability that both of them will not win ?

Q 9: One card is drawn from a well shuffled pack of 52 cards.Deduce the probability of drawing.

(i) a red card or an ace or both

(ii) a diamond or an ace or both.

(iii) king of diamond or king of heart or both.

Q 10: 4 men play with a pack of cards containing 52 cards. What is the probability that the 4 kings are held by only one of the players ?

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UNIT 9: RANDOM VARIABLES AND ITS PROBABILITY DISTRIBUTION

UNIT STRUCTURE

9.1 Learning Objective

9.2 Introduction

9.3 Random Variable

- 9.3.1 Definition or Random Variable
- 9.3.2 Discrete Random Variable
- 9.3.3 Continuous Random Variable

9.4 Probability Distribution

- 9.4.1 Probability Mass Function
- 9.4.2 Probability Density Function

9.5 Mathematical Expectation and Variance

- 9.5.1 Mathematical Expectation
- 9.5.2 Properties of Mathematical Expectation
- 9.5.3 Variance of Random Variable
- 9.5.4 Properties of Variances
- 9.5.5 Illustrated Examples

9.7 Let Us Sum up

9.8 Further Readings

9.9 Answers to Check Your Progress

9.10 Model Questions

9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- distinguish between two types of experiments viz deterministic experiments and random experiments
- define what is a random variable and discuss when we get a discrete random variable or a continuous random variable
- have knowledge about probability distributions
- discuss the concept of probability mass function and probability

density function along with their properties

- know about mathematical expectations and variance of a random variable which are vital in the study of any distribution, along with their properties and applications
- realise how different measures of location and dispersion as discussed in earlier chapters can be expressed with the help of moments.

9.2 INTRODUCTION

The experiments that are performed can be classified into two broad categories. In one type of experiment the outcomes of the experiments remain same whenever and wherever the experiment is performed. In the second type of experiment, the outcomes are found to vary every time the experiment is performed.

The first types of experiments are known as deterministic experiments, whereas the second types are known as random experiments. As we have already discussed, in this type of experiment, the experimenter knows the set of all possible outcomes, but cannot say with certainity which outcome will turn out at a particular time. If we want to denote thses various outcomes of a random experiment with the help of a variable, then it becomes a random variable.

In this unit we shall learn about random variables, the different types of random variables and their mathematical expectation and some related theorems. We shall also study about moments and try to understand the difference between raw moments, i.e. the moment about any arbitary point and central moment, the moment about mean.

9.3 RANDOM VARIABLE

Most of the random experiments we encounter have outcomes that can be interpreted in terms of real numbers, such as marks obtained by a student in an examination, number of accidents in a day etc. On the other hand, sometimes the outcomes of an experiment can take any value in an interval of real numbers. For example, the heights of students in a class

can take any value within a specified range. Sometimes, the outcomes of a random experiment may be qualitative, e.g. tossing of a coin may result in head or in tail. We may denote head by '1' and tail by '0' and in this way the outcomes of a random experiment, quantitaive or qualitative can be expressed by a real number. these numerical values of the outcomes that changes from experiment to experiment are called random variables. So we can formally define a random variable as follows:

9.3.1 Definition of Random Variable

The real numbers which are associated with the outcomes of a random experiment are called random variables.

It can also be defined as a variable which takes a definite set of values with a probability associated with each of its values is called a random variable. That is why it is also called a chance variable or a stochastic variable.

Random variables may be of two types:

(i) Discrete random variables

(ii) Continuous random variables

Note: if X and Y are two random variables, then X+Y, X-Y are also random variables. Again, if, a and b are constants, then ax+y, x+by, ax+by are also random variables.

9.3.2 Discrete Random Variable

If a random variable x assumes only a finite number or countably infinite number of values, then it is called a discrete random variable. So a random variable x can take finite number of values $x_1, x_2, \ldots\ldots\ldots\ldots, x_n$ or it can take countably infinite number of values $x_1, x_2, \ldots\ldots\ldots\ldots$. The number of letters received by a post office during a particular time period, the number of vehicles arriving at a toll bridge, the number of machines breaking down on a given day etc are some examples of discrete random variables.

9.3.3 Continuous Random Variable

If a random variable is such that it takes any value within a given interval, then it is called a continuous random variable. in this case we say that x takes any value in an interval, say. (a,b), i.e. $a \leq x \leq b$; i.e. x is a continuous random variable in the interval (a,b). Amount of rainfall in a rainy season, heights of individuals, the time between arrival of customers at a service system etc are some examples of continuous random varible.

9.4 PROBABILITY DISTRIBUTION

The distribution obtained by taking all possible values of a random variable along with their respective probabilities is called a probability distribution. When the random variable is discrete we get discrete probability distribution and when the variable under consideration is continuous, we get a continuous probability distribution. The discrete probability distribution is called probability mass function and the continuous probability distribution is called probability density function.

9.4.1 Probability Mass Function

Let x denote a discrete random variable which takes the values $x_1, x_2, \ldots\ldots\ldots, x_n$ with respective probabilities $p_1, p_2, \ldots\ldots\ldots, p_n$. Let, $p_i = P(x = x_i) = p(x_i)$, $i = 1, 2, \ldots\ldots$, n be the probability that x takes the value xi. Then p (x_i) is called the Probability Mass Function (p.m.f.) of x if it satisfies the following conditions.

(i) $p(x_i) \geq 0$ for all $i = 1, 2, \ldots\ldots\ldots$, n

(ii) $\sum_{i=1}^{n} p(x_i) = 1$.

In case, the discrete random variable x is countably infinite, then, $p(x_i) = P(x = x_i)$, $i = 1, 2, \ldots\ldots$,n is said to be the p.m.f. of x if

(i) $p(x_i) \geq 0$ for all $i = 1, 2, \ldots\ldots$

(ii) $\sum_{i=1}^{\infty} p(\upsilon_i) = 1$

Illustration: Let x denote the uppermost face while throwing a dice. Then x can take the values 1, 2, 3, 4, 5 or 6, each with probability 1/6. This can be written as,

$p(x_i) = P(x = x_i) = 1/6$, $i = 1, 2, \ldots\ldots\ldots\ldots, 6.$

Here, $p(x_i) > 0$

and $\sum_{i=1}^{6} p(x_i) = 1$

Thus $p(x_i)$ satisfies both the conditions for a p.m.f.

9.4.2 Probability Density Function

The probability distribution of a continuous random variable x is defined by $f(x)$ is called the probability density function (p.d.f.). Let, x be a continuous random variable in the interval (a,b), then the function $f(x) = P(X = x)$ is said to be the p.d.f. of x, if it satisfies the following conditions.

(i) $f(x) \geq 0 \ \forall \ x$ in (a,b)

(ii) $\int_a^b f(x)dx = 1$

Illustration: Consider the function, $f(x) = \begin{cases} a, 0 \leq x \leq 5 \\ 0, otherwise \end{cases}$

For $f(x)$ to be a p.d.f., the condition $\int_a^b f(x)dx = 1$ must be satisfied, which is true if,

$$\int_0^5 a dx = 1$$

$$\Rightarrow \quad a = \frac{1}{5}$$

Since a>0, the function $f(x) \geq 0$. Thus f(x) satisfies both the conditions for a p.d.f.

9.5 MATHEMATICAL EXPECTATION AND VARIANCE

9.5.1 Mathematical Expectation

The mathematical expectation of a discrete random variable is the sum of the product of the values taken by the random variable and their respective probabilities. Let us consider the discrete random variable x, which assumes the values $x_1, x_2, \ldots\ldots\ldots, x_n$ with respective probabilities $P_1, P_2, \ldots\ldots\ldots, P_n$, then the mathematical expectation of the random variable x is denoted by E(x) and is given by

$$E(x) = \sum_{i=1}^{n} x_i P_i$$

provided the series is convergent and $\sum_{i=1}^{n} P_i = 1$

If x is a continuous random variable with p.d.f. $f(x)$, $-\infty < x < \infty$, then the mathematical expectation of the random variable x is given by,

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx \text{, provided } \int_{-\infty}^{\infty} f(x) dx = 1$$

Note: The expectation of a random variable x represents the mean of the probability distribution of x.

Mathematical Expectation of a function of a random variable:

If $g(x)$ is a function of a random variable x, then the expectation of $g(x)$ is given by,

$$E[g(x)] = \sum_{i=1}^{n} g(x_i) p(x_i) = \sum_{i=1}^{n} g(x_i) p_i$$

where, x is a discrete random variable taking values $x_1, x_2, \ldots\ldots\ldots, x_n$ with respective probabilities $p_1, p_2, \ldots\ldots\ldots, p_n$.

Similarly, when x is a continuous random variable in the interval $(-\infty, \infty)$, with p.d.f. $f(x)$, $-\infty < x < \infty$, then,

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

9.5.2 Properties of Mathematical Expectation

The following are the important properties of an expected value of a random variable:

1. The expected value of a constant is the constant itself. i.e. E(c) = c, for every constant c.

2. The expected value of the product of a constant a and a random variable x is equal to the times the expected value of the random variable, i.e. E(ax) = a E(x).

3. The expected value of a linear function of a random variable is same as the linear function of its expectation, i.e. E (a+bx) = a+b E(x).

4. The expected value of the sum of the two random variables is equal to the sum of their → individual expected values. i.e. E(x+y) = E(x) + E(y).

5. The expected value of the product of two independent random variables is equal to the product of their individual expected values. i.e. E(xy) = E(x) E(y).

9.5.3 Variance of a Random Variable

Variance is an important characteristic of a random variable. It provides the measure of dispersion of the random variable. The variance of a random variable x is denoted by V(x) or σ^2 and is given by,

$$\sigma^2 = V(x) = E\{x - E(x)\}^2.$$

$$= E(x^2) - \{E(x)\}^2.$$

Note: The positive square root of variance is the standard deviation.

9.5.4 Properties of Variance

1. The variance of a constant is zero. i.e. V(c) = 0.
2. The variance of the product of a constant and a random variable is equal to the variance multiplied by the square of the constant. i.e. $V(cx) = c^2 V(x)$.

3. The variance of the sum or difference of two independent random variables. i.e. $V(x \pm y) = V(x) + V(y)$.

9.5.5 Illustrated Examples

1) A random variable x has the following probability distribution

x:	-2	-1	0	1	2	3
p(x):	0.1	0.1	0.2	0.2	0.3	0.1

Calculate E(x) and V(x)

Solution:

$$E(x) = \sum_x xp(x)$$

$= -2 \times 0.1 + (-1) \times 0.1 + 0 \times 0.2 + 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.1$

$= 0.8$

$$V(x) = E(x^2) - \{E(x)\}^2$$

Now, $E(x^2) = \sum_x x^2 p(x)$

$= (-2)^2 \times 0.1 + (-1)^2 \times 0.1 + 0 + 1^2 \times 0.2 + 2^2 \times 0.3 + 3^2 \times 0.1$

$= 2.8$

$\therefore V(x) = 2.8 - (-0.8)^2$

$= 2.8 - 0.64$

$= 2.16$

2) In an office there are 10 HCL and 20 HP computers. One computer is selected at random. Find the probability distribution of selecting an HP computer. What is the expected value of selecting a HP computer.

Solution: Let x be a random variable such that,

x = 1, if a HP computer is selcted

= 0, if a HP computer is not selected.

$\therefore$ the probability distribution of x is,

x:	0	1
P(X=x):	10/30	20/30

=1/3 = 2/3

$\therefore$ E (an HP computer is selected) = 0x1/3 + 1x2/3

= 2/3

3) Eighty percent patients of a hospital are suffering from hypertension. One patient is randomly selected. Find the probability distribution of patients who are suffering from hypertension and find mean of the distribution.

Solution: Let x be a random variable such that,

x = 1, if the patient suffers from hypertension

= 0, otherwise

Hence the probability distribution of x is

x:	0	1
p(X=x):	0.2	0.8

and E(x) = 0 x 0.2 + 1 x 0.8

= 0.8

4) The monthly demand for transistors have the following probability distribution

Demand (n):	1	2	3	4	5	6
Probability (p):	0.10	0.15	0.20	0.25	0.18	0.12

Determine the expected demand for transistors. If the cost of producing n transistors is given by C = 10,000+500n, determine the expected cost.

Solution: Expected demand for transistor

= E(n)

= $\sum np$

= 1 x 0.10 + 2 x 0.15 + 3 x 0.20 + 4 x 0.25 + 5 x 0.18 + 6 x 0.12

= 3.62

E (C) = E(10,000 + 500n) = 10,000 + 500 E(n)

= 10,000 + 500 x 3.62

= 11,810

5) Proprieter of a food stall has invented a new item of food delicacy. He has calculated that the cost per piece will be Rs. 1, but he wants

to sell it for Rs. 3 per piece. It is, however, perishable and the goods unsold at the end of the day is a total loss. If demand is a variable with the following frequency distribution, calculate his net profit if he makes 15 pieces.

No. of pieces demanded:	10	11	12	13	14	15
Probability :	0.07	0.10	0.23	0.38	0.12	0.10

Solution: Units produced = 15

Total cost price = 15 x 1 = Rs. 15

sell price peer unit = Rs. 3

Unit demanded	Profit
10	Rs. (10x3) - 15 = rs. 15
11	Rs. (11x3) - 15 = Rs. 18
12	Rs. (12x3) - 15 = Rs. 21
13	Rs. (13x3) - 15 = Rs. 24
14	Rs. (14x3) - 15 = Rs. 27
15	Rs. (15x3) - 15 = Rs. 30

$\therefore$ Expected net profit = $\sum$ profit x probability

= 15x0.07 + 18x0.10 + 21x0.23 + 24x0.38 + 27x0.12 + 30x0.10

= Rs. 3.45



CHECK YOUR PROGRESS

Q 1: Check true or false

(i) The probability distribution of a random variable x is as follows

X = x:	1	2	3	
P(X= x):	1/4	1/2	3/4	(T/F)

(ii) If c is a constant then, (a) E(c) = c (b) V(c) = c (T/F)

(iii) If x and y are two independent random variables, then,

(a) V(x+y) = V(x) + V(y) (T/F)

(b) V(x-y) = V(x) - V(y) (T/F)

(iii) If x and y are two independent random variables, then,

E(xy) = E(x) E(y)

(iv) A continuous random variable can take any value within a given range.

Q 2: Define random variable. What are the important properties of a random variable.



9.7 LET US SUM UP

- The experiments can be classified into two broad categories - deterministic experiment and random experiment.
- The real numbers which are associated with the outcomes of a random experiment are called random variables.
- Random variables may be of two types - discrete random variable and continuous random variable.
- The distribution obtained by taking all possible values of a random variable along with their respective probabilities is called a probability distribution.
- The probability distribution for discrete random variable is called probability mass function (p.m.f.) and the same for continuous random variable is known as probability density function (p.d.f.).
- The mathematical expectation of a discrete random variable is the sum of the product of the values taken by the random variable and their respective probabilities.
- The expectation of the random variable x represents the mean of the probability distribution of x.
- The variance of a random variable provides a measure of dispersion of the random variable.
- The expectation of a constant quantity is the constant itself, whereas variance of a constant is zero.
- The expectation of the sum of two or more random variables is the sum of the individual random variables.
- The expectation of the product of two independent random variables is the product of their expectations.

- The variance of the addition or subtraction of two independent random variables is equal to the sum of their variances.



9.8 FURTHER READINGS

1) Mukhopadhyay, P. (1996), *Mathematical statistics*. New Central Book Agency (P) Ltd.
2) Goon, A.M., Gupta, M.K. and Dasgupts, B. (1997), *An ouline of statistical Theory*. Vol. I, World Press.
3) Mukhopadhyay, P. (1991). *Theory of Probability*. New Central Book Agency (P) Ltd.
4) Gupta, S.C., Kapoor, V.K. (2009). *Fundamentals of Mathematical Statistics*, Sultan Chand & Sons.
5) Feller, W. (1968). *An Introduction to Probability Theory and its Application*, Vol I, Wiley Eastern.



9.9 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (i) False (Total probability exceeds 1

(ii) (a) True, (b) False

(iii) (a) True (b) False

(iv) True

(v) True

Ans to Q No 2: The real numbers which are associated with the outcomes of a random experiment are called random variables.

The properties of random variable are as follows:

(i) If X is a random variable there, $1/x$, x^2 are also random variables.

(ii) If x is a random variable and a and b are constants then ax+b is also a random variable.

(iii) If x and y are two random variables, then x+y and x-y are also random variables.



9.10 MODEL QUESTIONS

Q 1: Fill in the blanks

(i) Heights or weights of individuals can be represented by a ..... random variable.

(ii) If x and y are two random variables then ax+by is a ....., where a and b are constants.

(iii) If x and y are independent, then E(xy) = ...

(iv) $V(x \pm y) = V(x) + V(y)$, when x and y are ..... variables.

(v) V(ax+b) = ..., where a and b are constants.

(vi) E(a) = ... and V(a) = ..., where a is a constant.

Q 2: Can you find the mathematical expectation of a random variable x, if x takes the values 1,2,3 with corresponding probabilities $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$.

Q 3: Discuss discrete and continous random variables with examples.

Q 4: What do you mean by mathematical expectation of a function of a random variable. State addition and multiplication law of mathematical expectation.

Q 5: A random variable x has the following probability distribution,

x:	-2	-1	0	1	2	3	
f(x):		0.1	2k	0.2	2k	0.3	2k

Find the value of k.

Q 6: Write some important properties of mathematical expectation.

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UNIT 10: DISCRETE PROBABILITY DISTRIBUTIONS

UNIT STRUCTURE

10.1 Learning Objectives

10.2 Introduction

10.3 Moment Generating Function

- 10.3.1 Definition
- 10.3.2 Generation of Moments
- 10.3.3 Properties of Moment Generating Function

10.4 Binomial Distribution

- 10.4.1 Derivation of Binomial Distribution
- 10.4.2 Definition
- 10.4.3 Moments of Binomial Distribution
- 10.4.4 Moment Generating Function of Binomial Distribution
- 10.4.5 Fitting of Binomial Distribution
- 10.4.6 Properties of Binomial Distribution
- 10.4.7 Importance of Binomial Distribution

10.5 Poisson Process

- 10.5.1 Poisson Distribution as a limiting case of Binomial Distribution
- 10.5.2 Moments of Poisson Distribution
- 10.5.3 Moment Generating Function
- 10.5.4 Fitting of Poisson Distribution
- 10.5.5 Properties of Poisson Distribution
- 10.5.6 Application of Poisson Distribution

10.6 Let Us Sum Up

10.7 Further Readings

10.8 Answers To Check Your Progress

10.9 Model Questions

10.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- obtain moment generating function for any given distribution
- generate moments from any probability distribution
- learn some important properties of moment generating function
- derive binomial distribution
- derive poisson distribution as a limiting form of binomial distribution
- compute moments of binomial and poisson distribution
- fit binomial and poisson distribution to real life data
- understand the application of the concept of these probability distributions to real life data.

10.2 INTRODUCTION

So far we have brought out the nature of discrete and continuous probability distributions in general and also have defined their mean and variances. In this present unit we shall discuss about how the moments of a particular distribution as discussed in unit 4, can be generated by a special kind of function called the moment generating function.

Again we know that the rule generating the probability distribution is the same whether we want to find the probability of the number of heads when three coins are tossed simultaneously or for the number of red balls in case of an urn containing 3 red and 2 black balls, out of which, say, two balls are drawn at random with replacement. Given this basic characteristic of a discrete distribution, our main concern is to examine the outcoms generated by an experiment and choose a distribution that best describes those outcomes.

Under this consideration we shall discuss some discrete probability distributions applicable to many real life situations in the subsequent sections of this unit. The distributions we are going to discuss here are binomial and poisson distributions.

10.3 MOMENT GENERATING FUNCTION

The moments of a probability distribution play an important role in theoretical and in applied statistics. With the help of moments we can measure the central tendency of a set of observations, their variability, their symmetry and the height of the peak their curves would make. That is the knowledge of moments enable us to determine the probability distribution completely. So it would be very useful if a function could be found which will help in generating moments. One such function is moment generating function. So we can define a moment generating function as follows :

10.3.1 Definition

A function that is used to generate moments of a random variable X is called the moment generating function, denoted by M_x (t) and is given as

$$M_X(t) = E\left(e^{tx}\right).$$

When X is a discrete random variable with p.m. f.p(x)

$$M_x(t) = E(e^{tx}) = \sum_x e^{tx} p(x)$$

When x is a continous random variable in the interval (a,b) with p.d.f. f(x),

$$M_x(t) = E(e^{tx}) = \int_a^b e^{tx} f(x)dx.$$

10.3.2 Generation of Moments

We have, $M_X(t) = E\left(e^{tx}\right)$

$$= E(1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + ... + \frac{t^rx^r}{r!} + ...)$$

$$= E(1) + t\,E(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + ... + \frac{t^r}{r!}E(x^r) + ...$$

$\therefore$ μ_1' = co-efficient of t in $M_x(t)$

μ_2' = co-efficient of $\frac{t^2}{2!}$ in $M_x(t)$

μ_3' = co-efficient of $\frac{t^3}{3!}$ in $M_x(t)$

Thus in general we see that,

$\therefore \mu_r'$ = co-efficient of $\frac{t^r}{r!}$ in $M_x(t)$.

Another convenient way of obtaining moments is to differentiate $M_x(t)$ successively and then put t = 0.

So if we differentiate $M_x(t)$, one time, with respect to t and then put t = 0, we get μ_1',

i.e. $\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$

Differenting $M_x(t)$ times with respect to t and then put t = 0 we get μ_α'

$\mu_2' = \left[\frac{d^2}{dt^2} M_\lambda(t)\right]_{t=0}$

So in general, we get the raw moment of order r, μ_r' as

$\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}$

10.3.3 Preperties of Moment Generating Function

(i) If X is a random variable and C is a constant, there,

$M_{C_x}(t) = M_x(ct)$

Proof : L. H. S = $M_{C_x}(t) = E(e^{tcx})$

$= E(e^{ctx})$

$= M_x(ct)$

= R.H.S.

(ii) Additive property of moment generating function.

The moment generating function of sum of independent random variables is equal to the product of their moment generating functions. In symbol, if X_1, X_2, ..., X_n are independent random variables then,

$M_{X_1+X_2+\ldots+X_n}(t) = M_{X_1}(t).M_{X_2}(t) \ldots M_{X_n}(t).$

Proof : L.H.S. $= M_{X_1+X_2+\ldots+X_n}(t)$

$= E\left[e^{t(x_1+x_2+\ldots+x_n)}\right]$

$= E\left[e^{tx_1}.e^{tx_2}\ldots.e^{tx_n}\right]$

$= E(e^{tx_1}).E(e^{tx_2})\ldots E(e^{tx_n})$ $\quad \because X, X_2\ldots, X_n$ are independent

$= M_{X_1}(t).\ M_{X_2}(t)\ \ldots\ M_{X_n}(t)$

= R.H.S.

(iii) Effect of change of origin and scale on m. g. f.

$$M_{\frac{x-a}{h}}(t) = e^{\frac{-at}{h}} M_x\left(t/h\right)$$

Proof : $M_{\frac{x-a}{h}}(t) = E\left[e^{t\left(\frac{x-a}{h}\right)}\right]$

$$= E\left[e^{\frac{tx}{h}} e^{\frac{-at}{h}}\right]$$

$$= e^{\frac{-at}{h}} E\left[e^{\frac{t}{h}(x)}\right]$$

$$= e^{\frac{-at}{h}} M_x\left(t/h\right)$$

= R.H.S.

Note : (i) The moment generating function does not always exist.

(ii) In some cases the moment generating function exists, but it cannot generate moments.

Illustrated Example

1. Let X be a random variable with p. m. f.

$p(x) = q^x p, \quad x = 0, 1, 2, \ldots$

find the moment generating function and find its mean and variance.

Solution : $M_x(t) = E(e^{tx})$

$$= \sum_{x=0}^{\alpha} e^{tx}.q^x p$$

$$= p\sum_{x=0}^{\alpha}(qe^t)^x$$

$$= p\left[1 + qe^t + (qe^t)2 + \ldots\right]$$

$$= p\left(1 - qe^t\right)^{-1}$$

Mean = $\mu_1' = \left[\frac{d}{dt}M_x(t)\right]_{t=0}$

$$= \left[(-1)p(1-qe^t)^{-2}(-qe^t)\right]_{t=0}$$

$$= pq(1-q)^{-2}$$

$$= \frac{pq}{p^2}$$

$$= \frac{p}{p}$$

$$\mu_\alpha' = \left[\frac{d^2}{dt^2}M_x(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{pq\ e^t(1-ge^t)^{-2}\right\}\right]_{t=0}$$

$$= pq\left[et(1-ge^t)^{-2} + e^t(-2)(1-qe^t)^{-3}(-ge^t)\right]_{t=0}$$

$$= pq\left[(1-q)^{-2} + 2q(1-q)^{-3}\right]$$

$$= \frac{pq}{p^2} + \frac{2pq^2}{p^3}$$

$$= \frac{q}{p} + \frac{2q^2}{p^2}$$

The expansion used here is called binomial expansion with negative index, which is $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots$

$\therefore$ variance $= \mu_\alpha = \mu_\alpha' - \mu_1'^2$

$$= \frac{q}{p} + \frac{2q^2}{p^2} - \frac{q^2}{p^2}$$

$$= \frac{q}{p} + \frac{q^2}{p^2}$$

$$= \frac{q(p+q)}{p^2}$$

$$= \frac{q}{p^2}$$

2. Final moment generating function of a continuous random variable x having the p.d.f.

$$f(x) = \theta e^{-\theta x},\ x \geq 0$$

Hence find its mean and variance.

Solution : $M_x(t) = E(e^{tx})$

$$= \int^{\alpha} e^{tx}\, \theta e^{-\theta x} dx$$

$$= \theta^0 \int^{\alpha} e^{(t-\theta)x} dx$$

$$= \theta^0 \left[\frac{e(t-\theta)x}{t-\theta} \right]_0^{\alpha}$$

$$= \theta \left[\frac{1}{t-\theta}(0-1) \right]$$

$$= \frac{-\theta}{t-\theta}$$

$$= \frac{\theta}{\theta - t} = \left(1 - \frac{t}{\theta}\right)^{-1}$$

$$= 1 + \frac{t}{\theta} + \left(\frac{t}{\theta}\right)^2 + \left(\frac{t}{\theta}\right)^3 + \ldots.$$

$\therefore \mu_1'$ = mean - co-eff^t. of t. in $M_x(t) = \frac{1}{\theta}$

μ_2' = mean - co-eff^t. of t. in $M_x(t) = \frac{2}{\theta^2}$

$\therefore$ variance $= \mu_2 = \mu_2' - \mu_1'^2 = \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2 = \frac{1}{\theta^2}$



CHECK YOUR PROGRESS

Q 1: State true or false.

(i) Since M_x (t) generates moments, it is called moment generating function. (T / F)

(ii) If X an Y are independent r. v then $M_{x+y}(t) = M_x(t) + M_y(t)$ (T / F)

(iii) Moment generating function always exists. (T / F)

(iv) A moment generating function, if exists, determines the distribution function. (T / F).

10.4 BINOMIAL DISTRIBUTION

The binomial distribution, as derived by James Bernoulli in the year 1700, deals with population whose elements or outcomes can be divided into two categories with reference to presence or absence of a particular attribute or characteristic. For example, the answer to a question may be correct or incorrect, results in an examination may be pass or fail, the factory workers may be educated or uneducated.

That is the outcome in each of these situations may be viewed in terms of presence or absence of an attribute. That is we consider only the two outcomes one possessing and the other not possessing the attribute. A convenient method is to denote them as 'success' and 'failure' according as the presence or absence of the attribute.

The mathematical background for binomial distribution describes discrete data resulting from an experiment called Bernoulli trials, which are repeated finite number of times. Let us now derive binomial distribution from a finite number of Bernoulli trials.

Note : A random experiment whose outcomes has been classified into two categories, called 'success' and 'failure' is called Bernoulli trial. For example, tossing of a coin may result in a head or in a tail.

10.4.1 Derivation of Binomial Distribution

Consider a coin tossing experiment which is repeated, say, n number of times. When we get a head we call it a success and when we get a tail, we call it a failure, success is denoted by 's' and failure by 'f'. Probability of success is p and that of failure is q.

P(s) = p, P(f) = q, q = 1 – p.

suppose we are interested in getting x success out of n trials. Now the probability of x successes and consequently (n-x) failures in n independent trials, in a specified order, say, first x trials succes and the remaining (n-x) trials failure is

P(ss . . . s f f . . . f) = P(s) P(s) . . . P(s) P(f) . . . P(f)

$\because$ the trials are independent

$$= \underbrace{p.p...p}_{x\ factor}.\ \underbrace{q.q...q}_{(n-x)\ factor}$$

$= p^x q^{n-x}$

But this is simply one way of getting the probability of x successes out of n trials. As we know, x successes in n trials can occur in n_{C_x} ways, and the probability for each of these cases is $p^x q^{n-x}$, Hence by the addition theorem of probability the probability of getting x success in n trials is given by.

$p^x q^{n-x} + p^x q^{n-x} + \ldots + p^x q^{n-x}$, n_{C_x} times

$= n_{C_x} p^2 q^{n-x}$.

Which is the probability mass function of binomial distribution. Again, x, the number of successes out of n trials can take values 0, 1, 2, ..., n. Therefore, we may define a binomial distribution as follows.

10.4.2 Definition

A random variable X is said to follow binomial distribution if it assumes non-w values and if its p.m.f. is given by,

$$p(x) = p(X = x) = n_{C_x}\ p^x\ q^{n-x},\ x = 0, 1, 2, \ldots, n$$

= 0, otherwise.

The two constants n an p are the parameters of this distribution. It is also abbreviated as B(n, p).

Note : The probability of getting 0 success. 1 success, ..., n successes in this case, i.e. $n_{C_o} p^o q^n$, $n_{C_1} p\, q^{n-1}$, . . ., $n_{C_n} p^0 q^x$ ... are the successive terms in the expansion of $(q + p)^n$, a binomial expansion, that is why this distribution is also termed as 'binomial distribution'.

10.4.3 Moments of Binomial Distribution

We are interested mainly in the first two moments of binomial distribution which are obtained as follows.

$$\mu_1' = Mean = E(x) = \sum_x x\, p(x)$$

$$= \sum_{x=0}^{n} x\; n_{c_x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n}{x}^{n-1} c_{x-1} p^x q^{n-x} \qquad \because n_{c_x} = \frac{n}{x},\ n-1_{c_x}$$

$$= np \sum_{x=1}^{n} n-1\ _{C_{x-1}} p^{x-1} q^{n-x}$$

$$= np\left[{}^{n-1}C_o\, p^0 q^{n-1} + n-1_{C1}\, pq^{n-2} + \ldots + {}^{n-1}C_{n-1} p^{n-1} q^0\right]$$

$= np(q + p)^{n-1}$

$= np \qquad \because p + q = 1$

Hence the mean of binomial distribution is n p.

Again, $\mu_2' = E(\lambda^2) = E\{X(x-1)\} + E(x)$

$$E\{x(x-1)\} = \sum_{x=0}^{n} x(x-1)\; n_{c_x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n(n-1)}{x(x-1)}^{n-2} c_{x-2} p^x q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} n-2\; c_{x-2} p^{x-2} q^{n-x}$$

$$= n(n-1)p^2 (q+p)^{n-2}$$

$$= n(n-1)p^2.$$

$\therefore E(x^2) = n(n-1)p^2 + np$

$\therefore \mu_2$ = variance $= \mu_2' - \mu_1'^2 = n(n-1)p^2 + np - n^2p^2$

$= n^2p^2 - np^2 + np - n^2p^2$

$= np(1 - p)$

$= npq.$

So the variance of binomial distribution is npq.

Remark : For binomial distribution mean is always greater than variance. This is an important characteristic of binomial distribution.

10.4.4 Moment Generating Function of Binomial Distribution

If X ~ B (n, p), then

$$M_x(t) = E(e^{tx}) = \sum_x e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} \, {}^nc_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nc_x (e^t p)^2 q^{n-x}$$

$$= {}^nc_0 (e^t p)^0 q^n + {}^nc_1 (e^t p) q + \ldots + {}^nc_n (e^t p)^n q^r$$

$$= (q + e^t p)^n$$

which is the moment generating function of binomial distribution.

10.4.5 Fitting of Binomial Distribution

The recurrence relation for probabilities of binomial distribution is, $p(x+1) = \frac{n-x}{x+1} \frac{p}{q} p(x)$. x = 0, 1, 2, . . ., n–1

Putting x = 0, 1, 2, ..., (n-1) we get p(1), p(2), ..., p(n). Here, we need to calculate only p(0), which is.

$p(0) = {}^nc_0 \, p^0 q^n = q^n$

For felling a binomial distribution, we need to find the values of n and p. n is generally known, and $p = \frac{np}{n}$.

where np is the mean of the given set of data.

After calculating the value of p, we calculate the probabilities p(0), p(1), ... , p(n) as discussed above. Now multiplying these probabilities by N, the total frequency, we get the expected frequencies for x = 0, 1, 2, ..., n. Now we shall illustrate the fitting of biomial distribution with the help of an example.

Note : Calculation of expected frequency is called fitting.

Example : 3 coins are tossed 64 times and the following distribution of number of heads were obtained.

Number of heads :	0	1	2	3
Frequency :	10	18	20	16

Fit a binomial distribution under the hypothesis that the coins are unbiased.

Solution : Since the coins are unbiased

$p = q - \frac{1}{2}$ [in case coins are biased p is to be calculated from mean]

Here, N = 64, n = 3

Now, $p(0) = q^n = \left(\frac{1}{2}\right)^3 = \frac{1}{2^3} = \frac{1}{8}$

Now using recurrence relation, $p(x+1) = \frac{n-x}{x+1}\frac{p}{q}p(x)$

We calculate the probabilities as,

$$p(1) = \frac{3-0}{0+1}\frac{p}{q}p(0) = 3.\frac{1}{2^3} = \frac{3}{8}$$

$$p(2) = \frac{3-1}{1+1}\frac{p}{q}p(1) = \frac{3}{2}.\frac{3}{8} = \frac{3}{8}$$

$$p(3) = \frac{3-2}{2+1}\frac{p}{q}p(2) = \frac{1}{3}.\frac{3}{8} = \frac{1}{8}$$

Hence the expected frequenties are,

x	Expected frequency
0	$n \times p(0) = 64 \times \frac{1}{8} = 8$

1 $n \times p(1) = 64 \times \frac{3}{8} = 24$

2 $n \times p(2) = 64 \times \frac{3}{8} = 24$

3 $n \times p(3) = 64 \times \frac{1}{8} = 8$

10.4.6 Properties of Binomial Distribution

(i) The binomial distribution is a discrete distribution where the random variable X takes the values 0, 1, 2, ..., n.

(ii) The binomial distribution has two parameters, n and p.

(iii) The mean is equal to np and variance is given by npq.

(iv) The mean of binomial distribution is always greater than variance.

(v) If X_1 and X_2 follows binomial distribution with parameters (n_1, p) and (n_2, p) respectively, these $x_1 + x_2$ follows binomial distribution with parameter ($n_1 + n_2$, p). This is known as the additive property of binomial distribution.

(vi) Binomial distribution with parameters (n. p) tends to poisson distribution if n is very largs, p is very small and np is finite.

(vii) Binomial distributions also tends to Normal distribution if n is very large and neither non q is very small.

10.4.7 Importance of Binomial Distribution

Theoretical distribution like Binomial distribution provide data on the basis of which observed (empirical or experimental) results can be assessed. In fact, where theoretical distributions are available one need not have to bother for observed distributions. Theoretical distributions provide the decision maker with a sound basis for taking rational and dependable decisions. However in order to apply Binomial distribution, the assumptions of the distribution must hold true which are: two mutually exclusive outcomes, a fixed probability of outcome in any trial and independent trials.

Illustration Examples

1. A r. v. X follows binomial distribution with mean 5/3 and P(x = 2) = P (x = 1). Find variance, P(x = at least 1), P(x = at most 1).

Solution : If X follows binomial distribution with parameters n and p, the p.m.f. of X is,

$p(x) = {}^{n}c_{x} p^{x} q^{n-x}$, x = 0, 1, 2, . . . , n; q = 1–p

Given, mean = np = $\frac{5}{3}$

and P(x = 2) = P(x = 1)

$\Rightarrow {}^{n}c_{2} p^{2} q^{n-2} = {}^{n}c_{1} p q^{n-1}$

$\Rightarrow \frac{n(n-1)}{2} \frac{p}{q} = n$

$\Rightarrow (n-1)p = 2q$

$\Rightarrow np - p = 2q$

$\Rightarrow \frac{5}{3} - p = 2q$

$\Rightarrow 5 - 3p = 6q$

$\Rightarrow 6q + 3p = 5$

$\Rightarrow 6(1-p) + 3p = 5$

$\Rightarrow 6 - 6p + 3p = 5$

$\Rightarrow -3p = -1$

$\Rightarrow p = 1/3$

$\therefore q = 1 - p = 2/3$

Again, $np = \frac{5}{3}$ $\therefore n = \frac{5}{3} \Big/ \frac{1}{3} = 5$

Hence, variance $= npq = 5.\frac{1}{3}.\frac{2}{3} = \frac{10}{9}$

P(x = at least 1) = $p(x \geq 1) = 1 - p(x = 0)$ $\because$ $p(x = 0) + p(x \geq 1) = 1$

$= 1 - {}^{5}C_{0} \left(1/3\right)\left(2/3\right)^{5} = 1 - \left(\frac{2}{3}\right)^{5} = \frac{211}{243}$

P(x = at most 1) = P(x=0) + p(x=1)

$$= {}^5C_0\left(\tfrac{1}{3}\right)^0\left(\tfrac{2}{3}\right)^5 + {}^5C_1\left(\tfrac{1}{3}\right)^1\left(\tfrac{2}{3}\right)^4$$

$$= \frac{112}{243}$$

2. Probability of follows in statistics examination is 15%, 20 batches of 5 students each appear for the examination. Obtain in how many batches 3 or more students would pass.

Sol^n : Given q = probability of failure = 15% = 0.15

$\therefore$ q = probability of success = 0.85

n = 5.

If X denote the number of students who would pass, then

$p(x) = {}^nC_x p^x q^{n-x}$, x = 0, 1, . . ., 5.

We need $p(x \geq 3) = p(x=3) + p(x=4) + p(x=5)$

$= {}^5C_3 p^3 q^{5-3} + {}^5C_4 p^4 q^{5-4} + {}^5C_5 p^5 q^0$

$= 10x\ (.85)^3\ (.15)^2 + 5(0.5)^4\ (0.15) + (0.15)^5$

$= 0.1381 + 0.3915 + 0.4437$

$= 0.9733.$

$\therefore$ Required number of batches in which 30 or more students would pass $= 20 \times 0.9733$

$= 19.468 \approx 19.$



CHECK YOUR PROGRESS

Q 2: If $X \sim B\left(12, \frac{1}{3}\right)$ what is E(x)?

Q 3: Examine the following statement and add your comment, "The mean of a binomial distribution is 5 and s.d. is 3".

Q 4: What are the parameters of binomial distribution?

Q 5: Find the varience of the binomial distribution

$p(x) = {}^{10}C_x\left(\frac{2}{5}\right)^x\left(\frac{3}{5}\right)^{10-x}$, x = 0, 1, 2, . . ., 10.

Q 6: Define Bernoulli trials. Find its distributions and also its mean and variance.

Q 7: Derive Binomial distribution

or

Obtain the probability of exactly x successes in n independent Bernoulli treats.

10.5 POISSON PROCESS

The Poisson process was first development by French mathematician and physicist serieon Devis Poisson, who published if in the year 1837. In the class of univariate distributions, the poisson distribution is perhaps the most useful one, because of its wide applicability in the description of natural phenomena, for example, demand for service in a big departmental store, major accident rate per day in a certain highway etc. A typical application of poisson distribution is for analysing queing (or waiting live) problems in which arriving customers during an interval of time arrive independently and the number of arrivals depends on the length of the time interval.

10.5.1 Poisson Distribution as a limiting care of Binomial Distribution

The binomial probability law of rare events gives rise to poisson probability law. That is poisson distribution can be derived as a limiting case of binomial distribution under the following conditions.

(i) n, the number of independent Bernoulli trials is very large, i, e $n \to \infty$

(ii) p, the probability of success in each trial is very small, i.e. $p \to \infty$

(iii) np = λ (say), is finits, where λ is a positive real number. under the above conditions, we can derive the p.m.f. of poisson distribution as follows:

Let us consider the p.m.f. of poisson distribution as,

$$p(x) = \binom{n}{x} p^x q^{n-x}, \qquad x = 0, 1, 2, \ldots, n; \qquad q = 1 - p.$$

Now, $\lim_{n\to\infty} p(x) = \lim_{n\to\infty} \binom{n}{x} p^x q^{n-x}$

$$= \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} \quad \because np = \lambda$$

$$= \frac{\lambda^x}{n!} \lim_{n\to\infty} \frac{n!}{(n-x)!n^x} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{n!} \lim_{n\to\infty} \frac{n!}{(n-x)!n^x} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{n!} \lim_{n\to\infty} \frac{n(n-1)(n-2)\ldots(n-x-1)(n-x)!}{(n-x)!\,n^x}$$

$$= \frac{\lambda^x}{n!} \lim_{n\to\infty} \frac{n^x\left\{1.\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)\right.}{n^x},$$

$$= \frac{e^{-\lambda}\lambda^x}{x!} \quad \because \lim_{n\to\infty} \frac{1}{n} \to 0$$

Which is the p.m.f of poisson distribution. We give below the formal definition of poisson distribution as,

Definition :

A discrete random variable X is said to follow poisson distribution, if it assumes non-negative values only and its p.m.f is given by,

$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, x = 0, 1, 2, . . ., λ

= 0, else where.

λ is the only parameter of this distribution.

10.5.2 Moments of Poisson Distribution

The arithmetic mean of poisson distribution is given by,

$$\mu_1' = E(x) = \sum_x x\, p(x)$$

$$= \sum_{x=0}^{\alpha} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\lambda \sum_{x=1}^{\alpha} \frac{x\,\lambda^{x-1}}{x(x-1)!}$$

$$= e^{-\lambda}.\lambda \sum_{x=1}^{\alpha} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda}.\lambda \left[1 + \lambda + \frac{\lambda^2}{2!} + \ldots\right]$$

$$= e^{-\lambda}.\lambda.e^{\lambda}$$

$$= \lambda .$$

$\therefore$ the mean of poisson distribution is λ.

$\mu_2' = E(x^2) = E\{x(x-1)\} + E(x)$

Now, $E\{x(x-1)\} = \sum_x x(x-1)p(x)$

$$= \sum_{x=0}^{\alpha} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!}$$

The exponential expansion,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$= e^{-\lambda}\lambda^2 \sum_{x=2}^{\alpha} x(x-1)\frac{\lambda^{x-2}}{x.(x-1)(x-2)!}$$

$$= e^{-\lambda}\lambda^2 \sum_{x=2}^{\alpha} \frac{\lambda^{x-2}}{(x-2)!}$$

$$= e^{-\lambda}\lambda^2 \left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right]$$

$$= e^{-\lambda}\lambda^2.e^{\lambda}$$

$$= \lambda^2$$

$\therefore\ E(\lambda^2) = \lambda^2 + \lambda$

Now, $v(x) = \mu_2 = \mu_2' - \mu_1'^2$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda .$$

Hence the variance of poisson distribution is also.

Remark : Poisson distribution is the only discrete distribution where mean is equal to variance.

10.5.3 Moment Generating Function

The moment generating function of poisson distribution is given by,

$$Mx(t) = E(e^{tx}) = \sum_{x=0}^{\alpha} e^{tx} . \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\alpha} \frac{(e^t\lambda)^2}{x!}$$

$$= e^{-\lambda}\left[1 + e^t\lambda + \frac{(e^t\lambda)^2}{2!} + \frac{(e^t\lambda)^3}{3!} + \ldots\right]$$

$$= e^{-\lambda} e^{e^t\lambda}$$

$$= e^{-\lambda}(1 - e^t)$$

10.5.4 Fitting of Poisson Distribution

The recurrence relation for probabilities of poisson distribution is,

$$p(x+1) = \frac{\lambda}{x+1} p(x), \quad x = 0, 1, 2, \ldots . \qquad (1)$$

Putting x = 0, 1, 2, ... , in (1) we get the probabilities p(1), p(2), ... , etc. so we need to calculate p(0) only, which is,

$$p(0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda} .$$

where, λ is the mean of poisson distribution is to be obtained from data to be fitted.

Ultimately, the expected frequencies are calculated by multiplying the corresponding probabilities by N, the total frequency. We shall explain the fitting of poisson distribution to a given data set.

Example : 200 drivers made the following road accidents

Accidents :	0	1	2	3	4
No. of drivers :	122	60	15	2	1

Fit a theoretical distribution to the data.

Sol^n : The suitable theoretical distribution to be fitted here may be the poission distribution,

Let, x_i = number of road accidents

f_i = no. of drivers.

Total number of drivers = N = 200

Here, λ = mean of poisson distribution

$$= \frac{1}{N} \sum_{i=1}^{n} f_i x_i$$

$$= \frac{100}{200} = 0.5$$

$\therefore\ p(0) = e^{-0.5} = 0.60653$

Using recurrence relation we get,

$$p(1) = \frac{\lambda}{0+1} p(0) = \lambda . p(0) = 0.5 \times 0.60653 = 0.30327$$

$$p(2) = \frac{\lambda}{2} p(1) = \frac{0.5}{2} \times 0.60653 = 0.07582$$

$$p(3) = \frac{\lambda}{3} p(2) = \frac{0.5}{3} \times 0.60653 = 0.01264$$

$$p(4) = \frac{\lambda}{4} p(3) = \frac{0.5}{4} \times 0.60653 = 0.00158$$

$\therefore$ the expected frequencies are,

f(0) = N x p(0) = 200 x 0.60653 = 121.306 $\approx$ 121

f(1) = N x p(1) = 200 x 0.30327 = 60.654 $\approx$ 61

f(2) = N x p(2) = 200 x 0.07582 = 15.164 $\approx$ 15

f(3) = N x p(3) = 200 x 0.01264 = 2.528 $\approx$ 3

f(4) = N x p(4) = 200 x 0.00158 = 0.316 $\approx$ 0

Total frequency = 200

10.5.5 Properties of Poisson Distribution

(i) Poisson distribution is a discrete distribution where the random variable X lakes the values 0, 1, 2, ... , λ.

(ii) The poisson distribution has only one parameter λ.

(iii) The mean and the variance of poisson distribution are equal

and is equal to λ.

(iv) The poisson distribution has only one or two modes.

(v) Poisson distribution can be obtained as a limiting case of binomial distribution.

(vi) If X and Y are independent poisson variates with paremeters λ and λ_2 respectively, then x + y is also a poisson variate with parameter $\lambda_1 + \lambda_2$.

10.5.6 Application of Poisson Distribution

Physical situations in daily life where poisson distribution is applicable is

(i) Number of telephone calls recieved per minute at a particular switch board.

(ii) Number of accidents in a city per unit time.

(iii) Number of defective parts produced in a factory per unit time.

(iv) Number of printing mistakes in each page of a book.

Illustrated Examples

Example 1 : Find the mean and variance of poisson distribution such that, P(x = 2) = P(x = 1)

Solution : It X follows poisson distribution with parameter λ,

then, $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, x = 0, 1, 2, ...

Now, p (x = 2) = p (x = 1)

$$\Rightarrow \frac{e^{-\lambda}\lambda^2}{2!} = \frac{e^{-\lambda}.\lambda}{1!}$$

$$\Rightarrow \frac{\lambda}{2} = 1$$

$$\Rightarrow \lambda = 2$$

$\therefore$ the man and variance of poisson distribution is equal to $\lambda = 2$.

Example 2 : The number of ticketters traceallers deteched per checking follows poisson distribution with mean 10. Find the

probability of getting two ticketters tracallers.

Solution : Let X denote the no. of ticketters traceallers in a checking.

so X follows poisson distribution with given mean.

$\lambda = 10$

$\therefore p(x) = \frac{e^{-10}\, 10^x}{x!}$, x = 0, 1, 2, ...

$\therefore$ P(getting two ticketters tracallers during a checking)

= P(x = 2)

$= \frac{e^{-10}\, 10^2}{2!}$

= 0.0025.

3. What probability model is appropriats to describes a situation where 100 misprints are distributed randomly throughout the 100 pages of a book? For this model what is the probability that a page observed at random will contain at least 3 misprints.

Solution : Since 100 misprints are distributed randomly throughout the 100 pages of the book, the probability that there being a misprint is $p = \frac{1}{100} = 0.01$, which is very small.

n = 100 pages is quite large.

Hence poisson distribution is but reaited in this case.

Here, $\lambda = np = 0.01 \times 100 = 1$

$\therefore$ Probability that there are alteast three misprints in a page is

$$P(x \geq 3) = 1 - p(x < 3) = 1 - \{P(x=0) + P(x=1) + P(x=2)\}$$

$$= 1 - \left\{ e^{-1} + e^{-1} + \frac{e^{-1}}{2!} \right\}$$

$$= 1 - 2.5 e^{-1}$$

$$= 1 - 2.5 \times 0.3679$$

$$= 0.0802 .$$



CHECK YOUR PROGRESS

Q 8: Fill in the blanks :

(i) For a poisson distribution mean is ..... to variance (less/equal/greater)

(ii) The s.d. of a poisson distribution with mean 4 is ..... (4/2).

(iii) For a poisson distribution $\frac{e^{-1.5}(1.5)^x}{x!}$, x = 0, 1, 2, ... the varience is (1.5/2.5).

(iv) If x follows poisson distribution is .....

$e^{-\lambda(1-e^t)} / e^{-\lambda}(1-2)^t$

(v) If x_i, i = 1, 2, ... , n are n poisson variates then the distribution of $\sum_{i=1}^{n} x_i$ is also a ..... variate. (Binomial / Poisson)

Q 9: "Poisson distribution is also known as probability distribution of rare events" – Explain.

Q 10: Give some examples of poisson distribution.



10.6 LET US SUM UP

- A function, which is used to generate moments of a probability distribution is known as moment generating function.
- The importance of moment generating function lies in the fact that, a probability distribution can be completely determined if its moments are known.
- The moment generating function of a random variable X is given by, $M_x(t) = E(e^{tx})$, t > 0.
- A random experiment whose outcoms are classified into two categories, called 'success' and 'failure' is called a Bernoulli triats.
- The p. m. f. of binomial distribution with parameters n and p is,

$p(x) = {}^nC_x p^x q^{n-x}$, x = 0, 1, 2, . . ., n.

- The mean of binomial distribution is np and variance is npq.
- For a binomial distribution, mean is always greater than variance.
- Sum of independent binomial variates is also a binomial variate.
- Poisson distribution can be derived as a limiting case of binomial distribution when, $n \to \alpha$, $p \to o$ and $np \to \lambda$ (say).
- The p. m. f. of poisson distribution with parameter λ is given by,

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots, \lambda.$$

- Poisson distribution plays an important role in the description of random phenomena in a continuous interval of any kind. For example, telephone calls at a telephone exchange within a specified period of time.
- Sum of independent poisson variates is also a poisson variate.



10.7 FURTHER READINGS

1) Das, K. K, Bhattacharjee, D (2008). A Treatere on statistical Inference and Distributions. Asian Books Pvt. Ltd.
2) Gupta, S. C. and Kapoor, V. K. (2002). Fundementals of Mathematical statistics; Sultan Chand & Sons.
3) Goon, A. M. Gupta. M. P and Dasgupta, B (2002). Fundementals of statistics (Vol I), Seventh Revised Edition; The World Press Pvt. Ltd. Kolkata.
4) Saxena, H. C. (1994). Probability and its application, sixth edition; S. Chand and Company, New Delhi.
5) Singh, R (2012). An Introduction to Probability and Probability Distributions, Books and Ellied (P) Ltd.

10.8 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (i) True, (ii) False, (iii) False, (iv) True.

Ans to Q No 2: We know that if X ~ B (n, p) then E(x) = np.

Here, n = 12, $p = \frac{1}{3}$

$\therefore E(x) = np = 12.\frac{1}{3} = 4$

Ans to Q No 3: Here, given, mean = 5, s. d. = 3

$\therefore$ variance = 9

But we know that for a binomial distribution, mean is always greater than variance. Hences, the above statement is false.

Ans to Q No 4: The parameters of a binomial distribution are n and p.

Ans to Q No 5: Given, $p(x) = {}^{10}C_x \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{10-x}$, x = 0, 1, 2, ... , 10.

Here, n = 10, $p = \frac{2}{5}$, $q = \frac{3}{5}$

Hence, variance = $npq = 10.\frac{2}{5}.\frac{3}{5} = \frac{12}{5}$

Ans to Q No 6: A particular trial having two outcomes only, viz success and failure occurring with probability p and (1–p) respectively is called a Bernoulli trial. e.g. tossing of a coin.

A random variable X is said to follow Bernoulli distribution if its p.m.f. is given by,

$p(x) = p^x (1-p)^{1-x}, \quad x = 0, 1.$

p is the only parameter of this distribution

$\therefore$ Mean = E(x) = 0. (1–p) + 1 . p = p.

$V(x) = E(x^2) - \{E(x)^2\}$.

$E(x^2) = 0^2 . (x-p) + 1^2 . p = p$

$\therefore V(x) = p - p^2 = p(1-p)$.

Ans to Q No 7: Consider n independent Bernoulli trials each having two and two outcomes only called success and failure, Probability of success is p and the probability of failure is (1 – p) = q.

The probability of x successes and (n–x) failures in n independent trials in a specific order, say, s s s... f.f.f is given by,

P (s. s. . . . s. f . f. . . . f) = p(s) p(s) s . . . p(s) p(f) . . . p(f) ∵ trials are independent

$$= \underbrace{p \cdot p \ldots p}_{x \text{ times}} \underbrace{q\, q \ldots q}_{(n-x) \text{ times}}$$

$$= p^x q^{n-x}.$$

But, x successes in n trials can occur in ${}^{n}C_{x}$ times which are mutually exclusive and the probability for each of these is $p^x q^{n-x}$. Hence by addition theorem of probability,

$p(x=x) = {}^{n}C_{x}\, p^x q^{n-x}$, x = 0, 1, 2, . . ., n.

Ans to Q No 8: (i) equal, (ii) 2, (iii) 1.5, (iv) $e^{-\lambda(1-e^t)}$ (v) poisson.

Ans to Q No 9: Poisson distribution is also known as probability distribution of rare events, because poisson distribution is a positively skewed distribution, i.e probabilities tend to zero for large number of occurrences. For these poisson distribution is also known as probability distribution of rare events.

Ans to Q No 10: Some examples of poisson distribution are,

(i) number of deaths from suicide reported in a particular city.

(ii) number of defective items in a lot of manifactures items.

(iii) number of printing mistakes in each page of a book.



10.9 MODEL QUESTIONS

Q 1: Define moment generating function and state its properties.

Q 2: Write a note on binomial distribution.

Q 3: Define a binomial distribution distribution and also obtain its mean and variance.

Q 4: Obtain poisson distribution as a limiting case of binomial distribution.

Q 5: If X follows poisson distribution with $V(\lambda) = 2$, find its p.m.f.

Q 6: Define poisson distribution, state the conditions under which the binomial p.m.f. can be approximated by the poisson p.m.f.

Q 7: Obtain m.g.f. of poisson distribution and hence find its mean.

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UNIT 11: INDEX NUMBERS

UNIT STRUCTURE

11.1 Learning Objectives

11.2 Introduction

11.3 Uses of Index Numbers

11.4 Types of Simple Index Numbers

11.5 Properties of Relatives

11.6 Construction of Simple Index Numbers

11.7 Construction of Weighted Index Numbers

11.8 Test of Adequacy of Index Numbers

11.9 Let Us Sum Up

11.10 Further Readings

11.11 Answers To Check Your Progress

11.12 Model Questions

11.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- discuss about the simple and weighted index numbers
- explain the uses of index numbers
- discuss the different types of index numbers
- explain the properties of different index numbers
- discuss the test of accuracy of index numbers.

11.2 INTRODUCTION

Index number is a statistical device (measure) with a purpose of showing average changes in one or more related variables (like price or quantity) between two periods of time (say, between 1991 and 1996) or two places like cities (say, Delhi and Mumbai) or countries (say, India and Japan). For example, we may be interested in knowing which city of India out of Delhi, Mumbai, Kolkata and Chennai is the costliest or the cheapest in terms of price level. A tourist may be interested in knowing about the cost of living

at different tourist places. Consumer price index number or cost of living index number helps in taking such decisions.

Different commodities are measured in different units. For example, rice and wheat are measured in kilograms, cloth in meters and milk in litres etc. Index numbers attempt some averages relating to commodities which are measured in different units. For purpose of comparison, we are interested in knowing relative changes. These relative changes are expressed in percentage terms.

Index numbers are the indicators of the various trends in an economy. Price index numbers indicate the position of prices whether they are rising or falling and at what rate. Similarly, index numbers regarding agricultural production indicates the trend of change whether it is rising or falling at what rate over a period of time. An index number is an economic data figure reflecting price or quantity compared with a standard or base value. The base usually equals 100 and the index number is usually expressed as 100 times the ratio to the base value. For example, if a commodity costs twice as much in 1970 as it did in 1960, its index number would be 200 relative to 1960. Index numbers are used especially to compare business activity, the cost of living, and employment.

An index number is specialized average. Index numbers may be simple or weighted depending on whether we assign equal importance to every commodities or different importance to different commodities according to the percentage of income spent on them or on the basis of some other criteria. In this chapter, we shall discuss both simple and weighted index numbers.

11.3 USES OF INDEX NUMBERS

In business an index is a statistical measure of changes in a representative group of individual data points.

Index numbers are constructed with various aims and purposes. Index numbers constructed are put into several uses. Some of them are described below:

1) The most commonly used types of index numbers are those that relate to changes in level in prices of a commodity or a group of commodities. Such index numbers are price index numbers. Price index numbers are useful in studying price movements with a view to analyzing their causes, as well as to determine their effects on the economy. To determine economic relationships, it is often useful to compare changes in the market price level with changes in other series, such as gold, bank deposits, bank loans, and the physical volume of production.

2) Indices of physical changes over a period of time in production, in marketing, in sales, in imports and exports, and so on have important uses. Such index numbers over a period of time are known as index time series. These index numbers are extremely useful in the study of the movement of the characteristics over time, in the study of seasonal and cyclic behaviour, and in the study of business cycles, etc.

3) Index numbers can be used for forecasting future trend in demand of commodities or in volume of production etc. Index numbers of industrial and agricultural production not only reflect the trend but also are useful in forecasting production. Index numbers of unemployment which reflect the trend of unemployment are useful in determining factors leading to unemployment and in forecasting future trends.

4) Index numbers which reflect the general price level, and in particular, the consumer price index numbers are useful in determining the quantum of additional wages or dearness allowances to compensate for the changes in the cost of living. The Government of India and most of the State Governments have index linked formulae for grant of dearness allowances to their employees.

5) The changes in consumer price index can be used to determine the real income over a period. Suppose that a person's income in 1980 was 150 times than that in 1970, that is, there is an increase of 50% in money income. Suppose that the cost of living index number in 1980 was 175 times than that in 1970, that is, there is an increase of 75%.

Then the person's real income in 1980 is $\frac{150}{175} = 85.7\%$ of his income in 1970. Thus index numbers are useful in determining the real income.

11.4 TYPES OF SIMPLE INDEX NUMBERS

1) Price Relatives:

One of the simplest types of index numbers is a price relative. It is the ratio of the price of a single commodity in a given period or point of time to its price in another period or point of time, called the reference period or base period. If the prices for a period, instead of a point of time, are considered, then suitable price average for the period is taken and these prices are expressed in the same units. If p_0 and p_1 denote the price of a commodity during the base period or reference period (0) and the given period (1) then the price relative of the period 1 with respect to the base period 0 is defined by

Price relative in percentage (of period 1 with respect to 0)

$$= \frac{p_1}{p_0} \times 100 \quad \ldots\ldots(11.1)$$

We denote price relative in percentage or without percentage by $p_{0/1}$.

Remark: In the formula of price relative we multiply $\frac{p_1}{p_0}$ by 100 only to get a better expression. This expression is therefore called price relative in percentage.

Exercise 1: If the retail price of fine quality of rice in the year 1980 was Rs.3.75 and that for the year 1983 was Rs.4.50, then find the price relative.

Solution: Here the base period is 1980. The price of rice in the base period was Rs.3.75. Also in the given period 1983, the price of rice was Rs.4.50. Using the formula (11.1), the required price relative is

$$p_{1980/\ 1983} = \frac{Rs.4.50}{Rs.3.75} \times 100 = 120\%$$

Exercise 2: The exchange rate of a US dollar was Rs.40.00 in July 2008 and was Rs.50.00 in February 2009. Find the price relative.

Solution: The price relative of a dollar in February 2009 (say period F) with respect to that in July 2008 (say period J) is given by

$$p_{F/J} = \frac{Rs.50.00}{Rs.40.00} \times 100 = 125\%$$

2) Quantity Relatives:

Another simple type of index number is a quantity relative. This is useful when we are interested in changes in quantum or volumes of a commodity such as quantities of production or sale or consumption. Here the commodity is used in a more general sense. It may mean the volume of goods (in tonnes) carried by roadways or the volume of export to a country or import from a country. In such cases, we consider of quantity or volume relatives. If quantities or volumes are taken for a period instead of a point of time, a suitable average is to be taken and the quantities or volumes are to be expressed in the same units. If q_0 and q_1 denote the quantity or volume produced, consumed or transacted during the base period (0) and the given period (1) then quantity relative of the period 1 with respect to the base period 0 is defined by

Quantity relative in percentage (of period 1 with respect to 0)

$$= \frac{q_1}{q_0} \times 100 \quad \ldots\ldots(11.2)$$

We denote quantity relative in percentage or without percentage by $q_{0/1}$

Exercise 3: The production of tea in Assam in the month of January 2009 was 3608 tonnes and that in February 2009 was 3700 tonnes. Find the quantity relative.

Solution: The quantity relative of tea in February 2009 (say period F) with respect to that in January 2009 (say period J) is given by

$$q_{F/J} = \frac{3700}{3608} \times 100 = 102.55\%$$

3) Value Relatives:

A value relative is another type of simple index number. It is usable when we wish to compare changes in the money value of the transaction,

consumption or sale in two different periods or points of time. Multiplying the quantity q by the price p of the commodity produced, transacted or sold gives the total money value pq of the production, transaction or sale. If instead of point of time, period of time is considered, a suitable average is to be taken and is to be expressed in the same units.

Suppose p_0 and q_0 denote the price and quantity of the commodity during the base period (0) and p_1 and q_1 denote the corresponding price and quantity during a given period (1). Then the total value of the commodity during the base period is $v_0 = p_0q_0$ and the corresponding total value during the given period is $v_1 = p_1q_1$. The value relative of the period 1 with respect to the base period 0 is defined by

Value relative in percentage (of period 1 with respect to 0)

$$= \frac{v_1}{v_0} \times 100 = \frac{p_1q_1}{p_0q_0} \times 100 \quad \text{.....(11.3)}$$

We denote value relative in percentage or without percentage by $v_{0/1}$.

Exercise 4: The production of tea in Assam in the month of January 2009 was 3608 tonnes and that in February 2009 was 3700 tonnes. The corresponding prices of tea were Rs.100.00 and Rs.120.00 per kilogram, respectively. Find the value relative.

Solution: Here the base period is January 2009 (say period J) and the given period is February 2009 (say period F). Then, $q_J = 3608000$ kg, $p_J = Rs.100.00$ and $q_F = 3700000$ kg, $p_F = Rs.120.00$. Hence

$v_J = q_J \times p_J = Rs.360800000.00$ $v_F = q_F \times p_F = Rs.444000000.00$.

Using formula (7.3), we find the required value relative

$$v_{F/J} = \frac{444000000}{360800000} \times 100 = 123\%.$$



CHECK YOUR PROGRESS

Q 1: Write True (T) or False (F).

(i) Index numbers are specialized averages to measure relative changes. (T / F)

(ii) Index numbers measure net change in related variables over time. (T / F)

(iii) Simple index numbers require weights in their construction. (T / F)

(iv) Index numbers do not indicate the trend of change in the economy. (T / F)

(v) Price relative is defined as the ratio of current year's price to some reference year's price. (T / F)

Q 2: Fill in the blanks:

(i) Index number is a statistical device to express ..... change in related variable.

(ii) An index number is a ..... average.

(iii) Index number measures net change in a group of ..... variables.

(iv) In the index number represented by $P_{1990/\ 1998}$, the base year is ..... and the current year is .....

(v) Simple index numbers have ..... weights.

(vi) The price relative means ratio of current price to .....

(vii) If the price relative increases from 100 to 300, it shows ..... increase in prices.

11.5 PROPERTIES OF RELATIVES

Let $p_a, p_b, p_c, \cdots$ denote the prices in the periods $a, b, c, \cdots$ respectively. Also, let $q_a, q_b, q_c, \cdots$ denote the quantities and $v_a, v_b, v_c, \cdots$ denote the values for the periods respectively. The relatives satisfy some properties which are directly obtained from the definitions. We shall state the results for price relatives and write similar results for the quantity and value relatives.

1) **Identity property**: The price relative of a given period with respect to the same period is 1, that is, $p_{a/\ a} = 1$.

2) **Time reversal property**: If the base period and the reference period are interchanged, then the product of the corresponding relatives is unity. That is, one is the reciprocal of the other. In other words,

$$P_{a/b} = \frac{1}{P_{b/a}}.$$

We know that $P_{a/b} = \frac{p_b}{p_a} = \frac{1}{p_a / p_b} = \frac{1}{P_{b/a}}$

3) Circular or Cyclic property: This property states that $P_{a/b} \times P_{b/c} \times P_{c/a} = 1$. We can prove this property as follows:

We know that $P_{a/b} = \frac{p_b}{p_a}$, $P_{b/c} = \frac{p_c}{p_b}$, $P_{c/a} = \frac{p_a}{p_c}$.

The required property follows by multiplying the above three terms.

4) Modified circular or cyclic property: This property states that $P_{a/b} \times P_{b/c} = P_{a/c}$. We prove this property as follows:

We have $P_{a/b} \times P_{b/c} = \frac{p_b}{p_a} \times \frac{p_c}{p_b} = \frac{p_c}{p_a} = P_{a/c}$.

11.6 CONSTRUCTION OF SIMPLE INDEX NUMBERS

In general we are interested in general price level rather than on the price level of any particular item. So we have to consider a number of items at the same time. If each commodity is of different qualities then all such qualities have also to be taken into consideration. We have to consider simultaneously a number of items or commodities together for measuring change in price or production.

The sources of data must be reliable and care should be taken in selecting the sources of data. For example, official publications as well as other reliable unofficial reports from producers and merchants should be considered for computing index for production. If retail food prices are being considered, then prices in stores, not in wholesale markets, are to be taken into consideration.

Now we shall discuss how to construct simple index numbers from a given set of data, like prices of certain commodities, production of certain commodities etc.

There are two methods of constructing simple index numbers. These are:

1) Simple aggregate method and

2) Method of simple average of relatives.

Remember that all items in a simple index number are assigned equal weights. These are unweighted index numbers.

1) Simple Aggregate Method: In this case each item is given equal weights. If we give an equal weight to each item, it means the same thing, whether each item is given a weight or not. It is the simplest method of constructing index numbers.

1.1) Simple aggregate method to find price index numbers:

We use the following three steps to find price index number.

Step 1: Find the sum of the current year (that is, given period) prices of all the items included in the list. If 1 denote the current year, then find $\sum p_1$.

Step 2: Find the sum of the base year prices of all the items included in the list. If 0 denote the base year, then find $\sum p_0$.

Step 3: Use the formula $P_{0/1} = \frac{\sum p_1}{\sum p_0} \times 100$.

Exercise 5: Find the price index number, using simple aggregate method, for the following information.

Commodities	Units	Prices (in Rupees)	
		2000	2005
Wheat	Quintal	800	1200
Rice	Quintal	1200	1600
Milk	Litres	12	18
Tea	Kilograms	100	120
Clothing	Metres	30	40
Meat	Kilograms	80	120

Solution: Represent prices of 2005 (current year) as p_1 and that of 2000 (base year) as p_0 and reconstruct the table as follows:

Commodities	p_0	p_1
Wheat	800	1200
Rice	1200	1600
Milk	12	18
Tea	100	120

Clothing	30	40
Meat	80	120
	$\sum p_0 = 2222$	$\sum p_1 = 3098$

Therefore, the price index of 2005 with base year 2000 is:

$$p_{0/1} = \frac{3098}{2222} \times 100 = 139.4$$

Thus, the increase in prices (or price level) is 139.40 - 100 = 39.4%.

Exercise 6: The prices of three commodities A, B and C increased from Rs.50, Rs.7 and Rs.3 in 2005 to Rs.52, Rs.8 and Rs.5 respectively in 2006. Using the simple aggregate method, find by how much on an average the prices have increased?

Solution: We know construct the following table:

Commodities	Price in 2005 (p_0)	Price in 2006 (p_1)
A	50	52
B	7	8
C	3	5
	$\sum p_0 = 60$	$\sum p_1 = 65$

Therefore, the price index of 2006 with respect to the base period 2005 is:

$$p_{2005/2006} = \frac{\sum p_1}{\sum p_0} \times 100 = \frac{65}{60} \times 100 = 108.33$$

This shows that the price index of current year 2006 with respect to base year 2005 is 108.33%. Hence the price increase in one year is 108.33 - 100 = 8.33%.

1.2) Simple aggregate method to find quantity index numbers:

We use the following three steps to find quantity index number.

Step 1: Find the sum of the current year (that is, given period) quantity or volume produced, consumed or transacted of all the items included in the list. If 1 denote the current year, then find $\sum q_1$.

Step 2: Find the sum of the base year quantity or volume produced, consumed or transacted of all the items included in the list. If 0 denote the base year, then find $\sum q_0$.

Step 3: Use the formula $q_{0/1} = \frac{\sum q_1}{\sum q_0} \times 100.$

2) Method of Simple Average of Relatives: In this method, average of the relatives is obtained by using any one of the measures of central tendency. In particular, here we shall use arithmetic mean for averaging the relatives. We shall discuss briefly this method only for price relatives. Similar formulas on quantity index number and value index number can be obtained considering quantity and value relatives instead of price relative. There are few questions to find quantity and value index numbers in the section "Possible Questions". You should work out those problems with the help of the problems on price index numbers which are solved below.

2.1) Method of Simple Average of Price Relatives:

We know that a price relative is nothing but the ratio of current year prices to those in the base year. Taking the arithmetic mean of the price relatives, we get the following formula for price index number:

$$p_{01} = \frac{\sum\left(\frac{p_1}{p_0} \times 100\right)}{N}$$

where N stands for number of commodities included in the index numbers. Steps to calculate price index numbers by Simple Average of Price Relatives Method:

Step 1: Find percentage price relative for each commodity, that is, $\frac{p_1}{p_0} \times 100$

Step 2: Find the sum of these percentage price relatives, that is, $\sum\left(\frac{p_1}{p_0} \times 100\right)$.

Step 3: Divide $\sum\left(\frac{p_1}{p_0} \times 100\right)$ by the total number of commodities included in the list.

Exercise 7: Give the solution of Exercise 5 using the Simple Average of Price Relatives.

Solution: We construct the following table:

Sl No.	Commo-dities	Units	Prices (in Rupees)		Price Relative $\frac{p_1}{p_0}\times 100$
			2000 p_0	2005 p_1	
1	Wheat	Quintal	800	1200	$\frac{1200}{800}\times 100 = 150.00$
2	Rice	Quintal	1200	1600	$\frac{1600}{1200}\times 100 = 133.33$
3	Milk	Litres	12	18	$\frac{18}{12}\times 100 = 150.00$
4	Tea	Kilograms	100	120	$\frac{120}{100}\times 100 = 120.00$
5	Clothing	Metres	30	40	$\frac{40}{30}\times 100 = 133.33$
6	Meat	Kilograms	80	120	$\frac{120}{80}\times 100 = 150.00$
N = 6					$\Sigma\left(\frac{p_1}{p_0}\times 100\right) = 836.66$

Hence, the price index number of 2005 with respect to base year 2000 is:

$$p_{0/1} = \frac{\Sigma\left(\frac{p_1}{p_0}\times 100\right)}{N} = \frac{836.66}{6} = 139.44.$$

Exercise 8: Give the solution of Exercise 6 using the Simple Average of Price Relatives.

Solution: We first prepare the following table:

Sl No.	Commo-dities	Prices (in Rupees)		Price Relative
		2005 p_0	2006 p_1	$\frac{p_1}{p_0}\times 100$
1	A	50	52	$\frac{52}{50}\times 100 = 104.00$
2	B	7	8	$\frac{8}{7}\times 100 = 114.29$
3	C	3	5	$\frac{5}{3}\times 100 = 166.66$

N=3				$\sum\left(\frac{p_1}{p_0}\times 100\right)=384.95$

Hence, the price index number of 2006 with respect to base year 2005 is:

$$p_{0/1}=\frac{\sum\left(\frac{p_1}{p_0}\times 100\right)}{N}=\frac{384.95}{3}=128.32.$$

11.7 CONSTRUCTION OF WEIGHTED INDEX NUMBERS

In weighted index number each item is given weight according to the importance it occupies in the list. There are two groups of methods to calculate index number of this category: (a) Weighted aggregate method and (b) Weighted average of price relative methods. In this unit, we will study three types of weighted aggregate method. These methods are popularly known as (1) Laspeyre's Method, (2) Paasche's Method and (3) Fisher's Method. All the methods are used to calculate weighted index numbers. The main difference between the Laspeyre's Method and Paasche's Method is that Laspeyre's uses base year quantities of commodities as their relative weights, while Paasche's uses current year quantities of commodities as their relative weights for preparing a price index.

(1) Laspeyre's Method:

Laspeyre's Price Index Number: It uses base year quantities as the weights. Accordingly, the formula is

$$P_L=\frac{\sum p_1q_0}{\sum p_0q_0}\times 100$$

Laspeyre's Quantity Index Number: If we multiply the quantities by the base year price, then we get Laspeyre's quantity index number. Accordingly the formula for Laspeyre's quantity index number is:

$$Q_L=\frac{\sum q_1p_0}{\sum q_0p_0}\times 100$$

Steps to Calculate Weighted Index Number by Laspeyre's Method:

Step 1: Multiply current year price (p_1) with the base year quantity (q_0) to get p_1q_0 for each item/ commodity.

Step 2: Multiply base year price (p_0) with the base year quantity to get p_0q_0 for each item/ commodity.

Step 3: Add all p_1q_0 and p_0q_0 separately to get $\sum p_1q_0$ and $\sum p_0q_0$.

Step 4: Divide $\sum p_1q_0$ by $\sum p_0q_0$ and multiply by 100 to obtain Laspeyre's price index number.

Exercise 9: Find Laspeyre's price index number from the following data.

Commodity	Price (in Rs)		Quantity bought in units	
	1995	1996	1995	1996
A	50	52	10	12
B	7	9	3	4
C	3	5	7	6

Solution: Represent prices and quantities in current year by p_1 and q_1 respectively and represent prices and quantities in base year by p_0 and q_0 respectively. Let us construct the following table following the steps 1-3.

Commodities	p_0	q_0	p_1	p_0q_0	p_1q_0
A	50	10	52	500	520
B	7	3	9	21	27
C	3	7	5	21	35
				$\sum p_0q_0 = 542$	$\sum p_1q_0 = 582$

Therefore, the Laspeyre's price index number is:

$$P_L = \frac{582}{542} \times 100 = 107.38$$

(2) Paasche's Method:

Paasche's Price Index Number: It uses current year quantities (q_1) as the weights. Accordingly, the formula is

$$P_P = \frac{\sum p_1q_1}{\sum p_0q_1} \times 100$$

Paasche's Quantity Index Number: It uses current year prices (p_1) as the

weights. Accordingly, the formula is

$$Q_P = \frac{\sum q_1 p_1}{\sum q_0 p_1} \times 100$$

Exercise 10: Find Paasche's price index number from the data given in Exercise 9.

Solution: Represent prices and quantities in current year by p_1 and q_1 respectively and represent prices and quantities in base year by p_0 and q_0 respectively. Let us construct the following table.

Commodities	p_0	q_1	p_1	p_1q_1	p_1p_1
A	50	10	52	500	520
B	7	3	9	21	27
C	3	7	5	21	35
				$\sum p_0q_1 = 646$	$\sum p_1q_1 = 690$

Therefore the Paasche's price index number is:

$$P_P = \frac{690}{646} \times 100 = 106.8$$

Remark: You have noticed that the value of Laspeyre's index number (107.38) and the value of Paasche's index number (106.8) are different for the same data. The difference has arisen because of differences in weights.

(3) Fisher's Method: The Fisher's price index number is nothing but the geometric mean of the Laspeyre's and Paasche's price index numbers. Hence the formula for Fisher's price index number is:

$$P_F = \sqrt{P_L \times P_P} = \sqrt{\left(\frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100\right) \times \left(\frac{\sum p_1 q_1}{\sum p_0 q_1} \times 100\right)}$$

$$= \sqrt{\frac{\sum p_1 q_0}{\sum p_0 q_0} \times \frac{\sum p_1 q_1}{\sum p_0 q_1}} \times 100 \qquad \cdots(1)$$

Similarly, the Fisher's quantity index number is given by:

$$Q_F = \sqrt{Q_L \times Q_P} = \sqrt{\left(\frac{\sum q_1 p_0}{\sum q_0 p_0} \times 100\right) \times \left(\frac{\sum q_1 p_1}{\sum q_0 p_1} \times 100\right)}$$

$$= \sqrt{\frac{\sum q_1 p_0}{\sum q_0 p_0} \times \frac{\sum q_1 p_1}{\sum q_0 p_1}} \times 100 \qquad \cdots(2)$$

Exercise 11: Find Fisher's price index number from the data given in Exercise 9.

Solution: We have already found that for the given data,

$P_L = 107.38$ and $P_P = 106.8$.

Hence the Fisher's price index number is:

$$P_F = \sqrt{107.38 \times 106.8} = 107.09$$



CHECK YOUR PROGRESS

Q 3: Write true (T) or False (F).

(i) There are two types of index numbers: simple or unweighted and weighted index numbers. (T / F)

(ii) There are two methods of constructing simple index numbers: simple aggregate method and simple average relatives.(T/F)

(iii) We get the same index number whether we use simple aggregate method or simple average of price relatives' method. (T / F)

(iv) Laspeyre's uses current year quantities as weights. (T / F)

(v) Paasche's uses base year quantities as weights. (T / F)

Q 4: Fill up the blanks.

(i) In Laspeyre's method of finding price index number we use ..... quantities as weights.

(ii) In Paasche's method of finding price index number we use ..... quantities as weights.

(iii) $I = \frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100$ is ..... weighted index number.