GCA S2 01

BACHELOR OF COMPUTER APPLICATION

SEMESTER - II

Discrete Mathematics

BLOCK 2



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January, 2018

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Printed and published by Registrar on behalf of the Krishna Kanta Handiqui State Open University.

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The University acknowledges with thanks the financial support provided by the **Distance Education Bureau**, **UGC** for the preparation of this study material.

BLOCK INTRODUCTION

This is the *Second Block* of the course '***Discrete Mathematics***'. After completing this block, which consists of six units, you will definitely be able to solve numerical problems of Discrete Mathematics.

This block comprises the following six units :

UNIT 8 introduces you to the preliminary concepts of Group. The semi group, abelian groups, finite and infinite groups are discussed in this unit.

UNIT 9 deals with the concept of an another algebraic system namely Ring.

UNIT 10 concentrates on a special kind of Ring called Integral Domain. Some of its basic properties are discussed in this unit.

UNIT 11 introduces you to the one of the most powerful tools in Mathematics namely Matrix. In addition, this unit also includes types of matrix, operations on matrix and transpose of a matrix.

UNIT 12 includes more concepts on matrix including the discussion Determinant of a square matrix, Adjoint and Inverse of a square matrix, Rank and Nullity of a matrix.

UNIT 13 is the last unit of this block. This unit concentrates on solving a system of linear equations, by Cramer's Rule and Gaussian Elimination method.

The structure of Block 2 is as follows :

UNIT 8 : Basic Algebraic Structure
UNIT 9 : Ring
UNIT 10 : Integral Domains and Fields
UNIT 11 : Matrices
UNIT 12 : Determinant and Inverse of a matrix
UNIT 13 : System of linear equations.

Each unit of these blocks includes some along-side boxes to help you know some of the difficult, unfamiliar terms. Some "EXERCISE" have been included to help you apply your own thoughts. You may find some boxes marked with: "LET US KNOW". These boxes will provide you with some additional interesting and relevant information. Again, you will get "CHECK YOUR PROGRESS" questions. These have been designed to self-check your progress of study. It will be helpful for you if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with "ANSWERS TO CHECK YOUR PROGRESS" given at the end of each unit.

BACHELOR OF COMPUTER APPLICATION

Discrete Mathematics

BLOCK 2

CONTENTS

		Page No.
UNIT 8 :	**Basic Algebraic Structure** Definition and basic properties of semi groups and groups	**5–21**
UNIT 9 :	**Ring** Definition and basic properties of rings	**22–33**
UNIT 10 :	**Integral Domains and Fields** Definition and basic properties of integral domains and fields	**34–42**
UNIT 11 :	**Matrices** Types of matrices - Matrix operations - canonical forms	**43–70**
UNIT 12 :	**Determinant and Inverse of a Matrix** Inverse of a matrix - Rank and nullity	**71–102**
UNIT 13 :	**System of Linear equations** Methods of solution to Linear systems (Cramer's Rule, Gaussian Elimination Method)	**103–135**

UNIT 8 : BASIC ALGEBRAIC STRUCTURE

UNIT STRUCTURE

8.1 Learning Objectives

8.2 Introduction

8.3 Binary Operation

8.4 Definition of Group

- 8.4.1 Abelian Group
- 8.4.2 Finite and Infinite Groups
- 8.4.3 Order of a Group
- 8.4.4 Semi-Group
- 8.4.5 Examples of Groups and Semi-Groups
- 8.4.6 Properties of Groups
- 8.4.7 Laws of Indices in a Group

8.5 Let Us Sum Up

8.6 Answers to Check Your Progress

8.7 Further Readings

8.8 Model Questions

8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- define a group and a semi-group
- learn about abelian groups, finite and infinite groups
- define order of a group
- learn about some elementary properties of groups
- define laws of indices of group-elements
- find examples of both abelian and non-abelian groups.

8.2 INTRODUCTION

The theory of groups plays an important role in present day mathematics and different branches of science, including computer science.

The structure of group is one of the simplest mathematical structures. It is the starting point in the study of various algebraic systems such as Rings, Fields, Vector Spaces, Linear operators, etc. In this present unit we shall discuss groups, semi-groups, subgroups, examples of these structures and their simple properties. We shall first define a **binary operation** on a non-empty set, a pre-requisite for defining a group.

8.3 BINARY OPERATION

A **binary operation** or **binary composition** denoted by $*$ in a set S is a mapping from $S \times S$ into S such that for all $(a, b) \in S \times S$, the image of (a, b) under the mapping $*$, denoted by $a * b$, belongs to S.

In short, an operation or composition $*$ defined on a non-empty set S is called a **binary operation** or **binary composition** if $a * b \in S$ for all $a, b \in S$. We shall henceforth use the symbol '$\forall$' to represent the phrase 'for all'. Thus, $*$ is a binary operation on S if $a * b \in S \; \forall \; a, b \in S$. If '$*$' is a binary operation on a set S, then S is said to be **closed** under '$*$' or the **closure property** holds in S.



NOTE : For the operation $*$ to be a binary operation in G, the **closure property** must hold in G, i.e., for all $a, b \in G$, $a * b$ must belong to G.

Example1 : Take S = { –1, 1}, Then '×' is a binary operation on S, but; '+' is not a binary operation on S. To check it, we construct the following tables, showing addition and multiplication of the elements of S.

Table 1:

×	–1	1
–1	1	–1
1	–1	1

Table 2 :

+	–1	1
–1	–2	0
1	0	2

Table 1 shows that the resulting elements belong to S again. So, '×' is a binary operation on S. But table-2 shows that the resulting elements do not belong to S, and so, '+' is not a finary operation on S.

Example 2 : We know that sum of any two integers is again an intiger. So, '+' is a binary operation on Z. Similary '+' is a binary operation on Q, R, C also. '×' is also a binary operation on Z, Q, R, C. '÷' is not a binary operation on these number sets, since division by 0 is inadmissible.

8.4 DEFINITION OF GROUP

A non-empty set G together with a binary operation $*$, denoted by $(G, *)$ is called a group if the following postulates are satisfied.

1. Associativity : $a * (b * c) = (a * b) * c \; \forall \; a, b, c \in G$
2. Existence of identity : There exists an element $e \in G$, called identity element, such that
$$a * e = a = e * a \; \forall \; a \in G$$
3. Existence of inverse : For each $a \in G$, there exists an element, denoted by a^{-1} in G such that
$$a * a^{-1} = e = a^{-1} * a$$
a^{-1} is called the **inverse element** of a.

8.4.1 Abelian Group

A group $(G, *)$ is called an **Abelian group** or a **Commutative group** if the commutative law holds in G, i.e. $a * b = b * a \; \forall \; a, b \in G$ otherwise, a group is called a non-abelian or non-commutative group.

8.4.2 Finite and Infinite Groups

A group $(G, *)$ is called a **finite** group if it has a finite number of elements, otherwise it is called an **infinite** group.

8.4.3 Order of a Group

The number of elements in a group $(G, *)$ is defined as the order of the group and is denoted by O (G) or | G |. If G is a finite group, then O (G) is a finite number. An infinite group is said to be of infinite order.

8.4.4 Semi Group

A non-empty set G together with a finary operation '$*$' is called a **semi group** if **associativity** holds in G, i.e,



NOTE : Henceforth, unless specifically defined, we shall use the symbol '.' (dot) in place of '$*$' for a binary operation in a group, as '.' is more convenient to write rather than '$*$'. In fact, we can simply say that G is a group instead of saying $(G, *)$ is a group and we can write ab instead of $a * b$ if there is no likelihood of any confusion regarding the binary composition in the group.

$$a * (b * c) = (a * b) * c \; \forall \; a, b, c \in G$$

In otherwords, G is a semi-group if **closure** and **associativity** with respect to an operation '$*$' holds in G.

Similar to groups, we can also define abelian semi-group, finite and infinite semi-group, order of a semi-group. From definitions of Group and Semi-groups, it is obvious that **every group is a semi-group.** Later on, we shall find examples of semi-groups which are not groups. In otherwords, every semi-group may not be a group.

8.4.5 Examples of Groups and Semi Group

Example 1 : (Z, +) is a group, called **the groups of integers** under usual addition.

1. We know $a+b \in Z \; \forall \; a, b \in Z$ since sum of two integers is again an integer. Thus '+' is a binary operation in Z.
2. We know $a + (b+c) = (a+b) + c \; \forall \; a, b, c \in Z$, i.e., associativity holds in Z under addition.
3. $0 \in Z$ and for any $a \in Z$, we know
$$a + 0 = a = 0 + a$$
4. For $a \in Z$, $-a \in Z$, and we know
$$a + (-a) = 0 = (-a) + a$$
So, –a is the inverse element of a in Z. It is called the additive inverse of a.
5. For $a, b \in Z$, we know $a + b = b + a$, i.e., commutativity under addition holds in Z.
Hence (Z, +) is an infinite abelian group.

Example 2 : As in example 1, it can be easily shown that (Q, +), (R, +), (C, +) are all infinite abelian groups. These are respectively called **additive groups of rational numbers, real numbers and complex numbers**.

Example 3 : Let $Q^0 = Q - \{0\}$. Then (Q^0, x) is an infinite abelian group, called **the multiplicative group of non-zero rational numbers**.

1. Let $a, b \in Q^0$. Then $a, b \in Q$ and $a \neq 0$, $b \neq 0$. Clearly, $ab \in Q$ and $ab \neq 0$. Hence $ab \in Q^0$, i.e., the closure property under multiplication holds in Q^0.
2. The associativity under multiplication holds in Q^0, since it holds in Q.
3. $1 \in Q^0$ and for any element $a \in Q^0$,

   $$a \times 1 = a = 1 \times a$$

   So, 1 is the multiplicative identity in Q^0.
4. $a \in Q^0 \Rightarrow a \in Q, \ a \neq 0$

   $$\Rightarrow \frac{1}{a} \in Q, \ \frac{1}{a} \neq 0$$

   $$\Rightarrow \frac{1}{a} \in Q^0$$

   Clearly $a \times \frac{1}{a} = 1 = \frac{1}{a} \times a$

   Hence, $\frac{1}{a}$ is the multiplicative inverse of a in Q^0.
5. Clearly, commutativity holds in Q^0. Thus (Q^0, x) is an infinite abelian group.

Example 4 : Let $R^0 = R - \{0\}$. Then as in example 3, it can be shown that (R^0, x) is an infinite abelian group. It is called the multiplicative group of non-zero real numbers.

Example 5 : Let $C^0 = \{x+iy : x, y \in R \text{ and } x+iy \neq 0\}$. Then (C^0, x) is an infinite abelian group.

1. Let $a+ib, c+id \in C^0$. Then $a+ib \neq 0$, $c+id \neq 0$. It can be shown that

   $$(a+ib)(c+id) = (ac-bd) + i(ad+bc) \in C^0$$

   So, closure property under multiplication holds in C^0.
2. Product of complex numbers obeys the associative law and so, the associativity also holds in C^0. [Prove yourselves]
3. $1 = 1 + i.0 \in C$ and it is the multiplicative identity in C^0.
4. If $a + ib \in C^0$, $a + ib \neq 0$ and hence

   $\frac{1}{a+ib} = \frac{1}{a^2+b^2} - i.\frac{b}{a^2+b^2} \in C^0$. Clearly

$$(a+ib)\times\frac{1}{(a+ib)} = 1 = \frac{1}{(a+ib)} \times (a+ib)$$

Thus $\frac{1}{a+ib}$ is the multiplicative inverse of a + ib in C^0.

5. The commutative law (a+ib)(c+id) = (c+id)(a+ib) for a + ib, c + id $\in C^0$ is obvious.

   Hence (C^0, x) is an infinite abelian group.

   It is called the **multiplicative group of non-zero complex numbers**.

Example 6 : (Q^+, x), (R^+, x) are multiplicative groups of positive rational numbers and positive real numbers respectively. But (Z^+, x) is not a group, i.e., the set of positive integers is not a group under multiplication, because except –1 & 1, no element of Z^+ has multiplicative inverse. Clearly (Z^+, x) is a **semi-group** since colsure and associativity under multiplication holds in Z^+.

Example 7 : Let G = {1, w, w^2}, where w is an imaginary cube root of unity. It can be easily checked that G is a multiplicative group where 1 is the identity, inverse of w is w^2 and inverse of w^2 is w. It is a finite abelian group where 0(G) = 3.

Example 8 : Let G = {±1, ±i, ±j, ±k}. Let us define product in G as follows :

1.1 = 1, (–1).(–1) = 1, $i^2 = j^2 = k^2 = -1$,

ij = –ji = k, jk = –kj = i, ki = –ik = j.

It can be verified that G satisfies all the group postulates except commutativity, since ij ≠ ji. Hence G is a finite **non-abelian group**. This is called the **Quaturnion Group**.

Example 9 : Let G = {0, 1, 2, 3, 4, 5}. For any two elements a, b ∈ G, let us define an operation denoted by ⊕ such that a ⊕ b = c, where c is the least non-negative integer obtained on dividing (a+b) by 6. The following table gives us the results when ⊕ is applied on G.

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Clearly, 1) closure property holds in G

2) associativity holds in G

3) 0 is the identity element.

4) additive inverses of 1, 2, 3, 4, 5 are 5, 4, 3, 2, 1 respectively.

5) the commutativity also holds.

Hence, $(G, \oplus)$ is a finite abelian group.

It is called the **'Group of Residues Modulo 6'**.

In general, $Z_n = \{0, 1, 2, ..., n - 1\}$ is called the **'Group of Residues Modulo n'**, where the operation $\oplus$ is defined as

$$a \oplus b = c$$

where c is the least non-negative integer obtained on dividing a+b by c. It is a finite group, $0(Z_n) = n$. The operation $\oplus$ is called the operation of **'addition modulo n'**.

Example 10 : Let G be the set of all 2×2 non-singular matrices over real numbers, that is,

$$G = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} : x_i \in R \text{ and } \begin{vmatrix} x_1 & x_2 \\ x_3 & x_4 \end{vmatrix} \neq 0 \right\}$$

Then G is a group under matrix multiplication.

1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ be any two elements of G.

Then $a, b, c, d, p, q, r, s \in R$ and $|A| \neq 0$, $|B| \neq 0$.

Now $AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$

Clearly $ap + br, aq + bs, cp + dr, cq + ds \in R$.

Also $|AB| = |A|.|B| \neq 0$ as $|A| \neq 0$, $|B| \neq 0$.

Hence $AB \in G$, i.e., closure property holds in G.

2. As matrix multiplication is associative, so the associativity holds in G.

3. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G$ as $|I| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

For any $A \in G$, A.I = A = I.A. Hence I is the identity element in G.

4. Since $A \in G \Rightarrow |A| \neq 0$, so $A^{-1} = \frac{1}{|A|}$.adj $A \in G$ such that $A.A^{-1} = I = A^{-1}.A$.

Thus every element of G has its inverse in G. Hence G is a group under matrix multiplication.

Take $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 16 & 19 \\ 36 & 43 \end{pmatrix}$$

$$BA = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 19 & 28 \\ 27 & 40 \end{pmatrix}$$

So, $AB \neq BA$, i.e., the commutative law does not hold in G.

Hence, G is a non-abelian infinite group.

8.4.6 Properties of Groups

We shall now prove some properties of groups and semi-groups. We shall not mention the binary operation in considering a group G for establishing these properties. It should be understood that when we write the product ab for two elements $a, b \in G$; there exists a binary operation between a and b.

Property 1 : The identity element in a group is unique.

Proof : Suppose a group G has two identities e and e′.

Taking e as identity, we get

$e'e = e' = ee'$ .....(1)

Again taking e′ as the identity, we get

ee′ = e = e′e .....(2)

From (1) and (2) we get

e = e.e′ = e′

Hence the identity in G is unique.

Property 2 : The inverse of an element in a group is unique.

Proof : Let G be a group and a $\in$ G. Suppose b and c are two inverse elements of G.

Then ab = e = ba .....(1)

ac = e = ca .....(2)

where e is the identity in G.

Now b = be

= b(ac), from (2)

= (ba)c, using associativity

= ec, from (1)

= c, using definition of identity.

Hence the inverse of a is unique.

Property 3 : In a group G, i) $(a^{-1})^{-1} = a \; \forall \; a \in G$

ii) $(ab)^{-1} = b^{-1}a^{-1} \; \forall \; a, b \in G$

Proof : Let e be the identity element in G.

i) Let a $\in$ G. Then

$aa^{-1} = e = a^{-1}.a$ .....(1)

Again $a^{-1} \in G$. So,

$a^{-1}(a^{-1})^{-1} = e = (a^{-1})^{-1}a^{-1}$ .....(2)

Now $(a^{-1})^{-1} = e(a^{-1})^{-1}$

$= (aa^{-1})(a^{-1})^{-1}$, using (1)

$= a[a^{-1}(a^{-1})^{-1}]$, using associativity

= ae, using (2) = a

ii) Let a, b $\in$ G. Then $a^{-1}, b^{-1} \in G$

Now $(ab)(b^{-1}a^{-1}) = [(ab)b^{-1}]a^{-1}$, using associativity

$= [a(bb^{-1})]a^{-1}$

$= (ae)a^{-1}$

$= aa^{-1} = e$



NOTE : If G be an abelian group, then $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$. In otherwords, for a non-abelian group, we cannot write $(ab)^{-1} = a^{-1}b^{-1}$.

Similarly, $(b^{-1}a^{-1})(ab) = e$

Thus $(ab)(b^{-1}a^{-1}) = e = (b^{-1}a^{-1})(ab)$

Hence, $(ab)^{-1} = b^{-1}a^{-1}$.

Property 4 : The concellation laws hold in a group.

That is, if a, b, c $\in$ G, then

i) $ab = ac \Rightarrow b = c$ [left cancellation law]

ii) $ba = ca \Rightarrow b = c$ [right cancellation law]

Proof : Let e be the identity in a group G and a, b, c $\in$ G.

i) $ab = ac \Rightarrow a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow (a^{-1}a)b = (a^{-1}a)c$, associativity

$\Rightarrow eb = ec$

$\Rightarrow b = c$

ii) $ba = ca \Rightarrow (ba)a^{-1} = (ca)a^{-1}$

$\Rightarrow b(aa^{-1}) = c(aa^{-1})$

$\Rightarrow be = ce$

$\Rightarrow b = c$.

Property 5 : For elements a, b in a group G, the equations ax = b and ya = b have unique solutions in G.

Proof : $ax = b \Rightarrow a^{-1}(ax) = a^{-1}b$

$\Rightarrow (a^{-1}a)x = a^{-1}b$

$\Rightarrow ex = a^{-1}b$

$\Rightarrow x = a^{-1}b$,

which is a solution of the equation ax = b, since $a^{-1}b \in G$.

If possible, suppose x_1, x_2 are two solutions of ax = b in G.

Then $ax_1 = b$, $ax_2 = b \Rightarrow ax_1 = ax_2$

$\Rightarrow x_1 = x_2$,

using the left cancellation law.

Hence the solution of the equation ax = b is unique in G.

Similarly, ya = b has the unique solution $y = ba^{-1}$ in G.



NOTE : In general, in a group G,
$ab = ca \not\Rightarrow b = c$.
This is due to the fact that we cannot change the order of ca as ac unless G is abelian. For an additive group (G, +) a^{-1} charges to –a, $(a^{-1})^{-1}$ changes to –(–a), the cancellation law
'$ab = ac \Rightarrow b = c$'
changes to
'$a+b = a+c \Rightarrow b = c$', etc.

8.4.7 Laws of indices in a Group

Leg G be a group and a $\in$ G.

We define i) $a^0 = e$, the identity of G

ii) $a^n = a.a.a. \ldots a$, n factors where $n \in N$

iii) $a^{-n} = a^{-1}.a^{-1}.a^{-1}. \ldots a^{-1}$, n factors where $n \in N$

It can be shown that for all $m, n \in Z$,

$a^{-n} = (a^n)^{-1} = (a^{-1})^n$,

$a^m.a^n = a^{m+n}$

and $(a^m)^n = a^{mn}$

These are called laws of indices for group elements.

ILLUSTRATIVE EXAMPLES

Example 1 : In 'Q^+', the set of positive rational numbers, define an operation $*$ such that

$a * b = a, b \in Q^+$.

Show that $(Q^+, *)$ is an infinite abelian group.

Solution :

1) $a, b \in Q^+ \Rightarrow \frac{ab}{2} \in Q^+$

$\Rightarrow a * b \in Q^+$

Hence $*$ is a binary operation in Q^+.

2) For $a, b, c \in Q^+$,

$$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$$

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4}$$

Thus $a * (b * c) = (a * b) * c$, showing that the associativity holds in Q^+.

3) For $a \in Q^+$, $a * 2 = a = 2 * a$ and so 2 plays the role of the identity element in Q^+ under '$*$'.

4) $a \in Q^+ \Rightarrow \frac{4}{a} \in Q^+$, and

$$a * \frac{4}{a} = 2 = \frac{4}{a} * a$$

Hence $\frac{4}{a}$ is the inverse of a in Q^+, i.e. $a^{-1} = \frac{4}{a}$ $\forall$ $a \in Q^+$.

Thus, all the group postulates are satisfied. Moreover, the commutative law hold in Q^+, since

$$a * b = \frac{ab}{2} = \frac{ba}{2} = b * a.$$

Hence $(Q^+, *)$ is an infinite abelian group.

Example 2 : Show that the set E of all even integers is a group under addition.

Solution :

1) Let $a, b \in E$. Then $a = 2m$, $b = 2n$, where $m, n \in Z$.
   Now, $a + b = 2m + 2n = 2(m+n) = 2l \in E$, since $l = m+n \in Z$.
   Thus E is closed under addition.
2) The associativity holds in E, since it holds in Z.
3) $0 \in E$ and it is the additive identity in E.
4) $a = 2m \in E \Rightarrow -a = -2m = 2(-m) \in E$ and $a + (-a) = 0 = (-a) + a$.
   So, every element in E has its additive inverse in E.
   Hence (E, +) is a group.

Example 3 : Show that a group G is abelian if and only if

$$(ab)^2 = a^2b^2 \ \forall\ a, b \in G.$$

Solution : Let G be abelian and $a, b \in G$.

Then $(ab)^2 = (ab)(ab)$

$= a(ba)b$, using associativity

$= a(ab)b$, as G is abelian

$= (aa)(bb)$

$= a^2b^2$.

Conversely, suppose $(ab)^2 = a^2b^2$ $\forall$ $a, b \in G$.

Then $(ab)(ab) = (ab)(bb)$

$\Rightarrow a(ba)b = a(ab)b$

$\Rightarrow (ba)b = (ab)b$, using left cancellation law

$\Rightarrow ba = ab$, using right cancellation law

Thus $ab = ba$ $\forall$ $a, b \in G$ which shows that G is abelian.



CHECK YOUR PROGRESS

Q.1. Show that the set of integers Z is an abelian group under the operation $*$ defined by $a * b = a + b + 1 \; \forall \; a, b \in Z$.

Q.2. Show that Z is not a group under the operation $*$ defined by $a * b = a - b \; \forall \; a, b \in G$.

Q.3. Show that the set M of all 2×2 matrices over integers is a semi-group but not a group under matrix multiplication.

Q.4. Show that cancellation laws may not hold in a semi-group.



8.5 LET US SUM UP

- An operation $*$ defined in a non-empty set S is called a binary operation if $a * b \in S$ for all $a, b \in S$.
- A non-empty set G together with a binary operation $*$ is called a group if the following postulates are satisfied :

  Associativity : $a * (b * c) = (a * b) * c \; \forall \; a, b, c \in G$

  Existence of identity : $e \in G$ such that for any $a \in G$, $a * e = a = e * a$

Existence of inverse : for every $a \in G$, there exists $a^{-1} \in G$ such that $a * a^{-1} = e = a^{-1} * a$

The group is denoted by $(G, *)$.

- $(G, *)$ is an abelian or commutative group if $a * b = b * a \forall a, b \in G$
- A group is called a **finite group** if it has finite number of elements, otherwise a group is called infinite.
- The number of elements in a group $(G, *)$ is called the **order** of G, denoted by $0(G)$ or $|G|$.
- A non-empty set G together with a binary operation $*$ is called a semi-group if the associativity holds in G, i.e.,
  $a * (b * c) = (a * b) * c \forall a, b, c \in G$
- Every group is a semi-group, but every semi-group may not be a group.
- The identity element in a group is unique.
- The inverse of an element in a group is unique.
- In a group G, $(a^{-1})^{-1} = a \forall a \in G$ and $(ab)^{-1} = b^{-1}a^{-1} \forall a, b \in G$.
- The cancellation laws hold in a group G, i.e., if $a, b, c \in G$, then
  $ab = ac \Rightarrow b = c$
  $ba = ca \Rightarrow b = c$



8.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : We show that Z under the given operation satisfies all the group postulates.

1) For $a, b \in Z$, $a + b + 1 \in Z \Rightarrow a * b \in Z$.
   So $*$ is a binary operation in Z.
2) For $a, b, c \in Z$
   $a * (b * c) = a + (b * c) + 1 = a + (b + c + 1) = a + b + c + 2$
   $(a * b) * c = (a * b) + c + 1 = (a + b + 1) + c + 1 = a + b + c + 2$
   So, $a * (b * c) = (a * b) + c$ and thus the associativity holds.
3) $-1 \in Z$ and $a * (-1) = a + (-1) + 1 = a$,
   $(-1) * a = (-1) + a + 1 = a$
   Thus $a * (-1) = a = (-1) * a$, showing that -1 is the identity in Z.

4) $a \in Z \Rightarrow -(a + 2) \in Z$ and $a * [-(a + 2)] = a - (a + 2) + 1 = -1$
and $[-(a + 2)] * a = -(a + 2) + a + 1 = -1$.
Hence, $a * [-(a+2)] = -1 = [-(a + 2)] * a$
[showing that $a^{-1} = -(a + 2)$]

5) For $a, b \in Z$, $a * b = a + b + 1 = b + a + 1 = b * a$, showing that commutativity holds.
Hence $(Z, *)$ is an abelian group.

Ans. to Q. No. 2 : The operation $*$ defined in Z is given by

$a * b = a - b \; \forall \; a, b \in Z$.

Now $2, 3, 4 \in Z$ and

$2 * (3 * 4) = 2 * (3-4) = 2 * (-1) = 2 - (-1) = 3$

$(2 * 3) * 4 = (2-3) * 4 = (-1) * 4 = -1 - 4 = -5$

Thus we get $2 * (3 * 4) \neq (2 * 3) * 4$, i.e., the associativity under $*$ does not hold in Z.

Hence Z cannot be a group under the given operation.

Ans. to Q. No. 3 : $M_2(Z) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z \right\}$

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$

be any three elements of $M_2(Z)$ where $a_i, b_i, c_i \in Z$

1) $AB = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix}$

$$= \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$$

where $\alpha_1 = a_1b_1 + a_2b_3 \in Z$, $\alpha_2 = a_1b_2 + a_2b_4 \in Z$

$\alpha_3 = a_3b_1 + a_4b_3 \in Z$, $\alpha_3 = a_3b_2 + a_4b_4 \in Z$

Since $a_i, b_i, c_i \in Z$.

Thus AB is again a 2×2 matrix over Z, that is $AB \in M_2(Z)$.

2) Since A, B, C are 2×2 matrices, clearly the products AB, BC exist and it can be easily shown that A(BC) = (AB)C.
Thus the associativity holds in $M_2(Z)$.
Hence $M_2(Z)$ is a semi-group under matrix multiplication.

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the multiplicative identity in $M_2(Z)$. But existence of inverse element for every element does not hold in $M_2(Z)$.

For example, $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \in M_2(Z)$

where $|A| = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$, and so A^{-1} does not exist.

Hence $M_2(Z)$ is not a group.

Ans. to Q. No. 4 : Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix}$.

Then $A, B, C \in M_2(Z)$.

Now $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$, the zero matrix.

Also $AC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$.

Thus $AB = AC$, but $B \neq C$.

Hence cancellation laws may not hold in a semi-group.



8.7 FURTHER READINGS

1. *Modern Algebra* – S. Singh & Q. Zameeruddin, Vikas Pub. House Pvt. Ltc.
2. *A course in Abstract Algebra* – V. K. Khanna & S. K. Bhambri, Vikas Pub. House Pvt. Ltc.



8.8 MODEL QUESTIONS

Q.1. Show that (Q, +), (R, +), (P, +) are infinite abelian groups.

Q.2. Show that the set of non-zero real numbers R^0 is a group under multiplication.

Q.3. Show that the set of natural numbers N is a semi-group under the operations (i) addition, (ii) multiplication.

Q.4. Show that G = {–1, 1, –i, i} is an abelian group under multiplication. Write down 0(G).

Q.5. Let m be a positive integer greater than 1 and mZ = {0, ±m, ±2m, ±3m, ...} be the set of all integral multiples of m. Show that (mZ, +) is an abelian group.

Q.6. Give examples of :

i) a finite abelian group

ii) a finite non-abelian group

iii) an infinite abelian group

iv) an infinite non-abelian group.

Q.7. Let G = R – {–1}. Define an operation $*$ on G by $a * b = a+b+ab \; \forall \; a, b \in G$. show that (G, $*$) is an abelian group.

Q.8. Examine whether R is a group under the operation $*$ defined by $a * b = 2(a+b) \; \forall \; a, b \in R$.

Q.9. Show that if every element of a group is its own inverse, then the group is abelian.

Q.10. If in a group G, $a^2 = e \; \forall \; a \in G$, then show that G is abelian.

Q.11. Let G be any set having atleast two elements. For $a, b \in G$, define $a * b = b$. Show that G is a semi-group under $*$ but not a group.

Q.12. Show that the set Q^+ of all positive rational numbers is a group under usual multiplication of numbers.

Q.13. Show that the set R^+ of all positive real numbers is a group under usual multiplication of numbers.

Q.14. Show that $M_2(R)$, the set of all 2×2 matrices over real numbers is an abelian group under the operation of matrix addition.

UNIT 9 : RING

UNIT STRUCTURE

9.1 Learning Objectives

9.2 Introduction

9.3 Definition of a Ring

- 9.3.1 Commutative Ring
- 9.3.2 Ring with Unity
- 9.3.3 Ring with or without Zero Divisors
- 9.3.4 Examples of Rings

9.4 Properties of a Ring

9.5 Let Us Sum Up

9.6 Answers to Check Your Progress

9.7 Further Readings

9.8 Model Questions

9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- define a ring
- know commutative ring
- know ring with unity
- know ring with zero divisors
- know elementary properties of a ring.

9.2 INTRODUCTION

In the preceding unit, are have discussed **Groups** – an algebraic system consisting of a non-empty set together with one binary operation. The most common of these groups are the groups of numbers (Z, +), (Q, +), (R, +), (C, +), (Q^0, x), (R^0, x), (C^0, x), etc. But we know that multiplication is also a binary operation in these sets, which is associative as well as distributive over addition. This leads us to the study of algebraic system

with two binary operations. Such algebraic systems are **Rings**, **Integral Domains**, **Fields**, **Linear Spaces**, etc. In this unit we shall study Rings and elementary properties of a ring.

9.3 DEFINITION OF A RING

An algebraic system (R, +, .) consisting of a non-empty set R and two **binary operations**, denoted by '+' and '.' in R is called a **ring** if the following postudates are satisfied :

R_1 : (R, +) is an abelian group, i.e.,

i) $a + (b+c) = (a+b) + c \; \forall \; a, b, c \in R$

ii) There exists an element, donoted by 0, in R such that $a+0 = a = 0+a \; \forall \; a \in R$

'0' is called the **additive identity** or the **zero-element** of the ring.

iii) For every $a \in R$, there exists an element, denoted by –a, in R such that $a + (-a) = 0 = (-a) + a$

This element '–a' is called the **additive inverse** of a.

iv) $a+b = b+a \; \forall \; a, b \in R$

R_2 : (R, .) is a semi-group, i.e.,

v) $a.(b.c) = (a.b).c \; \forall \; a, b, c \in R$

R_3 : Multiplication is distributive over addition in R, i.e., for all a, b, $c \in R$

vi) $a.(b+c) = a.b + a.c$ (left distributive law)

vii) $(b+c).a = b.a + c.a$ (right distributive law)

If R has finite number of elements, then (R, +, .) is called a **finite ring**, otherwise it is called an **infinite ring**.

From group-properties we know that the identity element of a group is unique and every element of a group has unique inverse. If (R, +, .) is a ring, then by R_1, (R, +) is a group and hence the zero element, $0 \in R$ is unique. Similarly, for $a \in R$, the additive inverse '–a' is also unique.

Let us caution ourselves that whenever we use '+' and '.' for the two binary operations called 'addition' and 'multiplication' respectively in a ring (R, +, .), we should not confuse them with the arithmetic operations of

addition and multiplication of real numbers. These operations may stand for addition and multiplication of ordered pairs, addition and multiplication of matrices, etc. Similarly, we should not confuse with the zero element of a ring with the real number 0 and the additive inverse –a of an element in a ring with the negative integer –a.

Basic Conventions :

1. If this is no scope for confusion regarding the binary operations, instead of writing '(R, + , .) is a ring', we shall simply write 'R is a ring'. It should be understood that + and . are the binary operations in the ring R.
2. The product of two elements a and b is a ring R will be simply written as ab, instead of a.b.
3. If a, b are two elements in a ring, then the element a + (–b) will be written as a – b.

9.3.1 Commutative Ring

If the commutativity under multiplication holds in a ring, that is, $ab = ba \ \forall\ a, b \in R$ then R is called a **commutative ring**, otherwise R is called a **non-commutative ring**.

9.3.2 Ring with Unity

If in a ring R, there exists an element denoted by 1 such that $a1 = a = 1a \ \forall\ a \in R$, then R is called a **ring with unity**, and 1 is called the **multiplicative identity** or the **unity** in R. If $1 \notin R$, we say that R is a ring without unity.

We should note that 1 is just a symbol to denote the multiplicative identity of a ring R, we should not confuse it with the integer 1. Some authors also use the symbol 'e' for unity in a ring.

9.3.3 Ring with or without Zero-Divisors

Let R be a ring and $a, b \in R$. If $a \neq 0, b \neq 0$ but $ab = 0$, then a and b are called **divisors of zero** or **zero-divisors** in R.

If there exists atleast one pair of zero-divisors in a ring R, then R is called **a ring with zero-divisors**.

R is called a **ring without zero-divisors** if for a, b $\in$ R, ab = 0 $\Rightarrow$ either a = 0 or b = 0.

9.3.4 Examples of Rings

Example 1 : (Z, +, .) is a commutative ring with unity and without zero-divisors.

We already know that (Z, +) is an abelian group.

Also a, b $\in$ Z $\Rightarrow$ ab $\in$ Z and so, multiplication is a binary operation in Z.

Clearly, for a, b, c $\in$ Z,

a(bc) = (ab)c,

a(b+c) = ab + ac,

(b+c)a = ba + ca

Also 1 $\in$ Z and a.1 = a = 1.a $\forall$ a $\in$ Z.

Again ab = ba $\forall$ a, b $\in$ Z.

Moreover, ab = 0 $\Rightarrow$ a = 0 or b = 0 is Z.

Hence Z is a commutative ring with unity 1 and without zero-divisors under the operations of usual addition and multiplication of numbers.

Example 2 : Similar to example 1, it can be easily shown that Q, R, C are commutative rings with unity and without zero-divisors under usual addition and multiplication.

Example 3 : If m be a positive integer greater than 1 and mZ = {0, $\pm$m, $\pm$2m, $\pm$3m, ...}, then mZ is a commutative ring **without unity**, without zero-divisors.

Example 4 : Let $M_2(Z)$ be the set of all 2×2 matrices over integers. Then it is a non-commutative ring with unity under addition and multiplication of matrices. It is a ring with zero-divisors.

It is a routine work to verify that R_1, R_2 and R_3 holds in $M_2(Z)$ under the operations of matrix addition and multiplication,

where $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the zero-element and the additive inverse of an element

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is } -A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}.$$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the unity in $M_2(Z)$. Hence it is a ring with unity.

Now, let us take $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ in $M_2(Z)$. Then

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 13 \\ 22 & 29 \end{pmatrix} \&$$

$$BA = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 11 & 16 \\ 19 & 28 \end{pmatrix}$$

and so, $AB \neq BA$.

Thus, commutativity under multiplication does not hold in $M_2(Z)$. Hence $M_2(Z)$ is a non-commutative ring.

Again, let us take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ so that $A \neq 0$, $B \neq 0$.

But $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

Thus $A \neq 0$, $B \neq 0$ but $AB = 0$.

Hence A and B are zero-divisors, and so $M_2(Z)$ is a ring with zero-divisors.

Examples 5 : Let S be any set and P(S) be the power set of S. For $A, B \in P(S)$, let us define $A + B = (A–B) \cup (B–A)$, $AB = A \cap B$.

Then P(S) is a commutative ring with unity.

Clearly $A+B \in P(S)$, $AB \in P(S) \forall A, B \in P(S)$.

So + and . are binary operations in P(S).

It can be shown that for $A, B, C \in P(S)$

$A + (B+C) = (A+B) + C$

$A + \phi = A = \phi + A$, and so ϕ is the zero-element in P(S).

$A + A = \phi = A + A$), showing that A is its own additive inverse.

$A + B = B + A$

$A(BC) = (AB)C$

$A(B+C) = AB + AC$, $(B+C)A = BA + CA$

Moveover, $AB = BA$

$AS = A \cap S = A = S \cap A = SA$.

Hence P(S) is a commutative ring with unity S.

Examples 6 : Let R = {0, a, b, c} and let us define '+' and '.' in R by the following tables :

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

It can be easily checked from the above tables that R is a non-commutative ring without unity having zero-divisors.

Example 7 : In $Z_n = \{0, 1, 2, 3, ..., n-1\}$, let us define for $a, b \in Z_n$. $a \oplus b = c$ where c is the least non-negative remainder obtained on dividing a+b by c. Then from unit 8, we know that $(Z_n, \oplus)$ is an abelian group.

Let us now define $a \otimes b = c$, where c is the least non-negative remainder obtained on dividing ab by c. Then it can be proved that $(Z_n, \otimes)$ is a semi-group and also the distributive laws of $\otimes$ over $\oplus$ holds in Z_n. Hence $(Z_n, \oplus, \otimes)$ is a ring.

Moreover, 1 is the unity in Z_n and also $a \otimes b = b \otimes a$ for $a, b \in Z_n$.

Hence Z is a commutative ring with unity. This ring is called the **'Ring of Integers Modulo n'**.



NOTE : Let us take the ring of integers modulo 6, $Z_6 = \{0, 1, 2, 3, 4, 5\}$. Clearly $2 \otimes 3 = 0$ but $2 \neq 0$, $3 \neq 0$, i.e., 2 and 3 are zero-divisors in Z_6. So, Z_6 is a ring with zero-divisors. In general, Z_n is a ring with or without zero-divisors according as n is a composite number or a prime number respectively.

9.4 PROPERTIES OF A RING

Let R be a ring. Then for all $a, b \in R$ the following properties hold

i) $a0 = 0 = 0a$

ii) $a(-b) = (-a)b = -(ab)$



NOTE : We already obtained laws of indices for group elements. Since for a ring $(R, +, \cdot)$, both + and $\cdot$ are binary operations, that is, R is closed under + and $\cdot$, so the laws of indices for ring elements hold under certain restrictions. These are
$na + ma = (n+m)a$
$na + nb = n(a+b)$
where n, m are any integers.
If n, m are positive integers, then
$a^n.a^m = a^{n+m}$,
$(a^n)^m = a^{nm}$.

iii) $(-a)(-b) = ab$

iv) $a(b-c) = ab - ac$, $(b-c)a = ba - ca$

Proof :

i) For all $a \in R$,

$a0 = a(0+0)$

$\Rightarrow a0 + 0 = a0 + a0$, using definition of zero element and the left-distributive law.

$\Rightarrow 0 = a0$, using the left-cancellation law in the group (R, +)

Similarly, $0a = 0$

Hence $a0 = 0 = 0a$

ii) $a[b+(-b)] = a0 = 0$, by (i)

$\Rightarrow ab + a(-b) = 0$, by left distributive law

$\Rightarrow ab + a(-b) = 0 = a(-b) + ab$, as (R, +) is abelian

$\Rightarrow a(-b) = -(ab)$

Similarly, $(-a)b = -(ab)$

Hence $a(-b) = (-a)b = -(ab)$.

iii) $(-a)(-b) = -[a(-b)]$, using (ii)

$= -[-(ab)]$, again using (ii)

$= ab$ [See note after property 4, 8.4.6]

iv) $a(b-c) = a[b+(-c)]$

$= ab + a(-c)$

$= ab + [-(ac)]$

$= ab - ac$.

Similarly, $(b-c)a = ba - ca$.



CHECK YOUR PROGRESS

Q.1. Give an example of a commutative ring with unity having 2 elements.

.

Q.2. Show that R = {a+ib : a, b $\in$ Z} is a ring with unity under usual addition and multiplication of complex numbers.

Q.3. Show that Z_7 = {0, 1, 2, 3, 4, 5, 6} forms a ring under addition and multiplication modulo 7.

Q.4. Show that the cancellation laws hold in a commutative ring if and only if it has no zero-divisors.

Q.5. Prove that if R is a ring with unity, then the unity is unique.

Q.6. If the unity and the zero element of a ring R are equal, then show that R = {0}.

Q.7. If in a ring R, $a^2 = a \forall a \in R$, prove that :

i) $2a = 0 \forall a \in R$

ii) R is a commutative ring.



9.5 LET US SUM UP

- A non-empty set R together with two binary operations + and . is a ring if (R, +) is an abelian group, (R, .) is a semi-group and if the distributive laws of multiplication (.) holds over addition (+) is R.
- A ring R is called a commutative ring if $ab = ba \forall a, b \in R$.
- R is called a ring with unity if there exists an element $1 \in R$ such that $a1 = a = 1.a \forall a \in R$.
- If there exist elements a, b in a ring R such that $a \neq 0$, $b \neq 0$ but $ab = 0$, then a, b are called zero-divisors and R is called a ring with zero-divisors.
- The following prosperties hold in a ring R :
  i) $a0 = 0 = 0a \forall a \in R$
  ii) $a(-b) = (-a)b = -(ab) \forall a, b \in R$
  iii) $(-a)(-b) = ab \forall a, b \in R$
  iv) $a(b-c) = ab - ac, (b-c)a = ba - ca \forall a, b \in R$.



9.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : Take R = {0, 1} and define + and . is R as in the following tables.

+	0	1
0	0	1
1	1	0

.	0	1
0	0	0
1	0	1

It is now easy to verify that (R, +, .) is a commutative ring with unity 1.

Ans. to Q. No. 2 : R = {a+ib : a, b ∈ Z}

Let $a + ib, c + id \in R$ where $a, b, c, d \in Z$.

Then $(a+ib) + (c+id) = (a+c) + i(b+d) \in R$, since $a+c, b+d \in Z$.

Also $(a+ib)(c+id) = (ac-bd) + i(ad+bc) \in R$,

since $ac-bd, ad+bc \in Z$.

Hence, both addition and multiplication are binary operation is R.

If can be easily verified that the associative law under + and × holds in R, the commutative laws under + and × also holds in R, 0 = 0+i.0 is the zero element in R and 1 = 1+i.0 is the unity in R. Also the distributive laws of multiplication over addition holds in R. Hence, R is a commutative ring with unity under usual addition and multiplication.

Ans. to Q. No. 3 : $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$. The operations of addition and multiplication modulo 7 is shown in the tables below :

$\oplus$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	7	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

$\otimes$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

From the tables it can be easily verified that Z_7 is a commutative ring with unity 1.

Ans. to Q. No. 4 : Let R be a commutative ring without zero-divisors.

Let $a, b, c \in R$ such that $a \neq 0$ and $ab = ac$.

Then $ab - ac = 0$

$\Rightarrow a(b-c) = 0$, by the left distributive law

$\Rightarrow b - c = 0$, since $a \neq 0$ and R has no zero-divisors

$\Rightarrow b = c$.

Thus $a \neq 0, ab = ac \Rightarrow b = c$ Similarly, $a \neq 0$ and $ba = ca \Rightarrow b = c$.

Hence the cancellation laws hold in R.

Conversely, suppose the cancellation laws hold in R and let a, $b \in R$ such that $ab = 0, a \neq 0$.

Then $ab = a0$, since $a0 = 0$ in $R \Rightarrow b = 0$, by the left cancellation law.

Similarly $ab = 0, b \neq 0 \Rightarrow a = 0$.

Thus, $ab = 0 \Rightarrow a = 0$ or $b = 0$ and so R has no zero-divisors.

Ans. to Q. No. 5 : If possible, let 1 and e be two unities in a ring R with unity.

Then $1e = e$, taking 1 as unity.

Also 1e = 1, taking e as unity.

Hence e = 1 and so the unity in R is unique.

Ans. to Q. No. 6 : Let R be a ring where 1 = 0, i.e., the unity and the zero element are equal. Then $x \in R \Rightarrow x = 1x = ox = 0$. So, R = {0}.

Ans. to Q. No. 7 : Let R be a ring such that $a^2 = a \; \forall \; a \in R$.

i) $a \in R \Rightarrow a + a \in R$

$\Rightarrow (a+a)^2 = a + a$

$\Rightarrow (a+a)(a+a) = a + a$

$\Rightarrow (a+a)a + (a+a)a = a + a$, by the right distributive law

$\Rightarrow a^2 + a^2 + a^2 + a^2 = a + a$

$\Rightarrow (a+a) + (a+a) = (a+a) + 0$

$\Rightarrow a + a = 0$, by the left cancellation law in (R, +)

$\Rightarrow 2a = 0$.

ii) $a, b \in R \Rightarrow a + b \in R$

$\Rightarrow (a+b)^2 = a + b$

$\Rightarrow (a+b)(a+b) = a + b$

$\Rightarrow (a+b)a + (a+b)b = a + b$

$\Rightarrow a^2 + ba + ab + b^2 = a + b$

$\Rightarrow a + ba + ab + b = a + 0 + b$

$\Rightarrow ba + ab = 0$, using cancellation laws in (R, +)

$\Rightarrow ba + ab = ab + ab$, by (i)

$\Rightarrow ba = ab$, again by the right cancellation law.

Thus $ab = ba \; \forall \; a, b \in R$ and so R is a commutative ring.



9.7 FURTHER READINGS

1. *Modern Algebra* – S. Singh & Q. Zameeruddin.
2. *A course in Abstract Algebra* – V. K. Khanna & S. K. Bhambri.



9.8 MODEL QUESTIONS

Q.1. Show that $G = \{a + b\sqrt{3} : a, b \in Q\}$ is a commutative ring with unity under usual addition and multiplication of numbers.

Q.2. Define $\oplus$ an $*$ is Z as follows :

$a \oplus b = a + b - 1$, $a * b = a + b - ab$ for all $a, b \in Z$.

Show that $(Z, \oplus, *)$ is a commutative ring. Find the unity of this ring, if it exists.

Q.3. Let $R \times R = \{(a, b) : a, b \in R\}$. For $(a, b), (c, d) \in R \times R$, define addition and multiplication by $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac - bd, ad + bc)$

Show that $R \times R$ is a commutative ring with unity.

Q.4. Give examples of :

i) A commutative ring with unity

ii) A non-commutative ring with unity

iii) A commutative ring without unity

iv) A non-commutative ring without unity.

Q.5. Give examples of :

i) A finite commutative ring with zero-divisors

ii) A finite commutative ring without zero-divisors.

Q/6. Show that $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in Z \right\}$ is a non-commutative ring without unity.

Q.7. Give an example of a non-commutative ring R such that $(ab)^2 = a^2b^2$ for all $a, b \in R$.

Q.8. Prove that a ring R is a commutative ring if and only if $(a+b)^2 = a^2 + 2ab + b^2$ for all $a, b \in R$.

Q.9. Let F be the set of all functions $f : R \to R$. For $f, g \in F$, let us define f+g and fg as follows :

$(f+g)(x) = f(x) + g(x) \; \forall \; x \in R$

$(fg)(x) = f(x)g(x) \; \forall \; x \in R$.

Show that F is a commutative ring with unity under the above defined operations.

Q.10. Show that the set of even integers forms a commutative ring without unity under usual addition and multiplication.

UNIT 10 : INTEGRAL DOMAINS AND FIELDS

UNIT STRUCTURE

10.1 Learning Objectives

10.2 Introduction

10.3 Integral Domain (I.D.)

- 10.3.1 Definition of an I.D.
- 10.3.2 Examples of Integral Domains
- 10.3.3 Necessary and sufficient condition for an I.D.

10.4 Division Rings and Fields

- 10.4.1 Definition and examples of units in a ring
- 10.4.2 Definition and examples of Division ring
- 10.4.3 Definition and examples of Fields

10.5 Properties of I.D. and Fields

10.6 Let Us Sum Up

10.7 Answers to Check Your Progress

10.8 Further Readings

10.9 Model Questions

10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- define an integral domain
- learn about the necessary and sufficient condition for a ring to be an I.D.
- define Division ring and Field
- learn about basic properties of I.D. and Field.

10.2 INTRODUCTION

In the preceding unit we defined zero-divisors in a ring and we come across rings with or without zero-divisors. This leads us to a special kind of ring called Integral domain. Similarly, existence of multiplicative inverses of

non-zero elements in a commutative ring with unity leads to a special kind of ring called Field. In this unit we shall discuss Integral domain and Field and know some basic properties.

10.3 INTEGRAL DOMAIN (I.D.)

Before defining an I.D., let us recall zero-divisors in a ring discussed in the preceding unit.

A non-zero element 'a' of a ring R is called a zero-divisor if there exists another non-zero element 'f' in R such that ab = 0.

10.3.1 Definition of an I.D.

A commutative ring with unity in called on **Integral Domain (I.D.)** if it has no zero-divisors.

Alternatively, a commutative ring R with unity is called an I.D. if for all a, f $\in$ R, ab = 0 $\Rightarrow$ a = 0.

10.3.2 Examples of I.D.

Example 1 : Z, Q, R, C, i.e., the ring of integers, the ring of rational numbers, the ring of real numbers and the ring of complex numbers under usual addition and multiplication are all Integral domains since.

i) all these are commutative rings with unity 1,

and, ii) for any two elements a, b in each of these rings,

ab = 0 $\Rightarrow$ a = 0 or b = 0.

All these are infinite I.D's.

Example 2 : Z_6 = {0, 1, 2, 3, 4, 5}, the ring of integers modulo 6 is not an I.D. even though it is a commutative ring with unity 1.

Because 2, 3 $\in$ Z_6, both non-zero; but 2 $\otimes$ 3 = 0, as 2.3 = 6 leaves the remainder 0 when divided by 6. Thus Z_6 has zero-divisors.

Example 3 : Z_p = {0, 1, 2, ... , p – 1}, the ring of integers modulo p is an I.D., if p is a prime.

We already know that Z_p is a commutative ring with unity 1.

Let $a, b \in Z_p$ such that $a \otimes b = 0$. Then

$p \mid ab$

$\Rightarrow$ $p \mid a$ or $p \mid b$, since p is a prime

$\Rightarrow$ $a = 0$ or $b = 0$, since $a, b < p$

Thus $a \otimes b = 0 \Rightarrow a = 0$ or $b = 0$ and hence Z has no zero-divisors. Consequently, Z_p is an I.D.



NOTE : In some books, an I.D. is simply defined as a commutative ring without zero-divisors. The existence of unity is not taken into consideration.
But most of the infinite commutative rings without zero divisors have unity. Also, every finite commutative ring without zero divisors contains the unity. Because of these two reasons, we have defined an I. D. to be a commutative ring with unity and without zero-divisors.

10.3.3 Necessary and Sufficient Condition for I.D.

Theorem 1 : A commutative ring with unity is an I.D. if and only if the cancellation laws hold in it.

Proof : Let R be a commutative ring with unity 1.

Suppose R is an I.D. Then R has no zero-divisors.

Let $a, b, c \in R$ such that $ab = ac$, $a \neq 0$.

Then $a(b - c) = 0 \Rightarrow b - c = 0$,

since $a \neq 0$ and R has no zero-divisors

$\Rightarrow b = c$.

Thus, $ab = ac$, $a \neq 0 \Rightarrow b = c$.

Similarly, $ba = ca$, $a \neq 0 \Rightarrow b = c$.

Hence the cancellation laws hold in R.

Conversely, let the cancellation laws hold in R.

Then for $a, b, c \in R$ and $a \neq 0$.

$ab = 0$

$\Rightarrow$ $ab = a0$ $[\because a0 = 0]$

$\Rightarrow$ $b = 0$, by the left cancellation law.

Thus $ab = 0$, $a \neq 0$

$\Rightarrow b = 0$

Similarly $ab = 0$, $b \neq 0$

$\Rightarrow a = 0$

Hence R has no zero-divisors and so, R is an I.D.

This completes the proof.

10.4 DIVISION RING AND FIELD

Let us consider the ring of integers Z and the ring of real numbers R under usual addition and multiplication. In Z, only –1 and 1 have multiplicative inverses. In R, every non-zero element has a multiplicative inverse, viz, if $a \in R$ and $a \neq 0$, then $\frac{1}{a} \in R$ and $\frac{1}{a}.a = 1 = a.\frac{1}{a}$, i.e., $\frac{1}{a}$ is the multiplicative inverse of 'a' in R. Thus, R is different from Z with respect to existence of multiplicative inverse of non-zero elements. Similarly, we shall find non-commutative ring with unity when every non-zero element has multiplicative inverse. This leads us to the definition of new classes of rings, such as Division Rings and Fields.

10.4.1 Definition and Examples of Units in a Ring

Definition : A non-zero element a of a ring R is called a **unit** if there exists on element b is R such that ab = ba = 1. In this ease, b is the multiplicative inverse of a in R.

In other words, a non-zero element of a ring is called a unit if its multiplicative inverse exists.

Examples : 1) In Z, only –1 and 1 are units.

2) In Q or R or C, every non-zero element is a unit.

3) In Z_7 = {0, 1, 2, 3, 4, 5, 6}, the ring of integers modulo 7 under addition and multiplication modulo 7, we can easily find

$1 \otimes 1=1, 2 \otimes 4=1, 3 \otimes 5 =1, 6 \otimes 6 =1.$

Hence, every non-zero element of Z_7 is a unit.

10.4.2 Division Ring

Definition : A ring with unity is called a **division ring** or a **skew field** if every non-zero element is a unit.

Alternatively, a ring with unity is called a division ring if every non-zero element has multiplicative inverse.

Examples : 1) The rings Q, R, C are division rings.

2) Z, 2Z etc. are not division rings.

3) Let R = {a + bi + cj + dk : a, b, c, d $\in$ R}

Where i, j, k are just three abstract symbols which can be multiplied by the following rules :

$i^2 = j^2 = k^2 = ijk = -1$,

ij = –jk = k, jk = –kj = i, ki = –ik = j

It can be proved that R is a ring under addition and multiplication with

0 = 0 + 0i + 0j + 0k as the zero element and

1 = 1 + 0i + 0j + 0k as the unity. It is a **non-commutative ring** since ij $\neq$ ji.

Now, let q = a + bi + cj + dk $\in$ R.

Then $q \neq 0 \Rightarrow$ atleast one of a, b, c, d is non-zero.

$\Rightarrow a^2 + b^2 + c^2 + d^2 \neq 0$

So, $q^1 = \dfrac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2} \in R$

It can be easily shown that

$qq^1 = 1 = q^1q$, i.e., $q^{-1} = q^{-1}$.

Hence, every non-zero element of R is a unit an so R is a division ring.

This ring is called the **Ring of Real Quaternions**.



NOTE : An integral domain is generally denoted by D. Similarly, a field is generally denoted by F.

10.4.3 Fields

Definition : A commulative division ring is called a **field**.

Alternatively, a commutative ring with unity is called a field if every non-zero element has multiplicative inverse.

Obviously, the non-zero elements of a field form a multiplicative group. Hence, we may also define a field as follows :

A commutative ring R with unity is called a field if (R–{0}, .) is a group.

Examples : 1) Q, R, C are all fields

2) Z is not a field since non-zero integers except –1, 1 have no multiplicative inverses.
3) The ring of real quaternions
   $R = \{a + bi + cj + dk; a, b, c, d \in R\}$ is not a field, since it is non-commutative.
4) $Z_p = \{0, 1, 2, ..., p–1\}$, where p is a prime, is a field under addition and multiplication modulo p.

10.5 PROPERTIES OF I.D. AND FIELD

The basic properties of I.D. and field are the following :

1) The cancellation laws hold in an I.D.
2) Every field is an I.D. and,
3) Every finite I.D. is a field.

1) has been proved in 10.3.3 Theorem 1. We shall now prove 2) and 3) in the following theorems.

Theorem 2 : Every field is an I.D.

Proof : Let F be a field. Then F is a commutative ring with unity and every non-zero element of F has multiplicative inverse.

Let $a, b \in F$ such that $a \neq 0$. Then $a^{-1} \in F$.

Now $ab = 0 \Rightarrow a^{-1}(ab) = a^{-1}0$

$\Rightarrow (a^{-1}a)\, b = 0$

$\Rightarrow 1b = 0$

$\Rightarrow b = 0$

Thus $a \neq 0, ab = 0 \Rightarrow b = 0$

Similarly $b \neq 0, ab = 0 \Rightarrow a = 0$

Hence F has no zero divisors and so F is an I.D. This completes the proof.

Theorem 3 : Every finite integral domain is a field.

Proof : Let R be a finite I.D. and so R is a commutative ring with unity. To prove that R is a field, it is enough to show that every non-zero element of R has multiplication inverse.

Since R is finite, we can take $R = \{0, 1, a_1, a_2 ..., a_n\}$.

Let a be a non-zero element of R, i.e., $a \neq 0$. If $a = 1$, $a^{-1} = 1$.

So, We take $a \neq 1$. By closure property.

$R' = \{a1, aa_1, aa_2, ... aa_n) \subset R$

Since R has non-zero divisors, all the elements of R′ are non-zero. Moreover, the cancellation laws hold in R and so for $i \neq j$,

$aa_i = aa_j \Rightarrow a_i = a_j$, a contradiction.

This shows that all the elements of R′ are distinct.

Thus R′ has (n+1) distinct non-zero elements and so

$R' = R - \{0\} = \{1, a_1, a_2, ..., a_n\}$

Hence, $1 \in R' \Rightarrow 1 = aa_i$ for some $a_i \in R$.

The commutativity of R shows that $aa_i = 1 = a_ia$

Hence a^{-1} exists and $a^{-1} = a_i$.

Thus every non-zero element of R has multiplicative inverse, which completes the proof.



CHECK YOUR PROGRESS

Q.1. Give an example to show that every I.D. is not a field.

Q.2. If p is a prime, prove that Z_p, the ring of integers modulo p is a field.

Q.3. Give an example to show that Z_n, the ring of integers modulo n is not a field if n is a composite number.

Q.4. Show that a field has no zero-divisors.



10.6 LET US SUM UP

- An I. D. is a commutative ring with unity and without zero-divisors.
- A commutative ring with unity is an I.D. if and only if the cancellation laws hold in it.
- A non-zero element of a ring is called a **unit** if its multiplicative inverse exists.
- A ring with unity is called a division ring if every non-zero element is a unit.
- A commutative division ring is called a field.
- Every field is an I.D., but the converse is not true.
- Every finite I.D. is a field.



10.7 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : The ring of integers Z in an I.D., but not a field.

Ans. to Q. No. 2 : $Z_p = \{0, 1, 2, ..., p-1\}$, p is a Prime. Clearly Z_p is a commutative ring with unity 1. It is finite as it has p elements.

Suppose $a \otimes b = 0$ for $a, b \in Z_p$.

Then $p \mid ab$

$\Rightarrow p \mid a$ or $p \mid b$, since p is prime

$\Rightarrow a = 0$ or $b = 0$, since $0 \leq a, b < p$

Thus $a \otimes b = 0 \Rightarrow a = 0$ or $b = 0$, showing that Z_p has non-zero divisors. Hence Z_p is a finite integral domain and so, it is a field

Ans. to Q. No. 3 : Consider $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, the ring or integers modulo 8, where 8 is a composite number.

Here, $4 \otimes 6 = 0$ where as $4 \neq 0, 6 \neq 0$.

Thus Z_8 has zero-divisors and so it cannot be a field.

Ans. to Q. No. 4 : See proof of Theorem 2.



10.8 FURTHER READINGS

1. *A course in Abstract Algebra*– V. K. Khanna & S. K. Bhambri.
2. *Modern Algebra*– S. Singh, Q. Zameeruddin.



10.9 MODEL QUESTIONS

Q.1. Find out the units of $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, the ring of integers modulo 8. Is it an I. D.? Is it a field?

Q.2. Show that M, the set of all 2×2 matrices of the type $\begin{pmatrix} Z_1 & Z_2 \\ -Z_2 & Z_1 \end{pmatrix}$ where Z_1, Z_2 are complex numbers is a division ring, but not a field.

Q.3. Show that $R = \{a + b\sqrt{2} : a, b \in Q\}$ is a field under usual addition and multiplication.

Q.4. Let R be the set of real numbers. Show that $R \times R$ forms a field under addition and multiplication defined by

$(a, b) + (c, d) = (a + c, b + d)$

$(a, b).(c, d) = (ac – bd, ad + bc)$

[Hints : additive identity is (0, 0), multiplicative identity is (1, 0)

If $(a, b) \neq (0, 0)$, then $(a, b)^{-1} = \left(\frac{a}{a^2 + b^2}, -\frac{b}{a^2 + b^2}\right)$]

Q.5. Write true of false :

i) Every field is a ring.

ii) Every ring has a multiplicative identity.

iii) Every ring with unity has atleast two units.

iv) Multiplication in a field is commutative.

v) The non-zero elements of a field form a multiplicative group.

vi) Every division ring is a field.

vii) Every integral domain is a field.

Q.6. Prove that Z_n is not a field, if n is a composite number.

UNIT 11 : MATRICES

UNIT STRUCTURE

11.1 Learning Objectives

11.2 Introduction

11.3 Definition of Matrix

11.4 Types of Matrix

11.5 Equality of Matrices

11.6 Addition of Matrices

11.6.1 Properties of Matrix Addition

11.7 Multiplication of a Matrix by a Scalar

11.7.1 Properties of Multiplication of a Matrix by a Scalar

11.8 Multiplication of Matrix

11.8.1 Properties of Multiplication of Matrices

11.9 Transpose of a Matrix

11.10 Symmetric and Skew-Symmetric Matrices

11.11 Canonical Form of Matrices

11.12 Let Us Sum Up

11.13 Answers to Check Your Progress

11.14 Further Readings

11.15 Model Questions

11.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- define a matrix
- describe about types of matrix
- learn about the matrix operations i.e. addition, subtraction and multiplication
- define transpose of a matrix.

11.2 INTRODUCTION

Matrix is one of the most powerful tools in modern mathematics. They provide an algebraic structure slightly different from that of real numbers. The method of solving linear equations becomes easy with the help of matrices. Moreover, matrix notation and operations are used in computer graphics programming and implementation of electronic spreadsheet programs. They are widely used in modern algebra, applied mathematics, atomic physics, mathematical problems of economics as well as in computer science also. In this unit, we will introduce you to the fundamentals of matrix. We shall also discuss about the various types of matrices along with matrix operations.

11.3 DEFINITION OF MATRIX

A matrix is an ordered rectangular array of numbers.

A set of *'mn'* numbers (real or complex) arranged in the form of a rectangular array having *'m'* rows and *'n'* colums is called an $m \times n$ matrix (to be read *'m'* by *'n'* matrix).

An $m \times n$ matrix is usually written as :

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{bmatrix}$$

In a compact form the above matrix is represented by

$A = \left[a_{ij}\right]$, $i = 1, 2, ..., ..., m$

$j = 1, 2, ..., ..., n$

Or simply as $A = \left[a_{ij}\right]_{m \times n}$

We write the general elements of the matrix and enclose it in brackets of the type [ ] or ( ). The numbers $a_{11}, a_{12}, ...,$ etc. of this rectangular array are called the elements or entries of the matrix. The element a_{ij} belongs

to the i^{th} row and j^{th} column and is sometimes called the $(i, j)^{th}$ element of the matrix.

We denote matrices by capital letters such as A, B, C, etc. The following are some examples of matrices :

$A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ is a 2×2 matrix i.e. it has 2 rows and 2 column, whose elements are 2, 1, 3, 5.

$B = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 4 & -5 \end{bmatrix}$ is a 2×3 matrix. Whose elements are 3, –2, 0, 1, 4, –5.

$C = \begin{bmatrix} 1-i & 3 & 2/3 \\ 2.7 & -1 & 1 \\ \sqrt{2} & 5 & 4/7 \end{bmatrix}$ is a 3×3 matrix.



NOTE : A matrix having *m*-rows and *n* column is called a matrix of order *m*×*n* or simply *m*×*n* matrix (read as *m* by *n* matrix). In the example, the matrix A is of order 2×2, B is of order 2×3 and C is of order 3×3.

11.4 TYPES OF MATRIX

Now, we are going to discuss about the different types of matrices.

i) Row Matrix : Any *1*×*n* matrix which has only one row and *n* columns is called a row matrix. e.g.

$A = [4 \;\; 6 \;\; -3 \;\; \sqrt{5} \;\; 0]$ is a row matrix of order 1×5.

In general, $A = \left[a_{1j}\right]_{1\times n}$ is a row matrix of order *1*×*n*.

ii) Column Matrix : Any *m*×*1* matrix which has *m* rows and only one column is called a column matrix. e.g.

$B = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ is a column matrix of order 3×1.

In general, $B = \left[b_{i1}\right]_{m\times 1}$ is a column matrix of order *m*×*1*.

iii) Square Matrix : A matrix in which the number of rows are equal to the number of columns is called a square matrix.

An *m*×*n* matrix for which *m* = *n* (i.e. the number of rows are equal to the number of columns) is called a square matrix of order *n*. For example,

$A = \begin{bmatrix} 1 & 2 & -1 \\ 1/2 & \sqrt{5} & 1 \\ 3 & 0 & -5 \end{bmatrix}$ is a square matrix of order 3.

In a square matrix $A = [a_{ij}]_{n\times n}$, the elements $a_{11}, a_{22}, ..., a_{nn}$ are called the **Diagonal Elements**.

iv) Diagonal Matrix : A square matrix $B = [b_{ij}]_{m\times m}$ is said to be a diagonal matrix if all its non-diagonal elements are zero, that is a matrix $B = [b_{ij}]_{m\times m}$ is said to be a diagonal matrix if $b_{ij} = 0$, when $i \neq j$, e.g.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

are diagonal matrices of order 2, 3 respectively.

v) Scalar Matrix : A square matrix in which all the diagonal elements are equal and other elements are zero is called a scalar matrix.

Thus, the square matrix $A = [a_{ij}]_{m\times m}$ is a scalar matrix if

$a_{ij} = 0$ when $i \neq j$ and $a_{11} = a_{22} = ... = a_{mm} = \alpha$ (say).

$$\text{e.g. } A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

are scalar matrices of order 2 and 3 respectively.

vi) Identity Matrix : A square matrix in which the diagonal elements are all 1 and rest are all zero is called an identity matrix.

Thus, a square matrix $A = [a_{ij}]_{m\times m}$ is an identity matrix if

$a_{ij} = 1$ when $i = j$

$a_{ij} = 0$ when $i \neq j$.

An $n\times n$ identity matrix is denoted by I_n. Thus

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are identity matrices of orders 2, 3 respectively. An identity matrix is also called a unit matrix. If there is no scope for confusion regarding the order of an identity matrix, then it is simply denoted by I.

vii) Null Matrix or Zero Matrix : A matrix of order $m \times n$ whose elements are all *0* is called a null matrix (or Zero matrix). It is usually denoted by *0*.

$$[0 \quad 0], \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are all zero matrices of orders 1×2, 2×1, 2×2 respectively.

11.5 EQUALITY OF MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if :

i) they are of the same size or order.

ii) the elements in the corresponding places of the two matrices are equal i.e.,

$a_{ij} = b_{ij}$ for all i and j.

If two matrices A and B are equal, we write A = B. If two matrices A and B are not equal, we write $A \neq B$. If two matrices are not of the same size, they cannot be equal, e.g.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

If $\begin{bmatrix} x & y \\ z & a \\ b & c \end{bmatrix} = \begin{bmatrix} -1.5 & 0 \\ 2 & \sqrt{6} \\ 3 & 2 \end{bmatrix}$

Then x = –1.5, y = 0, z = 2, a = $\sqrt{6}$, b = 3, c = 2



CHECK YOUR PROGRESS

Q.1. If a matrix has 12 elements, what are the possible orders it can have?

Q.2. Find the values of a, b, c and d from the following equation.

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

Q.3. In a matrix A = $\begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & 5/2 & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$

Find i) the order of the matrix,

ii) the number of elements,

iii) write the elements a_{13}, a_{21}, a_{33}, a_{24}, a_{23}.

11.6 ADDITION OF MATRICES

Suppose A and B be two matrices of the same order $m \times n$, then their sum, denoted by A+B, is defined to be the matrix of the order $m \times n$ obtained by adding the corresponding elements of A and B.

Thus, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, then $A + B = [a_{ij} + b_{ij}]_{m\times n}$

The resultant matrix should be of the order of $m \times n$.

More clearly, if $A = \begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{bmatrix}_{m\times n}$

and $B = \begin{bmatrix} b_{11} & b_{12} & \dots & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & \dots & b_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & \dots & b_{mn} \end{bmatrix}_{m\times n}$,

$$\text{then } A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & \dots & a_{2n}+b_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & \dots & a_{mn}+b_{mn} \end{bmatrix}_{m\times n}$$

Again, if, A and B are two matrices of order $m\times n$, then the difference 'A – B' is obtained as A – B = A + (–B), where –B is obtained on multiplying all the elements of B by –1.

It follows that $A - B = [a_{ij} - b_{ij}]_{m\times n}$.

whenever $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$.

Example : Given $A = \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 \\ 1 & 7 \end{bmatrix}$,

find A + B and A – B.

Solution : $A + B = \begin{bmatrix} 1+3 & 5+6 \\ 0+1 & 3+7 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ 1 & 10 \end{bmatrix}$

$$A - B = A + (-B) = \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix} + (-1)\begin{bmatrix} 3 & 6 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -3 & -6 \\ -1 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(-3) & 5+(-6) \\ 0+(-1) & 3+(-7) \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -4 \end{bmatrix}$$

$$\text{In short, } A - B = \begin{bmatrix} 1-3 & 5-6 \\ 0-1 & 3-7 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -4 \end{bmatrix}$$

11.6.1 Properties of Matrix Addition

The addition of matrices satisfy the following properties :

i) **Commutative Law :** If $A = [a_{ij}]$, $B = [b_{ij}]$ are matrices of the same order, say $m\times n$, then A + B = B + A

Now, $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}]$ (addition of numbers is commutative)

$= [b_{ij}] + [a_{ij}]$

$= B + A$

ii) Associative Law : For any three matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ of the same order, say $m \times n$,

$(A + B) + C = A + (B + C)$

Now, $(A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$= [a_{ij} + b_{ij}] + [c_{ij}]$

$= [(a_{ij} + b_{ij}) + c_{ij}]$

$= [a_{ij} + (b_{ij} + c_{ij})]$

$= [a_{ij}] + ([b_{ij}] + [c_{ij}])$

$= A + (B + C)$

iii) Existence of Additive Identity : If 0 be the $m \times n$ zero matrix and $A = [a_{ij}]$ be an $m \times n$ matrix, then

$A + 0 = [a_{ij} + 0]_{m \times n} = [a_{ij}]_{m \times n} = A$

also, $0 + A = [0 + a_{ij}]_{m \times n} = [a_{ij}]_{m \times n} = A$

In other words, the null matrix plays the role of additive identity in the set of all $m \times n$ matrices.

iv) Existence of Additive Inverse : Let $A = [a_{ij}]_{m \times n}$ be any matrix. Then we have another matrix $-A = [-a_{ij}]_{m \times n}$, such that

$A + (-A) = (-A) + A = 0$

So, $-A$ is the additive inverse of A or negative of A.

Example : If $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ find A + B.

Solution : $A + B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$= \begin{bmatrix} a+a & b-b \\ -b+b & a+a \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 0 & 2a \end{bmatrix}$

Example : If $A = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 \\ 2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 5 \\ 6 & 4 \end{bmatrix}$,

Show that $(A + B) + C = A + (B + C)$.

Solution : $A + B = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+3 \\ 3+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 9 \end{bmatrix}$

$$\therefore (A + B) + C = \begin{bmatrix} 1 & 3 \\ 5 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 1+3 & 3+5 \\ 5+6 & 3+4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 11 & 13 \end{bmatrix}$$

$$\text{Again, } B + C = \begin{bmatrix} 0 & 3 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 0+3 & 3+5 \\ 2+6 & 5+4 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 9 \end{bmatrix}$$

$$\therefore A + (B + C) = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 8 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 1+3 & 0+8 \\ 3+8 & 4+9 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 11 & 13 \end{bmatrix}$$

$\therefore$ (A + B) + C = A + (B + C).

11.7 MULTIPLICATION OF A MATRIX BY A SCALAR

Multiplication of a matrix by a scalar is defined as follows :

If $A = [a_{ij}]_{m\times n}$ is a matrix of order $m\times n$ and k is a scalar, then kA is an another matrix which is obtained by multiplying each element of A by the scalar k.

In other words, $kA = k[a_{ij}]_{m\times n} = [k.a_{ij}]_{m\times n}$

i.e. $(i, j)^{th}$ element of kA is ka_{ij} for all possible values of i and j.

For example, if $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$

then $2A = 2\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 \\ 4 & 0 & 6 \\ 6 & -2 & 4 \end{bmatrix}$

$$\frac{1}{2}A = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

Conversely, if all the elements of a matrix have a common factor, then that common factor can be taken out. For example,

$$\begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} = \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

11.7.1 Properties of Multiplication of a Matrix by a Scalar

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of same order, say $m \times n$ and k and l are two scalars, then the following are true.

i) $k(A + B) = kA + kB$

ii) $(k + l)A = kA + lA$

iii) $k(lA) = (kl)A$

iv) $(-k)A = k(-A)$

Example : If $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$, then find the matrix X.

Solution : We have $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

$$\Rightarrow 2X = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & 0-2 \\ -3-1 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{2}\begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix} \quad \therefore X = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

Example : If $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 6 & -3 \\ 1 & 0 & -2 \end{bmatrix}$, then find the matrix X such that $2A - 3B - X = 0$

Solution : We have $2A - 3B - X = 0$

$$\therefore X = 2A - 3B$$

$$= 2\begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} - 3\begin{bmatrix} 4 & 6 & -3 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{bmatrix} - \begin{bmatrix} 12 & 18 & -9 \\ 3 & 0 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 0-12 & 4-18 & 6-(-9) \\ 4-3 & 2-0 & 8-(-6) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -14 & 15 \\ 1 & 2 & 14 \end{bmatrix}$$

11.8 MULTIPLICATION OF MATRICES

Two matrices A and B are conformable for the product AB only when the number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]_{m\times n}$ i.e. A is of order $m\times n$ and $B = [b_{jk}]_{n\times p}$ i.e. B is of order $n\times p$. Then the product AB of the matrices A and B is the matrix C of order $m\times p$, i.e.

$C = [c_{ik}]_{m\times p}$

Where, $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + \ldots \ldots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}.b_{jk}$

It means, the $(i, k)^{th}$ element c_{ik} of the matrix C = AB is obtained by multiplying the corresponding elements of the i^{th} row of A and the k^{th} column of B and then adding the products. The rule of multiplication is row by column multiplication i.e. in the process of multiplication we take the rows of A and the columns of B.

In the product AB the matrix A is called the *prefactor* and the matrix B is called the *post factor*.

Example : If $A = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix}$ find AB. Also find BA. Are they equal?

Solution : Here, the matrix A is of order 3×1 and B is of order 1×3. Since the number of columns in A is equal to the number of rows in B, so the two matrix will be comformable for the product AB. The resultant matrix will be of order 3×3.

$$AB = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3\times 1 & 3\times 0 & 3\times 4 \\ 5\times 1 & 5\times 0 & 5\times 4 \\ 2\times 1 & 2\times 0 & 2\times 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 12 \\ 5 & 0 & 20 \\ 2 & 0 & 8 \end{bmatrix}$$

Again, B is 1×3 and A is 3×1. Here, column number in B is equal to the row number in A. The resultant matrix will be of order 1×1.

$$BA = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = [1\times 3 + 0\times 5 + 4\times 2] = [11]$$

$\therefore AB \neq BA$.



NOTE :

1) The product AB of two matrices A and B exists if and only if the number of columns in A is equal to the number of rows in B.
2) If A be of order m×n and B be of order n×p, then C = AB is defined and will be of order m×p.



NOTE : Matrix multiplication is not commutative i.e. for two matrices A and B which are comformable for multiplication, $AB \neq BA$.

Example : If $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ find AB.

Can you find BA?

Solution : Here, matrix A is of order 2×2 and matrix B is of order 2×3. Number of columns in matrix A is equal to the number of rows in B, so the product AB is defined and the resultant matrix will be of order 2×3.

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 3+1\times 1 & 2\times 0+1\times(-2) & 2\times 2+1\times 3 \\ 1\times 3+0\times 1 & 1\times 0+0\times(-2) & 1\times 2+0\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6+1 & 0-2 & 4+3 \\ 3+0 & 0 & 2+0 \end{bmatrix} = \begin{bmatrix} 7 & -2 & 7 \\ 3 & 0 & 2 \end{bmatrix}$$

Since number of columns of B is not equal to the number of rows of A, the product BA is undefined.

11.8.1 Properties of Multiplication of Matrix

The multiplication of matrices possesses the following properties :

i) The multiplication of matrices is not always commulative :
   a) Whenever AB exists, it is not always necessary that BA should also exist.
   b) Whenever AB and BA both exist, it is always not necessary that they should be matrices of the same order.
   c) Whenever AB and BA both exist and are matrices of the same order, it is not necessary that AB = BA.

      For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

      then we can show that $AB \neq BA$ although both AB, BA exist.
   d) It however does not imply that AB is never equal to BA.

For example, if $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ and

$B = \begin{bmatrix} 10 & -4 & -1 \\ -11 & 5 & 0 \\ 9 & -5 & 1 \end{bmatrix}$, we can show that AB = BA.

ii) Matrix multiplication is associative i.e. A(BC) = (AB)C
where A, B and C are matrices of order $m \times n$, $n \times p$ and $p \times q$ respectively.

iii) Matrix multiplication is distributive with respect to addition of matrices i.e. A(B + C) = AB + AC
where A, B and C are matrices of order $m \times n$, $n \times p$ and $n \times p$ respectively.

iv) For every square matrix A, there exist an identity matrix *I* of same order such that, AI = IA = A.



CHECK YOUR PROGRESS

Q.4. Find X and Y if $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

Q.5. Find the values of x and y from the following equation :

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Q.6. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$.

Q.7. Using examples, show that A(BC) = (AB)C.

Q.8. Using examples, show that A(B+C) = AB + AC.

11.9 TRANSPOSE OF A MATRIX

Let $A = [a_{ij}]_{m\times n}$ be a matrix of order $m\times n$. Then the matrix obtained from A by changing its rows into columns and columns into rows is called the transpose of A and is denoted by the symbol A′ or A^T.

i.e. if $A = [a_{ij}]_{m\times n}$ then $A' = [b_{ij}]_{n\times m}$, where $b_{ij} = a_{ji}$

i.e. the $(i, j)^{th}$ element of A′ is the $(j, i)^{th}$ element of A.

For example : If $A = \begin{bmatrix} 1 & 9 & 7 \\ 4 & 3 & 5 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 4 \\ 9 & 3 \\ 7 & 5 \end{bmatrix}$

Some important properties of transpose of matrices are given below :

1. The transpose of the transpose of a matrix is the matrix itself i.e. if A is a matrix of order $m\times n$, then $(A')' = A$.
2. If A is matrix of order $m\times n$ and k is a scalar, then $(kA)' = kA'$

3. If A and B are two matrices suitable for addition, then
   $(A+B)' = A' + B'$
4. If A and B are two matrices such that AB is defined, then
   $(AB)' = B'A'$

Example 1 : If $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$ show that $(A')' = A$.

Solution : We have $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$ $\therefore A' = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$

Now $(A')' = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix} = A$

Hence $(A')' = A$

Example 2 : If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 5 & 1 \\ 6 & 7 & -1 \end{pmatrix}$,

then show that $(AB)' = B'A'$.

Solution : $A' = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $B' = \begin{pmatrix} 4 & 6 \\ 5 & 7 \\ 1 & -1 \end{pmatrix}$,

and so $B'A' = \begin{pmatrix} 4 & 6 \\ 5 & 7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 16 & 36 \\ 19 & 43 \\ -1 & -1 \end{pmatrix}$

Also $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & 5 & 1 \\ 6 & 7 & -1 \end{pmatrix} = \begin{pmatrix} 16 & 19 & -1 \\ 36 & 43 & -1 \end{pmatrix}$

$\therefore (AB)' = \begin{pmatrix} 16 & 36 \\ 19 & 43 \\ -1 & -1 \end{pmatrix}$ Thus $(AB)' = B'A'$.

11.10 SYMMETRIC AND SKEW-SYMMETRIC MATRICES

1. **Symmetric Matrix :** A square matrix $A = [a_{ij}]$ is said to be symmetric if $A' = A$ i.e. $[a_{ij}] = [a_{ji}]$ for all possible values of *i* and *j*, its *(i, j)*th element is the same as its *(j, i)*th element.

The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 7 \\ 3 & 7 & 6 \end{bmatrix}$ is a symmetric

For $A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 7 \\ 3 & 7 & 6 \end{bmatrix} = A$

Examples : $\begin{bmatrix} 2 & 7 \\ 7 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & i & -2i \\ i & -2 & 4 \\ -2i & 4 & 3 \end{bmatrix}$ are symmetric matrices.



NOTE :

i) A square matrix A is said to be symmmetric if $A = A'$

ii) A square matrix A is said to be skew-symmetric if $A' = -A$

iii) The diagonal elements of a skew-symmetric matrix are all zero.

2. **Skew-Symmetric Matrix :** A square matrix $A = [a_{ij}]$ is said to be skew-symmetric if $A' = -A$ that is $a_{ij} = -a_{ji}$ for all possible values of i, j.
In particular, for $i = j$, we have $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$
Thus, the diagonal elements of a skew-symmetric matrix are all zero.

$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{bmatrix}$ is skew-symmetric

for $A' = \begin{bmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{bmatrix} = -\begin{bmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{bmatrix} = -A$

Examples : $\begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & 8 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ are skew-symmetric matrices.

11.11 CANONICAL FORM OF MATRICES

Definition 1 :

ECHELON MATRIX : A matrix A is called an **echelon matrix**, or is said to be in **echelon form** if :

i) any zero rows, that is rows with all elements zero, are on the bottom of the matrix.

ii) the first non-zero entry, called the **leading entry** in a row, occurs to the right of the leading non-zero entry in the preceding row.

Example : The matrices $\begin{pmatrix} 1 & 2 & -3 \\ 0 & 4 & -20 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & 4 & 5 \end{pmatrix}$

are echelon matrices.

Definition 2 :

ROW CANONICAL FORM OF MATRIX : A matrix A is said to be in **row canonical form** if :

i) A is an echelon matrix,

ii) each leading non-zero entry in a row is 1,

iii) each leading non-zero entry is the only non-zero entry in its column.

A matrix in Row Canonical Form is also called **Reduced Echelon Matrix**.

Example : The matrix $\begin{pmatrix} 1 & 0 & 5 & 0 & 2 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is in row canonical form

where as the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is not in row canonical form.

Any given matrix can be reduced to row canonical form with the help of some operations performed in its rows. These are called **Elementary Row Operations** which we state below.

ELEMENTARY ROW OPERATIONS : The following are called elementary row operations or a matrix $A = [a_{ij}]_{m\times n}$ with respective symbols :

i) Interchanging the i^{th} row and the j^{th} row : $R_i \leftrightarrow R_j$.

ii) Multiplying the i^{th} row by a non-zero scalar k : $R_i \rightarrow kR_i$.

iii) Replacing the i^{th} row by adding to this row k times the j^{th} row : $R_i \rightarrow R_i + kR_j$.

EQUIVALENT MATRICES : If A and B are two m×n matrices, then B is said to be **row-equivalent** to A if it can be obtained from A by a finite number of elemantary row operations. This row-equivalence is then denoted by $A \xrightarrow{R} B$ or by $A \sim B$.

It can be shown that $A \sim B \Leftrightarrow B \sim A$

$A \sim B, B \sim C \Rightarrow A \sim C.$

Example 1 : Using elementary row operations, show that

$$B = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 2 \\ 0 & 3 & 0 & 5 \end{pmatrix} \text{ is row equivalent to } A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 1 & 0 & 2 \end{pmatrix}.$$

Solution : $A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 3 & 0 & 5 \\ -1 & 1 & 0 & 2 \end{pmatrix},$

by $R_2 \to R_2 + 2R_3$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 2 \\ 0 & 3 & 0 & 5 \end{pmatrix} = B, \text{ by } R_2 \leftrightarrow R_3$$

Thus $A \sim B$.

Example 2 : Reduce the matrix $A = \begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$

to echelon form.

Solution : $A = \begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{pmatrix}, \text{ by } R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7 & -10 \end{pmatrix}, \text{ by } R_3 \to 3R_3 - 7R_2$$

Thus A $\sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7 & -10 \end{pmatrix}$ which is in echelon form.

Example 3 : Reduce A = $\begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{pmatrix}$ to echelon form.

Solution : A $= \begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{pmatrix}$

$\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & -1 \end{pmatrix}$, by $R_1 \leftrightarrow R_2$

$\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -4 \end{pmatrix}$, by $R_3 \to R_3 - R_1$

$\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, by $r_3 \to R_3 - 2R_2$

Thus A $\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which is in echelon form.

Example 4 : Reduce the matrix A = $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 7 & 4 \\ 4 & 7 & 10 & 13 & 16 \end{bmatrix}$

to row canonical form.

Solution : A $= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 7 & 4 \\ 4 & 7 & 10 & 13 & 16 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -1 & -3 & -5 & -11 \\ 0 & -1 & -2 & -3 & -4 \end{bmatrix}, \text{ by } R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 3R_1,\ R_4 \to R_4 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -5 & -11 \\ 0 & -1 & -2 & -3 & -4 \end{bmatrix}, \text{ by } R_2 \to (-1).R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -2 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + R_2,\ R_4 \to R_4 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ by } R_3 \to (-1).R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ by } R_2 \to R_2 - 2R_3$$

Which is in row canonical form.

Example 5 : Reduce to row canonical form :

$$A = \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 4 & -1 & 3 \\ 2 & 5 & 9 & -2 & 8 \end{bmatrix}$$

Solution : $A = \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 4 & -1 & 3 \\ 2 & 5 & 9 & -2 & 8 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 9 & 3 & -4 & 4 \end{bmatrix}, \text{ by } R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}, \text{ by } R_3 \to R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}, \text{ by } R_3 \to \tfrac{1}{2}R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 0 & \frac{3}{2} \\ 0 & 3 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}, \text{ by } R_1 \to R_1 - R_3,\ R_2 \to R_2 + 2R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}, \text{ by } R_2 \to \frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & -2 & \frac{11}{3} & 0 & \frac{17}{6} \\ 0 & 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}, \text{ by } R_1 \to R_1 + 2R_2$$

Which is the row canonical form of A.



11.12 LET US SUM UP

- A rectangular array of *mn* numbers, arranged in *m*-rows and *n*-columns and enclosed in a square [ ] or a round bracket ( ), is called a *matrix* of order *m* by *n* (denoted by *m*×*n*).

- The numbers occuring in a matrix are called the elements of a matrix.
- The element a_{ij} appearing in the i^{th} row and the j^{th} column of A is called the *(i, j)th* element of A.
- A matrix with exactly one column and any number of rows is called a column *matrix*. The order of a column matrix is of the type *m×1*.
- A matrix with exactly one row and any number of columns is called a row matrix. The order of a row matrix is of the type *1×n*.
- A matrix $A = [a_{ij}]_{m\times n}$ is a square matrix if $m = n$.
- The square matrix $[a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.
- The square matrix $[a_{ij}]$ is a scalar matrix if $a_{ij} = \begin{cases} \alpha & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$
- The square matrix $[a_{ij}]$ is a unit matrix if $a_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$

  It is denoted by I_n or I.
- A matrix in which each element is zero is called a zero matrix or a null matrix, denoted by 0.
- Two matrix are said to be equal if and only if they are of the same order and their corresponding elements are equal.
- If A and B are two matrices of the same order, then their sum, denoted by A + B, is the matrix obtained by adding the corresponding elements of A and B.
- Commutative property of addition. For two matrix A and B, A+B = B+A.
- Associativity of addition : (A+B) + C = A + (B+C)
- Existence of additive identity : A + 0 = 0 + A = A
- Existence of additive inverse : A + (–A) = (–A) + A = 0
- Scalar multiplication of matrix : $A = [a_{ij}]_{m\times n} \Rightarrow kA = [k.a_{ij}]_{m\times n}$
  a) $k(A + B) = kA + kB$
  b) $(k+l)A = kA + lA$
  c) $k(lA) = (kl)A$
  d) $(–k)A = k(–A)$
- If $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$, then $AB = C = [c_{ik}]_{m\times p}$,

where $c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$

- Matrix multiplication is not commutative in general, i.e. $AB \neq BA$.
- Matrix multiplication is associative, i.e. (AB)C = A(BC).
- Matrix multiplication is distributive over addition,
  i) A(B+C) = AB + AC
  ii) (A+B)C = AC + BC
- If A is a square matrix and *I* is an identity matrix having the order same as A then A*I* = *I*A = A.
- If $A = [a_{ij}]_{m\times n}$, then A′ or $A^T = [b_{ji}]_{n\times m}$ where $b_{ji} = a_{ij}$, is called the transpose of A.
- Matrix A is called a symmetric matrix if A′ = A.
- A is skew symmetric matrix if A′ = –A.
- A matrix A is in echelon form if :
  i) any zero rows are on the bottom of the matrix.
  ii) The first non-zero entry, called the leading entry in a row occurs to the right of the leading non-zero entry in the preceding row.
- A matrix A is said to be in row canonical form if :
  i) A is an echelon matrix,
  ii) each leading non-zero entry in a row is 1,
  iii) each leading non-zero entry is the only non-zero entry in its column.
- Elementary row operations are :
  i) interchanging any two rows : $R_i \leftrightarrow R_j$
  ii) multiplying a row by a non-zero scalar : $R_i \rightarrow kR_j$
  iii) replacing the i^{th} row by adding to this row k times the j^{th} row : $R_i \rightarrow R_i + kR_j$.
- Two matrices A and B are row-equivalent, denoted by A ~ B, if B can be obtained from A using finite sequence of elementary row operations.

11.13 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : We have already discussed that if a matrix is of order $m \times n$, it has *'mn'* elements. Now to find all possible orders of a matrix with 12 elements, we will find all ordered pairs of natural numbers, whose product is 12. Thus, all possible ordered pairs are :

(1, 12), (12, 1), (2, 6), (6, 2), (3, 4), (4, 3)

Hence, possible orders are :

1×12, 12×1, 2×6, 6×2, 3×4, 4×3.

Ans. to Q. No. 2 : By equality of two matrices, equating the corresponding elements we get $2a + b = 4$ $\quad a - 2b = -3$

$5c - d = 11$ $\quad 4c + 3d = 24$

Solving these equations we will get $a = 1, b = 2, c = 3, d = 4$

Ans. to Q. No. 3 : i) 3×4; ii) 12; iii) 19, 35, –5, 12, $\frac{5}{2}$

Ans. to Q. No. 4 : We have $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$, $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Adding the two, we get

$$X + Y + X - Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow 2X = \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 4+3 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{2}\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

Now, $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$

$$\Rightarrow Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - X$$

$$= \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$ [Putting the value of X]

$$= \begin{bmatrix} 7-5 & 0-0 \\ 2-1 & 5-4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}, Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Ans. to Q. No. 5 : We have, $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$\Rightarrow 2x + 3 = 7$ and $2y - 4 = 14$

$\Rightarrow 2x = 7 - 3$ and $2y = 14 + 4$

$$\Rightarrow x = \frac{4}{2} \text{ and } y = \frac{18}{2}$$

$\therefore x = 2$ and $y = 9$

Ans. to Q. No. 6 : We have $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$\text{Now } A^2 = A.A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$$

$$7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence, $A^2 - 5A + 7I = 0$.

Ans. to Q. No. 7 : Take $A = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix}$

Clearly AB, BC are defined and we get.

$$AB = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 15 \\ 4 & 12 \end{pmatrix}$$

$$BC = \begin{pmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ -3 & -5 \\ 1 & 5 \end{pmatrix}$$

$$\text{Now,} \quad A(BC) = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ -3 & -5 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 30 \\ -8 & 0 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 15 & 15 \\ 4 & 12 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 30 \\ -8 & 0 \end{bmatrix}$$

Thus A(BC) = (AB)C.

Ans. to Q. No. 8 : Take $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$

Clearly AB, AC, B+C and A(B+C) are defined.

$$B+C = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$$

$$\therefore A(B+C) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & -1 \\ 5 & 3 \end{bmatrix}$$

Again, $AB = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 6 & -1 \end{bmatrix}$

$$AC = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 0 \\ -1 & 4 \end{bmatrix}$$

$$\therefore AB + AC = \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 6 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ -1 & 0 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & -1 \\ 5 & 3 \end{bmatrix}$$

Thus A(B+C) = AB + AC.



11.14 FURTHER READINGS

1. *Matrix Algebra*, S. K. Jain, Eastern Book House.
2. *Matrices*, A. R. Vasishtha, Krishna Prakashan Mandir, Meerut.



11.15 MODEL QUESTIONS

Q.1. If $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$ find A – 3B.

Q.2. If $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 4 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$.

Verify the following : i) A + B = B + A ii) 2(A – B) = 2A – 2B

Q.3. Solve for x, $\begin{bmatrix} x^2 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2x & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & 7 \end{bmatrix}$

Q.4. Find x and y if $x-2y = \begin{bmatrix} -1 & 4 \\ -3 & 1 \end{bmatrix}$ and $2x-y = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$

Q.5. If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 4 \\ 2 & 1 \end{bmatrix}$

find AB and BA and show that $AB \neq BA$.

Q.6. If $A = \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$ find $-A^2 + 6A$.

Q.7. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, show that $A^2 - 2A = 3I$.

Q.8. If $A = \begin{pmatrix} 2 & -3 & 5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}$, show that $A^2 = A$.

Q.9. If $A = \begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix}$, show that $A^2 = 0$.

Q.10. If n is any positive integer, show that

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}^n = \begin{bmatrix} \cos nx & \sin nx \\ -\sin nx & \cos nx \end{bmatrix}$$

Q.11. Give examples to show that, $(A+B)' = A' + B'$, $(AB)' = B'A'$.

Q.12. If A be a square matrix, show that :

i) AA' is symmetric matrix.

ii) $A + A'$ is symmetric and $A - A'$ is skew-symmetric.

iii) A is the sum of a symmetric and a skew-symmetric matrix.

Q.13. Reduce to echelon form :

i) $\begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{pmatrix}$

ii) $\begin{pmatrix} 2 & -2 & 2 & 1 \\ -3 & 6 & 0 & -1 \\ 1 & -7 & 10 & 2 \end{pmatrix}$

Q.14. Reduce to row-canonical form :

i) $\begin{pmatrix} 1 & 2 & -3 & 0 & 1 \\ 0 & 0 & 5 & 2 & -4 \\ 0 & 0 & 0 & 7 & 3 \end{pmatrix}$

ii) $\begin{pmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 6 & 6 & 0 & 20 & 19 \end{pmatrix}$

UNIT 12 : DETERMINANT AND INVERSE OF A MATRIX

UNIT STRUCTURE

12.1 Learning Objectives

12.2 Introduction

12.3 Determinant of a Square Matrix

- 12.3.1 Minors and Cofactors of Elements of a Matrix
- 12.3.2 Determinant of a Diagonal Matrix and that of Identity Matrix
- 12.3.3 Determinant of Product of two Matrices

12.4 Adjoint of a Square Matrix

12.5 Inverse of a Square Matrix

- 12.5.1 Illustrative Examples on Inverse Matrices
- 12.5.2 Finding Inverse by Elementary Row Operations

12.6 Rank and Nullity of a Matrix

- 12.6.1 Finding Rank by Elementary Row Operations

12.7 Let Us Sum Up

12.8 Answers to Check Your Progress

12.9 Further Readings

12.10 Model Questions

12.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- know about determinants of a square matrix, minors and cofactors of elements of a matrix
- define adjoint of a square matrix and know how to find it
- define inverse of a square matrix
- find inverse of a square matrix provided it exists
- define rank and nullity of a matrix
- find rank using definition and using elementary row operations.

12.2 INTRODUCTION

One of the important topics in Algebra is that of solution of a system of linear equations. The system is expressed in matrix form as a single equation in matrices. To determine the existence of solution of the system, the concept of rank of matrix is necessary. Also the concept of inverse of a matrix is used to find the solution set. In this unit, we shall learn about the inverse of a matrix and rank and nullity of a matrix. For this purpose the concept of determinant of a square matrix and that of adjoint of a square matrix is necessary which will be discussed at the beginning of the unit.

12.3 DETERMINANT OF A SQUARE MATRIX



NOTE : Let F be a field and $M_n(F)$ be the set of all square matrices of order n with entries from the field F. Then $M_n(F)$ is a non-commutative ring. We can define a mapping from $M_n(F)$ to F such that the image of $A = (a_{ij})_{n\times n} \in M_n(F)$ is the determinant of A, i.e., $|A| \in F$.

$M_n(F) \to F$

$A \to |A|$

Thus, the determinant of a matrix can be considered as a function.

If $A = (a_{11})_{1\times 1}$
then $|A| = |a_{11}| = a_{11}$

1) Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2×2 square matrix.

We define its determinant, denoted by **detA** or $|\mathbf{A}|$, as

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

It is called a determinant of **order 2**.

For example, if $A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$, than $|A| = \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} = 1.3 - 2.(-2) = 7$

2) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a 3×3 square matrix, its determinant,

$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is given by

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

In general, $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= (-1)^{i+1}a_{i1}|A_{i1}| + (-1)^{i+2}a_{i2}|A_{i2}| + (-1)^{i+3}a_{i3}|A_{i3}|$$

Where A_{ij} represents the 2×2 matrix obtained on deleting the i^{th} row and j^{ih} column of A(i, j = 1, 2, 3).

Similarly, $|A| = (-1)^{1+i}a_{1j}|A_{1j}| + (-1)^{2+j}a_{2j}|A_{2j}| + (-1)^{3+j}a_{3j}|A_{3j}|$

This determinant is called a determinant of **order 3**.

For example, if $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -2 & 1 & 4 \end{pmatrix}$, then

$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -2 & 1 & 4 \end{vmatrix}$$

$$= 1.\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} - 2.\begin{vmatrix} 2 & 3 \\ -2 & 4 \end{vmatrix} + (-1).\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix}$$

$$= 1.(0-3) - 2(8+6) - 1.(2-0)$$

$$= -3 - 28 - 2 = -33$$

Similarly, we can define the determinant of a square matrix of order 4, 5, ..., n.

12.3.1 Minors and Cofactors

If $A = [a_{ij}]_{n\times n}$ be a square matrix, then the determinant of the (n–1)×(n–1) matrix A_{ij} obtained on deleting the i^{th} row and j^{th} column of A is called the **minor of a_{ij}**, i.e., minor of $a_{ij} = |A_{ij}|$,

where A_{ij} is the (n–1)×(n–1) matrix as defined above.

The **cofactor of a_{ij}**, denoted by C_{ij}, is defined as

$C_{ij} = (-1)^{i+j} \times$ minor of $a_{ij} = (-1)^{i+j}|A_{ij}|$.

Example 1 : Find minors & cofactors of the elements of

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Solution : The minor of a_{11} is $|A_{11}| = |a_{22}| = a_{22}$

The minor of a_{12} is $|A_{12}| = |a_{21}| = a_{21}$

The minor of a_{21} is $|A_{21}| = |a_{12}| = a_{12}$

The minor of a_{22} is $|A_{22}| = |a_{11}| = a_{11}$

So, the cofactors are

$C_{11} = (-1)^{1+1}|A_{11}| = a_{22}$

$C_{12} = (-1)^{1+2}|A_{12}| = -a_{21}$

$C_{21} = (-1)^{2+1}|A_{21}| = -a_{12}$

$C_{22} = (-1)^{2+2}|A_{22}| = a_{11}$

Clearly $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$= a_{11}a_{22} - a_{12}a_{21}$

$= a_{11}C_{11} + a_{12}C_{12}$.

Similarly, $|A| = a_{21}C_{21} + a_{22}C_{22}$

$= a_{11}C_{11} + a_{21}C_{21}$

$= a_{12}C_{12} + a_{22}C_{22}$

Also $a_{11}C_{21} + a_{12}C_{22} = a_{11}(-a_{12}) + a_{12}.a_{11} = 0$, etc.

Example 2 : Find the cofactors of the elements of

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Solution : $C = (-1)^{1+1}|A_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$$C_{12} = (-1)^{1+2}|A_{12}| = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{13} = (-1)^{1+3}|A_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \text{ etc.}$$

Clearly, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$, etc.

Also, we can show that $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$, etc.

We can summerize the findings in examples 2 and 3 in the following property.

Property : 1) The sum of the products of the elements of any row (column) and the corresponding cofactors of the row (column) is equal to the value of the determinant.

2) The sum of the products of the elements of any row (column) and the cofactors of the corresponding elements of a different row (column) is equal to zero.

12.3.2 Determinant of a Diagonal Matrix

Let $A = \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$ be a diagonal matrix of order n. Then

$|A| = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$

$= d_1C_{11}$, as $a_{12} = \dots = a_{1n} = 0$

$$= d_1 \begin{bmatrix} d_2 & 0 & 0 & \dots & 0 \\ 0 & d_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$$

$$= d_1d_2 \begin{bmatrix} d_3 & 0 & 0 & \dots & 0 \\ 0 & d_4 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$$

$= \dots \dots \dots \dots \dots$

$= d_1 d_2 d_3 \dots d_n$

As a corollary, we get $|I_n| = 1$.

12.3.3 Determinant of Product of Two Matrices

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ then $|A| = ad - bc$, $|B| = \alpha\delta - \beta\gamma$

$\therefore |A|\,|B| = (ad-bc)(\alpha\delta-\beta\gamma)$

$= ad\alpha\delta + bc\beta\gamma - ad\beta\gamma - bc\alpha\delta$ .....(1)

Also $AB = \begin{bmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha+d\gamma & c\beta+d\delta \end{bmatrix}$

$\therefore |AB| = \begin{vmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha+d\gamma & c\beta+d\delta \end{vmatrix}$

$= (a\alpha+b\gamma)(c\beta+d\delta) - (a\beta+b\delta)(c\alpha+d\gamma)$

$= ad\alpha\delta + bc\beta\gamma - ad\beta\gamma - bc\alpha\delta$ .....(2)

From (1) & (2), we get $|AB| = |A|.|B|$

In general, if $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ be two square matrices of same order, then $|AB| = |A|.|B|$.

12.4 ADJOINT OF A SQUARE MATRIX

Definition 1 : Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n. Then the adjoint of A, denoted by adjA, is the transpose of the matrix of the cofactors of the corresponding elements of A.

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$

then $\text{adj}A = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} & \dots & c_{n1} \\ c_{12} & c_{22} & \dots & c_{n2} \\ \dots & \dots & \dots & \dots \\ c_{1n} & c_{2n} & \dots & c_{nn} \end{bmatrix}$

Example 3 : Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix}$

Then, $C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix} = 7$ $\quad C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} = -11$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ $\quad C_{21} = (-1)^{2+1}\begin{vmatrix} 0 & -1 \\ 2 & 5 \end{vmatrix} = -2$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -1 \\ 1 & 5 \end{vmatrix} = 6 \qquad C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -2$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = 1 \qquad C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$$

$$\therefore \text{adjA} = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

Theorem 1 : If A be an n×n square matrix, then

$$A(\text{adj}A) = |A|\, I_n = (\text{adj}A)A.$$

Proof : Let $A = (a_{ij})_{n\times n}$.

Then $\text{adj}A = (b_{ij})_{n\times n}$, where $b_{ij} = C_{ji}$, the cofactor of a_{ji}.

Now, the $(i, j)^{th}$ element of $A(\text{adj}A) = \sum_{k=1}^{n} a_{ik}b_{kj}$

$$= \sum_{k=1}^{n} a_{ik}C_{jk}$$

$$= \begin{cases} |A|, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

$$\text{Hence, } A(\text{adj}A) = \begin{bmatrix} |A| & 0 & 0 & \ldots & 0 \\ 0 & |A| & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & |A| \end{bmatrix}$$

$$= |A|\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1| \end{bmatrix}$$

$$= |A|.I_n$$

Similarly, $(\text{adj}A) = |A|.I_n$

Thus, $A(\text{adj}A) = |A|\, I_n = (\text{adj}A)A$.

12.5 INVERSE OF A SQUARE MATRIX

Definition 2 : Let A be a square matrix of order n. If there exists another square matrix of order n such that $AB = I_n = BA$, then A is said to be **invertible** and B is called the **inverse** of A and it is denoted by A^{-1}.

Thus, $AA^{-1} = I_n = A^{-1}A$

Example 4 : Let $A = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$, Suppose $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ such that

$AB = I_n = BA$

$$\Rightarrow \begin{bmatrix} 2\alpha + 7\gamma & 2\beta + 7\delta \\ \alpha + 4\gamma & \beta + 4\delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $2\alpha + 7\gamma = 1$, $2\beta + 7\delta = 0$,

$\alpha + 4\gamma = 0$, $\beta + 4\delta = 1$

$\Rightarrow$ $\alpha = 4$, $\beta = -7$, $\gamma = -1$, $\delta = 2$ [on solving the equations]

Hence $A^{-1} = B = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$

Theorem 2 : The inverse of a matrix is unique.

Proof : Let A be an invertible matrix and suppose both B and C are inverses of A. Then $AB = I = BA$, $AC = I = CA$.

$\therefore B = BI$

$= B(AC)$

$= (BA)C$

$= IC$

$= C$

Hence the inverse of A is unique.

Theorem 3 : If A, B are invertible matrices, then

i) $(A^{-1})^{-1} = A$

ii) $(AB)^{-1} = B^{-1}A^{-1}$

Proof :

i) Since A is invertible, we have

$AA^{-1} = I = A^{-1}A$

$\Rightarrow A^{-1}A = I = AA^{-1}$

$\Rightarrow (A^{-1})^{-1} = A$.

ii) We have $AA^{-1} = I = A^{-1}A$, $BB^{-1} = I = B^{-1}B$

Now $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= A\,I\,A^{-1}$

$= AA^{-1} = I$

Similarly $(B^{-1}A^{-1})(AB) = I$

Thus $(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB)$

$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$.

Definition 3 : SINGULAR AND NON-SINGULAR MATRIX

A square matrix A is called a **non-singular** matrix if $|A| \neq 0$, otherwise A is called a **singular** matrix.

Example 5 : Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix}$. Then

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{vmatrix}$$

$= 1.(5+2) - 0.(10+1) - 1(4-1) = 4$

As $|A| \neq 0$, so A is a non-singular matrix.

Take $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$. Then $|B| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 0 & 5 \end{vmatrix} = 0$

So, B is a singular matrix.

Theorem 4 : The inverse of a square matrix exists if and only if it is non-singular.

Proof : Let A be a square matrix.

Suppose A^{-1} exists. Then

$AA^{-1} = I = A^{-1}A$

$\Rightarrow |AA^{-1}| = |I| = |A^{-1}A|$

$\Rightarrow |A|.|A^{-1}| = 1$

$\Rightarrow |A| \neq 0$

$\Rightarrow$ A is non-singular.



NOTE : Every matrix is not invertible, only square matrices may have inverse matrices. All square matrices are again not invertible. Only those square matrices whose determinant are non-zero, that is, those square matrices which are non-singular, are invertible.

Conversely, suppose A is non-singular. Then $|A| \neq 0$.

Now, by Theorem 1 we know

$$A(adjA) = |A|\, I = (adjA)A$$

$$\Rightarrow \frac{1}{|A|}A(adjA) = I = \frac{1}{|A|}(adjA)A, \text{ as } |A| \neq 0$$

$$\Rightarrow A\left(\frac{1}{|A|}adjA\right) = I = \left(\frac{1}{|A|}adjA\right)A$$

Hence A^{-1} exists and $A^{-1} = \frac{1}{|A|}adjA$

This completes the proof of the theorem.

12.5.1 Illustrative Examples on Inverse of Matrices

Example 6 : Show that $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ are inverses of each other.

Solution : We have $AB = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$$BA = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Thus $AB = I = BA$, and so

$A^{-1} = B$, $B^{-1} = (A^{-1})^{-1} = A$.

Example 7 : Examine the existence of inverse. If exists, find inverse of the matrix.

i) $A = \begin{pmatrix} -2 & 6 \\ 3 & -9 \end{pmatrix}$

ii) $A = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$

iii) $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{pmatrix}$

Solution :

i) $|A| = \begin{vmatrix} -2 & 6 \\ 3 & -9 \end{vmatrix} = (-2)(-9) - 6.3 = 18 - 18 = 0$

Since A is a singular matrix, A^{-1} does not exist.

ii) $|A| = \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} = 9 - 10 = -1$

Since $|A| \neq 0$, A is a singular matrix and so A^{-1} exists,

where $A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{(-1)}\text{adj}A = -\text{adj}A$.

Now $C_{11} = 3$, $C_{12} = -2$, $C_{21} = -5$, $C_{22} = 3$

$$\therefore \text{adj}A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}' = \begin{bmatrix} 3 & -2 \\ -5 & 3 \end{bmatrix}' = \begin{bmatrix} 3 & -5 \\ -2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = -\begin{bmatrix} 3 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}$$

iii) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{vmatrix} = 1.(5+2) - 0.(10+1) - 1(4-1) = 4$$

Since $|A| \neq 0$, A^{-1} exists and

$$A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{4}\text{adj}A$$

$$= \frac{1}{4}\begin{bmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{bmatrix}, \text{ [see Example 3, 12.4]}$$

$$= \begin{bmatrix} \frac{7}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{11}{4} & \frac{3}{2} & -\frac{1}{4} \\ \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$



NOTE : Inverse of a square matrix can be obtained by two methods :

1) Using the formula

$$A^{-1} = \frac{1}{|A|}\text{adj}A$$

for which |A| and adjA should have to enumerate.

2) Using elementary row operations which does not necessitate finding |A| and adjA.

12.5.2 Finding Inverse by Elementary Row Operations

Finding inverse of a square matrix by elementary row operations is based on a theorem stated below (without proof).

Theorem 5 : If a square matrix A of order n is reduced to I_n by successively performing elementary row operations, then the same successive elementary row operations reduces I_n to A^{-1}.

Thus, if we perform successive elementary row operations on $(A \mid I_n)$ which reduces A to I_n, then the resulting matrix is $(I_n \mid A^{-1})$. In symbol, $(A \mid I) \sim (I \mid A^{-1})$.

It should be noted that in performing elementary row operations on (A | I), if we get a zero-row in the A-part, then A does not reduce I and in such case A^{-1} does not exist.

ILLUSTRATIVE EXAMPLES :

Example 8 : Find inverse matrix using elementary row operations :

i) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix}$ ii) $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$

iii) $C = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 2 \\ -1 & 2 & 1 & -2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$ iv) $D = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{bmatrix}$

Solution :

i) $(A \mid I) = \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 6 & -1 & 0 & 1 \end{array}\right], \text{ by } R_2 \to R_2 - 2R_1;\ R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -2 & 1 \end{array}\right], \text{ by } R_3 \to R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right], \text{ by } R_3 \to \frac{1}{4}R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{3}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right] \text{by } R_1 \to R_1+R_3,\ R_2 \to R_2-R_3$$

$\sim (I \mid A^{-1})$

and hence, $A^{-1} = \begin{bmatrix} \frac{7}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{11}{4} & \frac{3}{2} & -\frac{1}{4} \\ \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$

ii) $(B \mid I) = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right], \text{ by } R_2 \to R_2-2R_1\ ;\ R_3 \to R_3-4R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right], \text{ by } R_3 \leftrightarrow R_3+R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & -1 & 0 & 4 & 0 & -1 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right], \text{ by } R_1 \to R_1+2R_3;\ R_2 \to R_2-R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right], \text{ by } R_2 \to (-1)R_2 \,;\, R_3 \to (-1)R_3$$

$$\sim (I \mid B^{-1})$$

$$\text{and hence, } B^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

iii) $$(C \mid I) = \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & -2 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 & -2 & 1 & 0 & 0 \\ 0 & 3 & 3 & -1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & -2 & -1 & 0 & 0 & 1 \end{array}\right],$$

$$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 + R_1,\ R_4 \to R_4 - R_1$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 3 & 3 & -1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & -2 & -1 & 0 & 0 & 1 \end{array}\right], \ R_2 \to -\frac{1}{3}R_2$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right],$$

$$R_1 \to R_1 - R_2,\ R_3 \to R_3 - 3R_2,\ R_4 \to R_4 + 2R_2$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 \end{array}\right], \; R_3 \leftrightarrow R_4$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 3 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 & -1 \\ 0 & 0 & 1 & -2 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \end{array}\right],$$

$$R_1 \to R_1 - R_3, \; R_2 \to R_2 - R_3, \; R_4 \to (-1)R_4$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -3 & 4 & 3 & -1 \\ 0 & 1 & 0 & 0 & -\frac{5}{3} & \frac{7}{3} & 2 & -1 \\ 0 & 0 & 1 & 0 & \frac{7}{3} & -\frac{8}{3} & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \end{array}\right],$$

$$R_1 \to R_1 - 3R_4, \; R_2 \to R_2 - 2R_4, \; R_3 \to R_3 + 2R_4$$

$$\sim (I \mid C^{-1})$$

and hence, $C^{-1} = \left[\begin{array}{cccc} -3 & 4 & 3 & -1 \\ -\frac{5}{3} & \frac{7}{3} & 2 & -1 \\ \frac{7}{3} & -\frac{8}{3} & -2 & 1 \\ 1 & -1 & -1 & 0 \end{array}\right] = \frac{1}{3}\left[\begin{array}{cccc} -9 & 12 & 9 & -3 \\ -5 & 7 & 6 & -3 \\ 7 & -8 & -6 & 3 \\ 3 & -3 & -3 & 0 \end{array}\right]$

iv) $(D \mid I) = \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 1 & 5 & -1 & 0 & 1 & 0 \\ 3 & 13 & -6 & 0 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 4 & 6 & -3 & 0 & 1 \end{array}\right], \; R_2 \to R_2 - R_1, \; R_3 \to R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right], \; R_3 \to R_3 - 2R_2$$

Since the D-part on the r.h.s. has a zero-row, the given matrix is not invertible, i.e., D^{-1} does not exist.



CHECK YOUR PROGRESS

Q.1. Find the adjoint of the following matrices :

i) $\begin{bmatrix} -2 & 3 & 2 \\ 6 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}$ ii) $\begin{bmatrix} -1 & 1 & 1 \\ 2 & 1 & 1 \\ 5 & -2 & -1 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

Q.2. For what values of x the following matrices are not invertible?

i) $\begin{pmatrix} 2 & x \\ x & 8 \end{pmatrix}$ ii) $\begin{pmatrix} 2 & -x & 3 \\ 0 & 1 & -1 \\ x & 0 & 1 \end{pmatrix}$

Q.3. If A be a non-singular matrix and B, C are matrices of same order such that the product AB, AC exist and AB = AC, then prove that B = C.

Q.4. Find A^{-1} using adjA :

i) $A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$ ii) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{pmatrix}$

Q.5. Find A^{-1} using elementary row operations :

i) $A = \begin{bmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix}$ ii) $A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 6 \\ 1 & 1 & 7 \end{bmatrix}$

12.6 RANK AND NULLITY OF A MATRIX

Definition 4 : MINORS OF A MATRIX

Let $A = [a_{ij}]_{m\times n}$ be a given matrix. Let r be a natural number which is less than or equal to the minimum of m, n. By deleting (m–r) rows and (n–r) columns from A, a square matrix of order r can be obtained which is called a square sub-matrix of A. The determinant of this square sub-matrix is called a **Minor** of the matrix A of **order r**. For example,

if $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & -1 & 1 & 0 \\ 3 & 1 & -2 & 1 \end{bmatrix}$;

then $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$, etc. are minors of order 2.

Similarly, $\begin{bmatrix} 2 & 0 & 4 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$ are minors of order 3.



NOTE : Rank and nullity of a matrix are used in solving a system of linear equations.

Definitions 5 : RANK AND NULLITY OF A MATRIX

Let $A = [a_{ij}]_{m\times n}$ be a given matrix. If all minors of order (r+1) are zero while at least one minor of order r is non-zero, then r is defined as the **rank** of the matrix A, generally denoted by $\rho(A)$.

If $A = [a_{ij}]_{n\times n}$ be a square matrix of rank r, then (n–r) is called the **nullity** of A, generally denoted by $\tau(A)$. Thus

$\rho(A) = r \Rightarrow \tau(A) = n - r$

where n is the order of the square matrix A.

If $A \neq 0$, then A has atleast one non-zero entry and so, atleast one non-zero minor of order 1. Hence, $A \neq 0 \Rightarrow \rho(A) \geq 1$

If A = 0, it has no non-zero entry and so, $\rho(A) = 0$.

Example 9 : Find rank and nullity of :

i) $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ ii) $A = \begin{bmatrix} 4 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & -4 & 3 \\ 2 & -1 & -3 & 5 \\ -1 & 8 & -6 & -1 \end{bmatrix}$

Solution :

i) A has one minor of order 3 which is |A|.

$$\text{Now } |A| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 1(-1-1) - 2(2-3) - 1(2+3) = -5$$

Thus $|A| \neq 0 \Rightarrow \rho(A) = 3$, $\tau(A) = 3 - 3 = 0$.

ii) Let us first evaluate the minors of order 3 :

$$\begin{vmatrix} 1 & 2 & -4 \\ 2 & -1 & -3 \\ -1 & 8 & -6 \end{vmatrix} = 1.(6+24) - 2(-12-3) - 4(16-1) = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ -1 & 8 & -1 \end{vmatrix} = 1.(1-40) - 2(-2+5) + 3(16-1) = 0$$

$$\begin{vmatrix} 1 & -4 & 3 \\ 2 & -3 & 5 \\ -1 & -6 & -1 \end{vmatrix} = 1.(3+30) + 4(-2+5) + 3(-12-3) = 0$$

$$\begin{vmatrix} 2 & -4 & 3 \\ -1 & -3 & 5 \\ 8 & -6 & -1 \end{vmatrix} = 2(3+20) + 4(1-40) + 3(6+24) = 0$$

Thus all minors of order 3 are zero and so $\rho(A) < 3$.

But $\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5$, showing that A has atleast one non-zero minor of order 2. Hence, $\rho(A) = 2$.

Since A is not a square matrix, its nullity is undefined.

iii) The highest order minor is of order 4, which is

$$|A| = \begin{vmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 \\ 2 & 0 & 3 & 0 \end{vmatrix},$$

using $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 + 2C_1$, $C_4 \to C_4 - 3C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 3 & 0 \end{vmatrix} = 1.1.(0-3) = -3 \neq 0$$

$\therefore \rho(A) = 4$ and $\tau(A) = 4 - 4 = 0$.

12.6.1 Finding Rank Using Elementary Row Operations

In example-9 we have seen that to find the rank of a matrix we have to calculate values of a number of determinants, which is a tedious process. We can avoid it by the application of elementary row operations on the matrix reducing it to reduced echelon matrix. This method is based on the following theorem, stated without proof.

Theorem 6 :

a) The rank of a matrix remains invariant under elementary row operations.

b) The rank of a matrix is equal to the rank of its reduced echelon matrix.

c) If a reduced echelon matrix has r non-zero rows, then its rank is r.

From this theorem, it is obvious that if A ~ R where R is the reduced echelon matrix, and if R has r non-zero rows, then $\rho(A) = \rho(R) = r$.

Example 10 : Find the rank of $A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

Solution : We find the reduced echelon matrix of A performing elementary row operations :

$$A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{8} & \frac{3}{8} & \frac{3}{4} \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}, \text{ by } R_1 \to \frac{1}{8}R_1$$

$$\sim \begin{bmatrix} 1 & \frac{1}{8} & \frac{3}{8} & \frac{3}{4} \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 10 \end{bmatrix}, \text{ by } R_2 \to \frac{1}{3}R_2,\ R_3 \to R_3 + 8R_1$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{7}{24} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ by } R_1 \to R_1 - \frac{1}{8}R_2,\ R_3 \to \frac{1}{10}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{7}{24} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ by } R_1 \to R_1 - \frac{2}{3}R_3,\ R_2 \to R_2 - \frac{2}{3}R_3$$

The right hand side matrix is in reduced echelon form having 3 non-zero rows. So, $\rho(A) = 3$.

Example 11 : Find the rank of $A = \begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 6 \end{bmatrix}$

Solution :

$$A = \begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 0 & 2 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 6 \end{bmatrix}, R_1 \to \tfrac{1}{2}R_1$$

$$\sim \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 0 & 2 \\ 0 & -\frac{7}{2} & \frac{1}{2} & -1 & -5 \\ 0 & -7 & 1 & -2 & -10 \\ 0 & -\frac{7}{2} & \frac{1}{2} & -1 & -4 \end{bmatrix}, R_2 \to R_2 - 3R_1,$$

$R_3 \to R_3 - 4R_1,\ R_4 \to R_4 - 5R_1$

$$\sim \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 0 & 2 \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} & \frac{10}{7} \\ 0 & -7 & 1 & -2 & -10 \\ 0 & -\frac{7}{2} & \frac{1}{2} & -1 & -4 \end{bmatrix}, R_2 \to -\frac{2}{7}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{5}{7} & -\frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} & \frac{10}{7} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; R_1 \to R_1 - \frac{3}{2} R_2,$$

$$R_3 \to R_3 + 7R_2, \; R_4 \to R_4 + \frac{7}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{5}{7} & -\frac{3}{7} & 0 \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; R_1 \to R_1 + \frac{1}{7} R_4, \; R_2 \to R_2 - \frac{10}{7} R_4$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{5}{7} & -\frac{3}{7} & 0 \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; R_3 \leftrightarrow R_4$$

The right hand side matrix is in reduced echelon form having 3 non-zero rows.

So, $\rho(A) = 3$.



CHECK YOUR PROGRESS

Q.6. Find rank of the following matrices using definition :

i) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$ ii) $\begin{pmatrix} 3 & 4 & 1 \\ 4 & 3 & 2 \\ 2 & 1 & 4 \end{pmatrix}$ iii) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Q.7. Find rank of the following matrices using elementary row operations :

i) $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & 4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$

Q.8. Find rank and nullity of $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$



12.7 LET US SUM UP

- If $A = [a_{ij}]_{n\times n}$ be a square matrix, its determinant is denoted by det A or $|A| = |a_{ij}|_{n\times n}$.
- In $A = [a_{ij}]_{n\times n}$, minor of $a_{ij} = |A_{ij}|$, where A_{ij} is the $(n-1)\times(n-1)$ matrix obtained on deleting the i^{th} row and the j^{th} column from A.
- In $A = [a_{ij}]_{n\times n}$, the cofactor of a_{ij} is $C_{ij} = (-1)^{i+j}.|A_{ij}|$.
- If $A = \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$, then $|A| = d_1 d_2 \dots d_n$.

  In particular, $|I_n| = 1$, $n \in N$.
- For two matrices A and B, $|AB| = |A|.|B|$.
- If $A = [a_{ij}]_{n\times n}$, then its adjoint matrix is $adjA = [C_{ij}]'_{n\times n}$, where C_{ij} is the cofactor of a_{ij}.

- A square matrix A is invertible if there exists another square matrix B such that AB = I = BA. In this case B is called the inverse matrix of A, denoted by A^{-1}.
- The inverse matrix A^{-1} of A is unique.
- If A, B are invertible matrices, then $(A^{-1})^{-1} = A$, $(AB)^{-1} = B^{-1}A^{-1}$.
- A square matrix A is non-singular if $|A| \neq 0$, otherwise it is called singular.
- The inverse of a square matrix exists if and only if it is non-singular.
- For a square matrix A, A(adjA) = |A| I = (adjA)A and hence

  $A^{-1} = \frac{1}{|A|}\text{adjA}$, if A is non-singular.
- The determinant of any square sub-matrix of a given matrix is called a minor of the matrix.
- If a matrix A has atleast one non-zero minor of order r and all other higher order minors are zero, then r is called the rank of A, denoted by $\rho(A)$.
- If A be a square matrix of order n and $\rho(A) = r$, then n–r is called the nullity of A, denoted by $\tau(A)$.



12.8 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : i) $A = \begin{bmatrix} -2 & 3 & 2 \\ 6 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}$

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = -1, \qquad C_{12} = (-1)^{1+2}\begin{vmatrix} 6 & 3 \\ 4 & -1 \end{vmatrix} = 18,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 6 & 1 \\ 4 & 0 \end{vmatrix} = -4, \qquad C_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 2 \\ 0 & -1 \end{vmatrix} = 3,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} -2 & 2 \\ 4 & -1 \end{vmatrix} = -6, \qquad C_{23} = (-1)^{2+3}\begin{vmatrix} -2 & 3 \\ 4 & 0 \end{vmatrix} = 12$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = 7, \qquad C_{32} = (-1)^{3+2}\begin{vmatrix} -2 & 2 \\ 6 & 3 \end{vmatrix} = 18,$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} -2 & 3 \\ 6 & 1 \end{vmatrix} = -20$$

$$\text{Hence, adjA} = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} -1 & 3 & 7 \\ 18 & -6 & 18 \\ -4 & 12 & -20 \end{bmatrix}$$

ii) & iii) Try yourselves.

Ans. to Q. No. 2 : i) $A = \begin{bmatrix} 2 & x \\ x & 8 \end{bmatrix}$

A is not invertible if $|A| = 0$

$$\text{i.e., } \begin{bmatrix} 2 & x \\ x & 8 \end{bmatrix} = 0$$

i.e., $16 - x^2 = 0$

i.e., $x = \pm 4$.

ii) $A = \begin{bmatrix} 2 & -x & 3 \\ 0 & 1 & -1 \\ x & 0 & 1 \end{bmatrix}$

$$|A| = 0 \Rightarrow \begin{bmatrix} 2 & -x & 3 \\ 0 & 1 & -1 \\ x & 0 & 1 \end{bmatrix} = 0$$

$\Rightarrow 2(1-0) + x(0+x) + 3(0-x) = 0$

$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow (x-1)(x-2) = 0$

$\Rightarrow x = 1, 2$

So, A is not invertible for $x = 1$ and $x = 2$.

Ans. to Q. No. 3 : Since A is non-singular, A^{-1} exists and $AA^{-1} = I = A^{-1}A$

Now $AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow (A^{-1}A)B = (A^{-1}A)C$

$\Rightarrow IB = IC$

$\Rightarrow B = C.$

Ans. to Q. No. 4 : i) $A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$, $|A| = \begin{vmatrix} 1 & 4 \\ 2 & 7 \end{vmatrix} = 7 - 8 = -1$

As $|A| \neq 0$, so A^{-1} exists.

$C_{11} = 7$, $C_{12} = -2$, $C_{21} = -4$, $C_{22} = 1$

$$\text{adj}A = \begin{bmatrix} 7 & -1 \\ -2 & 1 \end{bmatrix}$$

$$\text{Thus, } A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{-1}\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

ii) $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$$

$= 1.(21–20) – 2(14–4) + 3(10–3)$

$= 1– 20 + 21 = 2$

As $|A| \neq 0$, so A^{-1} exists and $A^{-1} = \frac{1}{|A|}\text{adj}A = ½\text{adj}A$.

Now $C_{11} = 1$, $C_{12} = -10$, $C_{13} = 7$

$C_{21} = 1$, $C_{22} = 4$, $C_{23} = -3$

$C_{31} = -1$, $C_{32} = 2$, $C_{33} = -1$

$$\text{So, adj}A = \begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix}$$

Ans. to Q. No. 5 : i) $(A \mid I) = \left[\begin{array}{ccc|ccc} -2 & 3 & 2 & 1 & 0 & 0 \\ 6 & 0 & 3 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 0 \\ 6 & 0 & 3 & 0 & 1 & 0 \\ 4 & 1 & -1 & 0 & 0 & 1 \end{array}\right], \; R_1 \to -½R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 3 & 1 & 0 \\ 0 & 7 & 3 & 2 & 0 & 1 \end{array}\right], \; R_2 \to R_2 - 6R_1, \; R_3 \to R_3 - 4R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 7 & 3 & 2 & 0 & 1 \end{array}\right], \; R_2 \to \frac{1}{9}R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{6} & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 0 & -4 & -\frac{1}{3} & -\frac{7}{9} & 1 \end{array}\right], \; R_1 \to R_1 + \frac{3}{2}R_2, \; R_3 \to R_3 - 7R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{6} & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 0 & 1 & -\frac{1}{12} & -\frac{7}{36} & -\frac{1}{4} \end{array}\right], \; R_3 \to -\frac{1}{4}R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{24} & \frac{5}{72} & \frac{1}{8} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{12} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{12} & \frac{7}{36} & -\frac{1}{4} \end{array}\right], \; R_2 \to R_1 - ½R_3, \; R_2 \to R_2 - R_3$$

$\sim (I \mid A^{-1})$

$$\text{and so, } A^{-1} = \begin{bmatrix} -\frac{1}{24} & \frac{5}{72} & \frac{1}{8} \\ \frac{1}{4} & -\frac{1}{12} & \frac{1}{4} \\ \frac{1}{12} & \frac{7}{36} & -\frac{1}{4} \end{bmatrix} = \frac{1}{72}\begin{bmatrix} -3 & 5 & 9 \\ 18 & -6 & 18 \\ 6 & 14 & -18 \end{bmatrix}$$

ii) $$(A \mid I) = \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & -3 & 6 & 0 & 1 & 0 \\ 1 & 1 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 5 & -1 & 0 & 1 \end{array}\right], \; R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 5 & -3 & 1 \end{array}\right], \; R_1 \to R_1 + 2R_2, \; R_3 \to R_3 - 3R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -5 & 3 & -1 \end{array}\right], \; R_3 \to (-1)R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 27 & -16 & 6 \\ 0 & 1 & 0 & 8 & -5 & 2 \\ 0 & 0 & 1 & -5 & 3 & -1 \end{array}\right], \; R_1 \to R_1 - 6R_3, \; R_2 \to R_2 - 2R_3$$

$$\sim (I \mid A^{-1})$$

$$\text{So, } A^{-1} = \begin{bmatrix} 27 & -16 & 6 \\ 8 & -5 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Ans. to Q. No. 6 : i) $$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$$

A has no 3rd order minor.

The second order minors are

$\begin{vmatrix}1 & 2\\ 2 & 4\end{vmatrix} = 0, \begin{vmatrix}1 & 3\\ 2 & 6\end{vmatrix} = 0, \begin{vmatrix}2 & 3\\ 4 & 6\end{vmatrix} = 0$

Hence, all 2nd order minors are zero.

But $A \neq 0$ and so $\rho(A) = 1$.

ii) $A = \begin{pmatrix}3 & 4 & 1\\ 4 & 3 & 2\\ 2 & 1 & 4\end{pmatrix}$

3rd order minor is $|A| = \begin{vmatrix}3 & 4 & 1\\ 4 & 3 & 2\\ 2 & 1 & 4\end{vmatrix}$

$= 3(12–2) – 4(16–4) + 1(4–6)$

$= –20 \neq 0$

So, $\rho(A) = 3$

iii) $A = \begin{bmatrix}0 & 1 & 2 & 3\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$

Cleary all 3rd order minors are zero.

It has a non-zero 2nd order minor $\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix}$ and so, $\rho(A) = 2$.

Ans. to Q. No. 7 : i) $A = \begin{bmatrix}1 & 0 & 2 & 1\\ 0 & 1 & -2 & 1\\ 1 & -1 & 4 & 0\\ -2 & 2 & 8 & 0\end{bmatrix}$

$\sim \begin{bmatrix}1 & 0 & 2 & 1\\ 0 & 1 & -2 & 1\\ 0 & -1 & 2 & -1\\ 0 & 2 & 12 & 2\end{bmatrix}, R_3 \to R_3–R_1, R_4 \to R_4+2R_1$

$\sim \begin{bmatrix}1 & 0 & 2 & 1\\ 0 & 1 & -2 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 16 & 0\end{bmatrix}, R_3 \to R_3+R_2, R_4 \to R_4–2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R_3 \rightarrow \frac{1}{16} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R_1 \rightarrow R_1 - 2R_3, R_2 \rightarrow R_2 + 2R_3$$

The right hand side matrix is reduced echelon matrix with three non-zero rows. Hence, $\rho(A) = 3$.

ii) Try yourselves.

Ans. to Q. No. 8 : $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$

$\rho(A) = 3$ [done in 2(i)]

Hence the nullity of A is given by $\tau(A) = 4 - 3 = 1$.



12.9 FURTHER READINGS

1. *Matrix Algebra*, S. K. Jain, Eastern Book House.
2. *Matrices*, A. R. Vasishtha, Krishna Prakashan Mandir, Meerut.
3. *Linear Algebra*, Seymour Lipschutz, Schaum's Solved Problems Series, Tata McGraw Hill.



12.10 MODEL QUESTIONS

Q.1. Find the adjoint of the following matrices

i) $\begin{bmatrix} 3 & -1 \\ 2 & -4 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 5 & -2 & 1 \end{bmatrix}$

Q.2. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$, verify that $A(adjA) = |A|\, I = (adjA)A$.

Hence, find A^{-1}.

Q.3. Find the adjoint and inverse of $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Q.4. Find the inverse of $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ verify that $A^3 = A^{-1}$.

Q.5. Given $A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix}$, show that $A^3 - 3A^2 - 7A + 18\, I = 0$.

Hence find A^{-1}.

Q.6. Show that A and B are inverses of each other :

(i) $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$, $B = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$

Q.7. Find inverse matrix using elementary row operations :

i) $A = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{bmatrix}$ ii) $A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ iv) $A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

Q.8. If A is α non-singular n×n square matrix, prove that $(A')^{-1} = (A^{-1})'$

Q.9. If A and B are two square matrices of order n×n and AB = I, then show that BA = I.

Q.10. Find rank of the following matrices using definition :

i) $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 1 \\ -3 & -1 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

iii) $\begin{bmatrix} 1-\sqrt{6} & \sqrt{3} & \sqrt{2} \\ 2 & \sqrt{6} & -\sqrt{2} \\ 1 & -\sqrt{3} & -\sqrt{6} \end{bmatrix}$

Q.11. Find rank using elementary row operations :

i) $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

iii) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

And find nullity of the matrices (i) and (iii).

UNIT 13 : SYSTEM OF LINEAR EQUATIONS

UNIT STRUCTURE

13.1 Learning Objectives

13.2 Introduction

13.3 System of *n* Non-Homogeneous Linear Equation in *n* Unknowns

- 13.3.1 Condition for Existence of Unique Solution
- 13.3.2 Solution by Cramer's Rule
- 13.3.3 Illustrative Examples

13.4 System of *m* Non-Homogeneous Linear Equations in *n* unknowns

- 13.4.1 Consistent and Inconsistent System
- 13.4.2 Equivalent Systems
- 13.4.3 Homogeneous System
- 13.4.4 Solution by Gaussion Elimination Method
- 13.4.5 Illustrative Examples
- 13.4.6 Solutions of Homogeneous System

13.5 Let Us Sum Up

13.6 Answers to Check Your Progress

13.7 Further Readings

13.8 Model Questions

13.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- write a system of linear equations in matrix form
- know condition for existence of unique solution of *n* linear equations in *n* unknowns.
- solve *n* linear equations in *n unknowns by Cramer's rule*
- know consistent and inconsistent system of *m* linear equations in *n* unknowns
- solution of linear equations by Gaussian elimination method.

13.2 INTRODUCTION

We are familiar with the method of solution of two linear equations in two variables. The equations ax + by = c, px + qy = d can easily be solved by the method of elimination of one of the two variables. But the method is difficult to work with when the number of variables and number of equations are more than two. It becomes more difficult when the number of equations and the number of variables are not equal. In this unit we shall discuss method of solution of such a system of linear equations, converting the system of equations to matrix form and then solving by applying Cramer's rule and the Gaussian elimination method.

13.3 A SYSTEM OF *n* LINEAR NON-HOMOGENEOUS EQUATIONS IN *n* UNKNOWNS

A system of equations

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ &\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \quad \ldots\ldots\ldots (1) \\ &\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ &a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{aligned}$$

is called a system of *n* **non-homogeneous linear equations** in *n* unknowns [where b_i, $a_{ij} \in$ R or any field F].

Let A be the matrix of the coefficients of the unknowns, X be the column matrix of the unknowns and B be the column matrix of the right hand side constants. That is,

$$A = [a_{ij}]_{n\times n},\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

It can be easily verified that the system (1) can be written as a matrix equation AX = B .....(2)

For example, the equations $x - z = 1$

$2x + y - z = 1$

$x + 2y + 5z = 2$

can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

13.3.1 Condition for Existence of Unique Solution

Theorem : The system of equations $AX = B$ has a unique solution if $|A| \neq 0$, i.e., A is a non-singular matrix.

Proof : Let $|A| \neq 0$. Then A^{-1} exists and $AA^{-1} = I = A^{-1}A$ .....(1)

Now $AX = B$

$\Leftrightarrow A^{-1}(AX) = A^{-1}B$

$\Leftrightarrow (A^{-1}A)X = A^{-1}B$

$\Leftrightarrow IX = A^{-1}B$, by (1)

$\Leftrightarrow X = A^{-1}B$.

This shows that $A^{-1}B$ is a solution of the system $AX = B$.

To prove uniqueness, let us assume that X_1 and X_2 are two solutions of the system.

Then $AX_1 = B, \; AX_2 = B$

$\Leftrightarrow AX_1 = AX_2$

$\Leftrightarrow A^{-1}(AX_1) = A^{-1}(AX_2)$

$\Leftrightarrow (A^{-1}A)X_1 = (A^{-1}A)X_2$

$\Leftrightarrow IX_1 = IX_2$, by (1)

$\Leftrightarrow X_1 = X_2$.

This shows that the system $AX = B$ has a unique solution.

Example : Let us consider the equations

$x - z = 1$,

$2x + y - z = 1$,

$x + 2y + 5z = 2$.

The matrix form is AX = B where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

$$|A| = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix} = 1.(5+2) - 0.(10+1) - 1(4-1) = 4$$

Since $|A| \neq 0$, the system has a unique solution given by

$$X = A^{-1}B = \frac{1}{|A|}.(\text{adj } A)B$$

$$= \frac{1}{4}(\text{adj } A)B$$

$$\text{Now, adj}A = \begin{bmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{bmatrix} \text{ (workout yourselves)}$$

$$\therefore X = \frac{1}{4}\begin{bmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 7 & -2 & +2 \\ -11 & +6 & -2 \\ 3 & -2 & +2 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ -\frac{7}{4} \\ \frac{3}{4} \end{bmatrix}$$

$$\text{Thus} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ -\frac{7}{4} \\ \frac{3}{4} \end{bmatrix}$$

Hence $x = \frac{7}{4}$, $y = -\frac{7}{4}$, $z = \frac{3}{4}$.

13.3.2 Solution by Cramer's Rule

Let us counsider the *n* equations in *n* unknowns

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$... ... ... \quad ..... (1)$$
$$... ... ...$$
$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

In matrix notation, the system of equations is equivalent to

$$AX = B$$

Where $A = [a_{ij}]_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If $|A| \neq 0$, then by theorem 1 the unique solution is given by

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj } A)B$$

$$= \frac{1}{|A|}\begin{bmatrix} c_{11} & c_{21} & ... & c_{n1} \\ c_{12} & c_{22} & ... & c_{n2} \\ ... & ... & ... & ... \\ c_{1n} & c_{2n} & ... & c_{nn} \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

where c_{ij} = cofactor of a_{ij} in A.

$$\text{Thus, } \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} c_{11}b_1 + c_{21}b_2 + ... + c_{n1}b_n \\ c_{12}b_1 + c_{22}b_2 + ... + c_{n2}b_n \\ ... ... \\ c_{1i}b_1 + c_{2i}b_2 + ... + c_{ni}b_n \\ ... ... \\ c_{n1}b_1 + c_{n2}b_2 + ... + c_{nn}b_n \end{bmatrix}$$

Hence, $x_i = \frac{1}{|A|}(c_{1i}b_1 + c_{2i}b_2 + ... + c_{ni}b_n)$ for i = 1, 2, ..., n.

i.e., $x_1 = \frac{1}{|A|}\sum_{j=1}^{n} b_i c_{ji}$ where i = 1, 2, ..., n.

From properties of cofactors we know that

$$\sum_{j=1}^{n} a_{ji}c_{ji} = |A| = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1i} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2i} & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{ni} & ... & a_{nn} \end{bmatrix}$$

Hence replacing a_{ji} by b_j on both sides for j = 1, 2, ..., n; we get

$$\sum_{j=1}^{n} b_j c_{ji} = \begin{bmatrix} a_{11} & a_{12} & ... & b_{1i} & ... & a_{1n} \\ a_{21} & a_{22} & ... & b_{2i} & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & b_{ni} & ... & a_{nn} \end{bmatrix}$$

or, $\sum_{j=1}^{n} b_j c_{ji} = |A_i|$,

where A_i is obtained from metrix A by replacing the i^{th} column by the

column $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

In otherwords, $x_i = \frac{1}{|A|}.|A_i| = \frac{|A_i|}{|A|}$

Hence the solution sof the system (1) are given by

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, ..., x_i = \frac{|A_i|}{|A|}, ..., x_n = \frac{|A_n|}{|A|}.$$

This method of solution is known as **Cramer's Rule**.

13.3.3 Illustative Examples

Example 1 : Solve by Cramer's rule :

$$x - y + 2z = 1$$

$$2x + y + z = 2$$

$$x - 3y + z = 1$$

Solution : $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix}$

$\therefore |A| = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix} = -9$

As $|A| \neq 0$, unique solution exists, where

$x = \frac{|A_1|}{|A|}$, $y = \frac{|A_2|}{|A|}$, $z = \frac{|A_3|}{|A|}$

Now $|A_1| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 1 \end{vmatrix} = -9$

$|A_2| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

$|A_3| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} = 0$

Hence $x = \frac{-9}{-9} = 1$,

$y = \frac{0}{-9} = 0$,

$z = \frac{0}{-9} = 0$.

Example 2 : Solve by Cramer's rule :

$x_1 + x_2 - 2x_3 = 1$

$2x_1 - 7x_3 = 3$

$x_1 + x_2 - x_3 = 5$

Solution : $|A| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 0 & -7 \\ 1 & 1 & -1 \end{vmatrix} = -2,$

$$|A_1| = \begin{vmatrix} 1 & 1 & -2 \\ 3 & 0 & -7 \\ 5 & 1 & -1 \end{vmatrix} = -31,$$

$$|A_2| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 3 & -7 \\ 1 & 5 & -1 \end{vmatrix} = 13,$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & 5 \end{vmatrix} = -8$$

The solutions are $x_1 = \frac{|A_1|}{|A|} = \frac{31}{2}$,

$$x_2 = \frac{|A_2|}{|A|} = -\frac{13}{2},$$

$$x_3 = \frac{|A_3|}{|A|} = 4.$$

Example 3 : Solve by Cramer's rule :

$$x_1 + x_2 + x_3 + x_4 = 2$$
$$x_1 + 2x_2 + 3x_3 + 4x_4 = 2$$
$$2x_1 + 3x_2 + 5x_3 + 9x_4 = 2$$
$$x_1 + x_2 + 2x_3 + 7x_4 = 2$$

Solution : $|A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 9 \\ 1 & 1 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 2 & 1 & 3 & 7 \\ 1 & 0 & 1 & 6 \end{vmatrix},$

by $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 - C_1$

$$= 1.\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 0 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 6 \end{vmatrix}, \; R_2 \to R_2 - R_1$$

$$|A_1| = \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \\ 2 & 3 & 5 & 9 \\ 2 & 1 & 2 & 7 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 9 \\ 1 & 1 & 2 & 7 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 8 \\ 1 & 0 & 1 & 6 \end{vmatrix}$$

$$= 2.\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 0 & 1 & 6 \end{vmatrix} = 4\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 0 & 1 & 6 \end{vmatrix} = 4\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 6 \end{vmatrix}$$

$$= 4.(0–1) = –4$$

$$|A_2| = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 2 & 5 & 9 \\ 1 & 2 & 2 & 7 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 2 & 1 & 5 & 9 \\ 1 & 1 & 2 & 7 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 7 \\ 1 & 0 & 1 & 6 \end{vmatrix} = 2.\begin{vmatrix} 0 & 2 & 3 \\ -1 & 3 & 7 \\ 0 & 1 & 6 \end{vmatrix}$$

$$= 2.(–1)(–1)(12–3) = 18$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 4 \\ 2 & 3 & 2 & 9 \\ 1 & 1 & 2 & 7 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 4 \\ 2 & 3 & 1 & 9 \\ 1 & 1 & 1 & 7 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 7 \\ 1 & 0 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 3 \\ 1 & -1 & 7 \\ 0 & 0 & 6 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 6 \end{vmatrix} = 2(–6) = –12$$

$$|A_4| = \begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 2 \\ 1 & 1 & 2 & 2 \end{vmatrix} = 2. \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 1 \\ 1 & 1 & 2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 3 & -1 \\ 1 & 0 & 1 & 0 \end{vmatrix}$$

$$= 2. \begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 0 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 2.(0+1) = 2$$

$$\therefore \quad x_1 = \frac{|A_1|}{|A|} = -\frac{4}{2} = -2,$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{18}{2} = 9,$$

$$x_3 = \frac{|A_3|}{|A|} = -\frac{12}{2} = -6,$$

$$x_4 = \frac{|A_4|}{|A|} = \frac{2}{2} = 1.$$

Example 4 : Investigate existence of solution of the system

$$2x_1 + 4x_2 - x_3 = 4$$
$$x_1 + 2x_2 - 2x_3 = 5$$
$$x_1 + 2x_2 - x_3 = -1$$

Solution : $|A| = \begin{vmatrix} 2 & 4 & -1 \\ 1 & 2 & -2 \\ 1 & 2 & -1 \end{vmatrix} = 2 \begin{vmatrix} 2 & 2 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & -1 \end{vmatrix} = 2.0 = 0$

Since |A| = 0, Cramer's rule cannot be used to solve the system. In fact, the system is inconsistent and has no solution. Because, subtracting 3 times the third equation from the sum of the first two equations we get 0 = 12, an absurdity.



CHECK YOUR PROGRESS

Q.1. Express in matrix form AX = B, examine uniqueness of solution and then solve calculating A^{-1} where ever unique solution exists :

i) $10x - 15y = 0$
$3x - 4y = 1$

ii) $3x + 2y - 2z = 1$
$-x + y + 4z = 1$
$2x - 3y + 4z = 8$

iii) $7x_1 + 6x_2 + 5x_3 = 1$
$x_1 + 2x_2 + x_3 = -1$
$3x_1 - 2x_2 + x_3 = 4$

Q.2. Solve by Cramer's rule :

i) $2x - y + 3z = 9$
$x + y + z = 6$
$x - y + z = 2$

ii) $2x_1 + x_2 + 5x_3 + x_4 = 5$
$x_1 + x_2 - 3x_3 - 4x_4 = -1$
$3x_1 + 6x_2 - 2x_3 + x_4 = 8$
$2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$

13.4 A SYSTEM OF *m* NON-HOMOGENEOUS EQUATIONS IN *n* UNKNOWNS

A system of equations :

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$... ... ... \quad ..... (1)$$
$$... ... ...$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

where a_{ij}, $b_i \in R$ or any field F, is called a system of *m* non-homogeneous equations in *n* unknowns. The system can be written as :

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \ (i = 1, 2, ..., m)$$

The matrix $A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & ... & ... & ... & a_{1n} \\ a_{21} & a_{22} & ... & ... & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & ... & ... & a_{mn} \end{bmatrix}$

is called the **co-efficient matrix**, and the matrix

$$(A|B) = (a_{ij}|b_i)_{m\times(n+1)} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & b_2 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} & b_m \end{bmatrix}$$

is called the **augmented matrix** of the system. Sometimes the last two columns of (A|B) are separated by a dotted line which is not obligatory.

An *n*-tuple $(\alpha_1, \alpha_2, ..., \alpha_n)$ which satisfies each of the *m* equations in (1) is called a **solution** of the equations.

13.4.1 Consistent and Inconsistent System

The system of equations :

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \text{ where } i = 1, 2, ..., m$$

is called a **consistent system** if it has at least one set of solution, otherwise the system is called **inconsistent**.

Example 1 : The system $x + 2y - 3z - 4t = 2$

$$2x + 4y - 5z - 7t = 7$$

$$-3x - 6y + 11z + 14t = 0$$

is a consistent system, since (3, 4, 3, 0) is a solution of the system.

Exampe 2 : The system $x_1 + 2x_2 - 3x_3 + 4x_4 = 2$

$$x_2 + 4x_3 - 7x_4 = -3$$

$$2x_2 + 8x_3 - 14x_4 = 3$$

is inconsistent, because subtracting twice the second equation from the third we shall get 0 = 9, an absurdity. So the system has no solution.

13.4.2 Equivalent Systems of Equations

The system of equations :

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \ (i = 1, 2, ..., m)$$

and $$\sum_{j=1}^{n} a'_{ij}x_j = b'_i, \ (i = 1, 2, ..., m)$$

are said to be **equivalent** if every solution of one system is also a solution of the other system.

Example : The systems of equations $x_1 - 2x_2 - 3x_3 = 4$

$$2x_1 - 3x_2 + x_3 = 5$$

and the system $x_1 + 11x_3 = -2$

$$x_2 + 7x_3 = -3$$

are equivalent. It can be verified that any set of solution of one system is also a solution of the other system. For example (9, 4, –1) is a solution of both the systems.

13.4.3 Homogeneous System of Equations

The system of equations :

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \ (i = 1, 2, ..., m)$$

is called a **homogeneous system** if $b_i = 0$ for i = 1, 2, ..., m. In other-words,

$$\sum_{j=1}^{n} a_{ij}x_j = 0, \ (i = 1, 2, ..., m)$$

is a homogeneous system.

Obviously, (0, 0, ..., 0) is a solution of any homogeneous system of equations. It may have other solutions also.

Example : $x_1 + 2x_2 - x_3 = 0$

$2x_1 + 5x_2 + 2x_3 = 0$

$x_1 + 4x_2 + 7x_3 = 0$

$x_1 + 3x_2 + 3x_3 = 0$ is a homogeneous system.

i) (0, 0, 0) is a solution of the system.

ii) (9, –4, 1) is another solution of the system.

13.4.4 Gaussian Elimination Method of Solution

The Gaussian elimination method of solving a system of linear equations is based on the following theorem, stated without proof.

Theorem : If $\sum_{j=1}^{n} a_{ij}x_j = b_i$, (i = 1, 2, ..., m) be a system of linear equations with augmented matrix $(A|B) = (a_{ij}|b_i)$ and $(R|S) = (\rho_{ij}|\delta_i)$ be its reduced echelon matrix, then the system

$\sum_{j=1}^{n} \rho_{ij}x_j = \delta_i$ (i = 1, 2, ..., n) is equivalent to the original system.

Thus, to solve a system of linear equations, its augmented matrix should be transformed into a reduced echelon matrix (or, row canonical form). Then the solution of the system determined by this reduced echelon matrix is also a solution to the original system.

GAUSS ELIMINATION METHOD FOR REDUCTION OF THE AUGMENTED MATRIX : The Gaussian elimination method for

reducing the augmented matrix (A|B) to the row canonical form (R|S) is as follows :

Step 1 : First reduce (A|B) to an echelon form. If the echelon matrix has a row (0, 0, ..., 0, b) where $b \neq 0$, then the system of equations is inconsitent, i.e., no solution exists. Otherwise, we follow step-2.

Step 2 : Let R_1, R_2, ..., R_r be the non-zero rows in the echelon matrix with leading entries α_{1j_1}, α_{2j_2}, ..., α_{rj_r} respectively. If $\alpha_{rj_r} \neq 1$, make it 1 by the elementary row operation $R_r \to \frac{1}{\alpha_{rj_r}} R_r$. Now produce as above $\alpha_{rj_r} = 1$ by applying row operations on R_1, R_2, ..., R_{r-1}.

Step 3 : Repeat step-2 in succession with R_{r-1}, R_{r-2}, ..., R_2. Finally, if necessary, multiply R_1 by $\frac{1}{a_{1j_1}}$ to make $a_{1j_1} = 1$.

The matrix is row in row canonical form, from which we find the solution of the equivalent system, and hence the solution of the original system.

13.4.5 Illustrative Examples

Example 1 : Solve by Gaussian elimination method :

$$x - 2y + z = 7$$
$$2x - y + 4z = 17$$
$$3x - 2y + 2z = 14$$

Solution : We change the augmented matrix (A|B) to echelon form and then to row canonical form as follows :

$$(A|B) = \begin{bmatrix} 1 & -2 & 1 & 7 \\ 2 & -1 & 4 & 17 \\ 3 & -2 & 2 & 14 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 3 & 2 & 3 \\ 0 & 4 & -1 & -7 \end{bmatrix}, \; R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & -\frac{11}{3} & -11 \end{bmatrix}, R_3 \to R_3 - \frac{4}{3}R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}, R_3 \to -\frac{3}{11}R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{bmatrix}, R_1 \to R_1 - R_3, R_2 \to R_2 - 2R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}, R_2 \to \frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}, R_1 \to R_1 + 2R_2$$

which is in row canonical form. This give the equivalent system of equations

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \Rightarrow x = 2, y = -1, z = 3$$

Hence (2, –1, 3) is the unique solution of the given system of equations.

Example 2 : Solve the system :

$$x + 2y - 3z - 4t = 2$$

$$2x + 4y - 5z - 7t = 7$$

$$3x + 6y - 11z - 14t = 0$$

Solution : We reduce the augonented matrix to echelon form and then to row canonical form :

$$\begin{pmatrix} 1 & 2 & -3 & -4 & 2 \\ 2 & 4 & -5 & -7 & 7 \\ 3 & 6 & -11 & -14 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & -4 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & -2 & -2 & -6 \end{pmatrix}, \; R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - 3R_1$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & -4 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; R_3 \to R_3 + 2R_2$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & -1 & 11 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; R_1 \to R_1 + 3R_2$$

which is in row canonical form. This equivalent system is

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $x + 2y - t = 11, \; z + t = 3$

$\Rightarrow$ $x = 11 - 2y + t, \; z = 3 - t$

Taking arbitrary values $y = \alpha$, $t = \beta$ we get

$x = 11 - 2\alpha + \beta, \;\; z = 3 - \beta.$

Hence, the system has **infinite number** of solutions given by $(11 - 2\alpha + \beta, \alpha, 3 - \beta, \beta)$ for arbitrarilly chosen values of α and β.

In particular, taking $\alpha = 1$, $\beta = 0$ we get a solution (9, 1, 3, 0).

Taking $\alpha = 0$, $\beta = 1$ we get another solution (12, 0, 2, 1), etc.

Example 3 : Solve the system :

$$3x_1 + 4x_2 - x_3 + 2x_4 = 1$$
$$x_1 - 2x_2 + 3x_3 + x_4 = 2$$
$$3x_1 + 14x_2 - 11x_3 + x_4 = 3$$

Solution : Augmented matrix :

$$= \begin{pmatrix} 3 & 4 & -1 & 2 & 1 \\ 1 & -2 & 3 & 1 & 2 \\ 3 & 14 & -11 & 1 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 3 & 4 & -1 & 2 & 1 \\ 3 & 14 & -11 & 1 & 3 \end{pmatrix}, R_1 \leftrightarrow R_2$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 10 & -10 & -1 & -5 \\ 0 & 10 & -10 & -1 & 3 \end{pmatrix}, R_2 \to R_2 - 3R_1, R_3 \to R_3 - R_2$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 10 & -10 & -1 & -5 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}, R_3 \to R_3 - R_2$$

It is in echelon form having the row (0, 0, 0, 0, 7). Hence the system has no solution.

[**Note :** In fact, the third row gives an equation

$0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 = 7$

i.e., 0 = 7; hence the system is inconsistent.]

Example 4 : Find the conditions under which the following system has (i) no solution (ii) a solution :

$$x + 2y - 3z = a$$
$$2x + 6y - 11z = b$$
$$x - 2y + 7y = c$$

Solution : Augmented matrix

$$= \begin{pmatrix} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & -4 & 10 & c-a \end{pmatrix}, R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & 2b+c-5a \end{pmatrix}, \; R_3 \to R_3+2R_2$$

The equivalent system is $x + 2y - 3z = 0$

$$2y - 5z = b - 2a$$

$$0 = 2b + c - 5a$$

i) The system will have no solution if $2b + c - 5a \neq 0$.

ii) If $2b + c - 5a = 0$, i.e., $5a = 2b + c$, the system will have solutions given by the equivalent system.

$$x + 2y - 3z = a$$

$$2y - 5z = b - 2a$$

$$\text{or, } x = a - 2y + 3z,$$

$$y = \frac{b-2a+5z}{2}.$$

Taking arbitrarilly chosen values of z and applying the condition

$5a = 2b + c$, we shall get in finite number of solutions.

Note : This system cannot have a unique solution.

From the illustrative examples discussed above, it is clear that for a system of linear equations any one of the following holds :

i) has a unique solution,

ii) has no solution

iii) has an infinite number of solutions.

13.4.6 Solutions of the Homogeneous System

For the homogeneous system of linear equations :

$$\sum_{j=1}^{n} a_{ij}x_j = 0, \; (i = 1, 2, ..., m)$$

the last column of the augmented matrix has all zeros, and so, whatever elementary row operations are performed to reduce the augmented matrix to echelon form, the last column does not

change. Hence we reduce the cofficient matrix $A = (a_{ij})_{m\times n}$ to canonical form to obtain the equivalent system. We explain the method in the following examples.

Example 1 : Solve the system : $x + 3y - 2z = 0$

$2x - y + 4z = 0$

$x - 11y + 14z = 0$

Solution : $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{pmatrix}, \; R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{pmatrix}, \; R_3 \to R_3 - 2R_2$$

$$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{8}{7} \\ 0 & 0 & 0 \end{pmatrix}, \; R_2 \to -\frac{1}{7} R_2$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{10}{7} \\ 0 & 1 & -\frac{8}{7} \\ 0 & 0 & 0 \end{pmatrix}, \text{ which is in row cononical form.}$$

So, the equivalent system is given by

$$\begin{pmatrix} 1 & 0 & \frac{10}{7} \\ 0 & 1 & -\frac{8}{7} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x + \frac{10}{7}z = 0,\ y - \frac{8}{7}y = 0$$

$$\Rightarrow x = -\frac{10}{7}z,\ y = \frac{8}{7}z.$$

Taking $z = \alpha$, an arbitrary value, we get $(-\frac{10}{7}\alpha, \frac{8}{7}\alpha, \alpha)$ as the general solution of the system and hence the system has infinite number of solutions.

Example 2 : Solve the system : $x_1 + 2x_2 + x_3 - 3x_4 = 0$

$2x_1 + 4x_2 + 3x_3 + x_4 = 0$

$3x_1 + 6x_2 + 4x_3 - 2x_4 = 0$

Solution : Coefficient matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -3 \\ 2 & 4 & 3 & 1 \\ 3 & 6 & 4 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 4 & 7 \end{pmatrix}, R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & -10 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_1 \to R_1 - R_2, R_3 \to R_3 - R_2$$

Hence, the system is equivalent to $x_1 + 2x_2 - 10x_4 = 0$

$x_3 + 7x_4 = 0$

Setting $x_2 = a$, $x_4 = b$,

we get the general solution as $(10b - 2a, a, -7b, b)$ where a, b are arbitrary. The system has infinite number of solutions.

Example 3 : Find a non-zero solution of the system :

$x_1 - x_2 + x_3 = 0$

$x_1 + x_2 + 2x_3 = 0$

$x_1 + 2x_2 - x_3 = 0$

Solution : The coefficient matrix

$$= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & -2 \end{pmatrix}, \; R_2 \to R_2 - R_1, \; R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 3 & -2 \end{pmatrix}, \; R_2 \to \tfrac{1}{2} R_2$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{7}{2} \end{pmatrix}, \; R_1 \to R_1 + R_2, \; R_3 \to R_3 - 3R_1$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \; R_3 \to \frac{2}{7} R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; R_1 \to R_1 - \frac{3}{2} R_3, \; R_2 \to R_2 - \tfrac{1}{2} R_3$$

Hence the system is equivalent to $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

In otherwords, the system has unique solution (0, 0, 0) and it has no non-zero solution.

Note : A system of homogeneous linear equations has either (i) a unique solution (0, 0, 0), called the **trivial solution** or (ii) an infinite number of solutions.



CHECK YOUR PROGRESS

Q.3. Solve by using Gaussian elimination method :

$x + 2y – z = 3$

$3x – y + 2z = 1$

$2x – 2y + 3z = 2$

$x – y + z = –1$

Q.4. Examine existence of solution of the system

$x + y – 2z + 3t = 4$

$2x + 3y + 3z – t = 3$

$5x + 7y + 4z + t = 5$

Q.5. For what value of a the following system is consistent? Find solutions when consistent.

$x – y + z = 1$

$x + 2y + 4z = a$

$x + 4 + 6z = a^2$.



13.5 LET US SUM UP

- $\sum_{j=1}^{n} a_{ij}x_j = b_i$, where i = 1, 2, ..., n is a system of *n* linear non-homogeneous equations in *n* unknowns. In matrix form the system can be expressed as AX = B,

  where $A = [a_{ij}]_{n\times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

- The system AX = B has a unique solution when $|A| \neq 0$.

- By Cramer's rule, the solution of the system is given by $x_i = \frac{|A_i|}{|A|}$ for i = 1, 2, ..., n, where A_i is obtained from A replacing the i^{th} column by the column $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

- The system of equations $\sum_{j=1}^{n} a_{ij}x_j = b_i$, where i = 1, 2, ..., m is called a system of a *m* linear non-homogeneous equations in *n* unknowns where $A = [a_{ij}]_{m\times n}$ is called the **Co-efficient matrix** and $(A|B) = [a_{ij}|b_i]_{m\times n}$ is called the **Augmented matrix**.

- The system is called **consistent** if it has atleast one soloution i.e., atleast one n-tuple $(\alpha_1, \alpha_2, ..., \alpha_n)$ which satisfies the equations in the system, otherwise the system is called inconsistent.

- Two systems $\sum_{j=1}^{n} a_{ij}x_j = b_i$ (i = 1, 2, ..., m) and $\sum_{j=1}^{n} a'_{ij}x_j = b'_i$ (i = 1, 2, ..., m) are said to be equivalent if every solution of one system is also a solution of the other system.

- The system can be solved by reducing the augmented matrix to echelon form and then to row canonical form using Gaussian Elimination Method; thereby finding solution of an equivalent system and hence that of the original system.
- Any one of the following three holds for the system :
  i) has a unique solution
  ii) has no solution
  iii) has an infinite number of solutions.
- The system of equations $\sum_{j=1}^{n} a_{ij}x_j = 0$, where i = 1, 2, ..., m is called a homogeneous system. The system is solved by reducing the co-efficient matrix to row canonical form.
- A homogeneous system has either (i) a unique solution (0, 0, 0), called the trivial solution or (ii) an infinite number of solution.



13.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : i) Matrix form is $\begin{pmatrix} 10 & -5 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

or, AX = B, where $A = \begin{pmatrix} 10 & -15 \\ 3 & -4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$|A| = \begin{vmatrix} 10 & -15 \\ 3 & -4 \end{vmatrix} = -40 + 45 = 5 \neq 0$

So, unique solution exists and is given by $X = A^{-1}B$.

Now, $A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{5}\begin{pmatrix} -4 & -3 \\ 15 & 10 \end{pmatrix}' = \frac{1}{5}\begin{pmatrix} -4 & 15 \\ -3 & 10 \end{pmatrix}$

$\Rightarrow X = \frac{1}{5}\begin{pmatrix} -4 & 15 \\ -3 & 10 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 15 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} \\ 2 \end{pmatrix} \Rightarrow x = 3,\ y = 2.$$

ii) Matrix form is AX = B,

$$\text{where } A = \begin{pmatrix} 3 & 2 & -2 \\ -1 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 & -2 \\ -1 & 1 & 4 \\ 2 & -3 & 4 \end{vmatrix} = 70,\ \text{adj } A = \begin{pmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{pmatrix}$$

$$\text{Hence, } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{70} \begin{pmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix}$$

$$= \frac{1}{70} \begin{pmatrix} 94 \\ -52 \\ 54 \end{pmatrix} = \begin{pmatrix} \frac{47}{35} \\ -\frac{26}{35} \\ \frac{27}{35} \end{pmatrix}$$

$$\Rightarrow x = \frac{47}{35},\ y = -\frac{26}{35},\ z = \frac{27}{35}.$$

iii) $AX = B,\ \text{where } A = \begin{pmatrix} 7 & 6 & 5 \\ 1 & 2 & 1 \\ 3 & -2 & 1 \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}.$

$$|A| = \begin{vmatrix} 7 & 6 & 5 \\ 1 & 2 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 0 \Rightarrow \text{the solution does not exist.}$$

Ans. to Q. No. 2 : i) $|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = -2$ $|A_1| = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = -2$

$$|A_2| = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -4 \qquad |A_3| = \begin{vmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{vmatrix} = -6$$

So, the solutions are $x = \frac{|A_1|}{|A|} = \frac{-2}{-2} = 1$,

$$y = \frac{|A_2|}{|A|} = \frac{-4}{-2} = 2,$$

$$z = \frac{|A_3|}{|A|} = \frac{-6}{-2} = 3.$$

ii) $$|A| = \begin{vmatrix} 2 & 1 & 5 & 1 \\ 1 & 1 & -3 & -4 \\ 3 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & -3 & -4 \\ 2 & 1 & 5 & 1 \\ 3 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix}, \; R_1 \leftrightarrow R_2$$

$$= -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 11 & 9 \\ 3 & 3 & 7 & 13 \\ 2 & 0 & 8 & 5 \end{vmatrix}; \; c_2 \to c_2 - c_1, \; c_3 \to c_3 + 3c_1, \; c_4 \to c_4 + 4c_1$$

$$= -1.\begin{vmatrix} -1 & 11 & 9 \\ 3 & 7 & 13 \\ 0 & 8 & 5 \end{vmatrix} = -120$$

$$|A_1| = \begin{vmatrix} 5 & 1 & 5 & 1 \\ -1 & 1 & -3 & -4 \\ 8 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = -1\begin{vmatrix} -1 & 1 & -3 & -4 \\ 5 & 1 & 5 & 1 \\ 8 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix}, \; R_1 \leftrightarrow R_2$$

$$= \begin{vmatrix} 1 & -1 & 3 & 4 \\ 5 & 1 & 5 & 1 \\ 8 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 5 & 6 & -10 & -19 \\ 8 & 14 & -26 & -31 \\ 2 & 4 & -4 & -11 \end{vmatrix};$$

$$c_2 \to c_2 + c_1, \; c_3 \to c_3 - 3c_1, \; c_4 \to c_4 - 4c_1$$

$$= 1.\begin{vmatrix} 6 & -10 & -19 \\ 14 & -26 & -31 \\ 4 & -4 & -11 \end{vmatrix} = \begin{vmatrix} 6 & 10 & 19 \\ 14 & 26 & 31 \\ 4 & 4 & 11 \end{vmatrix}$$

$$= 4.\begin{vmatrix} 3 & 5 & 19 \\ 7 & 13 & 31 \\ 2 & 2 & 11 \end{vmatrix} = -240$$

$$|A_2| = \begin{vmatrix} 2 & 5 & 5 & 1 \\ 1 & -1 & -3 & -4 \\ 3 & 8 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & -3 & -4 \\ 2 & 5 & 5 & 1 \\ 3 & 8 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 11 & 9 \\ 3 & 11 & 7 & 13 \\ 2 & 4 & 8 & 5 \end{vmatrix} = -1.\begin{vmatrix} 7 & 11 & 9 \\ 11 & 7 & 13 \\ 4 & 8 & 5 \end{vmatrix} = -24$$

$$|A_3| = \begin{vmatrix} 2 & 1 & 5 & 1 \\ 1 & 1 & -1 & -4 \\ 3 & 6 & 8 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & -1 & -4 \\ 2 & 1 & 5 & 1 \\ 3 & 6 & 8 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 7 & 9 \\ 3 & 3 & 11 & 13 \\ 2 & 0 & 4 & 5 \end{vmatrix} = -1.\begin{vmatrix} -1 & 7 & 9 \\ 3 & 11 & 13 \\ 0 & 4 & 5 \end{vmatrix} = 0$$

$$|A_4| = \begin{vmatrix} 2 & 1 & 5 & 5 \\ 1 & 1 & -3 & -1 \\ 3 & 6 & -2 & 8 \\ 2 & 2 & 2 & 2 \end{vmatrix} = -\begin{vmatrix} 2 & 2 & 2 & 2 \\ 2 & 1 & 5 & 5 \\ 1 & 1 & -3 & -1 \\ 3 & 6 & -2 & 8 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 5 & 5 \\ 1 & 1 & -3 & -1 \\ 3 & 6 & -2 & 8 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 3 & 3 \\ 1 & 0 & -4 & -2 \\ 3 & 6 & -5 & 5 \end{vmatrix}$$

$$= -2.\begin{vmatrix} -1 & 3 & 3 \\ 0 & -4 & -2 \\ 3 & -5 & 5 \end{vmatrix} = -96$$

$\therefore$ The solutions are

$$x_1 = \frac{|A_1|}{|A|} = \frac{-240}{-120} = 2, \qquad x_2 = \frac{|A_2|}{|A|} = \frac{-24}{-120} = \frac{1}{5},$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{0}{-120} = 0, \qquad x_4 = \frac{|A_4|}{|A|} = \frac{-96}{-120} = \frac{4}{5}.$$

Ans. to Q. No. 3 : Augmented matrix

$$= \begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{pmatrix}; R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - R_1$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & \frac{5}{7} & \frac{20}{7} \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} \end{pmatrix}, R_3 \to R_3 - \frac{6}{7}R_2, R_4 \to R_4 - \frac{3}{7}R_2$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & \frac{5}{7} & \frac{20}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_4 \to R_4 + \frac{1}{5}R_3$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & -\frac{5}{7} & \frac{8}{7} \\ 0 & 0 & \frac{5}{7} & \frac{20}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_2 \to -\frac{1}{7}R_2$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{3}{7} & \frac{5}{7} \\ 0 & 1 & -\frac{5}{7} & \frac{8}{7} \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; R_1 \to R_1 - 2R_2, \; R_3 \to \frac{7}{5} R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ The equivalent system is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 4 \\ 0 \end{pmatrix}$

$\Rightarrow x = -1, y = 4, z = 4.$

Ans. to Q. No. 4 : Augmented matrix

$$= \begin{pmatrix} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{pmatrix}, \; R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - 5R_1$$

$$\sim \begin{pmatrix} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix}, \; R_3 \to R_3 - 2R_2$$

It is now in echelon form having a row (0, 0, 0, 0, –5). Hence the system is inconsistent, i.e., it has no solution.

Ans. to Q. No. 5 : Augmented matrix

$$= \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & 4 & a \\ 1 & 4 & 6 & a^2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 3 & 3 & a-1 \\ 1 & 5 & 5 & a^2-1 \end{pmatrix}, R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & \frac{a-1}{3} \\ 0 & 5 & 5 & a^2-1 \end{pmatrix}, R_2 \to \frac{1}{3}R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & \frac{a+2}{3} \\ 0 & 1 & 1 & \frac{a-1}{3} \\ 0 & 0 & 0 & \frac{3a^2-5a+2}{3} \end{pmatrix}, R_1 \to R_1 + R_2, R_3 \to R_3 - 5R_1$$

The equivalent system is $x + 2z = \frac{a+2}{3}$,

$$y + z = \frac{a-1}{3}$$

$$0 = \frac{a^2-5a+2}{3},$$

The system is

i) inconsistent if $3a^2 - 5a + 2 \neq 0$,

ii) consistent if $\frac{3a^2-5a+2}{3} = 0$ i.e., $a = 1, \frac{2}{3}$

Case 1 : When $a = 1$, we get $x + 2z = 1$, $y + z = 0$ and so, taking $z = \alpha$ (arbitrary), we get infinite solution given by $(1-2\alpha, -\alpha, \alpha)$ for arbitrary values of α.

Case 2 : When $a = \frac{2}{3}$, we get $x + z = \frac{8}{9}$, $y + z = -\frac{1}{9}$.

Taking $z = \beta$ (arbitrary), we get another set of infinite solutions given by $(\frac{8}{9} - 2\beta, -\frac{1}{9} - \beta, \beta)$.



13.7 FURTHER READINGS

1. *Matrices* – J. N. Sharma & S. N. Goel.
2. *Linear Algebra*, Seymour Lipschutz.



13.8 MODEL QUESTIONS

Q.1. Solve by Cramer's rule :

i) $x - y + 2z = 1$
$2x + y + z = 2$
$x - 3y + z = 1$

ii) $x + y - 2z = 1$
$2x - 7z = 3$
$x + y - z = 5$

iii) $x + 2y - z = 3$
$3x - y + 2z = 1$
$2x - 2y + 3z = 2$

iv) $3x + 2y - 2z = 1$
$-x + y + 4z = 13$
$2x - 3y + 4z = 8$

Q.2. Solve by using Gaussian elimination method :

i) $x - 4y - 3z = -16$
$2x + 7y + 12z = 48$
$4x - y + 6z = 16$
$5x - 5y + 3z = 0$

ii) $x - 2y + z = 7$
$2x - y + 4z = 17$
$3x - 2y + 2z = 14$

iii) $x_1 + 2x_2 - 3x_3 = 1$
$x_2 - 2x_3 = 2$
$2x_2 - 4x_3 = 4$

iv) $2x_1 - 5x_2 + 3x_3 - 4x_4 + 2x_5 = 4$
$3x_1 - 7x_2 + 2x_3 - 5x_4 + 4x_5 = 9$
$5x_1 - 10x_2 - 5x_3 - 4x_4 + 7x_5 = 22.$

Q.3. Show that the following equations are inconsistent. Apply Gaussian elimination on the augmented matrix to show inconsistency.

i) $x_1 + 2x_2 - 3x_3 + 4x_4 = 2$
$x_2 + 4x_3 - 7x_4 = -3$
$2x_2 + 8x_3 - 14x_4 = 3$

ii) $x + 2y - 3z = 0$
$2x + 4y - 2y = z$
$3x + 6y - 4z = 3$

iii) $x + 2y - 3z = 1$
$2x + 6y - 11z = -1$
$x - 2y + 7z = 8$

Q.4. Determine the values of λ so that the following system has

i) a unique solution,

ii) no solution,

iii) infinite number of solutions.

a) $x + y - z = 1$
$2x + 3y + \lambda z = 3$
$x + \lambda y + 3z = 2$

b) $x + 2y + \lambda z = 0$
$2x + 3y - 2z = \lambda$
$\lambda x + y + \lambda^2 z = 3$

Q.5. Find solutions of the following homogeneous systems :

i) $x + 2y - 3z = 0$
$2x + 5y + 2z = 0$
$3x - y - 4z = 0$

ii) $x_1 + 2x_2 + 5x_3 + 2x_4 = 0$
$2x_1 + 4x_2 + x_3 - 5x_4 = 0$
$x_1 + 2x_2 + x_3 - 2x_4 = 0$

iii) $2x_1 + 4x_2 - 5x_3 + 3x_4 = 0$
$3x_1 + 6x_2 - 7x_3 + 4x_4 = 0$
$5x_1 + 10x_2 - 11x_3 + 6x_4 = 0$