UNIT1: RELATIONS AND FUNCTIONS

UNIT STRUCTURE

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1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to know:

- the concept of a relation
- different types of relation
- equivalence relation, equivalence class
- the concept of a function
- different types of function.

1.2 INTRODUCTION

Set theory may be called the language of modern mathematics. We know that a set is a well-defined collection of objects. Also we know the notion of subset of a set. If every element of a set B is in set A, then B

is a subset of A. Symbolically we denote it by $B \subseteq A$.

Also we note that if A is a finite set having n elements, the number of subsets of A is 2^n.

Again if A, B are two non-empty sets, the cartesian product of A and B is denoted by A x B and is defined by $A \times B = \{(a, b) : a \in A, b \in B\}$

(a, b) is called an ordered pair.

Let $a \in A$, $c \in A$; $b \in B$, $d \in B$.

$\therefore$ $(a, b) \in A \times B$, $(c, d) \in A \times B$

We know that $(a, b) = (c, d) \Leftrightarrow a=c, b=d$.

Also we know that if A, B are finite and n(A)=x, n(B)= y, then

$$n(AxB) = n(BxA)=xy$$

[Here n(A) denotes the number of elements of A]

If one of the sets A and B is infinite, then A x B and B x A are infinite.

In this unit we will study relations and functions which are subsets of cartesian product of two sets.

We will denote the set of natural numbers by IN, the set of integers by Z, the set of rational numbers by Q, the set of real numbers by IR, the set of complex numbers by C.

1.3 CONCEPT OF RELATION

Let us consider the following sentences.

(1) 11 is greater than 10.

(2) 35 is divisible by 7.

(3) New Delhi is the capital of India.

In each of the sentences there is a relation between two 'objects'.

Now let us see what is meant by relation in set theory.

Definition Let A and B be two non-empty sets. A subset R of A x B is said to be a **relation** from A to B.

If A=B, then any subset of A x A is said to be a relation on A.

If $R \subseteq A \times B$, and $(a, b) \in R$; $a \in A$, $b \in B$, it is also written as aRb and is read as 'a is R related to b'.

Note 1. The set of the first components of the ordered pairs of R is called the domain and the set of the second components of the ordered

pairs of R is called the range of R.

2. If A, B are finite sets and n(A)=x, n(B)=y; then n(AxB)=xy. So, the number of subsets of AxB is 2^{xy}.

Therefore, the number of relations from A to B is 2^{xy}.

Example 1: Let A ={1, 2, 3},

B={8, 9}

$\therefore$ A x B ={(1, 8), (1, 9), (2, 8), (2, 9), (3, 8), (3, 9)}

Let R = {(1, 8), (2, 9), (3, 9)}

Clearly R$\subseteq$ AxB

$\therefore$ R is a relation from A to B.

Here 1R8, 2R9, 3R9

Domain of R = {1, 2, 3}

Range of R = {8, 9}

Example 2 : Let IR be the set of real numbers.

Let R = {(x, y) : x, y $\in$IR, x<y} $\subseteq$ IR x IR

$\therefore$ R is a relation on IR.

$\because$ 3<5, $\therefore$ (3, 5)$\in$R i.e. 3R5

19<27, $\therefore$ (19, 27)$\in$R i.e. 19R27

But 5>3, $\therefore$ (5, 3)$\notin$ R, i.e. 5 $\not R$ 3.

Example 3: Let X be the set of odd integers.

Let R = {(x, y) : x, y $\in$ X and x+y is odd}

We know that the sum of two odd integers is an even integer.

$\therefore$ if x, y are odd, then x+y cannot be odd.

$\therefore$ R = $\phi \subseteq$ X x X

In this case R is called a null relation on X.

Example 4: Let E be the set of even integers.

Let R = {(x, y) : x, y $\in$ E and x+y is even}

We know that the sum of the even integers is an even integer.

$\therefore$ if x, y are even, then x+y is always even.

$\therefore$ R = E x E $\subseteq$ E x E

In this case R is called a universal relation on E.

1.3.1 Identity Relation

Let A be a non-empty set.

$I_A = \{(a, a) : a \in A\} \subseteq A \times A$

I_A is called the identity relation on A.

Example 5: Let A = {1, 2, 3, 4}

Then $I_A = \{(1, 1), (2, 2), (3, 3), (4, 4)\} \subseteq A \times A$.

Clearly I_A is the identity relation on A.

1.3.2 Inverse Relation

Let A, B be two non-empty sets. Let R be a relation from A to B, i.e. $R \subseteq A \times B$. The inverse relation of R is denoted by R^{-1}, and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\} \subseteq B \times A$

Clearly, domain of R^{-1} = range of R

range of R^{-1} = domain of R

Example 6: Let A = {1, 2, 3}

B = {5, 6}

$R = \{(1, 5), (2, 6), (3, 5)\} \subseteq A \times B$

$R^{-1} = \{(5, 1), (6, 2), (5, 3)\} \subseteq B \times A$.



CHECK YOUR PROGRESS 1

Q 1: Let A, B be two finite sets, and n(A)=4, n(B)=3. How many relations are there from A to B?

Q 2: Let A be a finite set such that n(A) =5. Write down the number of relations on A.

Q 3: Let IN be the set of natural numbers. Let $R = \{(x, y) : x, y \in IN, x>y\}$. Examine if the following ordered pairs belong to R.

(7, 4), (23, 32), (5, 5), (72, 27), (18, 19).

Q 4: Let E be the set of even integers. Let R = {(x, y) : x, y ∈ E and x+y is odd}

Is R a null relation on E ? Justify your answer.

Q 5: Let A = {1, 2, 3, 4, 5}

Write down the identity relation on A.

Q 6: Let A = {4, 5, 6}, B = {7, 8, 9}

Let R = {(4, 7), (5, 8), (6, 7), (6, 8), (6, 9)}

Write down R^{-1}.

1.4 TYPES OF RELATION

Let A be a non-empty set, and R be a relation on A, i.e. $R \subseteq A \times A$.

1. R is called reflexive if $(a, a) \in R$, i.e; aRa, for all $a \in A$.
2. R is called symmetric if whenever $(a, b) \in R$, then $(b, a) \in R$, i.e, if whenever aRb, then bRa; $a, b \in A$.
3. R is called anti-symmetric if $(a, b) \in R$, $(b, a) \in R \Rightarrow a = b$, i.e., if aRb, bRa $\Rightarrow$ a=b; $a, b \in A$.
4. R is called transitive if whenever (a, b), $(b, c) \in R$, then $(a, c) \in R$, i.e., if whenever aRb, bRc, then aRc; $a, b, c \in A$.

Example 7: Let A = {1, 2, 3}

R = (1, 1), (2, 2), (1, 2), (2, 1)}

Examine if R is reflexive, symmetric,anti-symmetric, transitive.

Solution: Here $(1, 1) \in R$, $(2, 2) \in R$; but $(3, 3) \notin R$.

$\therefore$ R is not reflexive.

Again, $(a, b) \in R \Rightarrow (b, a) \in R$

$\therefore$ R is symmetric.

Again, $(1, 2) \in R$, $(2, 1) \in R$, but $1 \neq 2$

$\therefore$ R is not anti-symmetric.

Again, if $(a, b) \in R$, $(b, c) \in R$, then $(a, c) \in R$

$\therefore$ R is transitive.

Example 8: Let Z be the set of integers, and $R = \{(x, y) : x, y \in Z, x \leq y\}$

Examine if R is reflexive, symmetric, anti-symmetric and transitive.

Solution : We have $a \leq a$, $\forall\ a \in Z$

i.e, $(a, a) \in R$, $\forall\, a \in Z$

$\therefore$ R is reflexive.

Again, if $a \leq b$, $b \not\leq a$

i.e; $(a, b) \in R \not\Rightarrow (b, a) \in R$

$\therefore$ R is not symmetric.

Again, $a \leq b$, $b \leq a \Rightarrow a = b$

i.e. $(a, b) \in R$, $(b, a) \in R \Rightarrow a = b$

$\therefore$ R is anti-symmetric.

Again $a \leq b$, $b \leq c \Rightarrow a \leq c$

i.e. $(a, b), (b, c), \in R \Rightarrow (a, c) \in R$

$\therefore$ R is transitive.

Example 9: Give an example of a relation which is transitive, but neither reflexive nor symmetric.

Solution : Let A = {1, 2}

Let $R = \{(1, 1), (1, 2)\} \subseteq A \times A$

Clearly R is a relation on A.

Here $(2, 2) \notin R$.

$\therefore$ R is not reflexive.

Again $(1, 2) \in R$, but $(2, 1) \notin R$

$\therefore$ R is not symmetric.

But R is transitive.



CHECK YOUR PROGRESS 2

Q 1: Let A = {1, 2, 3) and R be a reflexive relation on A having minimum number of ordered pairs. Write down R.

Q 2: Define a relation on the set

A = {1, 2, 3, 4} which is

(i) symmetric, but neither reflexive nor transitive;

(ii) reflexive and transitive, but not symmetric;

(iii) reflexive and symmetric, but not transitive.

1.5 EQUIVALENCE RELATION

Let A be a non-empty set. A relation R on A is called an equivalence relation if it is reflexive, symmetric and transitive.

Example 10: Let Z be the set of integers and

$R = \{(x, y) : x, y \in Z \text{ and } x+y \text{ is even}\}$

Examine if R is an equivalence relation on Z.

Solution : Let $x \in Z$

$\therefore$ x+x is even

$\Rightarrow (x, x) \in R, \forall\ x \in Z$

$\therefore$ R is reflexive.

$(x, y) \in R \Rightarrow$ x+y is even

$\Rightarrow$ y+x is even

$\Rightarrow (y, x) \in R$.

$\therefore$ R is symmetric.

$(x, y) \in R, (y, z) \in R$

$\Rightarrow$ x+y is even, y+z is even

$\Rightarrow$ (x+y)+(y+z) is even

$\Rightarrow$ (x+z)+2y is even

$\Rightarrow$ x+z is even

$(x, z) \in R$

$\therefore$ R is transitive

$\therefore$ R is an equivalence relation on Z.

Example 11: Let A be the set of all straight lines in a plane.

Let $R = \{(x, y) : x, y \in A \text{ and } x \perp y\}$

Examine if R is reflexive, symmetric and transitive.

Solution : A line cannot be perpendicular to itself, i.e. $x \not\perp x$

$\therefore (x, x) \notin R$

$\therefore$ R is not reflexive

If a line x is perpendicular to another line y, then y is perpendicular to x, i.e. $x \perp y \Rightarrow y \perp x$

$\therefore (x, y) \in R \Rightarrow (y, x) \in R$

$\therefore$ R is symmetric.

If $x \perp y$, $y \perp z$, then $x \not\perp z$

$\therefore (x, y), (y, z) \in R \not\Rightarrow (x, z) \in R$

If x is perpendicular to y, y is perpendicular to z, then x is parallel to z.

$\therefore$ R is not transitive.

Example 12: Let A be the set of all straight lines in a plane.

Let $R = \{(x, y) : x, y \in A \text{ and } x||y\}$

Examine if R is an equivalence relation on A.

Solution : A line is parallel to itself, i.e. $x||x \ \forall \ x \in A$

$\therefore (x, x) \in R, \forall x \in A$

$\therefore$ R is reflexive.

$x||y \Rightarrow y||x$

$\therefore (x, y) \in R \Rightarrow (y, x) \in R$

$\therefore$ R is symmetric.

$x||y, y||z \Rightarrow x||z$

$\therefore (x, y), (y, z) \in R \Rightarrow (x, z) \in R.$

$\therefore$ R is transitive

Thus, R is an equivalence relation on A.

Example 13: Let IN be the set of natural numbers. Let a relation R be defined on IN x IN by (a, b) R (c, d) if and only if ad = bc

Show that R is an equivalence relation on IN x IN.

Solution : We have ab = ba

$\therefore$ (a, b) R (a, b)

$\therefore$ R is reflexive

(a, b) R (c, d) $\Rightarrow$ ad = bc; a, b, c, d, $\in$ IN

$\Rightarrow$ cb = da

$\Rightarrow$ (c, d) R (a, b)

$\therefore$ R is symmetric

(a, b) R (c, d) and (c, d) R (e, f); a, b, c, d, e, f, $\in$ IN

$\Rightarrow$ ad = bc and cf = de

$\Rightarrow$ adcf = bcde

$\Rightarrow$ af = be

$\Rightarrow$ (a, b) R (e, f)

$\therefore$ R is transitive.

Thus, R is an equivalence relation on IN x IN.

Example 14: (Congruence modulo n)

Let Z be the set of integers, and n be any fixed positive integer.

Let a, b$\in$ Z

a is said to be congruent to b modulo n if and only if a–b is divisible by n. Symbolically, we write.

a $\equiv$b (mod n)

Show that the relation 'congruence modulo n' is an equivalence relation on Z.

Solution : We know that a–a is divisible by n, i.e., a $\equiv$ a (mod n)

$\therefore$ the relation is reflexive.

Let a $\equiv$b (mod n)

$\Rightarrow$ a–b is divisible by n.

$\Rightarrow$ b–a is divisible by n.

$\Rightarrow$ b$\equiv$a (mod n)

$\therefore$ the relation is symmetric.

Let a $\equiv$b (mod n), b $\equiv$c (mod n)

$\Rightarrow$ a–b is divisible by n, b–c is divisible by n

$\Rightarrow$ a–b+b–c is divisible by n

$\Rightarrow$ a–c is divisible by n

$\Rightarrow$ a $\equiv$c (mod n)

$\therefore$ the relation is transitive.

Thus, the relation 'congruence modulo n' is an equivalence relation on Z.

We know that 15-3 is divisible by 4.

$\therefore$15 is congruent to 3 modulo 4 i.e.

15 $\equiv$3 (mod 4)

15–3 is not divisible by 7

$\therefore$15 is not congruent to 3 modulo 7

i.e. 15 $\not\equiv$3 (mod 7)

Example 15: Let Z be the set of integers.

Let R = {(a, b) : a, b $\in$ Z, ab$\geq$0}

Examine if R is an equivalence relation on Z.

Solution : We have aa $\geq$0

$\therefore (a, a) \in R, \forall a \in Z$

$\therefore R$ is reflexive

Let $(a, b) \in R$

$\therefore ab \geq 0$

$\Rightarrow ba \geq 0$

$\Rightarrow (b, a) \in R$

$\therefore$ R is symmetric.

We have

$(-2) \times 0 = 0, 0 \times 2 = 0$

$\therefore (-2, 0), (0, 2) \in R$

But $(-2) \times 2 = -4 < 0$

$\therefore (-2, 2) \notin R$

$\therefore$ R is not transitive.

Thus, R is not an equivalence relation.

Theorem 1: The inverse of an equivalence relation is also an equivalence relation.

Proof. Let A be a non-empty set. Let R be an equivalence relation on A

R is reflexive.

$\therefore (x, x) \in R, \forall x \in A$

$\Rightarrow (x, x) \in R^{-1}, \forall x \in A$

$\therefore R^{-1}$ is reflexive.

Let $(x, y) \in R^{-1}$

This $\Rightarrow (y, x) \in R$ [by def^n of R^{-1}]

$\Rightarrow (x, y) \in R$ [$\therefore R$ is symmetric]

$\Rightarrow (y, x) \in R^{-1}$

$\therefore R^{-1}$ is symmetric.

Let $(x, y), (y, z) \in R^{-1}$

$\Rightarrow (y, x), (z, y) \in R$ [by def^n of R^{-1}]

$\Rightarrow (z, y), (y, x) \in R$

$\Rightarrow (z, x) \in R$ [$\therefore R$ is transitive]

$\Rightarrow (x, z) \in R^{-1}$

$\therefore R^{-1}$ is transitive

Thus, R^{-1} is an equivalence relation.

Theorem 2: The intersection of two equivalence relations is also an equivalence relation.

Proof. Let A be a non-empty set.

Let R and S be two equivalence relations on A.

R and S are reflexive

$\therefore (x, x) \in R$ and $(x, x) \in S$, $\forall\, x \in A$

$\Rightarrow (x, x) \in R \cap S$, $\forall\, x \in A$

$\therefore R \cap S$ is reflexive.

Let $(x, y) \in R \cap S$

This $\Rightarrow (x, y) \in R$ and $(x, y) \in S$

$\Rightarrow (y, x) \in R$ and $(y, x) \in S$ [$\therefore$ R, S are symmetric]

$\Rightarrow (y, x) \in R \cap S$

$\therefore R \cap S$ is symmetric.

Let $(x, y) \in R \cap S$, $(y, z) \in R \cap S$

$\Rightarrow (x, y) \in R$ and $(x, y) \in S$, $(y, z) \in R$ and $(y, z) \in S$

$\Rightarrow (x, z) \in R$ and $(x, z) \in S$ [$\therefore$ R, S are transitive]

$\Rightarrow (x, z) \in R \cap S$

$\therefore R \cap S$ is transitive.

Thus, $R \cap S$ is an equivalence relation.

Remarks : The union of two equivalence relations is not necessarily an equivalence relation.

Let us consider A = {1, 2, 3}

Now, R = {(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)}

S = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}

are two equivalence relations on A.

$R \cup S$ = {(1, 1), (2, 2), (3, 3), (1, 3), (3, 1), (1, 2), (2, 1)}

$(2, 1) \in R \cup S$, $(1, 3) \in R \cup S$)

But $(2, 3) \notin R \cup S$

$\therefore$ $R \cup S$ is not transitive

$\therefore$ $R \cup S$ is not an equivalence relation.

1.5.1 Equivalence Class

Let us consider the set Z of integers

Let R = {(a, b) : a, b $\in$ Z, a–b is divisible by 3}

i.e. R = {(a, b) : a, b$\in$Z, a $\equiv$b (mod 3)}

Clearly R is reflexive, symmetric and transitive.

$\therefore$ R is an equivalence relation on Z.

Let [0] denote the set of integers congruent to 0 modulo 3. Then

[0] = {...., –9, –6, –3, 0, 3, 6, 9, ...}

Let [1] denote the set of integers congruent to 1 modulo 3. Then

[1] = {..., –8, –5, –2, 1, 4, 7, 10, ...}

Let [2] denote the set of integers congruent to 2 modulo 3. Then

[2] = {...., –7, – 4, –1, 2, 5, 8, 11, ...}

We see that

..... = [0] = [3] = [6] =.....

..... = [1] = [4] = [7] =.....

..... = [2] = [5] = [8] =.....

Each of [0], [1], [2] is called an equivalence class. [0] is the equivalence class of 0, [1] is the equivalence class of 1, [2] is the equivalence class of 2.

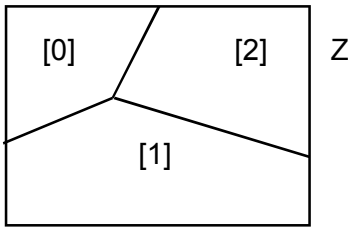
We see that there are 3 distinct equivalence classes, viz, [0], [1], [2].

Also we note that

[0] U [1] U [2] = Z

[0] $\cap$ [1] = ϕ , [1] $\cap$ [2] = ϕ [2]$\cap$[0] = ϕ

The set {[0], [1], [2]} is called a partition of Z.



Definition : Let A be a non-empty set and R be a relation on A.

For any a$\in$ A, the equivalence class [a] of a is defined by

$[a] = \{x \in A : xRa\}$

i.e. the equivalence class [a] of a is the collection of all those elements of A which are related to a under the relation R.



If A, B are two non-empty sets, then either $A \cap B \neq \phi$ or $A \cap B = \phi$

Note : $[a] \neq \phi$, $\because$ aRa

Definition : Let A be a non-empty set.

The set P of non-empty subsets of A is called a partition of A if

(i) A is the union of all members of P,

(ii) any two distinct members of P are disjoint.

Theorem 3 : Let A be a non-empty set and R be an equivalence relation of A. Let $a, b \in A$. Then [a] = [b] if and only if $(a, b) \in R$.

Proof. Let [a] = [b]

$\because$ R is reflexive, $\therefore$ aRa

$\therefore a \in [a]$

$\Rightarrow a \in [b]$ [$\because$ [a]=[b]]

$\Rightarrow$ aRb

$\Rightarrow (a, b) \in R$

Conversely : Let $(a, b) \in R$. $\therefore$ aRb

Let $x \in [a]$. $\therefore$ xRa

Now, xRa and aRb

$\therefore$ xRb [$\because$ R is transitive]

$\Rightarrow x \in [b]$

Thus, $x \in [a] \Rightarrow x \in [b]$

$\therefore [a] \subseteq [b]$ ... (1)

Again, let $y \in [b]$ $\therefore$ yRb

Now, yRb and bRa [$\because$ R is symmetric, aRb $\Rightarrow$ bRa]

$\therefore$ yRa [$\because$ R is transitive]

$\Rightarrow y \in [a]$

Thus, $y \in [b] \Rightarrow y \in [a]$

$\therefore [b] \subseteq [a]$ ... (2)

From (1) and (2), [a] = [b]

Theorem 4 : Two equivalence classes are either equal or disjoint.

Proof. Let A be a non-empty set. Let R be an equivalence relation on A.

Let a, b $\in$ A.

Then [a], [b] are either not disjoint or disjoint.

Let [a], [b] be not disjoint,

i.e. [a] $\cap$ [b] $\neq \phi$.

Let x $\in$ [a] $\cap$ [b]

$\Rightarrow$ x $\in$ [a] and x $\in$ [b]

$\Rightarrow$ xRa and xRb

$\Rightarrow$ aRx and xRb [$\because$ R is symmetric, xRa $\Rightarrow$ aRx]

$\Rightarrow$ aRb [$\because$ R is transitive]

Let y $\in$ [a]

$\Rightarrow$ yRa

Now yRa and aRb

$\therefore$ yRb [$\because$ R is transitive]

$\Rightarrow$ y $\in$ [b]

Thus, y $\in$ [a] $\Rightarrow$ y $\in$ [b]

$\therefore$ [a] $\subseteq$ [b]

Similarly, [b] $\subseteq$ [a]

$\therefore$ [a] = [b]

So, if [a], [b] are not disjoint, they are equal.

$\therefore$ [a], [b] are either equal or disjoint.



CHECK YOUR PROGRESS 3

Q 1: What is the name of the relation on a set . which is reflexive, symmetric and transitive?

Q 2: Is the relation "<" on the set of narural nimbers IN an equivalence relation? Justify your answer.

Q 3: Let Z be the set of integers.

Let R = {(a, b) : a, b $\in$ Z, a–b is divisible by 5}

Examine if R is an equivalence relation on Z ?

Q 4: Let A be the set of all triangles in a plane.

Let R = {(x, y) : x, y $\in$ A., x is similar to y}

Examine if R is an equivalence relation on A.

Q 5: let Z be the set of integers.

Let R = {(x, y) : x, y $\in$ Z, x–y is even}

Is R an equivalence relation on Z ?

Justify your answer.

1.6 CONCEPT OF FUNCTION

Let A and B be two non-empty sets, and f $\subseteq$ A xB such that

(i) (x, y) $\in$ f, $\forall$ x$\in$A and any y$\in$B

(ii) (x, y) $\in$ f and (x, y′)$\in$ f $\Rightarrow$ y=y′ .

In this case f is said to be a function (or a mapping) from the set A to the set B. Symbolically we write it as f : A $\rightarrow$ B.

Here A is called the domain and B is called the codomain of f.

Example 16 : Let A = {1, 2}, B = {7, 8, 9}

f = {(1, 8), (2, 7)} $\subseteq$ A x B.

Here, each element of A appears as the first component exactly in one of the ordered pairs of f.

$\therefore$f is a function from A to B.

Example 17 : Let A = {1, 2}, B = {7, 8, 9}

g = {(1, 7), (1, 9)} $\subseteq$ A x B

Here, two distinct ordered pairs have the same first component.

$\therefore$ f is not a function from A to B.

Example 18 : Let A = {1, 2, 3, 4}, B = {x, y, z, w}

Are the following relations from be functions A to B ?

(i) f_1 = {(1, x), (1, w), (2, x), (2, z), (4, w)}

(ii) f_2 = {(1, y), (2, z), (3, x), (4, w)}

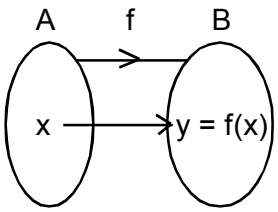
Solution : (i) No. Here two distinct ordered pairs (1, x), (1, w) have the same first component.

(ii) Yes.

Here, each element of A appears as the first component exactly in one of the ordered pairs of f_2.

Thus we see that.

Every function is a relation, but every relation is not a function.



We observe that if A and B are two non-empty sets and if each element of A is associated with a unique element of B, then the rule by which this association is made, is called a function from the set A to the set B. The rules are denoted by f, g etc. The sets A, B may be the same.

Let f be a function from A to B i.e. $f : A \to B$. The unique element y of B that is associated with x of A is called the image of x under f. Symbolically we write it as y = f(x). x is called the pre-image of y. The set of all the images under f is called the range of f.

Example 19 : Let IN be the set of natural members, and Z be the set of integer and $f : IN \to Z$, $f(x) = (-1)^x$; $x \in IN$

Clearly, domain of f = IN

codomain of f = Z

Now $f(1) = (-1)^1 = -1$, $f(2) = (-1)^2 = 1$, $f(3) = (-1)^3 = -1$ and so on.

$\therefore$ range of f = {–1, 1}

1.6.1 Identity Function

Let A be a non-empty set and $i : A \to A$, $i(x) = x$, $\forall x \in A$

i is called the identity function.

Note : In case of identity function, domain and codomain are the same.

1.6.2 Constant Function

Let A, B be two non-empty sets and $f : A \to B$ be a function such that $f(x) = k$, $\forall x \in A$

f is called a constant function.

Note : The range of a constant function is a singleton set.

1.7 TYPES OF FUNCTION

Let A, B be two non-empty sets and $f : A \to B$ be a function.

1. If there is at least one element in B which is not the image of any

element in A, then f is called an "into" function.

2. If each element in B is the image of at least one element in A, then f is called an "onto" function (or a surjective function or a surjection).

Note : In case of an onto function, range of f = codomain of f.

3. If different elements in A have different images in B, then f is called a one-one function (or an injective function or an injection).

4. If two (or more) different elements in A have the same image in B, then f is called a many-one function.

5. A function is said to be bijective if it is one-one (injective) and onto (surjective).

Note : Identity function is a bijective function.

How to prove that f is one-one?

Let $x_1, x_2, \in A$

If $f(x) = f(x_2) \Rightarrow x_1 = x_2$, then f is one-one.

or,

If $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, then f is one-one.

How to prove that f is onto?

Let $y \in B$ (codomain)

Let $y = f(x)$

We find x in terms of y

If $x \in A$, then f is onto; otherwise not.

Example 20: Let IN be the set of natural numbers,

Let $f : IN \to IN$, $f(x) = 3x+7$

Examine if f is a bijective function.

Solution : Let $x_1, x_2 \in IN$ (domain)

Now $f(x_1) = f(x_2)$

$\Rightarrow 3x_1+7=3x_2+7$

$\Rightarrow x_1 = x_2$

$\therefore$f is one-one (injection).

Let $y \in IN$ (codomain)

Let $y = f(x)$

$\Rightarrow y = 3x + 7$

$\Rightarrow x = \frac{y-7}{3}$

If y = 1, $x = \frac{1-7}{3} = -2 \notin N$

$\therefore$ f is not onto

$\therefore$ f is not a bijection.

Example 21: Let X be the set of real numbers excluding 1. Show that the function f : X→X, $f(x)=\frac{x+1}{x-1}$ is one-one and onto.

Solution: Let $x_1, x_2 \in$ X(domain)

Let $f(x_1) = f(x_2)$

$\Rightarrow \frac{x_1+1}{x_1-1} = \frac{x_2+1}{x_2-1}$

$\Rightarrow \frac{(x_1+1)+(x_1-1)}{(x_1+1)-(x_1-1)} = \frac{(x_2+1)+(x_2-1)}{(x_2-1)-(x_2-1)}$ (by componendo and dividendo)

$\Rightarrow \frac{2x_1}{2} = \frac{2x_2}{2}$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

Let y $\in$ X (codomain)

Let y = f(x)

$\Rightarrow y = \frac{x+1}{x-1}$

$\Rightarrow$ y(x–1) = x+1

$\Rightarrow$ yx–y = x+1

$\Rightarrow$ yx – x = y+1

$\Rightarrow$ x(y–1) = y+1

$\Rightarrow x = \frac{y+1}{y-1} \in$ X (domain)

$\therefore$ f is onto.

Example 22: Let A = {1, 2, 3,}. Write down all the bijective function from A to itself.

Solution :

i : A→ A, i(1) = 1, i(2) = 2, i(3) = 3

f_1 : A→ A, $f_1(1) = 1$, $f_1(2) = 3$, $f_1(3) = 2$

f_2 : A→ A, $f_2(1) = 2$, $f_2(2) = 1$, $f_2(3) = 3$

$f_3 : A \to A$, $f_3(1) = 3$, $f_3(2) = 2$, $f_3(3) = 1$

$f_4 : A \to A$, $f_4(1) = 2$, $f_4(2) = 3$, $f_4(3) = 1$

$f_5 : A \to A$, $f_5(1) = 3$, $f_5(2) = 1$, $f_5(3) = 2$

Note. There are 6 =3! bijective functions from A = {1, 2, 3} to itself.

If A has n elements, there are n! bijective functions from A to itself.

Theorem 5 : Let A be a finite set, and $f : A \to A$ be onto. Then f is one-one.

Proof. Let A be a finite set having n elements.

Let $A = \{a_1, a_2, ..., a_n\}$, where a_i' s are distinct.

Let $f : A \to A$ be onto.

Now, range of f

$= \{f(a_1), f(a_2), ..., f(a_n)\}$

$\therefore$ f is onto, range of f = codomain of f = A.

$\therefore A = \{f(a_1), f(a_2), ..., f(a_n)\}$

$\therefore$ A has n elements,

$\therefore f(a_1), f(a_2), ..., f(a_n)$ are distinct.

Thus, distinct elements in A (domain) have distinct images in A (codomain).

$\therefore$ f is one-one.

Note : The result does not hold good if A is an infinite set.

Let IN be the set of natural numbers.

Let $f : IN \to IN$, $f(x) = 1$, if $x = 1$ and $x = 2$,

$= x-1$, if $x \geq 3$

Clearly, f is onto, but not one-one.

Theorem 6 : Let A be a finite set, and $f:A \to A$ be one-one. Then f is onto.

Proof : Let A be a finite set having n elements.

Let $A = \{a_1, a_2, ..., a_n\}$, where a_i's are distinct.

Let $f : A \to A$ be one-one.

$\therefore f(a_1), f(a_2), ..., f(a_n)$ are n distinct elements of A (codomain).

Let $u \in A$ (codomain).

Let $u = f(a_i)$, $1 \leq i \leq n$

$\therefore$ there exists $a_i \in A$ (domain) such that $f(a_i) = u$.

$\therefore$ f is onto.

Note : The result does not hold good if A is an infinite set.

Let IN be the set of natural numbers.

Let f : IN $\rightarrow$IN, f(x) = 5x

Clearly, f is one-one, but not onto.



CHECK YOUR PROGRESS 4

Q 1: Let IN be the set of natural numbers.

Let f : IN $\rightarrow$IN, $f(x) = x^2+1$

Examine if f is (i) one-one, (ii) onto.

Q 2: Let IR be the set of real numbers and

f : IR$\rightarrow$IR be defined by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

Examine if f is (i) one-one, (ii) onto

Q 3: Let f : IR$\rightarrow$IR be defined by f (x) =|x|

Examine if f is (i) one-one, (ii) onto

$$\left[f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases} \right]$$

Q 4: If A = {1, 2}, write down all the bijective functions from A to itself.

Q 5: Write down the condition such that a constant function is onto.

Q 6: Write down the condition such that a constant function is one-one.



1.8 LET US SUM UP

- If A, B are two non-empty sets, a subset of A x B in said to be a relation from A to B.
- If A, B are two finite sets and n(A) = x, n(B) = y, the number of relations from A to B is 2^{xy}.
- If A is a non-empty set, $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A.

- If A, B are two non-empty sets and R is a relation from A to B, the inverse relation R^{-1} is defined by
  $R^{-1} = \{(b, a) : (a, b) \in R\}$, which is a relation from B to A.
- A relation R on a non-empty set A is called.
  (i) reflexive if $(a, a) \in R$, for all $a \in A$;
  (ii) symmetric if whenever $(a, b) \in R$, $(b, a) \in R$;
  (iii) anti-symmetric if $(a, b) \in R$, $(b, a) \in R \Rightarrow a = b$;
  (iv) transitive if whenever $(a, b), (b, c) \in R$, then $(a, c) \in R$.
- A relation R on a non-empty set A is called an equivalence relation if it is reflexive, symmetric and transitive.
- The inverse of an equivalence relation is also an equivalence relation.
- The intersection of two equivalence relations is also an equivalence relation.
- If R is a relation on a non-empty set A, then for any $a \in A$; the equivalence class [a] of a is the collection of all those elements of A which are related to a under the relation R.
- Two equivalence classes are either equal or disjoint.
- If A and B are two non-empty sets and if each element of A is associated with a unique element of B, then the rule by which this association is made, is called a function from A to B.
- Every function is a relation, but every relation is not a function.
- If different elements in domain have different images in codomain, then the function is one-one (injective).
- If each element in codomain is the image of at least one element in domain then the function is onto (surjective).
- A function is bijective if it is injective and surjective.
- If A is a finite set and $f : A \to A$ is onto, then f is one-one.
- If A is a finite set and $f : A \to A$ is one-one, then f is onto.



1.9 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1: 2^{12}

Ans to Q No 2: 2^{25}

Ans to Q No 3: (7, 4), (72, 27) belong to R.

Ans to Q No 4: Yes. The sum of two even integers cannot be odd.

Ans to Q No 5: $I_A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$

Ans to Q No 6: $R^{-1} = \{(7, 4), (8, 5), (7, 6), (8, 6), (9, 6)\}$

CHECK YOUR PROGRESS – 2

Ans to Q No 1: $R = \{(1, 1), (2, 2), (3, 3)\}$

CHECK YOUR PROGRESS – 3

Ans to Q No 1: Equivalence relation.

Ans to Q No 2: No. Not reflexive and symmetric.

Ans to Q No 3: Equivalence relation.

Ans to Q No 4: Equivalence relation.

Ans to Q No 5: Yes.

CHECK YOUR PROGRESS – 4

Ans to Q No 1: One-one, but not onto.

Ans to Q No 2: Neither one-one nor onto.

Ans to Q No 3: Neither one-one nor onto

Ans to Q No 4: $i : A \to A$, $i(1) = 1$, $i(2) = 2$

$f : A \to A$, $f(1) = 2$, $f(2) = 1$

Ans to Q No 5: Codomain is a singleton set.

Ans to Q No 6: Domain is a singleton set.



1.10 FURTHER READINGS

1) Lanski, Charles : *Concepts in Abstract Algebra*, (First Indian edition, 2010) American Mathematical Society.

2) Vasistha, A. R. : *Modern Algebra (Abstract algebra),* Krishna Prakashan Media (p) Ltd. Meerut-250001 (U. P).

3) Vinberg, E. B. : *A course in Algebra* (First Indian edition, 2009), American Mathematical Society.

4) Vatsa, B. S. and Vatsa, Suchi : *Modern Algebra*, New Age International (p) Ltd., New Delhi-110002.

5) Sen, M. K, Ghosh, Shamik and Mukhopadhyay, Parthasarthi : *Topics in Abstract Algebra,* Universities Press (India) Pvt. Ltd., Hyderabad, 500029.



1.11 MODEL QUESTION

Q 1: Let A = {1, 2, 3}, B = {3, 4, 5}.
How many relations are there from A to B ? Write down any four relations from A to B.

Q 2: Let A = {3, 4, 5}, B = {5, 6, 7}
Which of the following relations are functions from A to B ? If it is a function, determine whether it is one-one and whether it is onto ?
(i) {(3, 5), (4, 5), (5, 7)}
(ii) {(3, 7), (4, 5), (5, 6)}
(iii) {(3, 6), (4, 6), (5, 6)}
(iv) {(3, 6), (4, 7), (5, 5)}

Q 3: Let Q be the set of rational numbers and f : Q→Q be a function defined f(x) = 4x+5. Examine if f is bijective

Q 4: Let A be the set of all triangles in a plane.
Let R = {(x, y) : x, y ∈ A and x is congruent to y}
Examine if R is an equivalence relation on A.

Q 5: Let IN be the set of natural numbers. A relation R is defined on IN x IN by (a, b) R (c, d) if and only if a+d = b+c.
Show that R is an equivalence relation on IN x IN.

Q 6: Let A = {1, 2, 3}

R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)}

Show that R is reflexive but neither symmetric nor transitive.

Q 7: Let C be the set of complex numbers and IR be the set of real numbers. Let $f : C \to IR$, $f(z) = |z|$, $z \in C$.

Examine if f is (1) one-one, (ii) onto.

Q 8: Let A, B be two non-empty sets.

Let $f : A \times B \to B \times A$, $f(a, b) = (b, a)$; $(a, b) \in A \times B$

Show that f is bijective.

Q 9: Let $f : IN \times IN$, $f(a, b) = 3^a 4^b$, $(a, b) \in IN \times IN$

Examine if f is (i) one-one, (ii) onto.

Q 10: Let $f : Z \times Z \to Z$, $f(a, b) = ab$, $(a, b) \in Z \times Z$

Examine if f is (i) injective, (ii) surjective.

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