UNIT 2: BINARY OPERATION

UNIT STRUCTURE

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2.1 LEARNING OBJECTIVES

After going through this unit, you will be able to know :

- binary operation on a set

2.2 INTRODUCTION

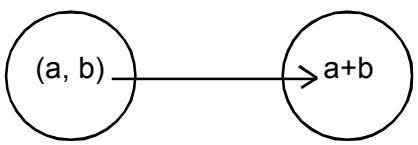
We have studied relations and functions. In this unit, we will introduce the notion of binary operations and its types.

2.3 BINARY OPERATION

Let us consider the set IN of natural numbers.

For every ordered pair $(a, b) \in IN \times IN$, we get a unique $a+b \in IN$. So we can define a funtion $f : IN \times IN \to IN$, $f(a, b) = a+b$.

IN x IN IN

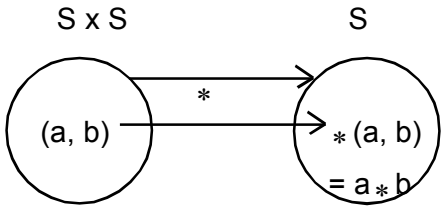


This function is known as a binary operation on IN. Thus, addition is a binary operation on IN.

Similarly, multiplication is a binary operation on IN; for every ordered pair (a, b) $\in$ IN x IN. We get a unique ab $\in$ IN. But subtraction is not a binary operation on IN; for every ordered pair (a, b) $\in$ IN x IN, we do not get a–b $\in$ IN. We have (2, 3) $\in$ IN x IN, but 2–3 = -1 $\notin$ IN.

Definition: A binary operation $*$ on a non-empty set S is a function from S x S to S, i.e.

$* : S \times S \to S$



Here the image of (a, b) $\in$ S x S under the function $*$, i.e $*$ (a, b) is denoted by a $*$ b.

Thus, $*$ is a binary operation on S if a $*$ b $\in$ S, for all a, b $\in$ S.

Definition: Let $*$ be a binary operation on a non-empty set S and H be a subset of S. H is said to be closed under $*$ if for all a, b $\in$ H, we have a $*$ b $\in$ H.In particular,if a $*$ b $\in$ A $\forall$ a, b $\in$ A,then A is said to be closed under the operation $*$.

Number of Binary operations on a finite set

Let A, B be two finite sets, and n (A) = m and n (B) = n.

Then the number of functions from A to B is n^m. [The first element of A can be associated in n ways, the second element in n ways and lastly, the m th element can be associated in n ways.

$\therefore$ the number of functions

= n. n ... ... n (m factors) = n^m]

We know that n (A x A) = mm = m^2

$\therefore$ number of binary operations on A

= number of functions from A x A to A

= m^{m^2}

Examples

1. Addition is a binary operation on IN.
2. Subtraction is not a binary operation on IN.

3. Multiplication is a binary operation on IN.
4. Addition is a binary operation on Z.
5. Addition is not a binary operation on the set S = Z – {0} of non–zero integers, for $3 \in S$, $-3 \in S$, but $3+(-3) = 0 \notin S$.
6. Subtraction is a binary operation on Z.
7. Let S be the collection of all real matrices. Matrix addition is not a binary operation on S. We cannot find the sum of a 2 x 3 matrix and a 3 x 4 matrix.
8. Let M_{mxn} be the collection of all m x n real matrices. Matrix addition is a binary operation on M_{mxn}.
9. Let M_{nxn} be the collection of all nxn real matrices. Matrix multiplication is a binary operation on M_{nxn}.
10. Addition (also multiplication) is a binary operation on the set C of complex numbers.

Example 1: Is $*$ defined by $a * b = a-b+10$ a binary operation on Z ?

Solution: Let $a, b \in Z$

Clearly $a-b+10 \in Z, \forall\ a, b \in Z$

i.e., $a * b \in Z, \forall\ a, b \in Z$

$\therefore *$ is a binary operation on Z

Example 2: Is $*$ defined by $a * b = \frac{ab}{5}$ a binary operation on Q ?

Solution: Let $a, b \in Q$

$\therefore ab \in Q$ (the product of two rational numbers is a rational number)

$\therefore \frac{ab}{5} \in Q$

i.e. $a * b \in Q, \forall\ a, b \in Q$

$\therefore *$ is a binary operation on Q

Example 3: Let P be the set of all subsets of a non-empty set S. Show that union and intersection are two binary operations on P.

Solution: Let $A, B \in P$ i.e. $A \subseteq S, B \subseteq S$

$\therefore A \cup B \subseteq S$ i.e $A \cup B \in P$

$\therefore \cup$ is a binary operation on P.

Similarly, $A \cap B \in P, \forall\ A, B \in P$

$\therefore \cap$ is a binary operation on P.

Operation (or composition) Table

Let S be a finite set and the number of elements of S be small. We can construct a table, known as the operation (or composition) table, as follows.

Let $A = \{a_1, a_2, a_3\}$.

We write the elements a_1, a_2, a_3 in a horizontal row as well as in a vertical column and write the binary operation $*$ in the top left corner. Now we enter each of the 9($= 3^2$) entries a_i x a_j (i = 1, 2, 3; j = 1, 2, 3) in the (i, j) th cell as shown in Table 2.1. Each of the 9 entries belongs to A, since $*$ is a binary operation on A.

$*$	a_1	a_2	a_3
a_1	$a_1 * a_1$	$a_1 * a_2$	$a_1 * a_3$
a_2	$a_2 * a_1$	$a_2 * a_2$	$a_2 * a_3$
a_3	$a_3 * a_1$	$a_3 * a_2$	$a_3 * a_3$

Table 2.1

Example 4: Is multiplication a binary operation on {–1, 0, 1} ?

Solution: Each of the 9 entries belongs to A

$\therefore$ multiplication is a binary operation on {–1, 0, 1}.

x	–1	0	1
–1	1	0	–1
0	0	0	0
1	–1	0	1

Table 2.2

Example 5: Is addition a binary operation on {–1, 0, 1} ?

Solution: $-2 \notin \{-1, 0, 1\}$

$2 \notin \{-1, 0, 1\}$

$\therefore$ addition is not a binary operation on {–1, 0, 1}.

+	–1	0	1
–1	–2	–1	0
0	–1	0	1
1	0	1	2

Table 2.3

Example 6: Is $*$ defined by $a * b=|a-b|$ a binary operation on {0, 1, 2} ?

Solution: $0 * 0 = |0-0|=0$,

$0 * 1 = |0-1|=1$,

$0 * 2 = |0-2|=2$;

$1 * 0 = |1-0|=1$, $1 * 1 = |1-1|=0$,

$1 * 2 = |1-2|=1$;

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

Table 2.4

$2 * 0 = |2-0|=2$, $2 * 1 = |2-1|=1$, $2 * 2 = |2-2|=0$,

From the table we see that $*$ is a binary operation on {0, 1, 2}.

2.3.1 Commutative Binary Operation

A binary operation $*$ on a non-empty set S is commutative if $a * b = b * a$, $\forall a, b \in S$.

Note : For a finite set, a binary operation is commutative if and only if the entries in the operation table are symmetric with respect to the principal diagonal.

For example, the entries in Table 2.3 are symmetric with respect to the principal diagonal 0, 0, 0. Therefore, $*$, defined in Example 6, is commutative on {0, 1, 2}.

Example 7: Is the binary operation $*$ defined by $a * b = a-b+5$ commutative on Z ?

Solution: We have $a * b = a-b+5$

$b * a = b-a+5$, $\forall a, b \in Z$

Clearly $a * b \neq b * a$

$\therefore *$ is not commutative on Z.

Example 8: Let Q be the set of rational numbers. Let $*$ be a binary operation on Q defined by $a * b = ab^2$, $\forall a, b \in Q$. Is $*$ commutative?

Solution: $a * b = ab^2$, $b * a = ba^2$, $\forall a, b \in Q$

$\therefore a * b \neq b * a$

$\therefore *$ is not commutative.

2.3.2 Associative Binary Operation

A binary operation $*$ on a non-empty set S is associative if $(a * b) * c = a * (b * c)$, $\forall$ a, b, c $\in$ S.

Example 9: Is the binary operation $*$ defined by $a * b = a–b+5$ associative on Z ?

Solution: Let a, b, c $\in$ Z.

$(a * b) * c = (a–b+5) * c$

$= (a–b+5) – c+5$

$= a–b–c+10$

$a * (b * c) = a * (b–c+5)$

$= a – (b–c+5)+ 5$

$= a – b + c – 5 + 5$

$= a – b + c$

$\therefore (a * b) * c \neq a * (b * c)$

$\therefore *$ is not associative.

Example 10: Let X be the set of rational numbers excluding 1. We define $*$ on X as follows.

$a * b = a + b – ab$, $\forall$ a, b $\in$ X.

Show that $*$ is a binary operation on X. Examine if $*$ is (i) commutative, (ii) associative.

Solution: Let a, b $\in$ X

$\therefore$ a, b are two rational numbers and $a \neq 1$, $b \neq 1$.

Now a + b – ab is a rational number and $a+b–ab \neq 1$, for if $a+b–ab=1$, then

$a+b–ab–1=0$

i.e. $(a–1)–b(a–1)=0$

i.e. $(a–1)(1–b)=0$

i.e. $a–1=0$ or $1–b=0$

i.e. a = 1 or b = 1, which is a contradiction.

$\therefore a+b–ab \in X$

i.e $a * b \in X$, $\forall$ a, b $\in$ X

$\therefore *$ is a binary operation on X

Second Part

$a * b = a+b-ab$

$= b+a-ba$

$= b * a$

$\therefore *$ is commutative on X.

Again $(a * b) * c$

$= (a+b-ab) * c$

$= (a+b-ab) + c - (a+b-ab)c$

$= a+b-ab+c - ac-bc+abc$

$= a+b+c-ab-bc-ca+abc$

$a * (b * c) = a * (b+c-bc)$

$= a+(b+c-bc) - a(b+c-bc)$

$= a+b+c-ab-bc-ca+abc$

$\therefore (a * b) * c = a * (b * c)$

$\therefore *$ is associative.

2.3.3 Identity Element

Let S be a non-empty set equipped with a binary operation $*$. An element $e \in S$ is called an identity element for $*$ if

$$a * e = a = e * a, \forall a \in S.$$

Note: Identity element for a binary operation if it exists, is unique.

Examples

1. Multiplication is a binary operation on IN. There exists $1 \in$ IN such that

$$a\ 1 = a = 1\ a, \forall\ a \in \text{IN}$$

$\therefore$ 1 is the identity for multiplication in IN.

2. Addition is a binary operation on Z.

There exists $0 \in Z$ such that

$$a + 0 = a = 0+a, \forall a \in Z$$

$\therefore$ 0 is the identity for addition in Z.We say that 0 is the additive identity in Z.

3. Addition is a binary operation on IN. But there does not exist

identity element for addition in IN.

4. Matrix addition is a binary operation on M_{mxn}, the set of all m x n real matrices. The m x n null matrix 0 is the identity for matrix addition in M_{mxn}.

2.3.4 Inverse of an Element

Let S be a non-empty set equipped with a binary operation $*$, and with identity element e. An element $b \in S$ is called inverse of a under $*$ if

$$a * b = e = b * a$$

Note : 1. Inverse of an element under a binary operation, if it exists, is unique.

2. Generally inverse of a is denoted by a^{-1}. In case of real numbers, inverse of a under addition (i. e. additive inverse) is denoted by –a, whereas inverse of a ($a \neq 0$) under multiplication (i.e. multiplicative inverse) is denoted by $\frac{1}{a}$.

Examples

1. Multiplication is a binary operation on IN and $1 \in$ IN is the identity for multiplication. But no element of IN, except 1, possesses inverse under multiplication in IN.
2. Addition is a binary operation on Z and $0 \in Z$ is the identity for addition. For every $a \in Z$, there exists $-a \in Z$ such that
   $a+(-a) = 0=(-a) + a$.
   $\therefore$ –a is the inverse of a under addition.



CHECK YOUR PROGRESS

Q 1: Let A be a finite set having 4 elements. What is the number of binary operations on A?

Q 2: Is $*$ defined by $a * b = \frac{a}{b}$, a binary operation on IR ?

Q 3: Is multiplication a binary operation on {–1, 1, i, –i} ? where $i = \sqrt{-1}$.

Q 4: Let Q be the set of rational numbers. Show that $*$ defined by

$a * b = a + ab$, is a binary operation on Q.

Examine if $*$ is (i) commutative, (ii) associative.



2.4 LET US SUM UP

- A binary operation $*$ on a non-empty set S is a function from S x S to S.
- If S is a finite set having m elements, the number of binary operations on S is m^{m^2}.
- A binary operation $*$ on a non-empty set S is commutative if $a * b = b * a$, $\forall\ a, b \in S$.
- A binary operation $*$ on a non-empty set S is associative if $(a * b) * c = a * (b * c)$, $\forall\ a, b, c \in S$
- An element e in a non-empty set S equipped with a binary operation $*$ is called an identity element for x if $a * e = a = e * a$, $\forall\ a \in S$.
- An element b in a non-empty set S equipped with a binary operation $*$ is called an inverse element of $a \in S$ if $a * b = e = b * a$.



2.5 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS

Ans to Q No 1: $4^{4^2} = 4^{16}$

Ans to Q No 2: No.

Ans to Q No 3: Yes

Ans to Q No 4: Not commutative, not associative.



2.6 FURTHER READINGS

1) Lanski, Charles : *Concepts in Abstract Algebra*, (First Indian edition, 2010) American Mathematical Society.
2) Vasistha, A. R. : *Modern Algebra (Abstract algebra),* Krishna Prakashan Media (p) Ltd. Meerut-250001 (U. P).
3) Vinberg, E. B. : *A course in Algebra* (First Indian edition, 2009), American Mathematical Society).
4) Vatsa, B. S. and Vatsa, Suchi : *Modern Algebra*, New Age International (p) Ltd., New Delhi-110002.
5) Sen, M. K, Ghosh, Shamik and Mukhopadhyay, Parthasarthi : *Topics in Abstract Algebra,* Universities Press (India) Pvt. Ltd., Hyderabad, 500029.



2.7 MODEL QUESTION

Q 1: Let IR be the set of real numbers. Examine if the binary operation $*$ defined by $a * b = a+5b$; $a, b \in IR$ is (i) commutative, (ii) associative.

Q 2: Let S be a set having 3 elements. How many binary operations can be defined on S?

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