UNIT 6: COSETS

UNIT STRUCTURE

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6.1 LEARNING OBJECTIVES

After going through this unit, you will be able to know :

- cosets of a subgroup
- Lagrange's subgroup order theorem

6.2 INTRODUCTION

We have already discussed group ,subgroup and their properties in unit 3 and unit 4.In this unit,we will discuss cosets of a subgroup of a group.

We will also discuss Lagrange's subgroup order theorem in this unit.

6.3 COSET

Let G be a group and $H \leq G$

Let $a \in G$

The set Ha = {ha : $h \in H$} is called a right coset of H in G.

Similarly, the set aH = {ah : $h \in H$} is called a left coset of H in G.

Note : (1) Right (or left) coset cannot be empty ,we have $e \in H$

$\therefore$ ea = a $\in$ H a $\quad\quad$ $\therefore$ Ha $\neq \phi$

Similarly ae = a $\in$ aH $\therefore$ aH $\neq \phi$

(2) $H \leq G$, but Ha or aH is not necessarily a subgroup of G.

(3) If H is a subgroup of the additive group G and $a \in G$, the right coset of H in G is given by $H+a = \{h+a : h \in H\}$

Similarly, the left coset of H in G is given by $a+H=\{a+h : h \in H\}$

(4) If G is abelian,then every left coset is a right coset.

Example : Let us consider the additive group of integers Z.

We know that $2Z = \{0, \pm 2, \pm 4, \pm 6,.....\} \leq Z$

We have

$0+2Z = \{0, \pm 2, \pm 4, \pm 6,.....\} = 2Z$

$1+2Z = \{\pm 1, \pm 3, \pm 5, \pm 7, ...\}$

$2+2Z = \{0, \pm 2, \pm 4, \pm 6, ...\}$

$3+2Z = \{\pm 1, \pm 3, \pm 5, \pm 7, ...\}$ and so on.

Thus, we see that

$2Z = 2+2Z = 4+2Z = ...$

$1+2Z = 3+2Z = 5+2Z = ...$

$\therefore$ 2Z has two distinct left cosets viz. 2Z and 1+2Z such that

$2Z \cup (1+2Z) = Z$

$2Z \cap (1+2Z) = \phi$

Similarly, 2Z has two distinct right cosets 2Z and 2Z + 1 such that

$2Z \cup (2Z+1) = Z$

$2Z \cap (2Z+1) = \phi$

Theorem 1 : Let G be a group and $H \leq G$. Then

(i) $aH = H \Leftrightarrow a \in H$; $Ha = H \Leftrightarrow a \in H$

(ii) $aH = bH \Leftrightarrow a^{-1}b \in H$; $Ha = Hb \Leftrightarrow ab^{-1} \in H$, where $a, b \in G$

Proof : (i) Let aH = H

We have $ae \in aH$, e is the identity of G

$\Rightarrow ae \in H$ $[\because aH = H]$

$\Rightarrow a \in H$

Conversely, let $a \in H$

Let $x \in aH$

This $\Rightarrow x = ah$, for some $h \in H$

Now $a \in H, h \in H$ $\therefore ah \in H \Rightarrow x \in H$

Thus, $x \in aH$ $\Rightarrow x \in H$

$\therefore aH \subseteq H$ ...(1)

Again, let $y \in H$

$\therefore a \in H, \therefore a^{-1} \in H$

$\therefore a^{-1}y \in H$

$\Rightarrow a^{-1}y = h_1$, for some $h_1 \in H$

$\Rightarrow a(a^{-1}y) = ah_1$

$\Rightarrow (aa^{-1})y = ah_1$

$\Rightarrow ey = ah_1$

$\Rightarrow y = ah_1 \in aH$

Thus, $y \in H \Rightarrow y \in aH$

$\therefore H \subseteq aH$ ...(2)

From (1) and (2), aH = H

Similarly $Ha = H \Leftrightarrow a \in H$.

(ii) aH = bH

$\Leftrightarrow a^{-1}(aH) = a^{-1}(bH)$

$\Leftrightarrow (a^{-1}a)H = (a^{-1}b)H$

$\Leftrightarrow eH = (a^{-1}b)H$

$\Leftrightarrow H = (a^{-1}b)H$

$\Leftrightarrow a^{-1}b \in H$ (using (1)

Similarly, $Ha = Hb \Leftrightarrow ab^{-1} \in H$.

Theorem 2 : (i) Any two left (or right) cosets of a subgroup are either disjoint or identical.

(ii) A group G is union of all its right (or left) cosets of a subgroup H of G.

Proof :(i) Let G be a group and $H \leq G$.

Let $a, b \in G$.

Then aH, bH are two left cosets of H in G.

Clearly aH, bH are either disjoint or not disjoint.

Let aH, bH be not disjoint, i.e, $aH \cap bH \neq \phi$.

Let $x \in aH \cap bH$

$\Rightarrow x \in aH$ and $x \in bH$

Now $x \in aH \Rightarrow x = ah_1$ for some $h_1 \in H$

$x \in bH \Rightarrow x = bh_2$ for some $h_2 \in H$

$\therefore ah_1 = bh_2$

$\Rightarrow (ah_1)h_1^{-1} = (bh_2)h_1^{-1}$

$\Rightarrow a\, h_1h_1^{-1} = b\, h_2h_1^{-1}$

$\Rightarrow ae = bh_2h_1^{-1}$, where e is the identity of G

$\Rightarrow a = bh_2h_1^{-1}$

Similarly $b = ah_1h_2^{-1}$

Let $p \in aH \Rightarrow p = ah_3$ for some $h_3 \in H$

$= bh_2h_1^{-1}h_3$

$\in bH, \quad \because h_2h_1^{-1}h_3 \in H$

Thus, $p \in aH \Rightarrow p \in bH \quad \therefore aH \subseteq bH \quad (1)$

Again, let $q \in bH \Rightarrow q = bh_4$ for some $h_4 \in H$

$\Rightarrow q = ah_1h_2^{-1}h_4$

$\in aH, \quad \because h_1h_2^{-1}h_4 \in H$

Thus $q \in bH \Rightarrow q \in aH \quad \therefore bH \subseteq aH \quad (2)$

From (1) and (2), aH = bH

Thus, aH, bH are either disjoint or identical.

(ii) Let $G = \{a_1, a_2, \cdots, a_n, \cdots\}$ be a group and let $H = \{h_1, h_2, \cdots, h_m, \cdots\}$ be a subgroup of G.Let $B = \cup(a_kH), a_k \in G$

Clearly $B \subseteq G$ ... (1)

Also $x \in G \Rightarrow x \in xH \subseteq B$

$\Rightarrow x \in B$

$\therefore G \subseteq B$ ... (2)

From (1) and (2),we get $G = B$

or $G = \underset{a_k \in G}{\cup}(a_kH)$

6.4 LAGRANGE'S SUBGROUP ORDER THEOREM

Theorem 3 : (Lagrange's subgroup Order Theorem) The order of a subgroup of a finite group divides the order of the group.

Proof : Let G be a finite group, and let O(G) = n

and H≤ G.

Let e be the identity of G.

Case 1 : Let H = {e},

There is nothing to prove. O(H) =1, and 1 divides n.

Case 2 : Let H = G. Hence O(H) = O(G).

There is nothing to prove.

Case 3 : Let $H \neq \{e\}$, $H \neq G$.

Let O(H) = m, m < n.

Let $H = \{h_1, h_2, h_3, \ldots h_m\}$, where the h_i's are distinct.

Let $a \in G$, $a \notin H$.

Let us consider the left coset.

$aH = \{ah_1, ah_2, ah_3, \ldots, ah_m\}$

We claim that all the elements in aH are distinct, for if $ah_i = ah_j$, $1 \leq i \leq m$, $1 \leq j \leq m$, $i \neq j$ then $h_i = h_j$, (by left cancellation law), which is a contradiction.

Again, no element of H is equal to any element of aH, for if $h_i = ah_j$, then $h_i h_j^{-1} = a$ i.e. $a = h_i h_j^{-1} \in H$

i.e. $a \in H$, which is a contradiction.

Thus, we have listed m+m = 2m elements of G. If this exhausts all the elements of G, then 2m = n. $\therefore$ m divides n.

If not, let $b \in G$, $b \notin H$, $b \notin aH$

Let us consider $bH = \{bh_1, bh_2, \ldots, bh_m\}$

Clearly, all the elements in bH are distinct. No element of H is equal to any element of bH.

Again, no element of aH is equal to any element of bH, for if $ah_i = bh_j$, then $ah_i h_j^{-1} = bh_j h_j^{-1}$ i.e $ah_i h_j^{-1} = b$ i.e $b = ah_i h_j^{-1} \in aH$, ($\because h_i h_j^{-1} \in H$) i.e. $b \in aH$, which is a contradiction.

Thus, we have listed m+m+m = 3m elements of G. If this exhausts all the elements of G, then 3m = n. $\therefore$ m divides n.

If not, we proceed as above. Since G is finite, we must stop somewhere, say, after k times. $\therefore$ mk = n $\Rightarrow$ m divides n.

Note : Joseph Louis Lagrange (1736 – 1813) was a famous mathematician, ''born at Turin of once prosperous parents with French and Italian backgrounds".

Remark : The converse of Lagrange's theorem is not true.



CHECK YOUR PROGRESS

Q 1: Find all cosets of the subgroup 4Z of 2Z.

Q 2: Find all cosets of the subgroup 6Z of 2Z.



6.5 LET US SUM UP

- If H is a subgroup of a group G, the set Ha = {ha : h $\in$ H}, a$\in$ G, is called a right coset of H in G. Similarly, a left coset is defined.
- Any two left (or right) cosets of a subgroup are either disjoint or identical.
- Lagrange's subgroup Order Theorem state that the order of a subgroup of a finite group divides the order of the group.



6.6 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS

Ans to Q No 1: Since 2Z is abelian,the left cosets and the right cosets are the same.The left coset containing the integer 0 is $0+4Z=\{\cdots,-8,-4,0,4,8,\cdots\}$ and left coset containing the integer 2 is $2+4Z=\{\cdots,-6,-2,2,6,10,\cdots\}$.

Ans to Q No 2: Left as an exercise



6.7 FURTHER READINGS

1) Lanski, Charles : *Concepts in Abstract Algebra*, (First Indian edition, 2010) American Mathematical Society.

2) Vasistha, A. R. : *Modern Algebra (Abstract algebra),* Krishna Prakashan

Media (p) Ltd. Meerut-250001 (U. P).

3) Vinberg, E. B. : *A course in Algebra* (First Indian edition, 2009), American Mathematical Society).

4) Vatsa, B. S. and Vatsa, Suchi : *Modern Algebra*, New Age International (p) Ltd., New Delhi-110002.

5) Sen, M. K, Ghosh, Shamik and Mukhopadhyay, Parthasarthi : *Topics in Abstract Algebra,* Universities Press (India) Pvt. Ltd., Hyderabad, 500029.



6.8 MODEL QUESTION

Q 1: Let G be a group and H be a subgroup of G. Let a, b $\in$ G. Show that the two right cosets Ha, Hb are equal if and only if the two left cosets $a^{-1}H$, $b^{-1}H$ are equal.

Q 2 : Find all cosets of the subgroup 4Z of Z.

Q 3: Index of a subgroup. Let G be a group and H be a subgroup of G. The number of distinct left (or right) cosets of H in G is called the index of H in G. Generally it is denoted by [G : H]

If G is a finite group, show that $[G : H] = \frac{o(G)}{o(H)}$.

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