UNIT 7: NORMAL SUBGROUP& QUOTIENT GROUP

UNIT STRUCTURE

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7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to know :

- normal subgroups
- quotient groups.

7.2 INTRODUCTION

We have already discussed group ,subgroup and their properties in unit 3 and unit 4.In this unit,we will discuss normal subgroup of a group.In this unit,we will also discuss quotient group.

7.3 NORMAL SUBGROUP

Definition : A subgroup H of a group G is said to be a normal subgroup of G if aH = Ha, for all a $\in$ G, i.e., if its left and right cosets coincide. Symbolically We write H Δ G.

Note : A group G has at least two normal subgroups, viz., G and {e}. A group, having no normal subgroups other than G and {e}, is called a simple group.

Example : Let G ={ –1, 1, –i, i}

We know that G is a group under multiplication.

Let H = {–1, 1}, and $H \leq G$

Now 1H = {–1, 1} = H1

(–1)H = {1, –1} = H (–1)

iH = {–i, i} = Hi

(–i)H = {i, –i} = H (–i)

Thus, aH = Ha, for all $a \in G$

$\therefore H \Delta G$.

Note : In the above example, H has two distinct left (or right) cosets, viz H, iH. We note that iH is not a subgroup of G.

Theorem 1 : Every subgroup of an Abelian group is normal.

Proof : Let G be an Abelian group.

Let $H \leq G$.

Let $a \in G$

Now aH = {ah : $h \in H$}

= {ha : $h \in H$} [$\because$ G ia Abelian]

= Ha

$\therefore H \Delta G$.

Theorem 2 : A subgroup H of a group G is a normal subgroup of G if and only if $ghg^{-1} \in H$, for all $g \in G$, $h \in H$

Proof : Let $H \Delta G$.

$\therefore$ aH = Ha, for all $a \in G$

Let $g \in G$, $h \in H$

$\therefore gh \in gH$

$\Rightarrow gh \in Hg$ [$\because$ gH = Hg, H being normal in G]

$\Rightarrow gh = h_1 g$, for some $h_1 \in H$

$\Rightarrow ghg^{-1} = h_1 gg^{-1}$

$\Rightarrow ghg^{-1} = h_1 \in H$

i.e. $ghg^{-1} \in H$

Conversely, let $H \leq G$ such that $ghg^{-1} \in H$, for all $g \in G$, $h \in H$.

Let $a \in G$ $\therefore ah \in aH$

$\therefore aha^{-1} \in H$, by the given condition

$\Rightarrow (aha^{-1})a \in Ha$

$\Rightarrow ah(a^{-1}a) \in Ha$

$\Rightarrow ah \in Ha$

$\therefore aH \subseteq Ha$ (1)

Now $ha \in Ha$.

We take $a^{-1} = b$.

$\therefore bhb^{-1} \in H$

$\Rightarrow a^{-1}h(a^{-1})^{-1} \in H$

$\Rightarrow a^{-1}ha \in H$

$\Rightarrow a(a^{-1}ha) \in aH$

$\Rightarrow (aa^{-1})ha \in aH$

$\Rightarrow ha \in aH$

$\therefore Ha \subseteq aH$ (2)

From (1) and (2), aH = Ha

$\therefore H \Delta G$.

Theorem 3 : The intersection of any two normal subgroup of a group is again a normal subgroup of the group.

Proof : Let G be a group, an $H \Delta G$, $K \Delta G$.

$\therefore H \leq G, K \leq G$ and so $H \cap K \leq G$.

Let $g \in G, h \in H \cap K$

$h \in H \cap K \Rightarrow h \in H$ and $h \in K$

Now $g \in G, h \in H$ and $H \Delta G$

$\therefore ghg^{-1} \in H$

Again $g \in G, h \in K$ and $K \Delta G$

$\therefore ghg^{-1} \in K$

Thus, $ghg^{-1} \in H$ and $ghg^{-1} \in K$.

$\therefore ghg^{-1} \in H \cap K$

$\therefore H \cap K \Delta G$.

Note : The union of two normal subgroups is not necessarily a normal subgroup.For example $2Z \Delta Z, 3Z \Delta Z$. But $2Z \cup 3Z \not\subset Z$ and so not a normal subgroup of Z.

7.4 QUOTIENT GROUPS

Let G be a group and H be a normal subgroup G.

$\because$ H is normal in G, therefore, there is no distinction between a left coset

and its corresponding right coset. So, by a coset we mean left coset (or right coset).

Let G/H denote the collection of all cosets of H in G.

Let $a, b \in G$ $\therefore aH, bH \in G/H$.

We define a binary operation in G/H by the rule $(aH)(bH) = abH$, $ab \in G$.

We show that the binary operation is well defined.

Let $aH = cH$,

$bH = dH$; $a, b, c, d \in H$.

Now $aH = cH \Rightarrow a^{-1}(aH) = a^{-1}(cH)$

$\Rightarrow (a^{-1}a)H = a^{-1}cH$

$\Rightarrow H = a^{-1}cH$

$\Rightarrow a^{-1}c \in H$ [Theorem 6 (i)]

$\Rightarrow a^{-1}c = h_1$, for some $h_1 \in H$

Similarly $bH = dH \Rightarrow b^{-1}d \in H$ $\Rightarrow b^{-1}d = h_2$, for some $h_2 \in H$.

Now $(ab)^{-1}(cd) = (b^{-1}a^{-1})(cd)$

$= b^{-1}(a^{-1}c)d$

$= b^{-1}h_1 d$

$= b^{-1}dh_1$ [$\because dH = Hd$]

$= h_2h_1 \in H$

$\therefore (ab)^{-1}(cd)H = H$

$\Rightarrow abH = cdH$

Thus, $\left.\begin{matrix} aH = cH \\ bH = dH \end{matrix}\right\} \Rightarrow abH = cdH$

$\therefore$ the binary operation defined above is well defined.

Theorem : Let G be a group and H be a normal subgroup G. The set G/H consisting of all the cosets of H in G is a group under the binary operation defined by

$(aH)(bH) = abH$; $a, b \in G$.

Proof : (i) It is given that the binary operation is $(aH)(bH) = abH$; $a, b \in G$

(ii) Let $a, b, c \in G$

$\therefore aH, bH, cH \in G/H$

Now $((aH)(bH))(cH) = (abH)(cH)$

$= (ab)c\,H$

= a(bc)H [∵G is a group, (ab)c = a(bc)]

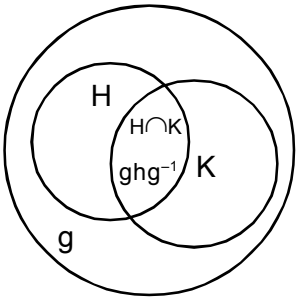
= (aH)((bc)H)

= (aH) ((bH)(cH))

∴ the binary operation is associative.

(iii) let e be the identity of G.

∴ eH = H $\in$ G/H



Now (aH)H = (aH) (eH) = aeH = aH

H (aH) = (eH) (aH) = eaH = aH.

∴ H is the identity.

(iv) a $\in$ G $\Rightarrow$ $a^{-1} \in$ G

$(aH)(a^{-1}H) = (aa^{-1})H = eH = H$

$(a^{-1}H)(aH) = (a^{-1}a)H = eH = H$

∴ $a^{-1}H$ is the inverse of aH.

∴ G/H is a group under the binary operation defined above.

Note : This group is known as the quotient group (or factor group) of G by H.

Theorem 5 : Every quotient group of an Abelian group is Abelian.

Proof : Let G be an Abelian group, and H $\leq$ G.

∴ G is Abelian, ∴ H is normal in G.

So, we can construct the quotient group G/H.

Let a, b $\in$ G. ∴aH, bH $\in$ G/H.

Now (aH)(bH) = abH

= baH [∵ G ia Abelian, ab = ba]

= (bH)(aH)

∴ G/H is Abelian.

Theorem 7 : Every quotient group of a cyclic group is cyclic.

Proof : Let G = <a> be a cyclic group.

Let H $\leq$ G

∵ G is cyclic, ∴G is Abelian and, therefore, H is normal in G.

So, we can construct the quotient group G/H.

Let x $\in$ G, ∴ xH $\in$ G/H.

Now $x = a^m$ for some integer m.

Let m > 0

Now $xH = a^m H$

$= (a\ a\ a \ldots\ a)H$

$= (aH)(aH)(aH) \ldots\ (aH)$

$= (aH)^m$

$\therefore\ G/H$ is a cyclic group generated by aH.

We can prove the result immediately for m = 0 or m < 0.



CHECK YOUR PROGRESS

Q 1: Can a non-Abelian group be cyclic?

Q 2: Show that a non-Abelian group must contain at least six elements.

Q 3: Let G = {1, ω , ω^2} be the multiplicative group; ω being a complex cube root of unity. Is the group simple?

Q 4: Can a group of order seven be non-Abelian?

Q 5: Let Z be the additive group of integers. The set 3 Z = {0, $\pm$ 3, $\pm$ 6, ...} is a subgroup of Z. Write down the number of elements of the quotient group $Z/3Z$.



7.5 LET US SUM UP

- A subgroup H of a group G is a normal subgroup of G if aH = Ha, for all a $\in$ G.
- If G is a group and H is a normal subgroup of G, the set G/H consisting of all the cosets of H in G is a group, called the quotient group (or the factor group).



7.6 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS

Ans to Q No 1: Every cyclic group is Abelian. Therefore, every non-Abelian group is non-cyclic.

Ans to Q No 3: Yes. It has no normal subgroups other than G, {1}.

Ans to Q No 4: No. Every group of prime order is cyclic and every cyclic group is Abelian.

Ans to Q No 5: Three.



7.7 FURTHER READINGS

1) Lanski, Charles : *Concepts in Abstract Algebra*, (First Indian edition, 2010) American Mathematical Society.

2) Vasistha, A. R. : *Modern Algebra (Abstract algebra),* Krishna Prakashan Media (p) Ltd. Meerut-250001 (U. P).

3) Vinberg, E. B. : *A course in Algebra* (First Indian edition, 2009), American Mathematical Society).

4) Vatsa, B. S. and Vatsa, Suchi : *Modern Algebra*, New Age International (p) Ltd., New Delhi-110002.

5) Sen, M. K, Ghosh, Shamik and Mukhopadhyay, Parthasarthi : *Topics in Abstract Algebra,* Universities Press (India) Pvt. Ltd., Hyderabad, 500029.



7.8 MODEL QUESTIONS

Q 1: If $N_1, N_2, \cdots, N_m$ are normal subgroups of a group G,Show that $N_1 \cap N_2 \cap \cdots \cap N_m$ is also a normal subgroup of G.

Q 2: Let G = $\left\{\begin{pmatrix} a & b \\ o & d \end{pmatrix} : a,b,d \in IR \text{ and } ad \neq 0\right\}$

Show that G is a non-Abelian group under matrix multiplication.

Let H = $\left\{\begin{pmatrix} 1 & b \\ o & 1 \end{pmatrix} : b \in IR\right\}$

Show that H is a normal subgroup of G.

Q 3: Show that a group of prime order is simple.

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