UNIT 8 : HOMOMORPHISM OF GROUPS

UNIT STRUCTURE

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8.1 LEARNING OBJECTIVES

After going through this unit, you will able to know-

- the concept of homomorphism of groups
- different types of homomorphism
- properties of homomorphism, and related theorems.

8.2 INTRODUCTION

The word homomorphism has been derived from two Greek words homos (meaning 'same') and morphe (meaning 'form' or 'shape'). Thus, homomorphism means 'of the same form or shape'.

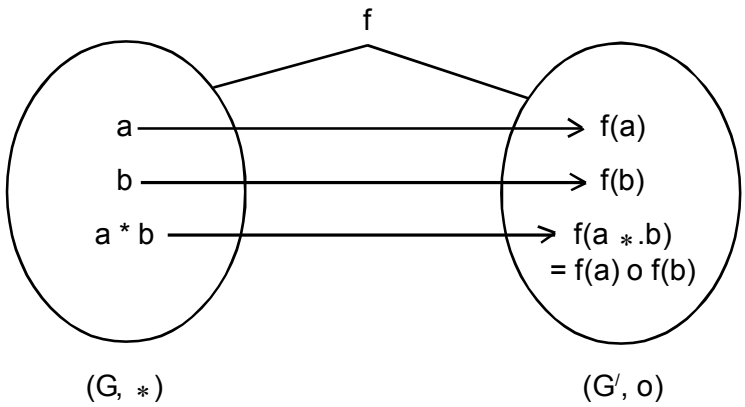
A homomorphism is a structure-preserving mapping between two algebraic structures. Thus, a group homomorphism is a homomorphism that preserves the group structure.

8.3 HOMOMORPHISM OF GROUPS

Definition : Let G be a group under the binary operation '$*$' and G' be a group under the binary operation 'o'.

Let $f : G \to G'$ be a mapping. f is said to be a homomorphism if

$f(a * b) = f(a)\ o\ f(b),\ \forall\ a, b \in G$



Note : 1. Here a is related to f(a), b is related f(b). Moreover, $a * b$ is related to f(a) o f(b). Thus, the mapping f not only relates two elements of G with two unique elements of G/, but also relates $a * b$ to f(a) of (b) in G/. Thus, the mapping f preserves the algebraic structure of the systems.

2. In our subsequent discussions, we will drop the symbols for binary operations. Simply we will write f(ab) = f(a) f(b).

It should be borne in mind that the binary operations in the two groups are not necessarily the same.

Example 1: Let $\underline{\underline{Z}}$ be the additive group of integers. We define a mapping

$$f : \underline{\underline{Z}} \to \underline{\underline{Z}},\ \ f(x) = -x, \forall x \in \underline{\underline{Z}}$$

Let $a,\ b \in \underline{\underline{Z}}$

Now f(a+b) = –(a+b), by definition of f

= – a – b

= (–a) + (–b)

= f(a) + f(b)

$\therefore$ f is a homomorphism.

Note : In the above example the binary operation is "+". Moreover, both the groups are the same with same binary operation.

Example 2 : Let R be the additive group of real numbers, and R_+ be the multiplicative group of positive real numbers. We define

f: $R \to R_+$, $f(x) = e^x$

We are to examine whether f is a homomorphism or not. Now, the

binary operation in R is " + ", and the binary operation in R_+ is " . ". So, we are to examine if f(a+b) = f(a) f(b), $\forall\ a, b \in R$

Now $f(a+b) = e^{a+b}$, by definition of f

$= e^a e^b$

$= f(a)\ f(b)$

$\therefore$ f is a homomorphism.

Example 3: Let $\underline{\underline{Z}}$ be the additive group of integers. We define

$$f : \underline{\underline{Z}} \to \underline{\underline{Z}},\ \ f(x) = x + 5$$

Let $a,\ b \in \underline{\underline{Z}}$

Now f(a+b) = (a+b) + 5

f(a) + f(b) = (a+5) + (b+5)

= (a+b) +10

$\therefore$ f(a+b) $\neq$ f(a) + f(b)

$\therefore$ f is not a homomorphism.

Example 4: Let $\underline{\underline{Z}}$ be the additive group of integers. Let G = {–1, 1} be the multiplicative group. We define

$$f(x) = \begin{cases} 1, & if\ x\ is\ even \\ -1, & if\ x\ is\ odd \end{cases}$$

Let $a,\ b \in \underline{\underline{Z}}$

Case I: Let a be even, b be even.

$\therefore$ f(a) = 1, f(b) = 1

Also, a + b is even

$\therefore$ f (a + b) = 1

= 1 x 1

= f(a) f(b)

Case 2: Let a be odd, b be odd

$\therefore$ f(a) = –1, f(b) = –1

$\therefore$ a, b are odd, $\therefore$ a + b is even

$\therefore$ f (a + b) = 1

= (–1) x (–1)

= f(a) f(b)

Case 3: Let a be even, be be odd $\therefore$ a + b is odd.

$\therefore$ f(a) = 1, f(b) = –1

$\because$ a is even and b is odd, $\therefore$ a + b is odd

$\therefore$ f(a + b) = –1

= 1 x (–1)

= f(a) f(b)

Similarly, if a is odd, and b is even, then f(a + b) = f(a) f(b)

Thus, in all the cases

f(a + b) = f(a) f(b)

$\therefore$ f is a homomorphism.

Example 5: Let G be a group, and e be the identity of G. We define

$f : G \to G$ such that f(x) = e, $\forall\ x \in G$

Let $a,\ b \in G$

$\therefore\ ab \in G$

Now f(a) = e, f(b) = e

Also f(ab) = e

= e e ($\because$ e is the identity)

= f(a) f(b)

$\therefore$ f is a homomorphism



CHECK YOUR PROGRESS

Q 1: Let $\underline{\underline{Z}}$ be the additive group of integers.

Define $f : \underline{\underline{Z}} \to \underline{\underline{Z}},\ f(x) = 5x$

Examine if f is a homomorphism.

Q 2: Let $\underline{\underline{Z}}$ be the additive group of integers.

Define $f : \underline{\underline{Z}} \to \underline{\underline{Z}},\ f(x) = x + 3$

Examine if f is a homomorphism.

Q 3: Let $\underline{\underline{R}}_+$ be the multiplicative group of positive reals. Define

$f : \underline{\underline{R}}_+ \to \underline{\underline{R}}_+,\ f(x) = \frac{1}{x}$

Examine if f is a homomorphism.

Q 4: Let $\underline{\underline{R}}_+$ be the multiplicative group of positive real numbers,

and $\underline{\underline{R}}$ be the additive group of real numbers. Define

$f : \underline{\underline{R}}_+ \to \underline{\underline{R}}$, $f(x) = \log_e x$

Examine if f is a homomorphism.

Q 5: Let S = {1, 2, 3}, and let S_3 be the group of all permutations of S. Let G = {–1, 1} be the multiplicative group. Define

$$f(\alpha)\begin{cases} = 1, \text{ if } \alpha \text{ is an even permutation} \\ = -1, \text{ if } \alpha \text{ is an odd permutation} \end{cases}$$

Examine if f is a homomorphism.

8.4 TYPES OF HOMOMERPHISM

Let f : G → G/ be a homomorphism from a group G to a group G/.

1. A homomorphism f is said to be a monomorphism if f is one-one (i.e. if f is injective).
2. A homomorphism f is said to be an epimorphism if f is onto (i.e. if f is surjective).
3. A homomorphism from a group to itself is called an endomorphism.
4. A homomorphism f is said to be an isomorphism if f is one-one and onto (i.e. if f is bijective).

   The word 'isomorphism' has also been derived from two Greek words "isos" (meaning "equal"). If there exists an isomorphism between two groups, they can be called "abstractly identical". They can be treated as one group, and not as two different groups.
5. An isomorphism from a group to itself is called an automorphism

Example 6: Let $\underline{\underline{Z}}$ be the additive group of integers. We define a mapping

f : $\underline{\underline{Z}} \to \underline{\underline{Z}}$, f(x) = – x, $\forall\ x \in \underline{\underline{Z}}$

(i) We have proved that f is a homomorphism (Example 1)

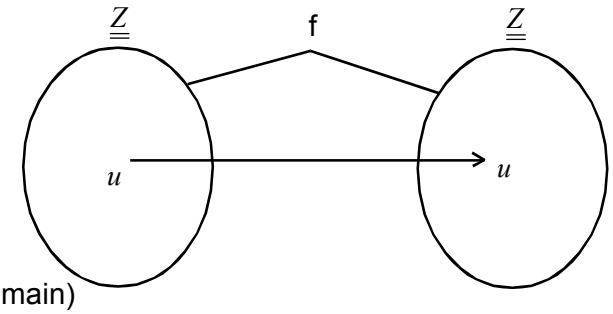
(ii) Let $a, b \in \underline{\underline{Z}}$

f(a) = f(b)

⇒ –a = – b

⇒ a = b

∴ f is one-one



(iii) Let $u \in \underline{\underline{Z}}$ (codomain)

Then there exists

$-u \in \underline{\underline{Z}}$ (domain) such that

$f(-u) = -(-u) = u$

$\therefore$ f is onto.

Thus, f is a homomorphism, f is one-one, and f is onto.

$\therefore$ f is an isomorphism. Moreover, f is an automorphism, for f is an isomorphism from $\underline{\underline{Z}}$ to itself.

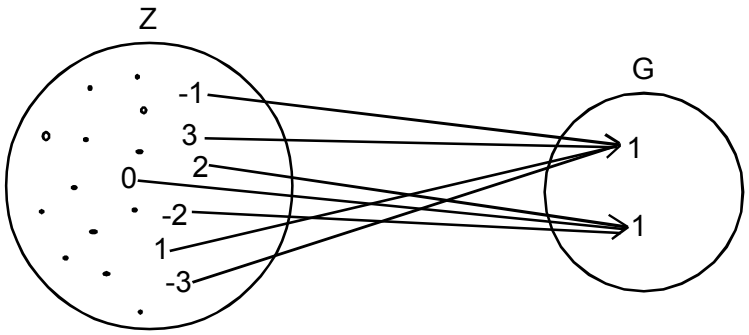
Example 7: Let $\underline{\underline{Z}}$ be the additive group of integers. Let G = {–1, 1} be the multiplicative group. We define

$$f(x)\begin{cases} = 1, \text{ if } x \text{ is even} \\ = -1, \text{ if } x \text{ is odd} \end{cases}$$

(i) We have proved that f is a homomorphism (Example 4).

(ii) Clearly f is not one-one.

(iii) Clearly f is onto.



1 is the image of all even integers, and –1 is the image of all odd integers.

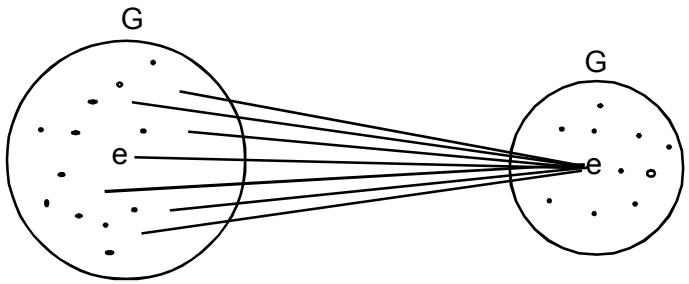
Thus, f is an epimorphism, but not an isomorphism.

Example 8: Let G be a group, and e be the identity of G. We define

$f(x) = e, \ \forall x \in G$

(i) We have proved that f is a homomorphism (Example 5).

(ii) Clearly f is not one-one. All the elements are related to e.



(iii) f is not onto. e is the only image.

∴ f is not an isomorphism. It is neither a monomorphism nor an epimorphism.

∵ f is a homomorphism from G to itself, it is an endomorphism.



CHECK YOUR PROGRESS

Q 6: Let C be the additive group of complex numbers.Define $f : \underline{\underline{C}} \to \underline{\underline{C}},\ f(z) = \bar{z},\ \forall z \in \underline{\underline{C}}$

Examine if f is an automorphism.

Q 7: Let $\underline{\underline{R}}$ be the additive group of real numbers, and $\underline{\underline{R}}_+$ be the multiplicative group of positive real numbers.

Define $f : \underline{\underline{R}} \to \underline{\underline{R}}_+,\ f(x) = e^x$

Examine if f is an isomorphism

Q 8: Let $\underline{\underline{Z}}$ be the additive group of integers, and $\underline{\underline{2Z}}$ be the additive group of even integers. Define $f : \underline{\underline{Z}} \to \underline{\underline{2Z}},\ f(x) = 2x,\ \forall x \in \underline{\underline{Z}}$

Examine if f is an isomorphism.

Q 9: Let $\underline{\underline{2Z}} = \{....., -8, -6, -4, -2, 0, 2, 4, 6, 8, ...\}$

$\underline{\underline{3Z}} = \{....., -9, -6, -3, 0, 3, 6, 9, ...\}$

$\underline{\underline{2Z}}$ and $\underline{\underline{3Z}}$ are additive groups.

Define $\underline{\underline{2Z}} \to \underline{\underline{3Z}}$, f(2x)=3x where x is an integer.

Examine if f is an isomorphism.

Q 10: Let $\underline{\underline{R}}_+$ be the multiplicative group of positive real numbers, and $\underline{\underline{R}}$ be the additive group of real numbers. Define

$f : \underline{\underline{R}}_+ \to \underline{\underline{R}},\ f(x) = \log_{10} x$

Examine if f is an isomorphism.

8.5 PROPERTIES OF HOMOMORPHISM

Theorem : Let G, G′ be two groups. Let e, e′ be the identities of G, G′ respectively. Let f : G → G′ be a homomorphism. Then

(i) f(e) = e′

(ii) $f(x^{-1}) = (f(x))^{-1},\ x \in G$

(iii) $f(x^n) = (f(x))^n$, $x \in G$, n is an integer.

Proof: (i) We have

$e\ e = e$

$\Rightarrow f(e\ e) = f(e)$

$\Rightarrow f(e)\ f(e) = f(e)$, $\because$ f is a homomerphism

$\Rightarrow f(e)\ f(e) = f(e)e'$, $f(e) \in G'$, and e' is the identity of G'

$\Rightarrow f(e) = e'$, by left cancellation.

(ii) We have

$$x x^{-1} = e = x^{-1} x$$

$$\Rightarrow f(x x^{-1}) = f(e) = f(x^{-1} x)$$

$$\Rightarrow f(x) f(x^{-1}) = e' = f(x^{-1}) f(x)$$

This $\Rightarrow f(x^{-1})$ is the inverse of f(x).

i.e. $(f(x))^{-1} = f(x^{-1})$

(iii) Let n be a positive integer :

Then $f(x^n) = f(x x x .... x)$

= $f(x) f(x) f(x) f(x) .... f(x)$, $\because$ f is a homomorphism

= $(f(x))^n$

Let n = 0 :

Then $x^0 = e$

$\therefore$ $f(x^0) = f(e)$

$= e'$

$= (f(x))^0$

Let n be a negative integer :

Let n = –m, where m is a positive integer.

Now $f(x^n) = f(x^{-m})$

$= f(x^{-1})^m)$

$= ((f(x))^{-1})^m$

$= (f(x))^{-m}$

$= (f(x))^n$

Thus, $f(x^n) = (f(x))^n$, for any integer n.

Theorem : Let $f : G \to G'$ be an epimorphism from a group G to a group G'. If G is Abelian, then G' is also Abelian

Proof : Here f : G $\to$ G/ is an epimorphism.

$\therefore$ f is a homomorphism and f is onto.

Let $u, v \in G'$

$\because$ f is onto, there exist a, b $\in$ G such that f(a) = u, f(b) = v

Now uv = f(a) f(b)

= f(ab), $\because$ f is a homomorphism

= f(ba), $\because$ G is Abelion

= f(b) f(a), $\because$ f is a homomorphism

= vu

Thus, $uv = vu, \ \forall u, v \in G'$

$\therefore$ G is Abelian

Note : The converse of the above theorem is not true.

Let us consider the group S_3 of all permutntions of S = {1, 2, 3}.

Let G = {–1, 1} be the multiplicative group.

We define $f : S_3 \to G$ by

$$f(\alpha)\begin{cases} = 1, \ if \ \alpha \ is \ an \ even \ permutation \\ = -1, \ if \ \alpha \ is \ an \ odd \ permutation \end{cases}$$

f is a homomorphism. Moreover f is onto, i.e, f is an epimorphism.

Now G is Abelian, but S_3 is not Abelian.

Theorem : Let f : G $\to$ G/ be an epimorphism from a group G to a group G/. If G is cyclic, then G/ is also cyclic.

Proof : Here f : G $\to$ G/ is an epimorphism.

$\therefore$ f is a homomorphism and f is onto. Let the cyclic group G be generated by $a \in G$.

Let $u \in G$

Then there exists $p \in G$ such that

u = f(p)

= $f(a^m)$, $\because$ p = a^m, for some integer m

= $(f(a))^m$

This $\Rightarrow$ G/ is a cyclic group generated by f(a).

Note : The converse of the above theorem is not true.

Let us consider the example given in the previous theorem.

Here G = {–1, 1} is cyclie,

but S_3 is not cyclic.

Theorem : Let f : G $\rightarrow$ G/ be a homomorphism from a group G to a group G/.

(i) If H is a subgroup G, f(H) is a subgroup of G/.

(ii) If K is a subgroup of G/, $f^{-1}(K)$ is a subgroup of G.

[f(H) = {f(h) : h $\in$ H}, called the image of H under f.

$f^{-1}(K) = \{a \in G : f(a) \in K\}$, called the inverse image of K under f.

It should be borne in mind that f^{-1} may not be a homomorphism, though f is a homomorphism]

Proof : Let e be the identity of G, and e/ be the identity of G/.

$\therefore$ f(e) = e/

(i) We have e/ = f(e) $\in$ f(H)

$\therefore\ f(H) \neq \phi$

Let $u, v \in f(H)$

$\therefore$ there exist a, b $\in$ H such that f(a) = u, f(b)= v

Now uv^{-1} = f(a) $(f(b))^{-1}$

= f(a) $f(b^{-1})$

= $f(ab^{-1})$

$\because$ H is a subgroup of G, $\therefore$ $ab^{-1} \in H$

$\therefore$ $f(ab^{-1}) \in f(H)$

$\therefore\ uv^{-1} \in f(H)$

Thus, $u,\ v \in f(H) \Rightarrow uv^{-1} \in f(H)$

$\therefore$ f(H) is a subgroup of G/

(ii) We have $e^{/} \in K$

$\Rightarrow f(e) \in K$

$\Rightarrow e \in f^{-1}(K)$

$\therefore\ f^{-1}(K) \neq \phi$

Let a, b $\in$ f^{-1}(K)

$\therefore$ f(a) $\in$ K, f(b) $\in$ K

$\Rightarrow$ f(a) $(f(b))^{-1} \in$ K, $\because$ K is a subgroup of G/

$\Rightarrow$ f(a) $f(b^{-1}) \in$ K

$\Rightarrow f(ab^{-1}) \in K$

$\Rightarrow ab^{-1} \in f^{-1}(K)$

Thus, $a, b \in f^{-1}(K) \quad \Rightarrow ab^{-1} \in f^{-1}(K)$

$\therefore f^{-1}(K)$ is a subgroup of G'.

Example 9 : Let G be a group. Let the mapping $f : G \to G$, $f(x) = x^{-1}$ be a homomorphism. Show that G is Abelian.

Solution : Let $a, b \in G$.

Now $f(ab) = f(a)\, f(b)$, $\because$ f is a homomorphism by definition of f

$\Rightarrow (ab)^{-1} = a^{-1} b^{-1}$

$\Rightarrow ((ab)^{-1})^{-1} = (a^{-1}b^{-1})^{-1}$

$\Rightarrow ab = (b^{-1})^{-1}(a^{-1})^{-1}$

$\Rightarrow ab = ba$

$\therefore$ G is Abelian



CHECK YOUR PROGRESS

Q 11: Let $f : G \to G$, $f(x) = x$ be a mapping from group G to itself. Show that f is an automorphism.

Q 12: Let G be an Abelian group. Show that the mapping.

$f : G \to G$, $f(x) = x^{-1}$ is an automorphism.



8.6 LET US SUM UP

Let G be a group under the binary operation $*$, and G' be a group under the binary operation o. The mapping $f : G \to G'$ is said to be a homomorphism if $f(a * b) = f(a)$ of (b)

- A homomorphism if said to be a monomorphism if it is one-one.
- A homomorphism is said to be an epimorphism if it is onto.
- A homomorphism is said to be an isomorphism if it is one-one and onto.
- A homomorphism from a group to itself is called an endomorphism.
- An isomorphism from a group to itself is called an automorphism.

- Let f : G $\to$ G/ be a homomorphism from a group G to a grant G/. Let e, e/ be the identities of G, G/ respectively. Then
  (i) f(e) = e/
  (ii) $f(x^{-1}) = (f(x))^{-1}$, $x \in G$
  (iii) $f(x^n) = (f(x))^n$, n is an integer.
- Let f G $\to$ G/ be an epimorphism from a group G to a group G/.
  (i) If G is Abelian, then G/ is Abelian.
  (ii) If G is cyclic, then G/ is cyclic.
  The converse of each of (i) and (ii) is not true.
- Let f : G $\to$ G/ be a homomorphism from a group G to a group G/.
  (i) If H is a subgroup of G, f(H) is a subgroup of G/.
  (ii) If K is a subgroup of G/, $f^{-1}(K)$ is a subgroup of G.



8.8 FURTHER READINGS

1) Fraleigh, Johu B : *A First Course in Abetract Algebra* (2003), Pearson Education (Singapure) Pvt. Ltd, Indian Branch, Delhi 110092.
2) Mapa, S.K. : *Higher Algebra* (Abstract and Linear) (1994), Asoke Prakasan, Calcutta 700007.
3) Khanna, V. K. and Bhambri, S. K. : *A Course in Abstract Algebra* (2006), Vikas Publishing House Pvt. Ltd. New Delhi 110014.



8.7 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: $f : \underline{\underline{Z}} \to \underline{\underline{Z}}$, $f(x) = 5x$

Let $a, b \in \underline{\underline{Z}}$

Now f(a+b) = 5(a+b)

=5a+ 5b

= f(a) + f(b)

$\therefore$ f is a homomorphism.

Ans to Q No 2: $f : \underline{\underline{Z}} \to \underline{\underline{Z}},\ f(x) = x + 3$

Let $a, b \in \underline{\underline{Z}}$

f(a+b) = (a+b) + 3 = a+b+3

f(a) +f(b) = (a+3) + (b+3)

= a + b + 6

$\therefore$ f(a+b) $\neq$ f(a) + f(b)

$\therefore$ f is not a homomorphism

Ans to Q No 3: $f : \underline{\underline{R}}_+ \to \underline{\underline{R}}_+, f(x) = \frac{1}{x}$

Let $a, b \in \underline{\underline{R}}_+$

Now $f(ab) = \frac{1}{ab}$

$= \frac{1}{a}\frac{1}{b}$

= f(a)f(b)

$\therefore$ f is a homomerphism.

Ans to Q No 4: $f : \underline{\underline{R}}_+ \to \underline{\underline{R}}, f(x) = \log_e x$

Let $a, b \in \underline{\underline{R}}_+$

Now f(ab) = $\log_e$ ab

= $\log_e$a + $\log_e$b

= f(a) + f(b)

$\therefore$ f is a homomorphism

Ans to Q No 5: f : $S_3 \to$ G = {–1, 1}, where

$$f(\alpha)\begin{cases} = 1, \text{ if } \alpha \text{ is an even permutation} \\ = -1, \text{ if } \alpha \text{ is an odd permutation} \end{cases}$$

Case I : Let α, β be even permutation.

$\therefore\ f(\alpha) = 1,\ f(\beta) = 1$

We know that the product of two even permutations is an even permutation.

$\therefore\ f(\alpha\beta) = 1$

$= 1 \times 1$

$= f(\alpha) f(\beta)$

Case 2: Let α, β be add permutations

$\therefore\ f(\alpha) = -1,\ f(\beta) = -1$

We know that the product of two odd permutations is an even permutation.

$\therefore\ f(\alpha\beta) = 1$

$= (-1) \times (-1)$

$= f(\alpha) f(\beta)$

Case 3 : Let α be an even permutation, and β be an odd permutation.

$\therefore\ f(\alpha) = 1,\ f(\beta) = -1$

We know that the product of an even permutation and an odd permutation is an odd permutation.

$\therefore\ f(\alpha\beta) = -1$

$= 1 \times (-1)$

$= f(\alpha) f(\beta)$

Similarly, if α is an odd permutation and β is an even permutation, then also

$f(\alpha\beta) = f(\alpha) f(\beta)$

Thus, in all the cases,

$f(\alpha\beta) = f(\alpha) f(\beta)$

$\therefore$ f is a homomorphism.

Ans to Q No 6: $f : \underline{\underline{C}} \to \underline{\underline{C}},\ f(z) = \bar{z}$

Let $z, w \in \underline{\underline{C}}$

Now $f(z + w) = \overline{z + w}$

$= \bar{z} + \overline{w}$

$= f(z) + f(w)$

$\therefore$ f is a homomorphism.

Let f(z) = f(w)

$\Rightarrow \bar{z} = \overline{w}$

$\Rightarrow z = w$

$\therefore$ f is one-one.

Let $\Rightarrow z \in \underline{\underline{C}}$ (codomain)

Then, there exists $\bar{z} \in \underline{\underline{C}}$ (domain) such that

$f(\bar{z}) = (\bar{\bar{z}}) = z$

$\therefore$ f is onto.

$\therefore$ f is an isomorphism.

$\because$ f is an isomorphism from $\underline{\underline{C}}$ to itself,

$\therefore$ f is an automorphism.

Ans to Q No 7: $f : \underline{\underline{R}} \to \underline{\underline{R}}_+,\ f(x) = e^x$

Let $a, b \in \underline{\underline{R}}$

Now f(a+b) = e^{a+b}

= e^a e^b

= f(a) f(b)

Let f(a) = f(b)

$\Rightarrow$ $e^a = e^b$

$\Rightarrow$ a = b

$\therefore$ f is one-one

Let $u \in \underline{\underline{R}}_+$ such that f(c) = u

$\therefore$ e^c = u

$\therefore$ e = $\log_e u \in \underline{\underline{R}}$

Thus, corresponding to every $u \in \underline{\underline{R}}_+$, there exists $\log_e u$ in $\underline{\underline{R}}$ such that

f($\log_e u$) = u

$\therefore$ f is onto.

$\therefore$ f is an isomorphism.

Ans to Q No 8: $f : \underline{\underline{Z}} \to 2\underline{\underline{Z}},\ f(x) = 2x$

Let $a, b \in \underline{\underline{Z}}$

Now f(a+b) = 2(a+b)

= 2a + 2b

= f(a) + f(b)

$\therefore$ f is a homomorphism.

Let f(a) = f(b)

$\Rightarrow$ 2a = 2b

$\Rightarrow$ a = b

$\therefore$ f is one-one

Let $u \in 2\underline{\underline{Z}}$

Then there exists $\frac{u}{2} \in \underline{\underline{Z}}$ such that $f\left(\frac{u}{2}\right) = 2\frac{u}{2} = u$

$\therefore$ f is onto.

$\therefore$ f is an isomorphism.

Ans to Q No 9: $f : 2\underline{\underline{Z}} \to 3\underline{\underline{Z}},\ f(2x) = 3x$, where x is an integer.

Let a, b be integers.

$\therefore$ 2a, 2b$\in 2Z$

Now f (2a+2b) = f(2(a+b))

= 3(a+b)

= 3a + 3b

= f(2a)+ f(2b)

$\therefore$ f is a homomorphism.

Let f(2a) = f(2b)

$\Rightarrow$ 3a = 3b

$\Rightarrow$ a = b

$\Rightarrow$ 2a = 2b

$\therefore$ f is one-one

Let $3u \in 3\underline{\underline{Z}}$, where u is an integer.

Then, there exists $2u \in 2Z$ such that f (2u) = 3u

$\therefore$ f is onto

$\therefore$ f is an isomorphism

Ans to Q No 10: $f : \underline{\underline{R}}_{+} \to \underline{\underline{R}},\ f(x) = \log_{10} x$

Let $a, b \in \underline{\underline{R}}_{+}$

Now f(ab) = $\log_{10}$ ab

= $\log_{10}$a + $\log_{10}$b

= f(a) + f(b)

$\therefore$ f is a homomorphism

Let f(a) = f(b)

$\Rightarrow$ $\log_{10}$a = $\log_{10}$b

$\Rightarrow$ a = b

$\therefore$ f is one-one

Let $u \in \underline{\underline{R}}$

Then there exists $10^u \in \underline{\underline{R}}_+$ such that

$$f(10^u) = \log_{10}{}^{10^u}$$
$$= u \log_{10}{}^{10}$$
$$= u$$

$\therefore$ f is onto.

$\therefore$ f is an isomorphism.

Ans to Q No 11: $f : G \to G, \quad f(x) = x$

Let $a, b \in G$

Now f(ab) = ab

= f(a) f(b)

$\therefore$ f is a homomorphism

f(a) = f(b)

$\Rightarrow$ a = b

$\therefore$ f is one-one

For every $a \in G$, f(a) = a

$\therefore$ f is onto

$\therefore$ f is an isomorphism from G to itself

$\therefore$ f is an authomorphism.

Ans to Q No 12: $f : G \to G, \quad f(x) = x^{-1}$

Let $a, b \in G$

Now f(ab) = $(ab)^{-1}$

= $b^{-1} a^{-1}$

= $a^{-1}b^{-1}$, $\because$ G is Abelion

= f(a) f(b)

$\therefore$ f is a homomorphism

Let f(a) = f(b)

$\Rightarrow$ a = b

$\therefore$ f is one-one

Let $u \in G$,

Then there exists $u^{-1} \in G$ such that

$f(u^{-1}) = (u^{-1})^{-1}$

$= u$

$\therefore$ f is onto

$\therefore$ f is an isomorphism from G to itself.

$\therefore$ f is an automorphism.



8.9 MODEL QUESTIONS

Q 1: Let $\underline{\underline{R}}$ be the additive group of real numbers. Let

$f : \underline{\underline{R}} \to \underline{\underline{R}}, f(x) = 5x$ be a mapping.

Examine if f is a homomorphism.

Q 2: Let $3\underline{\underline{Z}}$ ={. . ., –9, –6, –3, 0, 3, 6, 9, ...}

$5\underline{\underline{Z}}$ ={. . ., –20, –15, –10, –5, 0, 5, 10, 15, 20, ...}

$3\underline{\underline{Z}}$, and $5\underline{\underline{Z}}$ are additive groups.

Let $f : 3\underline{\underline{Z}} \to \underline{\underline{Z}}, \ f(3x) = 5x$, where x is an integer.

Examine if f is a homomorphism.

Q 3: Give an example of a homomorphism from an infinite group to a finite group.

Q 4: Let $f : G \to G'$ be an isomorphism from a group G to a group G′. If G′ is Abelian, prove that G is also Abelian.

Q 5: Let $f : G \to G'$ be an isomorphism from a group G to a group G′. If G′ is cyclic, prove that G is also cyclic.

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