UNIT 9: RING

UNIT STRUCTURE

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9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to know

- ring
- properties of ring.

9.2 INTRODUCTION

We have discussed group and its various properties, sub group, coset, cyclic group, normal sub groups and quotient group. In this unit we will study ring. Ring is an algebraic structure, i,e., a non-empty set equipped with two binary operations and satisfying certain postulates.

9.3 RING

A non-empty set R, equipped with two binary operations, called addition (+) and multiplication (·) is called a ring if the following postulates are satisfied.

(1) R is an additive Abelian group,

(2) R is an multiplicative semigroup

(3) The two distributive laws hold good, viz

a. (b + c) =a.b + a.c. (left distributive law)

(a + b). c = a.c + b.c, (right distributive law) for all a, b, $c \in R$

Note : 1. In order to be a ring, there must be a non-empty set equipped with two binary operations, viz, + and . , satisfying the postulates mentioned above. Later on we will simply write ''Let R be a ring''. It should be borne in mind that there are two binary operations, viz, + and . , on R.

2. It should also be borne in mind that the binary operations + and . may not be known usual addition and multiplication.

3. Since R is an additive Abelian group, the additive identity, denoted by 0, belongs to R. It is called the zero element of R. Here 0 is a symbol. It should not be confused with the real number zero.

4. If $a , b \in R$, then $- b \in R$ ($\because$ R is an additive group). a+ (– b) is, generally, denoted by a – b.

5. a.b is, generally, written as ab.

Commutative Ring: A ring R is said to be commutative if ab = ba, for all a, $b \in R$.

Ring with unity : A ring R is said to be a ring with unity (or identity) if there exists an element $e \in R$ such that

ae = a = ea, for all $a \in R$.

Note 1. A ring may or may not have a unity.

2. If R is a ring with unity, unity may be same as the zero element. For example, {0} is a commutative ring with unity . Here 0 is the additive as well as the multiplicative identity. It is known as the zero ring.

Theorem 1 : If R is a ring with unity, then unity is unique.

Proof Let R be a ring with unity. If possible, let e and f be two identities of R.

ef = e = fe [$\because$ f is unity]

ef = f = fe [$\because$ e is unity]

$\therefore$ e = f

Note: In view of the above theorem, we write ''the unity''.

Example 1. Let R = $\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \text{ are integers}\right\}$

I (i) Let $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in R$, $\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in R$,

a, b, c, d $\in$ Z

Now $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ 0 & 0 \end{pmatrix} \in R$

$\therefore$ Matrix addition is a binary operation on R.

(ii) We know that matrix addition is associative.

(iii) We have $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in R$,

and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(iv) For every $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in R$, we have $\begin{pmatrix} -a & -b \\ 0 & 0 \end{pmatrix} \in R$ such that

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -a & -b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -a & -b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

$\therefore \begin{pmatrix} -a & -b \\ 0 & 0 \end{pmatrix}$ is the additive inverse of $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$.

(v) We know that matrix additive is commutative.

$\therefore$ R is an Abelian group under matrix addition.

II. (i) We have $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix} \in R$

$\therefore$ matrix multiplication is a binary operation on R

(ii) We know that matrix multiplication is associative.

$\therefore$ R is a semigroup under matrix multiplication.

III. We know that matrix multiplication is distributive over matrix addition.

The two distributive laws are satisfied.

$\therefore$ R is a ring under matrix addition and matrix multiplication.

We have $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \in R$, $\begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \in R$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix}3 & 4\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}=\begin{pmatrix}3 & 6\\ 0 & 0\end{pmatrix}$$

$$\therefore \begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}3 & 4\\ 0 & 0\end{pmatrix}\neq\begin{pmatrix}3 & 4\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}$$

$\therefore$ R is a non- commutative ring.

Again, there does not exist any element $X \in G$ such that

$$x\begin{pmatrix}a & b\\ 0 & 0\end{pmatrix}=\begin{pmatrix}a & b\\ 0 & 0\end{pmatrix}=\begin{pmatrix}a & b\\ 0 & 0\end{pmatrix}x \text{, for all } \begin{pmatrix}a & b\\ 0 & 0\end{pmatrix} \in R$$

$\therefore$ R is a ring without unity.

Example 2 Let R be the set of all 2 x 2 matrices over integers.

I. Clearly R is an Abelian group under matrix addition.

II. R is a semigroup under matrix multiplication.

III. The two distributive laws are satisfied.

$\therefore$ R is a ring under matrix addition and matrix multiplication.

We know that matrix multiplication is not, in general, commutative.

$\therefore$ R is a non-commutative ring.

We have $I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \in R$, and

$AI = A = IA$, for all $A \in R$.

$\therefore$ R is a ring with unity.

Example 3: Let 2Z be the set of all even integers

I. 2Z is an Abelian group under usual addition.

II. 2Z is a multiplicative semigroup.

III. The two distributive laws are satisfied.

$\therefore$ 2Z is a ring under usual addition and multiplication.

Moreover, 2Z is a commutative ring.

2Z is a ring without unity, for $1 \notin 2Z$.

Example 4: Let Z be the set of integers.

I. Z is an Abelian group under usual addition.

II. Z is a multiplicative semigroup.

III. The two distributive laws are satisfied.

$\therefore$ Z is a ring under usual addition and scalar multiplication.

Moreover, Z is a commutative ring . It is also a ring with unity, $1 \in Z$.

Other examples of commutative rings with unity are

- the ring Q of rational numbers
- the ring IR of real numbers.
- the ring C of complex numbers.

Example 5: Let C [a, b] denote the set of all real-valued continuous functions defined on [a, b] ; a, $b \in R$

Let $f, g \in C[a, b]$

We define $(f + g)(x) = f(x) + g(x)$

$(fg)(x) = f(x)\, g(x),\ x \in [a, b]$

I. (i) We have $(f + g)(x) = f(x) + g(x) \in R$

Also, the sum of two continuous functions is continuous.

$\therefore$ f + g is continuous and so $f + g \in C[a, b]$

$\therefore$ addition of functions is a binary operation on C [a, b].

(ii) Let $f, g, h \in C[a, b]$, $x \in [a, b]$

$$\begin{aligned}((f + g) + h)(x) &= (f + g)(x) + h(x)\\ &= (f(x) + g(x)) + h(x)\\ &= f(x) + (g(x) + h(x))\\ &= f(x) + (g + h)(x)\\ &= (f + (g + h))(x), \forall x \in [a, b]\end{aligned}$$

$\therefore (f + g) + h = f + (g + h)$

(iii) We have $0 : [a, b] \to R$, $0(x) = 0$, $\forall x \in [a, b]$

Clearly $0 \in C[a, b]$, and

$0 + f = f = f + 0$

(iv) For every $f \in C[a, b]$, we define

$-f : [a, b] \to R$, $(-f)(x) = -f(x)$

Clearly $-f \in C[a, b]$, and

$$(f + (-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = 0(x)$$

$\therefore f + (-f) = 0$

Similarly $(-f) + f = 0$

(v) $(f + g)(x) = f(x) + g(x)$

$= g(x) + f(x)$

$= (g + f)(x),\ \forall x \in [a, b]$

$\therefore f + g = g + f$

$\therefore$ C [a, b] is an Abelian group under addition of functions.

II. (i) (fg) (x) = f (x) g (x) $\in \mathbb{R}$

Also, the product of two continous functions is continuous.

$\therefore$ fg is continuous and so fg $\in$ C [a, b].

$\therefore$ Multiplication of functions is a binary operation on C [a, b]

(ii) ((fg) h) (x) = (fg) (x) h (x)

= (f (x) g (x)) h (x)

= f (x) (g (x) h (x))

= f (x) (gh) (x)

= (f (gh)) (x), $\forall x \in [a, b]$

$\therefore$ (fg)h = f (gh)

$\therefore$ C [a, b] is a semigroup under multiplication of functions.

III. ((f+g) h) (x) = (f + g) (x) h (x)

= (f (x) + g (x)) h (x)

= f (x) h (x) + g (x) h(x)

= (f h) (x) + (gh) (x)

= (fh + gh)(x)

$\therefore$ (f + g) h = fh + gh

Similarly f (g + h) = fg + fh

$\therefore$ C [a , b] is a ring under addition and multiplication of functions.



CHECK YOUR PROGRESS - 1

Q 1: Give an example of a commutative ring with unity having only one element.

Q 2: Give an example of a non-commutative ring without unity.

Q 3: Give an example of a commutative ring without unity.

Q 4: Give an example of a non-commutative ring with unity.

Q 5: Give an example of an infinite commutative ring with unity.

Q 6: Let IR^2 = {(a, b) : a, b are reals}

We define

(a, b) + (c, d) = (a + c, b + d),

(a, b) (c, d) = (ac, bd);

a, b, c, d are reals.

Show that IR^2 is a commutative ring with unity.

9.4 PROPERTIES OF RING

Let R be a ring. Then

(i) $a0 = 0a = 0$; for all $a \in R$

(ii) $a(-b) = (-a)b = -(ab)$; for all $a, b \in R$

(iii) $(-a)(-b) = ab$; for all $a, b \in R$

(iv) $a(b - c) = ab - ac$; for all $a, b, c \in R$

Proof (i) we have

$a0 = a(0 + 0)$

$= a0 + a0$, by left distributive law

$\therefore a0 + 0 = a0 + a0$

$\Rightarrow 0 = a0$, by left cancellation law in the additive group R

i. e $a0 = 0$

Similarly $0a = 0$

(ii) We have $a0 = 0$

This $\Rightarrow a((-b) + b) = 0$, $[\because (-b) + b = 0]$

$\Rightarrow a(-b) + ab = 0$, by left distributive law

$\Rightarrow a(-b)$ is the additive inverse of ab

$\therefore$ i.e., $a(-b) = -(ab)$

Similarly $(-a)b = -(ab)$

(iii) We have $(-a)(-b) = l(-b)$, taking $l = -a$

$= -(lb)$, using (ii)

$= -((-a)b)$

$= -(-(ab))$, using (ii)

$= ab$

(iv) $a(b - c) = a(b + (-c))$

$= ab + a(-c)$, by left distributive law

$= ab - ac$, by (ii)

Boolean Ring: Let R be a ring and $a \in R$.

a is called an idempotent element of R if $a^2 = a$.

A ring R is called a Boolean ring, if every element of it is idempotent.

Note: George Boole (1815-1864) of the nineteenth century was ‘‘an essentially self-taught Britisher’’. He is famous for his epoch - making book The Laws of Thought.

Theroem 3: Let R be a Boolean ring. Then

(i) $a + a = 0$, for all $a \in R$

i. e. every element of R is its own additive inverse

(ii) $a + b = 0 \Rightarrow a = b$, for all $a, b \in R$

(iii) R is a commutative ring.

Proof : (i) Let $a \in R$. $\therefore a^2 = a$

Also $a + a \in R$

$\therefore (a + a)^2 = a + a$ [$\because$ R is a Boolean ring]

$\Rightarrow (a + a)(a + a) = a + a$

$\Rightarrow (a + a)a + (a + a)a = a + a$, By left distributive law

$\Rightarrow (a^2 + a^2) + (a^2 + a^2) = a + a$, by right distributive law

$\Rightarrow (a + a) + (a + a) = (a + a) + 0$

$\Rightarrow a + a = 0$, by left cancellation law in the additive group R

(ii) We have $a + a = 0$

$\therefore a + b = a + a$, by (i)

$\Rightarrow b = a$, by left cancellation law in the additive group R

i.e. $a = b$

(iii) Let $a, b \in R \therefore a^2 = a, b^2 = b$

Also $a + b \in R$

$\therefore (a + b)^2 = a + b$

$\Rightarrow (a + b)(a + b) = a + b$

$\Rightarrow (a + b)a + (a + b)b = a + b$, by left distributive law

$\Rightarrow (a^2 + ba) + (ab + b^2) = a + b$

$\Rightarrow (a + ba) + (ab + b) = a + b$

$\Rightarrow (a + b) + ba + ab = (a + b) + 0$, by associative and commutative law of addition.

$\Rightarrow ba + ab = 0$, by left cancellation law in the additive group R.

$\Rightarrow ab = ba$, by (ii)

$\therefore$ R is a commutative ring.



CHECK YOUR PROGRESS - 2

Q 1: Let $R = \{a + b\sqrt{2} : a, b \in Z\}$. Show that R is a commutative ring with unity under usual operations of addition and multiplication.

Q 2: If in a ring R, $a^2 = a$ for all $a \in R$, show that $2a = 0$ and $a + b = 0 \Rightarrow a = b$

Q 3: In z,define $\oplus$ and $\otimes$ as $a \oplus b = a + b + 1, a \otimes b = a + b + ab$. Show that Z is a commutative ring under the operations $\oplus$ and $\otimes$. Is it a ring with unity ?



9.5 LET US SUM UP

- A non-empty set R, equipped with two binary operations, called addition (+) and multiplication (·), is called a ring if the following postulates are satisfied.

(i) R is an additive Abelian group

(ii) R is a multiplicative semigroup

(iii) The two distributive laws hold good, viz.,

a . (b + c) = a.b + a.c, (left distributive law)

(a + b).c = a.c + b.c, (right distributive law)

for all a, b, $c \in R$

- A ring R is said to be commutative if ab = ba, for all a, $b \in R$
- A ring R is said to be a ring with unity (or identity) if there exists an element $e \in R$ such that

ae = a = ea, for all $a \in R$



9.6 ANSWERS TO "CHECK YOUR PROGRESS"

CHECK YOUR PROGRESS-1

Ans to Q No1: {0} is a commutative ring with unity. It should be noted that, in this ring, 0 is the additive as well as the multiplicative identity.

Ans to Q No 2: The ring R = $\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$: a, b are integers$\}$ is a non-commutative ring without unity.

Ans to Q No 3: The ring 2Z of even integers is a commutative ring without unity.

Ans to Q No 4: The ring R of all 2 x 2 matrices over integers is a non-commutative ring with unity.

Ans to Q No 5: The ring Z of integers is an infinite commutative ring with unity.

Ans to Q No 6: $IR^2 = \{(a, b) : a, b \text{ are reals}\}$

I. (i) Let $(a, b), (c, d) \in IR^2$, where a, b, c, d are reals

Now $(a, b) + (c, d) = (a + c, b + d) \in IR^2$

(ii) Let $(a, b), (c, d), (e, f) \in IR^2$

$((a, b) + (c, d)) + (e, f) = (a + c, b + d) + (e, f)$

$= ((a + c) + e, (b + d) + f)$

$= (a + (c + e), b + (d + f))$

$= (a, b) + (c + e, d + f)$

$= (a, b) + ((c, d) + (e, f))$

$\therefore$ addition is associative.

(iii) We have $(0, 0) \in IR^2$,

and $(0, 0) + (a, b) = (a, b) = (a, b) + (0, 0)$

$\therefore$ (0, 0) is the additive identity.

(iv) For every $(a, b) \in IR^2$, we have $(-a, -b) \in IR^2$ such that $(a, b) + (-a, -b) = (0, 0) = (-a, -b) + (a, b)$

$\therefore$ (–a, – b) is the additive inverse of (a, b).

(v) (a, b) + (c, d) = (a + c, b + d)

= (c + a, d + b)

= (c, d) + (a, b)

$\therefore$ addition is commutative.

$\therefore$ IR^2 is an additive Abelian group.

II. (i) (a, b) (c, d) = (ac, bd) $\in IR^2$

(ii) ((a, b) (c, d)) (e, f) = (ac, bd) (e, f)

= ((ac) e, (bd) f)

= (a (ce), b (df))

= (a, b) (ce, df)

= (a, b) ((c, d) (e, f))

$\therefore$ multiplication is associative.

$\therefore$ IR^2 is a multiplicative semigroup.

III. (a, b) ((c, d) + (e, f)) = (a, b) (c + e ,d + f)

= (a (c + e), b (d + f))

= (ac + ae, bd + bf)

= (ac, bd) + (ae, bf)

= (a, b) (c, d) + (a, b) (e, f)

Similarly ((a, b) + (c, d)) (e, f) = (a, b) (e, f) + (c, d) (e, f)

$\therefore$ IR^2 is a ring

Furthermore, (a, b) (c, d) = (ac, bd)

= (ca, db)

= (c, d) (a, b)

$\therefore$ IR^2 is a commutative ring.

Again, (1, 1) $\in IR^2$, and

(a, b) (1, 1) = (a1, b1) = (a, b) = (1, 1) (a, b)

$\therefore$ (1, 1) is the unity.

$\therefore$ IR^2 is a ring with unity.

Thus, IR^2 is a commutative ring with unity

CHECK YOUR PROGRESS - 2

Ans to Q No 1: It can be easily verified that R is a commutative ring with

$$0 = 0 + 0\sqrt{2}$$

as additive identity and $1 = 1 + 0\sqrt{2}$ as the multiplicative identity or unity R.

Ans to Q No 2: Properties of Rings ((i), (ii))

Ans to Q No 3: It can be easily verified that Z is a commutative ring under the given operations.Also $a \otimes 0 = 0 = 0 \otimes a \ \ \forall a \in Z$ .Hence,it is a ring with unity 0.



9.7 FURTHER READINGS

1) Lanski, Charles : *Concepts in Abstract Algebra*, (First Indian edition, 2010) American Mathematical Society.

2) Vasistha, A. R. : *Modern Algebra (Abstract algebra),* Krishna Prakashan Media (p) Ltd. Meerut-250001 (U. P).

3) Vinberg, E. B. : *A course in Algebra* (First Indian edition, 2009), American Mathematical Society).

4) Vatsa, B. S. and Vatsa, Suchi : *Modern Algebra*, New Age International (p) Ltd., New Delhi-110002.

5) Sen, M. K, Ghosh, Shamik and Mukhopadhyay, Parthasarthi : *Topics in Abstract Algebra,* Universities Press (India) Pvt. Ltd., Hyderabad, 500029.



9.8 MODEL QUESTIONS

Q 1: Let Z be the set of integers, We define (+) and (·) on Z as follows

a (+) b = a + b – 1,

a (·) b = a + b – ab, for all a, b $\in$ Z

Examine Z is a commutative ring with unity under (+), (·).

Q 2: Prove that a ring R is commutative if and only if

$(a + b)^2 = a^2 + 2ab + b^2$, for all a, b $\in$ R

[Here a^2 means aa]

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