UNIT 10: INTEGAL DOMAIN & FIELD

UNIT STRUCTURE

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10.1 LEARNING OBJECTIVES

After going through this unit you will be able to know

- Integral domain and its properties
- Field and their properties.

10.2 INTRODUCTION

In Unit 9, we have discussed ring and its various properties. In this unit we will study Integral domain and Field. Integral domain and Field are also a Ring satisfying certain postulates.

10.3 INTEGRAL DOMAIN

Let us consider the ring R of all 2 x 2 matrices over integers.

Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \in R$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \in R$

Here A is not a null matrix, B is not a null matrix, i.e. $A \neq 0$, $B \neq 0$

But $AB = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

i.e, AB is a null matrix.

Thus, we see that $A \neq 0$, $B \neq 0$; but AB = O.

Again, let us consider the ring Z of integers. If $a, b \in Z$ and

$a \neq 0$, $b \neq 0$, then $ab \neq 0$

Definition: An element $a (\neq 0)$ in a ring R is called a zero divisor if there exists an element $b (\neq 0)$ in R such that ab = 0 or ba = 0.

We see that the ring R of all 2 x 2 matrices over integers has zero divisors whereas the ring Z of integers has no zero divisors, for if $a \in Z$, $b \in Z$, $a \neq 0$, $b \neq 0$, then $ab \neq 0$, $ba \neq 0$.

Concellation Laws: Let R be a ring. The additive cancellation laws hold good in R, since R is an additive Abelian group.

So, by cancellation laws in a ring, we mean multiplication laws (cancellation of non zero elements).

The cancellation laws hold good in R if $ab = ac \Rightarrow b = c$, $ba = ca \Rightarrow b = c$; where $a, b, c \in R$, $a \neq 0$.

Theorem 1: The cancellation laws hold good in a ring R if and only if R has no zero divisors.

Proof: Let the cancellation laws hold good in a ring R.

Let $a, b \in R$ such that ab = 0

If $a \neq 0$, $ab = 0 \Rightarrow ab = a0 \Rightarrow b = 0$, by cancellation law.

Similarly, if $b \neq 0$, $ab = 0 \Rightarrow ab = 0b \Rightarrow a = 0$.

So there can be no zero divisor.

Conversely, let there be no zero divisors in R.

Let $a, b, c \in R$ and $ab = ac$, $a \neq 0$

$\therefore ab - ac = 0$

$a(b - c) = 0$

$\therefore a \neq 0$ and R is without zero divisors

$\therefore b - c = 0 \Rightarrow b = c$

Thus, if $a \neq 0$, $ab = ac \Rightarrow b = c$

similarly, if $a \neq 0$, $ba = ca \Rightarrow b = c$

Thus, the cancellation laws hold good in R.

Definition A commutative ring with unity $e \neq 0$ and containing no zero divisors is called an Integral Domain.

The simplest example of an integral domain is the ring Z of integers. It is a commutative ring, has unity and is without zero divisors.

Note: An integral domain may be considered as a generalisation of Z.

Example 1: Let us consider the set R = {a + bi : a, b are integers, $i = \sqrt{-1}$} of Gaussian integers, after the name of C. F. Gauss (1777-1855).

I. (i) Let a + bi, c + di $\in$ R,

a, b, c, d are integers, i = $\sqrt{-1}$

(a + bi) + (c + di) = (a + c) + (b + d)i $\in$ R

(ii) Addition of complex numbers is associative.

(iii) We have 0 = 0 + 0i $\in$ R

and 0 + (a + bi) = a + bi = (a + bi) + 0

(iv) For every a + bi $\in$ R, we have (–a) + (–b)i $\in$ R such that

(a + bi) + ((–a) + (–b)i) = 0 =((–a) + (–b)i) + (a + bi)

(v) Addition of complex numbers is commutative.

$\therefore$ R is an Abelian group under addition of complex numbers.

II. (i) (a + bi) (c + di) = (ac – bd) + (ad + bc)i $\in$ R

(ii) Multiplication of complex numbers is associative

$\therefore$ R is a semigroup under multiplication of complex numbers.

III. Clearly the two distributive laws hold good.

$\therefore$ R is a ring

Furthermore, multiplication of complex numbers is commutative.

$\therefore$ R is a commutative ring.

Also R possesses 1 (1=1 + oi)

$\therefore$ R is a ring with unity.

Again we know that the product of two non-zero complex numbers cannot be zero.

$\therefore$ R is without zero divisors.

Thus, R is an integral domain.

10.4 FIELD

Definition: Let R be a ring with unity e $\neq 0$. If every non-zero element of R has multiplicative inverse in R, then R is called a Division Ring (or a Skew

Field).

A commutative division ring is called a field.

The simplest example of a field is the ring IR of real numbers.

It is commutative, has unity and every non-zero element has multiplicative inverse in IR.

Therefore, IR is a field.

Similarly, the ring C of complex numbers is also a field.

Note: The ring z of integers is not a field. (why?)

Example 2: Examine if the ring

R = {a + bi : a, b, are integers, i = $\sqrt{-1}$ }

of Gaussian integers is a field.

Solution: We know that R is a commutative ring with unity.

(Ref. Example 6)

Let a + bi $\in$ R such that a + bi $\neq 0$

Now the multiplicative inverse of a + bi is $\frac{a}{a^2+b^2}+\left(-\frac{b}{a^2+b^2}\right)i$ which is not an element of R.

$\therefore$ R is not a field.

Theorem 2: Every field is an integral domain.

Proof: Let F be a field.

Let a, b $\in$ F such that a $\neq 0$ and ab = 0

$\therefore$ a $\neq$ 0, $a^{-1} \in$ F (F is a field)

Now ab = 0

$\Rightarrow$ a^{-1} (ab) = a^{-1} 0

$\Rightarrow$ (a^{-1} a) b = 0

$\Rightarrow$ b = 0

Similarly, if b $\neq 0$ and ab = 0, then a = 0

$\therefore$ F has no zero divisors.

Thus, F is a commutative ring with unity and containing no zero divisors.

$\therefore$ F is an integral domain.

Note: The converse of the above theorem is not true, i.e, every integral domain is not a field.

Let us consider the ring Z of integers. We know that Z is an integral domain. But only two elements, viz, –1 and 1 have multiplicative inverses in Z. $\therefore$ Z is not a field.

However we have the following

Theorem 3: A non-zero finite integral domain is a field.

Proof: Let R = $\{a_1, a_2, a_3, ... a_n\}$ be a finite non-zero integral domain, where all the a_i's are distinct.

Let a_1 be the zero element of R.

Let us consider the non zero elements $a_2, a_3, ..., a_n$.

Let a_p be not the unity of R ($p \neq 1$),i.e., $a_p \neq e,$ the unity of R.

Let us consider the products

$a_p a_2,\ a_p a_3, ...,\ a_p a_n$

All these elements belong to R. Also all the elements are distinct, for if

$a_p a_i = a_p a_j$, $2 \leq i \leq n$, $2 \leq j \leq n$, $i \neq j$

then $a_p a_i - a_p a_j = 0$

$a_p \neq e,$

$\Rightarrow a_p (a_i - a_j) = 0$

$\Rightarrow a_i - a_j = 0$, [$\because a_p \neq 0$ and R is without zero divisors]

$\Rightarrow a_i = a_j$, which is a contradiction

Now one of $a_p a_2, a_p a_3, ..., a_p a_n$ must be unity, Let $a_p a_q = e$, where e is the unity of R

Then $a_p a_q = e = a_q a_p$.

$\therefore a_p$ has multiplicative inverse in R. If a_p is the unity of R, then a_p has multiplicative inverse in R.

Thus, every non-zero element of R has multiplicative inverse in R.

$\therefore$ R is a field.

Theorem 4: A division ring has no zero divisors.

Proof: Let R be a division ring.

Let a, b $\in$ R such that $a \neq 0$ and ab=0

$\therefore a \neq 0$, $a^{-1} \in$ R ($\because$ R is a division ring)

Now ab = 0

$\Rightarrow a^{-1} (ab) = a^{-1} 0$

$\Rightarrow (a^{-1}a)b = 0$

$\Rightarrow b = 0$

Similarly, if $b \neq 0$ and $ab = 0$, then $a = 0$.

$\therefore$ R is without zero divisors



CHECK YOUR PROGRESS

Q 1: Does the ring Q of rational numbers contain zero divisors?

Q 2: Let R be the ring of all 2 x 2 matrices over integers, Does the cancellation laws hold good in R?

Q 3: Is the ring 2Z of even integers an integral domain?

Q 4: Give an example of an integral domain which is not a field.

Q 5: Does the ring $IR^2 = \{(a, b) : a, b \in IR\}$ contain zero divisiors?



10.5 LET US SUM UP

- An element $a(\neq 0)$ in a ring R is called a zero divisor if there exists an element b $(\neq 0)$ in R such that ab = 0 or ba = 0
- A commutative ring with unity $e \neq 0$ and containing no zero divisors is called an integral domain
- Let R be a ring with unity $e \neq 0$.
  If every non-zero element of R has multiplicative inverse in R, then R is called a division ring (or a skew field).
  A commutative division ring is called a field.
- Every field is an integral domain, but every integral domain is not a field.
- A non-zero finite integral domain is a field.

10.6 ANSWERS TO "CHECK YOUR PROGRESS"

CHECK YOUR PROGRESS

Ans to Q No 1: No. If a, b $\in$ Q and a $\neq 0$, b $\neq 0$, then ab $\neq 0$

Ans to Q No 2: No. Cancellation laws hold good in a ring if and only if it has no zero divisors. The ring of all 2 x 2 matrices over integers contains zero divisors. So cancellation laws do not hold good in the ring of all 2 x 2 matrices over integers.

For example

$$\begin{pmatrix}1 & 3\\ 3 & 9\end{pmatrix}\begin{pmatrix}3 & 3\\ 5 & 7\end{pmatrix}=\begin{pmatrix}18 & 24\\ 54 & 72\end{pmatrix}$$

$$\begin{pmatrix}1 & 3\\ 3 & 9\end{pmatrix}\begin{pmatrix}6 & 9\\ 4 & 5\end{pmatrix}=\begin{pmatrix}18 & 24\\ 54 & 72\end{pmatrix}$$

i. e. $\begin{pmatrix}1 & 3\\ 3 & 9\end{pmatrix}\begin{pmatrix}3 & 3\\ 5 & 7\end{pmatrix}=\begin{pmatrix}1 & 3\\ 3 & 9\end{pmatrix}\begin{pmatrix}6 & 9\\ 4 & 5\end{pmatrix}$,

i. e. AB = AC, but B $\neq$ C,

where $A=\begin{pmatrix}1 & 3\\ 3 & 9\end{pmatrix}, B=\begin{pmatrix}3 & 3\\ 5 & 7\end{pmatrix}, C=\begin{pmatrix}6 & 9\\ 4 & 5\end{pmatrix}$

Ans to Q No 3: The ring 2Z of even integers is not an integral domain, since it does not contain the multiplicative identity.

Ans to Q No 4: The ring Z of integers is an integral domain, but not a field. Only –1 and 1 have multiplicative inverses in Z,

Ans to Q No 5: Yes.

(7, 0) $\in$ IR^2, (0, 13) $\in$ IR^2 ,(7, 0) $\neq$ (0, 0), (0, 13) $\neq$ (0, 0).

But (7, 0) (0, 13) = (0, 0).



10.7 FURTHER READINGS

1) Lanski, Charles : *Concepts in Abstract Algebra*, (First Indian edition, 2010) American Mathematical Society.
2) Vasistha, A. R. : *Modern Algebra (Abstract algebra),* Krishna Prakashan Media (p) Ltd. Meerut-250001 (U. P).
3) Vinberg, E. B. : *A course in Algebra* (First Indian edition, 2009), American Mathematical Society).
4) Vatsa, B. S. and Vatsa, Suchi : *Modern Algebra*, New Age International (p) Ltd., New Delhi-110002.
5) Sen, M. K, Ghosh, Shamik and Mukhopadhyay, Parthasarthi : *Topics in Abstract Algebra,* Universities Press (India) Pvt. Ltd., Hyderabad, 500029.



10.8 MODEL QUESTIONS

Q 1: Let R = {(a, b) : a, b are reals}. We define

(a, b) + (c, d) = (a + c, b + d)

(a, b) (c, d) = (ac – bd, bc + ad)

Prove that R is a field.

Q 2: Let R = {a + b $\sqrt{2}$: a, b are rational numbers}. show that R is a field under usual addition and multiplication.

Q 3: Let R be a commutative ring with unity and is without zero divisors. Show that the only idempotent elements are the zero element and the unity element.

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