UNIT 11: VECTOR SPACE

UNIT STRUCTURE

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11.1 LEARNING OBJECTIVES

After going through this unit, you will be able to know :

- definition and examples of vector space
- elementary properties of vector space
- subspace of a vector space
- linear dependence and independence of vectors
- basis and dimension of a vector space

11.2 INTRODUCTION

We know the notion of a vector. The concept of a vector space comes from the notion of a vector.In vectors, we have addition of vectors, multiplication of a vector by a scalar, and the scalar product of vectors.In vector space also, we find the addition of vectors,multiplication of a vector by a scalar and the scalar product of vectors.Thus a vector space may be regarded as a generalisation of a vector that is found in mechanics or Analytical geometry.

11.3 DEFINITION AND EXAMPLES OF VECTOR SPACE

Definition : Let V be an Abelian group under " + ". Let F be a field under "+, $\bullet$ ".

For each element $\alpha \in F$, $x \in V$,we have $\alpha x \in V$, called scalar multiplication.

Then V is said to be a vector space over the field F if the following conditions are satisfied.

1. $\alpha(x+y) = \alpha x + \alpha y$
2. $(\alpha + \beta)x = \alpha x + \beta x$
3. $(\alpha\beta)x = \alpha(\beta x)$
4. $1x = x$

where $\alpha, \beta \in F$, $x, y \in V$ and 1 is the unit element of F.

We write "$V(F)$ is a vector space".

Note

1. The element of V are called vectors, and the elements of F are called scalars.
2. $x+y$ denotes the "addition " (+) of two vectors $x, y \in V$, whereas $\alpha + \beta$ denotes "addition " (+) of two scalars $\alpha, \beta \in F$. It should be borne in mind that these two additions are different, though they are denoted by the same symbol.
3. αx denotes the "multiplication" of a vector $x \in V$ by a scalar $\alpha \in F$;whereas $\alpha\beta$ denotes the "multiplication"of two scalars $\alpha, \beta \in F$. It should be borne in mind that these two multiplication are also different ,though they are denoted by the same symbol.
4. V is an Abelian group under " + ". So,we have the operation " + "in V .F is a field under "+, $\bullet$ ".Also,we have multiplication of a vector by a scalar.Thus ,we have altogether four operations in a vector space.
5. V is an Abelian group under " + ".So,we have the additive identity 0 under +,called the zero element.Again F is a field.So,we have the zero element $0 \in F$ .These two zeroes are different,though they are dented by the same symbol.Again F contains the unit element 1.It should be borne in mind that 1 is not necessarily the real number 1.

6. If $F = R$,i.e., if F is the field of real numbers,then $V(R)$ is called a real vector space.

   If $F = C$,i.e., if F is the field of complex numbers,then $V(C)$ is called a complex vector space.

7. We define the "subtraction"in V as follows $x - y = x + (-y)$,where $-y$ is the additive inverse of y ; $x, y \in V$.

Example 1 : Let $R^2 = \{(x_1, x_2) : x_1, x_2 \text{ are reals}\}$,and R be the field of real numbers.We define addition and scalar multiplication in R^2 as follows

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \quad \text{... (1)}$$

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2) \quad \text{... (2)}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2) \in R^2$, $\alpha \in R$.

Then R^2 is a vector space over R,i.e. R^2 is a real vector space.

Solution : I.First we prove that R^2 is an Abelian group under + defined by (1)

(i) $x + y = (x_1 + y_1, x_2 + y_2) \in R^2$

(ii) Let $x = (x_1, x_2)$, $y=(y_1,y_2), z=(z_1,z_2) \in R^2$

Now $(x + y) + z = (x_1 + y_1, x_2 + y_2) + (z_1, z_2)$

$$= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2)$$

$$= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2))$$

$$= (x_1, x_2) + (y_1 + z_1, y_2 + z_2)$$

$$= x + (y + z)$$

$\therefore$ "+" is associative.

(iii) We have $0 = (0,0) \in R^2$

and $x + 0 = (x_1, x_2) + (0,0) = (x_1 + 0, x_2 + 0) = (x_1, x_2) = x$

Similarly, $0 + x = x$

i.e., $x + 0 = x = 0 + x$

[Note that in $0 = (0,0)$, 0 in the L.H.S is a symbol,not the real number 0]

(iv) For every $x = (x_1, x_2) \in R^2$, there exists $-x = (-x_1, -x_2) \in R^2$ such that

$$x + (-x) = (x_1, x_2) + (-x_1, -x_2)$$

$$= (x_1 - x_1, x_2 - x_2)$$

$$= (0,0)$$

$$= 0$$

Similarly, $(-x) + x = 0$

i.e. $x+(-x)=0=(-x)+x$

(v) $x+y=(x_1,x_2)+(y_1,y_2)$

$=(x_1+y_1,x_2+y_2)$

$=(y_1+x_1,y_2+x_2)$

$=(y_1,y_2)+(x_1,x_2)$

$=y+x$

$\therefore R^2$ is an abelian group under "+".

II. we have $\alpha\in R, x=(x_1,x_2)\in R^2$

$\alpha x=\alpha(x_1,x_2)$

$=(\alpha x_1,\alpha x_2)\in R^2$

III. (i) $\alpha(x+y)=\alpha(x_1+y_1,x_2+y_2)$

$=(\alpha x_1+\alpha y_1,\alpha x_2+\alpha y_2)$

$=(\alpha x_1,\alpha x_2)+(\alpha y_1,\alpha y_2)$

$=\alpha(x_1,x_2)+\alpha(y_1,y_2)$

$=\alpha x+\alpha y$

(ii) $(\alpha+\beta)x=(\alpha+\beta)(x_1,x_2)$ $x+$

$=((\alpha+\beta)x_1,(\alpha+\beta)x_2)$

$=(\alpha x_1+\beta x_1,\alpha x_2+\beta x_2)$

$=(\alpha x_1,\alpha x_2)+(\beta x_1,\beta x_2)$

$=\alpha(x_1,x_2)+\beta(x_1,x_2)$

$=\alpha x+\beta x\ ,\alpha,\beta\in R, x=(x_1,x_2)\in R^2$

(iii) $(\alpha\beta)x=\alpha\beta(x_1,x_2)$

$=((\alpha\beta)x_1,(\alpha\beta)x_2)$

$=(\alpha(\beta x_1),\alpha(\beta x_2))$

$=\alpha(\beta x_1,\beta x_2)$

$=\alpha(\beta(x_1,x_2))$

$=\alpha(\beta x)$

(iv) We have $1\in R$

$1x=1(x_1,x_2)=(1x_1,1x_2)=(x_1,x_2)=x$

$\therefore R^2$ is a vector space over R,i.e., R^2 is a real vector space .

Example 2 : Let $R^n=\{(x_1,x_2,...,x_n): x_i\text{'s are reals}\}$, and R be the field

of real numbers. R^n is a real vector space if we define

$$x+y=(x_1,x_2,...,x_n)+(y_1,y_2,...,y_n)=(x_1+y_1,x_2+y_2,...,x_n+y_n)$$

and $\alpha x=\alpha(x_1,x_2,...,x_n)=(\alpha x_1,\alpha x_2,...,\alpha x_n)$

where $x=(x_1,x_2,...,x_n), y=(y_1,y_2,...,y_n)\in R^n$ and $\alpha\in R^n$.

It can be shown as in the previous example.

Example 3 : Let V be the set of all real-valued continuous functions defined on $[a,b]$.Then V is a real vector space if we define addition and scalar multiplication as follows

$$(f+g)(x)=f(x)+g(x),$$
$$(\alpha f)(x)=\alpha f(x),$$

where $f,g\in V$, and $\alpha\in R$.

I. First we prove that V is an Abelian group under addition.

(i) Let $f,g\in V$

$\therefore$ f and g are real valued continuous functions defined on $[a,b]$

$\therefore$ $f+g$ is also a real valued continous function defined on $[a,b]$

$y\in W_1$ $\therefore$ $f+g\in V$.

(ii) Let $f,g,h\in V$ and $x\in[a,b]$

Now, $((f+g)+h)(x)=(f+g)(x)+h(x)$

$$=(f(x)+g(x))+h(x)$$
$$=f(x)+(g(x)+h(x))$$
$$=f(x)+(g+h)(x)$$
$$=(f+(g+h))(x),\ \forall x\in[a,b]$$

$\therefore (f+g)+h=f+(g+h),\ \forall f,g,h\in V$

(iii) Let $\theta:[a,b]\to R$, $\theta(x)=0$, $\forall x\in[a,b]$.

Clearly $\theta\in V$

Now $(\theta+f)(x)=\theta(x)+f(x)$

$$=0+f(x)$$
$$=f(x)$$

$\therefore \theta+f=f$

Similarly, $f+\theta=f$

$\therefore f+\theta=f=\theta+f$.

This shows that θ is the additive identity in V.

(iv) For every $f \in V$, we define $-f$ by the rule

$$(-f)(x) = -f(x), \forall x \in [a,b]$$

Clearly, $(f + (-f))(x) = f(x) + (-f)(x)$

$$= f(x) - f(x)$$
$$= 0$$
$$= \theta(x), \quad \forall x \in [a,b]$$

$$\therefore f + (-f) = \theta$$

Similarly, $(-f) + f = \theta$

$$\therefore f + (-f) = \theta = (-f) + f$$

(v) $(f+g)(x) = f(x) + g(x)$

$$= g(x) + f(x)$$
$$= (g+f)(x), \forall x \in [a,b]$$

$$\therefore f + g = g + f$$

$\therefore$ V is an abelian group under addition.

II.Let $\alpha \in R$, $f \in V$, $x \in [a,b]$

$$(\alpha f)(x) = \alpha f(x) \in R$$

Since f is continuous, $\therefore \alpha f$ is also continous.

$$\therefore \alpha f \in V$$

III. (i) $(\alpha(f+g))(x) = \alpha(f+g)(x), \alpha \in R, f, g \in V$

$$= \alpha(f(x) + g(x)), x \in [a,b]$$
$$= \alpha f(x) + \alpha g(x)$$
$$= (\alpha f + \alpha g)(x), \quad \forall x \in [a,b]$$

$$\therefore \alpha(f+g) = \alpha f + \alpha g$$

(ii) $((\alpha + \beta)f)(x) = (\alpha + \beta)f(x)$

$$= \alpha f(x) + \beta f(x)$$
$$= (\alpha f)(x) + (\beta f)(x)$$
$$= (\alpha f + \beta f)(x) \quad , \forall x \in [a,b]$$

$$\therefore (\alpha + \beta)f = \alpha f + \beta f$$

(iii) $((\alpha\beta)f)(x) = (\alpha\beta)f(x)$

$$= \alpha(\beta f(x))$$
$$= \alpha(\beta f)(x), \quad \forall x \in [a,b]$$

$\therefore (\alpha\beta)f = \alpha(\beta f)$

(iv) $(1f)(x) = 1f(x)$

$= f(x), \forall x \in [a,b]$

$\therefore 1f = f$

$\therefore$ V is a vector space over R,i.e.,V is a real vector space.



CHECK YOUR PROGRESS 1

Q 1: Show that the field of real numbers R is a vector space over R.

Q 2: Show that the field of complex numbers C is a vector space over C.

Q 3: Show that the field of complex numbers C is a vector space over R. Is R is a vector space over C?

Q 4: Let V be the set of all $m \times n$ matrices over reals. Show that V is a vector space over R.

Q 5: Let $V = \{(x_1, x_2) : x_1, x_2 \text{ are reals}\}$, and $\alpha \in R$. Define addition and scalar multiplication in V as follows

$x + y = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

$\alpha x = \alpha(x_1, x_2) = (\alpha x_1, x_2)$

where $x = (x_1, x_2), y = (y_1, y_2) \in V, \alpha \in R$

Show that V is not a vector space.

11.4 ELEMENTARY PROPERTIES

Let V be a vector space over a field F.Let $\alpha, \beta \in F, x, y \in V$ .Then

(i) $\alpha 0 = 0$ (This"0" is the zero element of the Abelian group V)

(ii) $0x = 0$ (The "0" on the left is the zero element of the field F,and the"0" on the right is the zero element of V.)

(iii) if $\alpha x = 0$,then $\alpha = 0$ or $x = 0$

(iv) $(-\alpha)x = \alpha(-x) = -\alpha x$

(v) $\alpha(x - y) = \alpha x - \alpha y$

(vi) $(\alpha - \beta)x = \alpha x - \beta x$

Proof : (i) We have $0 + 0 = 0$

We note that this $0 \in V$

Now $\alpha 0 = \alpha(0 + 0)$

$= \alpha 0 + \alpha 0$

$\therefore \alpha 0 + (-\alpha 0) = \alpha 0 + \alpha 0 + (-\alpha 0)$

$\Rightarrow 0 = \alpha 0$

i.e. $\alpha 0 = 0$

(ii) We have $0 + 0 = 0$

We note that this $0 \in F$

Now $0x = (0 + 0)x$

$= 0x + 0x$

$\therefore 0x + (-0x) = 0x + 0x + (-0x)$

$\Rightarrow 0 = 0x$

i.e. $0x = 0$

(iii) Here $\alpha x = 0$

If $\alpha = 0$, there is nothing to prove.

Let $\alpha \neq 0$, α^{-1} exists.

Now $x = 1x$

$= (\alpha \alpha^{-1})x$

$= \alpha^{-1}(\alpha x)$

$= \alpha^{-1} 0$

$= 0$

(iv) We have $x + (-x) = 0$, the zero element of V

$\alpha + (-\alpha) = 0$, the zero element of F

Now, $\alpha 0 = \alpha(x + (-x))$

$\Rightarrow 0 = \alpha x + \alpha(-x)$

$\therefore (-\alpha x) + 0 = (-\alpha x) + \alpha x + \alpha(-x)$

$\Rightarrow -\alpha x = \alpha(-x)$

Again $0x = (\alpha + (-\alpha))x$

$\Rightarrow 0 = \alpha x + (-\alpha)x$

$(-\alpha x) + 0 = (-\alpha x) + \alpha x + (-\alpha)x$

$\Rightarrow -\alpha x = (-\alpha)x$

Thus, $(-\alpha)x = \alpha(-x) = -\alpha x$.

(v) We have $x - y = x + (-y)$

Also,we have $\alpha(-y) = -\alpha y$

Now $\alpha(x-y)=\alpha(x+(-y))$

$=\alpha x+\alpha(-y)$

$=\alpha x-\alpha y$

(vi) Proceed as above.

Example 4 : Let V be a vector space over the field F.Show that

$x+x=2x\,,\forall x\in V$

Solution : we have

$x+x=1x+1x$

$=(1+1)x$

$=2x\,.$



CHECK YOUR PROGRESS 2

Q 1: Let V be a vector space over the field F.Let $x,y\in V$ and $\alpha,\beta\in F$. Show that (i) if $\alpha x=\beta x$, Then $\alpha=\beta\,,x\neq 0$

(ii) if $\alpha x=\alpha y$, then $x=y\,,\alpha\neq 0$

(iii) if $\alpha x=0$, and $x\neq 0$, then $\alpha=0$

(iv) if $\alpha x=0$, and $\alpha\neq 0$, then $x=0$

(v) if $\alpha x=x$, then $\alpha=1$ or $x=0$.

11.5 SUBSPACE OF A VECTOR SPACE

Let V be a vector space over a field F.Let W be a nonempty subset of V.W is called a subspace of V if W itself is a vector space over F under the same addition and scalar multiplication in V.

Note: 1. V itself is a subspace of V.

2. The set $\{0\}$ consisting of the zero element of the additive group V is a vector space,called the null space. $\{0\}$ is a supspace of V.

Theorem : A non-empty subset W of a vector space V(F) is a subspace of V if and only if W is closed under addition and scalar multiplication.

Proof : Let W be a subspace of V(F).Then,by definition,W is closed under addition and scalar multiplication.

Conversely, let W be closed under addition and scalar multiplication.

Let $a,b\in W$

Now $1 \in F$

$\therefore -1 \in F$

Now $-1 \in F, b \in W$

$\therefore (-1)b \in W$ i.e. $-b \in W$

Thus, $a \in W, -b \in W$

$\therefore a - b \in W$

$\therefore$ W is a subgroup of the additive abelian group V.

All the other conditions for a vector space are satisfied in W, as they are satisfied in V.

$\therefore$ W is a vector space over F.

Theorem : A non-empty subset W of a vector space V(F) is a subspace of V if and only if $\alpha x + \beta y \in W$,for all $\alpha, \beta \in F; x, y \in W$.

Proof : Let W be a subspace of V.

Let $x, y \in W$, $\alpha, \beta \in F$

$\therefore \alpha x \in W$, $\beta y \in W$

$\therefore \alpha x + \beta y \in W$

Conversely, let W be a non-empty subset of V such that for all $\alpha, \beta \in F$; $x, y \in W$, $\alpha x + \beta y \in W$.

Now $1 \in F$

$\therefore 1x + 1y \in W$

i.e., $x + y \in W$

$\therefore$ W is closed under addition.

Again $\alpha \in F$, $0 \in F$

$\therefore \alpha x + 0y \in W$

i.e., $\alpha x \in W$

$\therefore$ W is closed under scalar multiplication.

$\therefore$ W is a subspace of V. (by previous theorem)

Example 5 : Let us consider the vector space $R^2(R)$.

Let $W = \{(a,0) : a \in R\}$

Clearly $W \subseteq R^2$,and $W \neq \phi$

Let $\alpha, \beta \in R$; $(a,0), (b,0) \in W$

$$\alpha(a,0) + \beta(b,0) = (\alpha a, 0) + (\beta a, 0)$$
$$= (\alpha a + \beta a, 0 + 0)$$

$$= (\alpha a + \beta a, 0) \in W$$

$\therefore$ W is a subspace of R^2.

Example 6 : Let us consider the vector space $R^3(R)$.

Let $W = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0, x_1, x_2, x_3 \in R\}$

Clearly, $W \subseteq R^3, W \neq \phi$

Let $\alpha, \beta \in R$; $x = (x_1, x_2, x_3) \in W$, $y = (y_1, y_2, y_3) \in W$

$$\therefore x_1 + x_2 + x_3 = 0$$

$$y_1 + y_2 + y_3 = 0$$

Now, $\alpha x + \beta y = \alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3)$

$$= (\alpha x_1, \alpha x_2, \alpha x_3) + (\beta y_1, \beta y_2, \beta y_3)$$

$$= (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \beta y_3 + \alpha x_3)$$

Now, $(\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) + (\beta y_3 + \alpha x_3)$

$$= \alpha(x_1 + x_2 + x_3) + \beta(y_1 + y_2 + y_3)$$

$$= \alpha 0 + \beta 0$$

$$= 0$$

$\therefore \alpha x + \beta y \in W$

$\therefore$ W is a subspace of V.

Theorem : Intersection of two subspaces of a vector space is a subspace.

Proof : Let V be a vector space over a field F.

Let W_1 and W_2 be two subspaces of V.

$$\therefore W_1 \neq \phi, W_2 \neq \phi$$

$\therefore W_1 \cap W_2 \neq \phi$, since at least the zero element 0 of W is in $W_1 \cap W$.

Let $\alpha, \beta \in F; x, y \in W_1 \cap W_2$

$x \in W_1 \cap W_2 \Rightarrow x \in W_1$ and $x \in W_2$

$y \in W_1 \cap W_2 \Rightarrow y \in W_1$ and $y \in W_2$.

Now $\alpha, \beta \in F$; $x, y \in W_1$ and W_1 is a subspace of V.

$$\therefore \alpha x + \beta y \in W_1$$

Again $\alpha, \beta \in F$; $x, y \in W_2$ and W_2 is a subspace of V.

$$\therefore \alpha x + \beta y \in W_2$$

Thus, $\alpha x + \beta y \in W_1$ and $\alpha x + \beta y \in W_2$

$$\therefore \alpha x + \beta y \in W_1 \cap W_2$$

$\therefore W_1 \cap W_2$ is a subspace of V.

Note : Union of two subspaces may not be a subspace.

We know R^3 is a vector space over R under componentwise addition and componentwise scalar multiplication.

Let $W_1 = \{(x,0,0) : x \in R\}$

$W_2 = \{(0,y,0) : x \in R\}$

Clearly, W_1 is a subspace of R^3, W_2 is a subspace of R^3.

Now $(5,0,0) \in W_1$, $(0,5,0) \in W_2$

But $(5,0,0)+(0,0,5) = (5,5,0) \notin W_1 \cup W_2$

i.e., $(5,0,0) \in W_1 \cup W_2$, $(0,5,0) \in W_1 \cup W_2$

but $(5,0,0)+(0,0,5) \notin W_1 \cup W_2$

$\therefore W_1 \cup W_2$ is not a subspace of R^3.

Theorem: Union of two subspaces is a subspace if and only if one of them is contained in the other.

Proof : Let V be a vector spae over a field F.

Let W_1, W_2 be two subspace of V.

Let $W_1 \subseteq W_2$ .Then $W_1 \cup W_2 = W_2$

$\because W_2$ is a subspace of V,

$\therefore W_1 \cup W_2$ is a subspace of V.

Conversely, let W_1, W_2 be a subspace of V such that $W_1 \cup W_2$ is also a subspace of V.

We are to prove that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

If possible,let it not be the true.

$\therefore W_1 \not\subset W_2$ and $W_2 \not\subset W_1$

$\therefore$ there exists $x \in W_1$ such that $x \notin W_2$ .Also, there exists $y \in W_2$ such that $y \notin W_1$.

Now $x \in W_1 \Rightarrow x \in W_1 \cup W_2$

$y \in W_2 \Rightarrow y \in W_1 \cup W_2$

$\because W_1 \cup W_2$ is a subspace of V.

$\therefore x + y \in W_1 \cup W_2$

This $\Rightarrow x + y \in W_1$ or $x + y \in W_2$

Let $x + y \in W_1$

Now $-x \in W_1$ $(\because x \in W_1), x + y \in W_1$

$\therefore (-x) + x + y \in W_1$

$\Rightarrow (-x+x)+y \in W_1$

$\Rightarrow 0+y \in W_1$

$\Rightarrow y \in W_1$

which is a contradiction.

Similarly,let $x+y \in W_2$

Now $x+y \in W_2, -y \in W_2 \ (\because y \in W_2)$

$\therefore (x+y)+(-y) \in W_2$

$\Rightarrow x+(y+(-y)) \in W_2$

$\Rightarrow x+0 \in W_2$

$\Rightarrow x \in W_2,$

which is a contradiction.

So,our assumption is wrong.

$\therefore$ either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.



CHECK YOUR PROGRESS 3

Q 1: Let $R^3(R)$ be the vector space under componentwise addition and componentwise scalar multiplication.

(i) Let $W=\{(a,2b,3c): a,b,c \in R\}$. Show that W is a subspace of R^3.

(ii) Let $W=\{(x,y,z): x-3y+4z=0, x,y,z \in R\}$. Show that W is a subspace of R^3.

(iii) Let $W=\{(x,y,z): x^2+y^2=z^2, x,y,z \in R\}$. Show that W is not a subspace of R^3.

Q 2: Let $R^n(R)$ be the vector space under componentwise addition and componentwise scalar multiplication.

Let $W=\{(x_1,x_2,\cdots,x_n): x_1+x_2+\cdots+x_n=1\}$.Show that W is not a subspace of R^n.

11.6 LINEAR DEPENDENCE AND INDEPENDENCE

Let V be a vector space over a field F. Vectors $x_1,x_2,x_3,\cdots,x_n \in V$ are said to be linearly dependent over F if there exist scalars

$\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ (not all of them zero) such that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$.

Thus,the vectors $x_1, x_2, x_3, \cdots, x_n \in V$ are linearly dependent if

$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$

$\Rightarrow$ at least one of $\alpha_1, \alpha_2, \cdots, \alpha_n$ is not zero.

If $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \cdots, \alpha_n = 0$;

the vectors $x_1, x_2, x_3, \cdots, x_n$ are linearly independent.

A finite set $\{x_1, x_2, x_3, \cdots, x_n\}$ is said to be linearly dependent or linearly independent according as its n elements are linearly dependent or linearly independent.

A subset W of V is called linearly independent if the elements of W are linearly independent .Otherwise it is said to be linearly dependent.

Note

1. Let $x \in V$,and $x \neq 0$

   Let $\alpha \in F$

   Now $\alpha x = 0$

   $\Rightarrow \alpha x = 0x$

   $\Rightarrow \alpha = 0$,[$\alpha x = \beta x \Rightarrow \alpha = \beta$, if $x \neq 0$]

$\therefore$ x is linearly independent i.e.a non-zero vector is always linearly independent.

2. We have $1.0 = 0$,but $1 \neq 0$,where 1 is the unity of F.

$\therefore$ zero vetor is always lineraly dependent.

3. A superset of a linearly dependent set of vectors is linearly dependent.
4. A subset of a linearly independent set of vectors is linearly independent.

Example 7 : Let us consider the vector space R^2 over R under componentwise addition and componentwise scalar multiplication.

Let us consider the vectors $e_1 = (1,0), e_2 = (0,1)$ in R^2.

Let $\alpha_1, \alpha_2 \in R$

Now, $\alpha_1 e_1 + \alpha_2 e_2 = 0$

$\Rightarrow \alpha_1(1,0) + \alpha_2(0,1) = (0,0)$

$\Rightarrow (\alpha_1, 0) + (0, \alpha_2) = (0,0)$

$\Rightarrow (\alpha_1, \alpha_2) = (0,0)$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0$

$\therefore$ the vectors $e_1 = (1,0), e_2 = (0,1)$ are linearly independent.

Example 8 : Let us consider the vector space R^3 over R under componentwise addition and componentwise scalar multiplication.

Let us consider the vectors $x = (1,2,-1), y = (2,2,1), z = (1,-2,3)$ in R^3.

Let $\alpha_1, \alpha_2, \alpha_3 \in R$

Now $\alpha_1 x + \alpha_2 y + \alpha_3 z = 0$

$\Rightarrow \alpha_1(1,2,-1) + \alpha_2(2,2,1) + \alpha_3(1,-2,3) = (0,0,0)$

$\Rightarrow (\alpha_1, 2\alpha_1, -\alpha_1) + (2\alpha_2, 2\alpha_2, \alpha_2) + (\alpha_3, -2\alpha_3, 3\alpha_3) = (0,0,0)$

$\Rightarrow (\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + 2\alpha_2 - 2\alpha_3, -\alpha_1 + \alpha_2 + 3\alpha_3) = (0,0,0)$

$\Rightarrow \alpha_1 + 2\alpha_2 + \alpha_3 = 0$ ... (1)

$2\alpha_1 + 2\alpha_2 - 2\alpha_3 = 0$

i.e. $\alpha_1 + \alpha_2 - \alpha_3 = 0$ ... (2)

$-\alpha_1 + \alpha_2 + 3\alpha_3 = 0$ ... (3)

Now $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}$

$= 1(1+1) - 2(1-1) + 1(1+1)$

$= 4 \neq 0$

$\therefore$ the only solution of the equations (1),(2),(3) is $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$.

$\therefore$ the vectors $(1,2,-1), (2,2,1), (1,-2,3)$ are linearly independent.

Example 9: Let us consider the vector space $R^3(R)$.Let us consider the vectors $x = (1,1,2), y = (1,2,5), z = (5,3,4)$ in R^3.

Let $\alpha_1, \alpha_2, \alpha_3 \in R$

Now $\alpha_1 x + \alpha_2 y + \alpha_3 z = 0$

$\Rightarrow \alpha_1(1,1,2) + \alpha_2(1,2,5) + \alpha_3(5,3,4) = (0,0,0)$

$\Rightarrow (\alpha_1, \alpha_1, 2\alpha_1) + (\alpha_2, 2\alpha_2, 5\alpha_2) + (5\alpha_3, 3\alpha_3, 4\alpha_3) = (0,0,0)$

$\Rightarrow (\alpha_1 + \alpha_2 + 5\alpha_3, \alpha_1 + 2\alpha_2 + 3\alpha_3, 2\alpha_1 + 5\alpha_2 + 4\alpha_3) = (0,0,0)$

$\Rightarrow \alpha_1 + \alpha_2 + 5\alpha_3 = 0$ ... (1)

$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$ ... (2)

$2\alpha_1 + 5\alpha_2 + 4\alpha_3 = 0$ ... (3)

Now $\begin{vmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{vmatrix}$

$= 1 \times (8-15) - 1 \times (4-6) + 5 \times (5-4)$

$= -7 + 2 + 5 = 0$

$\therefore$ the system of equations possesses non zero solution.

From (1) and (2)

$$\frac{\alpha_1}{3-10} = \frac{\alpha_2}{-(3-5)} = \frac{\alpha_3}{2-1}$$

$$\Rightarrow \frac{\alpha_1}{-7} = \frac{\alpha_2}{2} = \frac{\alpha_3}{1}$$

$\Rightarrow \alpha_1 = -7k, \alpha_2 = 2k, \alpha_3 = k, k \in R$

From (3), $2 \times (-7k) + 5 \times 2k + 4k = 0$

$\therefore$ $(-7k, 2k, k)$ satisfies (3)

Thus, $(-7,2,1)$ is a solution.

$\therefore$ $-7x + 2y + z = 0$

$\therefore$ the vectors $(1,1,2), (1,2,5), (5,3,4)$ are linearly dependent.

Example 10 : Let us consider the vector space R over Q.

We have $1, \sqrt{3} \in R$

Let $\alpha, \beta \in Q$ such that $1\alpha + \sqrt{3}\beta = 0$

If possible, let $\beta \neq 0$

$$\therefore \sqrt{3} = -\frac{\alpha}{\beta}$$

Now $\alpha \in Q, \beta \in Q$ $\therefore -\frac{\alpha}{\beta} \in Q$

$\therefore$ $\sqrt{3} \in Q$, which is a contradiction.

$\therefore$ $\beta = 0$

This $\Rightarrow \alpha = 0$

$\therefore$ $\{1, \sqrt{3}\}$ is linearly independent in R over Q.

Proposition : If two vectors of a vector space are linearly dependent,one of them is a scalar multiple of the other.

Proof : Let V be a vector space over a field F.

Let $x, y \in V$ such that x, y are linearly dependent.

Then there exist scalars α, β not both zero such that

$$\alpha x + \beta y = 0$$

Let $\alpha \neq 0, \therefore \alpha^{-1} \in F$

Now $\alpha x = -\beta y$

$\Rightarrow \alpha^{-1}(\alpha x) = \alpha^{-1}(-\beta y)$

$\Rightarrow (\alpha^{-1}\alpha)x = (-\alpha^{-1}\beta)y$

$\Rightarrow x = \delta y$,taking $-\alpha^{-1}\beta = \delta$

Thus, x is a scalar multiple of y.

Similarly,if $\beta \neq 0$,y is a scalar multiple of x.



CHECK YOUR PROGRESS 4

Q 1: Show that the vectors $(1,1,2),(1,2,5),(5,3,4)$ are linearly dependent in $R^3(R)$.

Q 2: Show that the vectors $(2,3,1),(2,1,3),(1,1,1)$ are linearly dependent in $R^3(R)$.

Q 3: Show that the vectors $(0,1,1),(1,0,1),(1,1,0)$ are linearly independent in $R^3(R)$.

Q 4: Show that the vectors $(1,2,-1),(2,2,1),(1,-2,3)$ are linearly independent in $R^3(R)$.

11.7 BASIS AND DIMENSION

Definition : Let V be a vector space over a field F.Let $x_1, x_2, x_3, \cdots, x_n \in V$.

Any vector $x = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$;where $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ is called a linear combination of the vectors $x_1, x_2, x_3, \cdots, x_n$.

Definition : Let V be a vector space over a field F.A subset S of V is called a basis of V if

(i) any finite number of elements in S are linearly independent,

(ii) each vector in V is a linear combination of a finite number of elements of S.

Example 11 : Let us consider the vector space $R^2(R)$.Let us consider the subset $S = \{e_1, e_2\}$ of R^2 ,where $e_1 = (1,0), e_2 = (0,1)$

Let $\alpha_1, \alpha_2 \in R$ such that

$\alpha_1 e_1 + \alpha_2 e_2 = 0$

$\Rightarrow \alpha_1(1,0) + \alpha_2(0,1) = (0,0)$

$\Rightarrow (\alpha_1, 0) + (0, \alpha_2) = (0,0)$

$\Rightarrow (\alpha_1, \alpha_2) = (0,0)$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0$

$\therefore$ S is linearly independent.

Let $x = (x_1, x_2) \in R^2$

Now $x = (x_1, x_2)$

$= x_1(1,0) + x_2(0,1)$

$= x_1 e_1 + x_2 e_2$

Thus, each vector in R^2 is a linear combination of the elements of S.

$\therefore$ S is a basis of R^2 ,known as the standard basis.

Example 12 : $S = \{e_1, e_2, e_3\}$ is the standard basis of $R^3(R)$,where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.

Example 13 : $S = \{e_1, e_2, \cdots, e_n\}$is the standard basis $R^n(R)$, where $e_1 = (1,0,0,\cdots,0)$, $e_2 = (0,1,0,\cdots,0)$, $\cdots, e_n = (0,0,0,\cdots,1)$.

Example 14 : Let us consider the vector space $R^3(R)$.

Let $T = \{u_1, u_2, u_3\}$be a subset of R^3 , where

$u_1 = (1,1,0), u_2 = (1,0,0), u_3 = (0,1,1)$.

Let $\alpha_1, \alpha_2, \alpha_3 \in R$ such that

$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0$

$\Rightarrow \alpha_1(1,1,0) + \alpha_2(1,0,0) + \alpha_3(0,1,1) = (0,0,0)$

$\Rightarrow (\alpha_1, \alpha_1, 0) + (\alpha_2, 0, 0) + (0, \alpha_3, \alpha_3) = (0,0,0)$

$\Rightarrow (\alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_3) = 0$

$\therefore \alpha_1 + \alpha_2 = 0$

$\alpha_1 + \alpha_3 = 0$

$\alpha_3 = 0$

$\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\therefore$ T is linearly independent.

Let $x = (x_1, x_2, x_3) \in R^3$ such that

$x = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3$, where $\beta_1, \beta_2, \beta_3 \in R$

$\Rightarrow (x_1, x_2, x_3) = \beta_1(1,1,0) + \beta_2(1,0,0) + \beta_3(0,1,1)$

$\Rightarrow (x_1, x_2, x_3) = (\beta_1, \beta_1, 0) + (\beta_2, 0, 0) + (0, \beta_3, \beta_3)$

$\Rightarrow (x_1, x_2, x_3) = (\beta_1 + \beta_2, \beta_1 + \beta_3, \beta_3)$

$\therefore \beta_1 + \beta_2 = x_1$

$$\beta_1 + \beta_3 = x_2$$

$$\beta_3 = x_3$$

$$\therefore \beta_1 = x_2 - \beta_3 = x_2 - x_3$$

$$\beta_2 = x_1 - \beta_1 = x_1 - x_2 + x_3$$

$$\therefore (x_1, x_2, x_3) = (x_2 - x_3)(1,1,0) + (x_1 - x_2 + x_3)(1,0,0) + x_3(0,1,1)$$

Thus, each vector in R^3 is a linear combination of the element of T.

$\therefore$ T is a basis of R^3.

Note : From Example 12 and Example 14,we see that a vector may have more than one basis.

Example 15 : Show that the set $\{(1,1,2),(1,2,5),(5,3,4)\}$ is not a basis of $R^3(R)$.

Solution : We have already proved that the set $\{(1,1,2),(1,2,5),(5,3,4)\}$ is linearly dependent. (Example 9)

$\therefore$ the given set is not a basis of R^3.

Theorem : Let V be a vector space over the field F. If $S = \{u_1, u_2, \cdots, u_n\}$ is a basis of V, then each element of V can be uniquely expressed as a linear combination of $u_1, u_2, \cdots, u_n$.

Proof : By definition of basis, we know that each element of V can be expressed as a linear combination of $u_1, u_2, \cdots, u_n$.

We show that the expression is unique.

Let $x \in V$

If possible, let

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

$$x = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$$

where $\alpha_i \in F$, $\beta_i \in F$, $i = 1,2,\cdots,n.$

$$\therefore \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$$

$$\Rightarrow (\alpha_1 - \beta_1)u_1 + (\alpha_2 - \beta_2)u_2 + \cdots + (\alpha_n - \beta_n)u_n = 0$$

$\because$ $\{u_1, u_2, \cdots, u_n\}$ is linearly independent,

$$\therefore \alpha_1 - \beta_1 = 0, \alpha_2 - \beta_2 = 0, ..., \alpha_n - \beta_n = 0$$

$$\Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_n = \beta_n$$

Thus, the expression is unique.

Definition : Let V be a vector space over a field F.

If the number of elements in a basis of V is finite, then V is said to be a finite dimensional vector space.It is said to be a of finite dimension.Other vector spaces are said to be infinite dimensional.We note that any two bases of finite dimensional vector space have the same number of elements.

The number of elements in a basis of a vector space V is said to be the dimension of V.Generally it is denoted by dimV.THe null space is said to be of 0 dimension.

Example 16 : We know that the standard basis $\{(1,0),(0,1)\}$ of the vector space $R^2(R)$ has two elements.

$\therefore$ dim $R^2 = 2$

Example 17: Dimension of $R^3(R)$ is 3.

Example 16: Dimension of $R^n(R)$ is n.

Example 19 : Show that $\{1, i\}$forms a basis of $C(R)$.What is the dimension of $C(R)$?

Solution : Let $\alpha, \beta \in R$ such that

$$1\alpha + i\beta = 0$$
$$\Rightarrow \alpha + i\beta = 0 + i0$$
$$\Rightarrow \alpha = 0, \beta = 0$$

$\therefore \{1, i\}$is linearly independent.

Now $\alpha + i\beta \in C$

$$\alpha + i\beta = 1\alpha + i\beta$$

Thus, each vector in C is a linear combination of the elements of $\{1, i\}$.

$\therefore \{1, i\}$is a basis.

Number of elements of $\{1, i\}$ is 2.

$\therefore$ dimension of $C(R)$ is 2.

Example 20 : If x, y are elements in a vector space $V(F)$,and $\alpha, \beta \in F$;Show that the set $\{x, y, \alpha x + \beta y\}$is linearly dependent.

Solution : We have

$$(-\alpha)x + (-\beta)y + 1(\alpha x + \beta y) = 0$$

Hence $1 \neq 0$

$\therefore$ the set is linearly dependent.



11.8 LET US SUM UP

- Let V be an Abelian group under " + ". Let F be a field under "+, $\bullet$ ". For each element $\alpha \in F$, $x \in V$, we have $\alpha x \in V$, called scalar multiplication.

  Then V is said to be a vector space over the field F if the following conditions are satisfied.

  1. $\alpha(x+y) = \alpha x + \alpha y$
  2. $(\alpha + \beta)x = \alpha x + \beta x$
  3. $(\alpha\beta)x = \alpha(\beta x)$
  4. $1x = x$

  where $\alpha, \beta \in F$, $x, y \in V$ and 1 is the unit element of F.
- The set $\{0\}$ consisting of the zero element of the additive group V is a vector space,called the null space.
- Let V be a vector space over a field F.Let W be a nonempty subset of V.W is called a subspace of V if W itself is a vector space over F under the same addition and scalar multiplication in V.

  V itself is a subspace of V.

  $\{0\}$ is a supspace of V.
- Intersection of two subspaces of a vector space is a subspace, whereas union of two subspaces may not be a subspace.
- The vectors $x_1, x_2, x_3, \cdots, x_n$ in a vector space V over a field Fare linearly dependent over F if $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$.

  $\Rightarrow$ at least one of $\alpha_1, \alpha_2, \cdots, \alpha_n$ is not zero.

  If $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$

  $\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \cdots, \alpha_n = 0$;

  The vectors $x_1, x_2, x_3, \cdots, x_n$ are linearly independent.
- A non-zero vector is always linearly independent.
- Zero vector is always linearly dependent.

- Let V be a vector space over a field F.Let $x_1, x_2, x_3, \cdots, x_n \in V$.
  Any vector $x = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$;where $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ is called a linear combination of the vectors $x_1, x_2, x_3, \cdots, x_n$.
- Let V be a vector space over a field F.A subset S of V is called a basis of V if
  (i) any finite number of elements in S are linearly independent,
  (ii) each vector in V is a linear combination of a finite number of elements of S.
- The number of elements in a basis of a vector space V is said to be the dimension of V.Generally it is denoted by dimV.The null space is said to be of 0 dimension.



11.9 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1: I. We know that R is an abelian additive group.

II. If $\alpha \in R, x \in R$, then $\alpha x \in R$.

III. (i). $\alpha(x+y) = \alpha x + \alpha y$

(ii). $(\alpha+\beta)x = \alpha x + \beta x$

(iii). $(\alpha\beta)x = \alpha(\beta x)$

(iv). $1x = x$

where $\alpha, \beta \in R$, $x, y \in R$ and 1 is the unit element of R.

$\therefore$ R is a vector space over R.

Ans to Q No 2: Left as an exercise.As above.

Ans to Q No 3: First part : Left as an exercise.

Second Part : R is not a vector space over C

$2+3i \in C$, $5 \in R$

But $(2+3i)5 \notin R$, i.e. R is not closed with repect to scalar multiplication.

Ans to Q No 4: I. We know that V be an Abelian group under matrix addition.

II.If $\alpha \in R, A \in V$,then $\alpha A \in V$.

III.(i). $\alpha(A+B)=\alpha A+\alpha B$

(ii). $(\alpha+\beta)A=\alpha A+\beta A$

(iii). $(\alpha\beta)A=\alpha(\beta A)$

(iv). $1A=A$

where $\alpha, \beta \in R$; $A, B \in V$,1 is the unity element of R.

$\therefore$ V is a vector space over R.

Ans to Q No 5 : Let $\alpha, \beta \in R$ **,** $x=(x_1, x_2) \in V$

Now $(\alpha+\beta)x=(\alpha+\beta)(x_1, x_2)$

$=((\alpha+\beta)x_1, x_2)$

$=(\alpha x_1+\beta x_1, x_2)$

$\alpha x+\beta x=\alpha(x_1, x_2)+\beta(x_1, x_2)$

$=(\alpha x_1, x_2)+(\beta x_1, x_2)$

$=(\alpha x_1+\beta x_1, x_2+x_2)$

$=(\alpha x_1+\beta x_1, 2x_2)$

$\therefore (\alpha+\beta)x \neq \alpha x+\beta x$

$\therefore$ V is not a vector space over R.

CHECK YOUR PROGRESS – 2

Ans to Q No 1: (i) $\alpha x=\beta x$, $x \in V; \alpha, \beta \in F, x \neq 0$

$\Rightarrow \alpha x+(-\beta x)=\beta x+(-\beta x)$

$\Rightarrow \alpha x-\beta x=0$

$\Rightarrow (\alpha-\beta)x=0$

$\Rightarrow \alpha-\beta=0 \quad [\because x \neq 0, 0x=0]$

$\Rightarrow \alpha=\beta$

(ii) $\alpha x=\alpha y$, $x, y \in V; \alpha \in F, \alpha \neq 0$

$\Rightarrow \alpha x-\alpha y=0$

$\Rightarrow \alpha(x-y)=0$

$\Rightarrow x-y=0$ $[\because \alpha \neq 0, \alpha 0=0]$

$\Rightarrow x=y$

(iii) $\alpha x=0, x \neq 0$

$\Rightarrow \alpha x=0x$

$\Rightarrow \alpha=0$

(iv) $\alpha x = 0, \alpha \neq 0$

$\Rightarrow \alpha x = \alpha 0$

$\Rightarrow x = 0$

(v) $\alpha x = x$

$\Rightarrow \alpha x - x = 0$

$\Rightarrow \alpha x - 1x = 0$

$\Rightarrow (\alpha - 1)x = 0$

$\Rightarrow \alpha - 1 = 0$ or $x = 0$

$\Rightarrow \alpha = 1$ or $x = 0$

CHECK YOUR PROGRESS – 3

Ans to Q No 1: (i) Let $\alpha, \beta \in R$, and $x = (a_1, 2b_1, 3c_1) \in W$,

$y = (a_2, 2b_2, 3c_2) \in W$

Now $\alpha x + \beta y = \alpha(a_1, 2b_1, 3c_1) + \beta(a_2, 2b_2, 3c_2)$

$= (\alpha a_1, 2\alpha b_1, 3\alpha c_1) + (\beta a_2, 2\beta b_2, 3\beta c_2)$

$= (\alpha a_1 + \beta a_2, 2\alpha b_1 + 2\beta b_2, 3\alpha c_1 + 3\beta c_2)$

$= (\alpha a_1 + \beta a_2, 2(\alpha b_1 + \beta b_2), 3(\alpha c_1 + \beta c_2))$

$\in W$

$\therefore W$. is a subspace of R^3.

(ii) Left as an exercise.

(iii) Left as an exercise.

Ans to Q No 2: Let $\alpha, \beta \in R$

$x = (x_1, x_2, \cdots, x_n) \in W, y = (y_1, y_2, \cdots, y_n) \in W$

$\therefore x_1 + x_2 + \cdots + x_n = 1$

$y_1 + y_2 + \cdots + y_n = 1$

Now $\alpha x + \beta y$

$= \alpha(x_1, x_2, \cdots, x_n) + \beta(y_1, y_2, \cdots, y_n)$

$= (\alpha x_1, \alpha x_2, \cdots, \alpha x_n) + (\beta y_1, \beta y_2, \cdots, \beta y_n)$

$= (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \cdots, \alpha x_n + \beta y_n)$

Now $(\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) + \cdots + (\alpha x_n + \beta y_n)$

$= \alpha(x_1 + x_2 + \cdots + x_n) + \beta(y_1 + y_2 + \cdots + y_n)$

$= \alpha + \beta$

$\neq 0$, if $\alpha \neq 1-\beta$

$\therefore \alpha x + \beta y \notin W$, if $\alpha \neq 1-\beta$

$\therefore$ W is not a subspace of R^n.

CHECK YOUR PROGRESS – 4

1. Left as an exercise.
2. Left as an exercise.
3. Left as an exercise.
4. Left as an exercise.



11.10 FURTHER READINGS

1) Lanski, Charles : *Concepts in Abstract Algebra*, (First Indian edition, 2010) American Mathematical Society.
2) Vasistha, A. R. : *Modern Algebra (Abstract algebra),* Krishna Prakashan Media (p) Ltd. Meerut-250001 (U. P).
3) Vinberg, E. B. : *A course in Algebra* (First Indian edition, 2009), American Mathematical Society.
4) Vatsa, B. S. and Vatsa, Suchi : *Modern Algebra*, New Age International (p) Ltd., New Delhi-110002.
5) Sen, M. K, Ghosh, Shamik and Mukhopadhyay, Parthasarthi : *Topics in Abstract Algebra,* Universities Press (India) Pvt. Ltd. Hyderabad 500029.



11.11 MODEL QUESTION

Q 1: Prove that a field F is a vector space over any subfield K of F.

Q 2: Let $R^3(R)$ be the vector space under componentwise addition and componentwise scalar multiplication.

Let $\alpha_1, \alpha_2, \alpha_3$ be three fixed elements of R.

Define $W = \{(x_1, x_2, x_3) : \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0, x_1, x_2, x_3 \in R\}$

Show that W is a subspace of $R^3(R)$.

Q 3: Let $R^3(R)$ be the vector space under componentwise addition and componentwise scalar multiplication.

Let $W = \{(x_1, x_2, x_3) : x_2 + 4x_3 = 0, x_1, x_2, x_3 \in R\}$

Examine if W is a subspace of $R^3(R)$.

Q 4: Let R(R) be the vector space of real numbers.Let $x, y, z \in R$. Show that the set $\{x, y, z\}$ is linearly dependent if the set $\{x + ay + bz, y, z\}$ is linearly dependent.

[$\{x + ay + bz, y, z\}$ is linearly dependent,there exist scalar α, β, γ ,not all zero such that $\alpha(x + ay + bz) + \beta y + \gamma z = 0$

This $\Rightarrow \alpha x + (\alpha a + \beta) y + (\alpha b + \gamma) z = 0$

If $\alpha \neq 0$, then clearly $\{x, y, z\}$ is linearly dependent.

If $\alpha = 0$, then at least one of β, γ is not zero; When $\alpha = 0$; $\alpha a + \beta = \beta$, $\alpha b + \gamma = \gamma$.

$\therefore$ at least one of $\alpha a + \beta$, $\alpha b + \gamma$ is not zero.

$\therefore$ the set $\{x, y, z\}$ is linearly dependent.

Q 5: Consider the vector space $R^3(R)$ under componentwise addition and componentwise scalar multiplication.

Examine if the set $\{(1,2,1), (2,1,0), (1,-1,2)\}$ is a basis of $R^3(R)$.

Q 6: Consider the vector space $R^4(R)$ under componentwise addition and componentwise scalar multiplication.

Show that $\{e_1, e_2, e_3, e_4\}$ is a basis of $R^4(R)$, where $e_1 = (1,0,0,0), e_2 = (0,1,0,0), e_3 = (0,0,1,0), e_4 = (0,0,0,1)$.

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