Unit 13: MATHEMATICAL LOGIC

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13.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define statements and examples of statements
- define truth tables about different statements
- know about negation of statements
- know about conjuction, disjunction, conditional and bi-conditional of two statements

13.2 INTRODUCTION

Mathematical logic is the discipline that deal with the methods of reasoning. As all of us know that the main asset that makes humans far more superior than other species is the ability of reasoning. Logic provides rules and techniques for determining whether a given argument or mathematical proof or conclusion in a scientific theory is valid or not. Logic

is concerned with studying arguments and conclusions. Logic is used in mathematics to prove theories and to draw conclusions from experiments in physical science in our every day life to solve many types of problem. Logic is used in computer science to verify the correctness of programs. The rules of logic or techniques of logic are called rules of inference because the main aim of logic is to draw conclusions inferences from given set of hypothesis. In this unit, we will introduce you to the definition and examples of statements, truth table of different statements. We will allso discuss about the logical connectives. Moreover, we will discuss about the converse, opposite and contrapositive of statements.

13.3 DEFINITION OF STATEMENTS

In logic we communicate our ideas or thoughts with the help of sentences in a particular language. The following types of sentences are normally used in our everyday communication.

Assertive sentence : A sentence that makes an assertion is called an assertive sentence or declerative sentence.

For example, " Mars supports life ".is an assertive or a declerative sentence

Imperative sentence : A sentence that expresses a request or a command is called an imperative sentence .

For example, "please bring me a cup of tea." is an imperative sentence.

Exclamatory sentence : A sentence that express some strong feeling is called an exclamatory sentence.

For example, "How big is the whale fish !." is an exclamatory sentence .

Interrogative sentence : A sentence that asks some questions is called an interrogative sentence.

For example, "What is your age ? "is an interrogative sentence .

STATEMENT : *A sentence or proposition is an assertive (or decleartive) sentence which is either true or false but not both. A sentence cannot both true and false at the same time .*

13.3.1 Examples of Statements

IIIUSTRATION 1 : Consider the following sentences :

i) Washington D .C is not in America.

ii) Every rectangle is a square.

iii) The earth is a planet.

iv) Three plus six is 9.

v) The sun is a star.

Each of sentences (iii), (iv) & (v) is a true declerative sentence and so each of them is a statement.

Each of sentences (i) & (ii) is a false declerative sentence and so each of them is a statement.



To denote statements we use the capital letters P, Q, R, ... etc

IIIUSTRATION 2 : Consider the following sentences :

i) Do your home work.

ii) Give me a glass of water.

iii) How are you ?

iv) Have you ever seen Taj Mahal ?

v) May god bless you !

vi) May you live long !

Sentences (i) & (ii) are imperative sentences. so they are not statements. Each of sentences (iii) & (iv) is asking a question. So they cannot be assigned true or false. Hence they are not statements. Each of sentences (v) & (vi) is an imperative optive. So we cannot assign true or false of them and hence none of them is a statement.



CHECK YOUR PROGRESS

Q 1: Find out which of the following sentences are statements and which are not - justify your answer.

i) Paris is in England.

ii) May god bless you ?

iii) 6 has three prime factors .

iv) 18 is less than 16 .

v) How far is chennai from here?

vi) Every rhombus is a square .

vii) There are 35 days in a month.

viii) Two plus three is five.

ix) x +2 = 9

x) The moon is made of green cheese.

13.4 LOGICAL CONNECTIVES

Till now, we have considered primary statements. We often combine simple (primary) statements to form compound statements by using certain connecting words known as logical connectives. Primary statements are combined by means of connectives : *AND , OR, IF---- THEN ,* and *IF AND ONLY IF,* lastly *NOT.*

Now we will discuss in details about logical connectives with their truth tables.

13.4.1 Negation

The denial of a statement P is called its negation and is written as **~P** and read as 'not P'. Negation of any statement P is formed by writing "It is not the case that -------" or "It is false that ---------" before P or inserting in P the word "not".

Let us consider the statement

P : All integers are rational numbers .

The negation of this statement is :

~ P : It is not the case that all integers are rational numbers.

or

~ P : It is false that all integer are not rational number.

or

~ P : At least one integer is not a rational number.

or

Consider now the statement ,

P : 7> 9

The negation of this statement is

~ P : 7 < 9

or

~ P : 7= 9

TRUTH TABLE OF NEGATION

If the truth value of "P " is T, then the truth value of ~P is F. Also if the truth value of "P " is F, then the truth value of ~p is T .

This definion of the negation is summarized as follows by a table .

The truth table of ~P is :

P	~ P
T	F
F	T

Table 2.1

IIUSTRATION EXAMPLES :

Write the negation of the following statements :

(i) $\sqrt{7}$ is a rational.

(ii) $\sqrt{2}$ is not a complex number.

(iii) Every natural number is greater than zero.

(iv) All primes are odd.

(v) All mathematicians are man .

Solution : (i) Let *r* denotes the given statement i.e

r : $\sqrt{7}$ is a rational.

The negation ~ r of this statement is given by

~ r : It is not the case that $\sqrt{7}$ is a rational.

or

~ r : $\sqrt{7}$ is not a rational.

or

~ r : It is false that $\sqrt{7}$ is rational .

(ii) Let the given statement be denoted by *u* i.e

u : $\sqrt{2}$ is not a complex number.

The negation ~ u of this statement is given by

~u : $\sqrt{2}$ is a complex number.

or

~u : It is false that $\sqrt{2}$ is not a complex number.

(iii) The negation of the given statement is :

It is false that every natural number is greater than 0.

or

There exists a natural number which is not greater than 0.

(iv) The negation of the given statement is

There exists a prime which is not odd.

or

Some primes are not odd.

or

At least one prime is not odd.

(v) The negation of the given statement is :

Some mathematician is not man.

or

There exists a mathematician who is not man.

or

At least one mathematician is not man.

or

It is false that all mathematicians are man.



CHECK YOUR PROGRESS

Q 2: Write the negation of the following statements -

i) Bangalore is the capital of Karnataka.

ii) The Earth is round.

iii) The Sun is cold.

iv) Some even integers are prime.

v) Both the diagonals of a rectangle have the same length.

13.4.2 Conjunction

The conjunction i.e joining of two statements P and Q is the statement P $\wedge$ Q which is read as "*P and Q*"

IIIUSTRATION EXAMPLES :

(i) The conjunction of the statements :

P : It is raining.

Q : 2+2 = 4 is P $\wedge$ Q.i.e

It is raining and 2+2=4 .

(ii) Consider the statement

P : The Earth is round and the Sun is cold.

Its components are :

Q : The Earth is round.

R : The Sun is cold.

Truth table : The statement P $\wedge$ Q has the truth value T whenever both P and Q have the truth value T ; Other wise it has the truth value F.

The truth table for conjunction as follows :

P	Q	P $\wedge$ Q
T	T	T
T	F	F
F	T	F
F	F	F

Table 2.2

IIIUSTRATION EXAMPLES :

Translate the following statement into symbolic form :

Ramu and Raghu went to school.

Solution : In order to write it as a conjunction of two statements, it is necessary first to paraphrase the statement as Ramu went to school and Raghu went to school.

Now write P : Ramu went to school.

Q : Raghu went to school

Then the given statement can be written in symbolic form as $P \wedge Q$.

13.4.3 Disjunction

The disjunction of the two statements P and Q is the statement $P \vee Q$ which is read as "*P or Q*".

IIUSTRATION 3 :

(i) Consider the compound statement

P : Two lines intersect at a point or they are parallel.

The component statements of this statement are :

Q : Two lines intersect at a point.

R : Two lines are parallel.

(ii) Consider another statement

P : 45 is a multiple of 4 or 6.

Its component statements are :

Q : 45 is a multiple of 4 .

R : 45 is a multiple of 6.

Truth table : The statement $P \vee Q$ has the truth value F only when both P and Q have the truth value F, $P \vee Q$ is true if either P is true or Q is true (or both Pand Q are true). Truth table for disjunction :

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 2.3



CHECK YOUR PROGRESS

Q 3: Write the following statements in symbolic form :

(i) Pavan is rich and Raghav is not happy.

(ii) Pavan is not rich and Raghav is happy.

(iii) Naveen is poor but happy.

(iv) Naveen is rich or unhappy

(v) Naveen and Amal are both smart .

(vi) It is not true that Naveen and Amal are both smart

(vii) Naveen is poor or he is both rich and unhappy

(vii) Naveen is neither rich nor happy.

IIIUSTRATIVES EXAMPLES :

Write the components statements of the following compound statements and check the compound statement is true or false.

(i) 50 is a multiple of both 2and 5.

(ii) All living things have two legs and two eyes .

(iii) Mumbai is the capital of Gujrat or Maharashtra.

(iv) A rectangle is a quadrilatateral or a 5 sided polygon.

(v) $\sqrt{2}$ is a rational number or an irrational



The connective is not always the same as the word "or "because the fact that the word "or " in English language can be used in two different senses :
i) Inclusive OR (one or the other or both) and
ii) Exclusive OR (one or the other but not both)

Solution : (i) The components statements of the given statement are

P : 50 is multiple of 2

Q : 50 is multiple of 5

we observe that both P and Q are true statements. Therfore the compound statement is true.

(ii) The component statements of the given statement are

P : All living things have two legs.

Q : All living things have two eyes.

we find that both P and Q are false statements. Therefore, the compound statement is false.

(iii) The components statements of the given statement are

P : Mumbai is the capital of Gujrat.

Q : Mumbai is the capital of Maharashtra.

we find that P is false and Q is true. Therefore, the compound statement is true.

(iv) The components statement are

P : A rectangle is a quadrilateral.

Q : A rectangle is a 5 sided polygon.

we observe that P is true and Q is false . Therefore, the compound statement is true.

(v) The compound statement are

P : $\sqrt{2}$ is a rational number.

Q : $\sqrt{2}$ is an irrational number.

clearly P is false and Q is true, therefore the compound statement is true.



CHECK YOUR PROGRESS

Q 4: Write the component statements of the following compound statements and check the compound statement is true or false.

(i) Delhi is in India and 2+2=4.

(ii) Delhi is in England and 2+2=4.

(iii) Delhi is in India and 2+2=5 .

(iv) Delhi is in England and 2+2=5.

(v) Square of an integer is positive or negative.

(vi) The sky is blue and the grass is green .

(vii) The earth is round or the sun is cold.

(viiI) All rational numbers are real and all real numbers are complex.

(ix) 25 is a multiple of 5 and 8.

(x) 125 is a multiple of 7or 8.

13.4.5 Conditional

If P and Q are any two statements, then the statement $P \rightarrow Q$ read as "*if P , then Q*", is called a conditional statement.

Example : Let P : Amulya works hard.

Q : Amulya will pass the examination.

Then $P \rightarrow Q$: If Amulya works hard, then he will pass the exam.

The statement P is called the *antecedent* and Q is called the *consequent* in $P \rightarrow Q$. The sign $\rightarrow$ is called the sign of implication. We will also write $P \rightarrow Q$ for P only if Q.

Truth table for conditional '$P \rightarrow Q$ ' as follows :

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

we will also write $P \rightarrow Q$ for (i) P only if Q (ii) Q If P (iii) Q provided that P, (iv) P is sufficient for Q, (v) Q is necessary conditions for P, (vi) P implies Q, (vii) Q is implied by P.

IIIUSTRATIVES EXAMPLES :

1. Write each of the following statements in the form " If--then"
   (i) You get job implies that your credentials are good.
   (ii) A quadrilateral is a parallogram if its diagonals bisect each other.
   (iii) To get A+ in the class, it is necessary that you do all the excercisesof the book.

Solution :

(i) we know that "If P, then Q" is equivalent to "$P \rightarrow Q$"
Therefore,the given statement can be written as
"If you get a job , then your credentials are good."

(ii) The given statement can be written as-
"If the diagonal of a quadrilateral bisect each other, then it is parallogrm".

(iii) The given statement can be written as

"If you get A+ in the class, then you do all the exercise of the books".

2. Write the component statements of each of the following statements. Also check whether the statements are true or not.

   (i) If a triangle ABC is equilateral , then it is isosceles.

   (ii) If a and b are integers , then ab is a rational number.

Solution :

(i) The component statements of the given statement are:

P : The triangle ABC is equilateral.

Q : The triangle ABC is isosceles.

Since an equilateral triangle is isosceles, so the given statement is true.

(ii) The component statements are :

P : a and b are integers.

Q : ab is a rational number.

Since the product of two integers is an integer and therefore a rational number. So, the compound statement is true.



CHECK YOUR PROGRESS

Q 5: Write down the truth value of each of the following implication.

(i) If 3+2=7, then Paris is the capital of india.

(ii) If 3+4=7,then 3 $\rangle$ 7.

(iii) If 4 $\rangle$ 5,then 5 $\langle$ 6.

(iv) If 7 $\rangle$ 3, then 6 $\langle$ 14.

(v) If two integers a and b are such that a>b, then a-b is always a positive integer.

13.4.6 Biconditional (If and only if implication)

If P and Q are any two statements , then the statement $P \leftrightarrow Q$ which is read as"P If and only if Q" and abbreviated as "Piff Q" is called a biconditional.

The following table defines the biconditional :

The statement $P \leftrightarrow Q$ has the truth value T whenever both P and Q have identical truth values.

Truth table for biconditional ($P \leftrightarrow Q$):

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

The biconditional $P \leftrightarrow Q$ is the conjunction of the conditionals $P \rightarrow Q$ and $Q \rightarrow P$ i.e $(P \rightarrow Q) \wedge (Q \rightarrow P)$ is an alternate notation for $P \leftrightarrow Q$

Also, truth table for $(P \rightarrow Q) \leftrightarrow (Q \rightarrow P)$

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note that both truth tables are identical.

IIIUSTRATIVES EXAMPLES :

1. Consider the statement :

   A triangle is equilateral if and only if it is equiangular.

   This is if and only if implication with the component statements:

   P : A triangle is equilateral.

   Q : A triangle is equiangular.

2. 8 $\rangle$ 4 if and only if 8 -4 is positive.
3. 2+ 2=4 if and only if it is raining.
4. Two lines are parallel if and only if they have the same slope.

Biconditional $P \leftrightarrow Q$ may be read by following way :

a. P if and only if Q

b. P is equivalent to Q

c. P is necessary and sufficient condition for Q

d. Q is necessary and sufficient condition for P

IIIUSTRATION EXAMPLES :

Write the truth value of each of the following biconditional statements.

(i) 4 $\rangle$ 2 if and only if 0 $\langle$ 4-2 .

(ii) 3 $\langle$ 2 if and only if 2 $\langle$ 1.

(iii) 3+5 $\rangle$ 7 if and only if 4+6 $\langle$ 10.

Solution :

(i) Let P : 4 $\rangle$ 2

Q : 0 $\langle$ 4-2

Then, the given statement is $P \leftrightarrow Q$.

Clearly, P is true and Q is true and therefore, $P \leftrightarrow Q$ is true. Hence, the given statement is true, and its truth value is T.

(ii) Let P : 3 < 2

Q : 2<1

Then, the given statement is $P \leftrightarrow Q$.

Clearly, P is false and Q is false and therefore, $P \leftrightarrow Q$ is true. Hence, the given statement is true, and its truth value is T.

(iii) Let P : 3+5 > 7

Q : 4+6<10

Then, the given statement is $P \leftrightarrow Q$



Clearly, P is true and Q is false and therefore, $P \leftrightarrow Q$ is false. Hence, the given statement is false and therefore, its truth value is F.

CHECK YOUR PROGRESS

Q 6: Write down the truth value of each of the following

(i) 3+5=8 if and only if 4+3=7.

(ii) 4 is even if and only if 1 is prime.

(iii) 6 is odd if and only if 2 is odd.

(iv) 2+3=5 if and only if 3>5.

(v) 4+3=8 if and only if 5+4=10.

(vi) 2<3 if and only if 3<4.



13.5 LET US SUM UP

- Mathematical logic is concerned with all kinds of reasoning, whether they be legal arguments or mathematical proofs or conclusions in a scientific theory based upon a set of hypotheses.
- Sentences are usually classified as declarative, exclamatory, interrogative and imperative. In our study of logic, we will confine ourselves to declarative sentences only i.e. we begin by assuming that the object language contains a set of declarative sentences. A primary statement is a declarative sentence which cannot be further broken down or analyzed into simpler sentences.
- New statements can be formed from primary statements through the use of sentential connectives. The resulting statements are called *compound statements.*
- The sentential connectives are called logical connectives.the different types of connectives are: *negation(~), AND(conjunction),OR (disjunction), IF---THEN (conditional), IF AND ONLY IF (Bi-conditional) etc.*

- Truth tables have already been introduced in the definitions of the connectives Our basic concern is to determine the truth value of a statement formula for each possible combination of the truth values of the component statements. A table showing all such truth values is called the truth table of the formula. we constructed the truth table for ~P, $P \vee Q$, $P \wedge Q$, $P \rightarrow Q$ and $P \leftrightarrow Q$. Observe that if the truth values of the components are known, then the truth value of the resulting statement can be readily determined from the truth table by reading along the row which corresponds to the correct truth value of the component statements.
- The statement P is called the antecedent and Q is called the consequent in $P \rightarrow Q$.



13.6 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: Statement : (i), (iii), (iv), (vi), (vii), (viii) & (x).

Ans to Q No 2: (i) Bangalore is not the capital of Karnataka.

(ii) The earth is not round.

(iii) The sun is not cold.

(iii) No even integer is prime.

(iv) There is at least one rectangle whose both diagonals do not have the same length.

Ans to Q No 3: (i) $P \wedge \sim Q$ where P : Pavan is rich
Q : Raghav is happy

(ii) $\sim P \wedge Q$

(iii) $\sim R \wedge H$ where R : Naveen is rich
H : Naveen is happy

(iv) $R \vee \sim H$

(v) $P \wedge Q$ where P : Naveen is smart

Q : Amal is smart

(vi) $\sim(P \wedge Q)$

(vii) $\sim R \vee (R \wedge \sim H)$ where R : Naveen is rich

R : Naveen is happy

(viii) $\sim R \wedge \sim H$

Ans to Q No 4: (i) P : Delhi is in india

Q : 2+2=4

the compound statement is true

(ii) P : Delhi is in England

Q : 2+2=4

the compound statement is false.

(iii) P : Delhi is in india

Q : 2+2=5

the compound statement is false

(iv) P : Delhi is in England

Q : 2+2=5

the compound statement is false

(v) P : Square of an integer is positive

Q : Square of an integer is negative

the compound statement is true.

(vi) P : The sky is blue

Q : The grass is green

The compound statement is true.

(vii) P : The earth is round

Q : The sun is cold

The compound statement is true.

(viii) P : All rational numbers are real

Q : All real numbers are complex.

The compound statement is true.

(ix) P : 25 is a multiple of 5

Q : 25 is a multiple of 8

The compound statement is false.

(x) P : 125 is a multiple of 7

Q : 125 is a multiple of 8

The compound statement is false.

Ans to Q No 5: (i) True (ii) False (iii) True (iv) False (v)True

Ans to Q No 6: (i) True (ii) False (iii) True (iv) False (v) True (vi) True.



13.7 FURTHER READING

1) C. L. Liu, Elements of Diserete Mathemathics, Tata McGraw-Hill Edition

2) Seymour Lipschutz, Marc Lars Lipson, Discrete Mathematics, Tata McGraw-Hill Edition.



13.8 MODEL QUESTIONS

Q 1: Find out which of the following sentences are statements and which are not. Justify your answer.

(i) The real number x is less than 2.

(ii) All real numbers are complex numbers.

(iii) Listen to me, Ravi !

Q 2: Find the component statements of the following and check whether they are true or not:

(i) A line is staight and extends indefinitely in both directions.

(ii) The sky is blue and the grass is green.

(iii) The earth is round or the sun is cold.

(iv) All rational numbers are real and all real numbers are complex

(v) 25 is a multiple of 5 and 8.

Q 3: Write the component statements of each of the following statements.Also, check whether the statements are true or not.

(i) Sets A and B are equal if and only if ($A \subseteq B$ and $B \subseteq A$).

(ii) $|a|<2$ if and only if ($a>-2$ and $a<2$)

(iii) Δ ABC is isosceles if and only if $\angle B = \angle C$.

(iv) $7<5$ if and only if 7 is not a prime number.

(v) Δ ABC is a triangle if and only if AB+BC>AC.

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