UNIT 10: DECISION PROCEDURES

UNIT STRUCTURE

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10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- explain Decision Procedure as an important concept in Symbolic Logic
- analyse the concept of decidability
- analyse the different types of decision procedures
- describe Truth-Table method with example
- discuss Indirect Truth-table method with example
- explain Conjunctive Normal Formal with example.

10.2 INTRODUCTION

As the name suggests, symbolic logic is a system of logic, not very different from traditional logic, but concerned with the use of symbols. Therefore, we can say that symbolic logic has two aims– (i) to symbolise the arguments and expressions, and (ii) to evaluate them. We are concerned with the latter objective. In symbolic logic, and more precisely in the logic of

propositions, there are many methods to test the validity of arguments and argument forms and also there are ways to test or examine whether statement forms are tautologies or not. Such ways or methods that examine whether a given argument or statement forms is valid or invalid, whether it is tautologies or not, is called a decision procedure. It is such a procedure, by which we come to a "decision" i.e. we decide if the argument is valid or invalid, a tautology or not.

There are some standard decision procedures in the logic of propositions. These are–

1) Truth-Table Method
2) Shorter Truth-Table Method
3) Method of Reduction to Conjunctive Normal Form (Cnf.)

10.3 DECIDABILITY

The concept of decision procedure is basically concerned with the concept of decidability, whether a well-formed formula (wff) is valid or not, an inference is sound or not. In other words, we decide the validity or invalidity of 'wff' through a procedure. This is what we roughly mean by decidability. So, decidability requires a decision procedure. Decision procedures are effective since they are believed to be stable, reliable and finite "which can be described in advance for providing in a finite number of steps a 'yes' or 'no' answer to any class of questions". Decision procedure is regarded as mechanical. It works like a computer. If a formula, is valid applied correctly, it would at some point give a positive answer. If the formula under consideration is in fact not valid, it would give a negative answer.

10.4 IS PROOF A DECISION PROCEDURE?

Often, we think that proof and decision procedures are synonymous terms. We need to understand that with regard to 'proof', there remains nothing to be proved as false. In case of proving a formula, we know prior to it's proving that it is logically true. We need to distinguish between 'proving something' and 'testing something'. To prove something, we have no option

to make it false. But in the case of testing something, the 'given' may be either true or false and it is the task of the doer to decide effectively whether the 'given' is true or false. So, this shows that proof establishes logical truth, but cannot establish logical falsity. However, Quin is of the opinion that just because a 'proof' fails to establish logical falsity, this does not prevent us from formulating a procedure to prove validity. But he distinctly shows the difference between a proof and a decision procedure, when he says that 'a decision procedure assures an affirmative and negative answer every time, while a proof procedure assures at best an eventual affirmative answer where an affirmative answer is in order". Further, according to Quine a complete proof procedure can be distinguished from a decision procedure because a 'complete decision procedure' enables us to give both affirmative and negative answers, while a complete proof procedure give a positive or affirmative answer; it cannot give us negative answers.

Following the argument given above about a 'proof' not being a decision procedure, we can say that both Conditional Proof (CP) and Indirect Proof (IP) are not decision procedures since, the rule of CP is used is dealing with valid arguments. But, at this juncture, we face a confusing situation. Indirect Proof is equated with Reductio Ad Absurdum (RAA). But RAA is considered as a decision procedure. If this be the case, then why IP is not considered as a decision procedure? Why do we say Indirect Proof is similar to Reductio Ad Absurdum? The similarity between the two is that in either we attempt to draw contradiction of that which is "given" or we make attempts to reduce an absurdity of the "given". Their difference lies in their procedures. In Reductio Ad Absurdum, the absurdity is derived in terms of truth-values; but in case of Indirect Proof, the absurdity is derived in terms of propositions or propositional variables. In both Reductio Ad Absurdum and Indirect Proof, if the given statement is valid, then its negation is a contradiction. Therefore, the "given" has to be a tautology. But what happens when the given statement is not valid? In this case, when the given statement is not valid then its negation does not lead into a contradiction nor it is proved as a tautology. So in such cases, the given statement is either contingent or inconsistent. With regard to the truth-values, RAA helps to productively decide contingent and

inconsistent statements. In Indirect Proof, the inferential rules fail to decide if the given statement is contingent or inconsistent. Since, IP as a proof procedure, gives only a positive answer to any statement, but fails to give a negative answer. Indirect Proof (IP) cannot be regarded as a decision procedure.

10.5 TRUTH-TABLE METHOD (TT)

The Truth-Table method is an ideal example of a decision procedure. This method helps us to show that one function is equivalent to another function by analyzing the components of the truth-tables. The validity of arguments involving the truth-functional compound statements can be tested by this, truth-table method. The phrase 'truth-function is an important concept in the truth-table method. We can define truth-function as a compound expression which contains variable, where the truth-values are determined solely by the truth-values of its component variables. Through the truth-table method logicians decide whether a given compound expression is a tautology or contradictory or contingency. A detail account of these three types of logical expressions will be given in the Chapter on Truth-table. Now let us find out the validity of an argument with the help of the Truth-table method. We have already stated that the Truth-table method help us to test the validity or invalidity of arguments. Let us consider the following argument.

Premise– If the student works hard, he will be successful.

Premise– The student works hard.

Conclusion– Therefore, the student will be successful.

The logical form of this argument is: $p \supset q$

p

$\therefore\ q$

We can test the validity of the above argument by arranging the premises and conclusion in the following as: $[(pq).p] \supset q$

Now let us construct a truth-table for the above symbolic expression. In this expression, p, q are known as variables, and the symbols '•', '⊃' are known as constants.

1	2	3	4	5
P	q	P ⊃ q	[(p ⊃ q).P]	[(p ⊃ q).P] ⊃ q
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The above Truth-table shows that the given symbolic expression is valid, since in the last column (5) of the truth-table, all the Truth-values are Ts (meaning true). This expression is a tautology. Similarly, an expression will be contradictory if the last column in the truth-table shows all the values as Fs (meaning false). A symbolic expression will be contingent, if the last column of the truth-table shows truth-values as having Ts and Fs.

Thus, we can say that the truth-table method is an effective mechanism, a decision procedure to decide if a given expression is valid or invalid. But this procedure may sometimes be inconvenient if we have more than three variables in a well formed formula (wff). But in spite of a symbolic expression being cumbersome, the truth-table method is regarded as an effective decision procedure.



CHECK YOUR PROGRESS

Q.1: Construct Truth-Tables for the following:

a) (p.q) ⊃ (p ∨ q)

.....

.....

.....

b) (p ⊃ q).~(p ⊃ q)

.....

.....

.....

Q.2: Fill in the blanks:

a) The Truth-table method decides whether a logical expression is a ..... and .....

10.6 THE METHOD OF REDUCTION AD ABSURDUM OR THE INDIRECT TRUTH TABLE METHOD

Another convenient method to test the validity of arguments containing large number of statements is the method of Reductio Ad Absurdum, also known as the Indirect Truth-table method. This method is shorter than the ordinary or direct Truth-table method. This method is mechanical and is used to test the validity of expression containing more number of variables and constants. In Indirect Truth-table method we count backwards. In the direct Truth-table method, we reach the main column at the end, but in the Indirect Truth table method, we start with the main column.

In this method of RAA, we begin with the hypothesis or assumption that the given argument is invalid. To do this, we place an 'F' (meaning false) below the main connective conjoining the premises and the conclusion. With the help of the principles governing the construction of the truth-tables, we assign values to the components of the argument. In the course of doing this activity of assigning values, if we find any discrepancies, then our original hypothesis about the given argument as wrong must be invalid. If such be the case, then the argument itself must be valid. But if the case be that there are no discrepancies when we assign values on the basis of our original hypothesis, then argument must be invalid. Now, let us apply the technique of RAA to test the validity or invalidity of an argument.

$$(p \supset q) \supset (q \supset p)$$

Step I F

$$(p \supset q) \supset (q \supset p)$$

Step 2 T F F

$$(p \supset q) \supset (q \supset p)$$

Step 3 T F T F F

$$(p \supset q) \supset (q \supset p)$$

Step 4 F T T F T F F

In the above example, we begin with the assumption or hypothesis that the argument is invalid. To do this, we put F under the constant $\supset$ joining $p \supset q$ and $q \supset p$ as shown in Step I. The Step 2 shows that main column can

be F when the antecedent ($p \supset q$) is true (T) and the consequent ($q \supset p$) is false (F). In Steps 3 and 4 we show that the consequent $q \supset p$ can be false (F) only if q is true (T) and p is false (F). Here we have the truth-values of p and q as false (F) and true (T) respectively. These values are assigned to p and q in the antecedent ($p \supset q$). After doing so, we find that there is no inconsistency in the expression $(p \supset q) \supset (q \supset p)$. So, our original assumption of putting F under main column is justified. But if this assumption is justified and no inconsistency arises thereby, it means that the argument is invalid.

Thus, we have seen that Reductio Ad Absurdum or the Indirect Truth-table method, when applied to argument or argument forms, is a complete decision procedure. It is a simpler method for testing arguments. Therefore, it is a convenient method to test the validity of complex logical expressions. It gives us a concrete answer to a complex logical statement, either as valid or invalid.



CHECK YOUR PROGRESS

Q.3: Prove the validity of the following arguments /expressions by applying the method of Reductio Ad Absurdum.

a) $[(p \supset q).\sim q) \supset \sim p$

.....

.....

b) $[p \supset q).\sim p) \supset \sim q$

.....

.....

c) The Direct Truth-table method/indirect direct table method is convenient to test the validity of arguments with more variables. (Tick the correct answer).

.....

d) Give the other name for Indirect Truth-Table Method.

.....

10.7 REDUCTION TO CONJUNCTIVE NORMAL FORM (CNF)

Apart from Direct and Indirect truth-table methods, we have another method to test the truth-value of complex logical expressions. This method is the Method of Reduction to Conjunctive Normal Form or what is known in brief as CNF. This method helps us to decide whether a given truth functional expression is a tautology or not. Like the other methods discussed above, this method of CNF is also a decision procedure and be conveniently applied to complex logical expressions having a large number of variables. Here, we need to introduce quite a few reference formulae for purpose of reduction to conjunctive normal form. By reference formulae, we mean equivalent logical expressions which are logically true expressions. The reference formulae are required for CNF in order to reduce a given truth functional expression to its equivalent expression. The reference formulae are:

1) **Principle of Commutation (Comm.)**
   i) $(p.q) \equiv (q.p)$
   ii) $(p \vee q) = (q \vee p)$
2) **Principle of Association (Assoc.)**
   i) $(p.(q.r)) \equiv ((p.q).r)$
   ii) $(p \vee (q \vee r) \equiv ((p \vee q) \vee r)$
3) **Principle of Distribution (Dist.)**
   i) $(p.(q \vee r) \equiv (p.q) \vee (p.r)$
   ii) $(p \vee (q.r) \equiv (p \vee q).(p \vee r)$
4) **Principle of Double Negation (D.N.)**
   $\sim\sim p \equiv p$
5) **De Morgan's Laws (De.M.)**
   i) $\sim(p.q) \equiv \sim p \vee \sim q$
   ii) $\sim(p \vee q) \equiv \sim p.\sim q$
6) **Definition of Material Implication (Impl.)**
   $p.q \equiv (\sim p \vee q)$

Let us understand CNF as a decision procedure by using the Reference Formulae to test the validity of a logical expression.

(p v q).~p) ⊃ q

~((p v q).~p) v q –by Impl.

(~(p v q) v ~~p) v q –by De.M.

((~ p.~q) v p) v q –by De.M., D.N.

((p v ~p).(p v ~q) v q –by Dist

((p v ~p) v q).(p v (q v ~q) –by Dist.

In the illustration of the given example through reduction to conjunctive Normal form, we conclude that the given truth – functional expression is a Tautology. Because, the last equivalent expression is a conjunction of two tautological disjunctions– ((p v ~p) v q) and ((p v (q v ~q). [A detailed account with illustrations will be given in the chapter on Reduction to conjunctive Normal Form]. In CNF, an argument is considered to be valid if its equivalent expression is a tautological Conjuncture Normal Form (CNF). But the argument will be invalid, if it cannot be reduced to a conjunction of tautologies.



CHECK YOUR PROGRESS

Q.4: Reduce the formulae/expressions to C.N.F.

a) ((p ⊃ q).~ q) ⊃ ~p

.....

.....

.....

b) ((p ⊃ q).(q ⊃ r)) ⊃ (p ⊃ r)

.....

.....

.....

c) What do you mean by Reference formulae?

.....

.....

d) State the method that is used to test whether a given argument is a tautology?

.....



10.8 LET US SUM UP

- Decision procedure is a purely mechanical or computing system to test the validity or invalidity of arguments. But we should remember that the Truth value assignments, is no doubt, an effective mechanism to test the validity of an argument, we can arrive at certain conclusions from certain premises. But in no way, we can ascertain how to draw the conclusions. A computer or a machine generates its actions only if its operation is conducted by certain rules. In a similar fashion, we can say that Decision procedure will be operative and adequate only if certain rules are followed. The rules alone, and not the method or process, can show how one statement is deduced from another.
- Logic is often shown to be dealing with mathematical reasoning, but this does not amount to asserting that logic exclusively relies on the basis of matrices and truth value assignments. We, no doubt, apply matrices and truth-value assignments to test Mathematical and Logical reasoning. But they can, in no way, be equivalent to mathematics and logic. We can derive a conclusion from certain premise/premises, not through assignment of truth values, but only on the basis of some rules. We can, thus, conclude that decision procedures, does not make rules redundant, but that all decision procedures operate on the basis of certain rules. The three types of Decision Procedures are convenient in allowing us to draw a conclusion, whether an argument is valid or invalid, and a tautology or not.



10.9 FURTHER READING

1) Basson, A. H., O'Connor, D. J.; *Introduction to Symbolic Logic*; Kolkata: Oxford University Press.
2) Copi, Irving M., Cohen and Irving; *Introduction to Logic*, (Ninth Edition); New Delhi: Prentice-Hall of India Pvt. Ltd.

3) Copi, Irving M.; *Symbolic Logic*, New Delhi: Prentice Hall of India Pvt. Ltd.
4) Eberle, Rolf A.; *Logic And Proof Techniques*; Kolkata: New Central Book Agency (P) Ltd.
5) Halberstad, T.; William H.; *Introduction to Modern Logic*; Harper & Row, Publishers New York, Evanston And London.
6) Jain, Krishna; *Logic an Introduction*; New Delhi: Ajanta Books.
7) Quine, W. V.; *Philosophy of Logic*; New Delhi: Prentice Hall of India Pvt. Ltd.



10.10 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 4: I) DIRECT TRUTH-TABLE METHOD

a) $(p.q) \supset (p \vee q)$

1	2	3	4	5
P	Q	P.q	P v q	$(p.q) \supset (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T

The above Truth-Table proves that the given expression is valid and is a tautology. Because, the last column (5) of the table shows that the expression is true (T) under all circumstances and is therefore valid.

b) $(p.q).\sim (p \supset q)$

1	2	3	4	5
P	Q	$P \supset q$	$\sim(p \supset q)$	$(p \supset q).\sim(\sim p \supset q)$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	F	F

From the above truth-table and as shown in the last column (5) where all are Fs, the expression is invalid and is therefore contradictory.

Ans. to Q. No. 2: Valid or invalid.

Ans. to Q. No. 3: a) Indirect Truth-table method or the Method of Reductio Ad Absurdum

$((p \supset q).\sim q) \supset \sim P$

Step 1 F

$((p \supset q).\sim q) \supset \sim P$

Step 2 T F F

$((p \supset q).\sim q) \supset \sim P$

Step 3 T T T F F

$((p \supset q).\sim q) \supset \sim P$

Step 4 T T T F F F T

$((p \supset q) . \sim q) \supset \sim P$

Step 5 T T F T T F F F T

In the Indirect truth-table constructed above we put an F under the main variable in Step 1 as required to do so in the method of Reduction Ad Absurdum. By following the logical rules we continue with the different steps. In Step 4, we find that the value of p is true and q is false. But if we assign these values to ($p \supset q$), where $p \supset q$ is already stated to be true, we are not permitted to put T under P and F under q because it violates the truth-functional use of '$\supset$'. Thus if we put F under the main column of the given expression, then it violates the basic rules of the truth table procedure. In other words, the original expression is valid.

b) $((p \supset q).\sim P) \supset \sim q$

Step 1 F

$((p \supset q).\sim P) \supset \sim q$

Step 2 T F F

$((p \supset q).\sim P) \supset \sim q$

Step 3 T T T F F

$((p \supset q).\sim P) \supset \sim q$

Step 4 F T T T T F F F T

In the Indirect truth table constructed for the given expression, we begin with Step 1 by putting an F under the main variable/ column. We find that in the subsequent steps, there is no violation of truth-functional procedures and the basic rules of truth-table method. Thus we can conclude that an F assumption in the main column leads to no violation of rules and therefore the given expression is invalid.

c) Indirect Truth Table Method

d) Method of Reductio Ad Absurdum.

Ans. to Q. No. 4: a) $[(p \supset q).\sim q) \sim P$

$\sim((\sim p \vee q) \vee \sim\sim q) \vee \sim P$ –by Mat Impln.

$(\sim (\sim p \vee q) \vee \sim\sim q) \vee \sim p$ –by D.N. and De.M.

$((P. \sim q) \vee q) \vee \sim P$ –by D.N. and De.M.

$((q \vee P).(q \vee \sim q)) \vee \sim P$ –by Comm. and Dist.

$((p \vee \sim P) \vee q).((\sim P \vee q) \vee \sim q$ –by Comm. and Dist.

This expression is now in Conjuncture Normal Form.

b) $((p \supset q).(q \ r)) \supset (p \supset r)$

$\sim(\sim(p \vee q).(\sim q \vee r) \vee (\sim p \vee r)$ –by Mat. Impln.

$(\sim(\sim p \vee q) \vee \sim(\sim q \vee r)) \vee (\sim p \vee r)$ –by De.M. and dropping double negation.

And a second application of the rule gives:

$((p.\sim q) \vee (q.\sim r)) \vee (\sim P \vee r)$

$((P \vee q).(p \vee \sim r).(q \vee \sim q).(\sim q \vee \sim r) \vee (\sim P \vee r)$

–by application of Distributive law to the first two members of the disjunction.

$(p \vee \sim p \vee q \vee r).(p \vee \sim p \vee r \vee \sim r).\ (\sim P \vee q \vee \sim q \vee r.(\sim P \sim q \vee r \vee \sim r)$

–by applying commutative law to the second part of the disjunction to bring it to the front of the expression and then apply Distributive Law.

The expression is shown to be in conjunctive Normal form.

c) Reference Formulae are equivalent logical expressions which are logically true expressions.

d) Reduction to Conjuncture Normal Form.



10.11 MODEL QUESTIONS

A) Very Short Questions:

Q.1: What do you mean by a Decision Procedure?

Q.2: What are the different types of Decision Procedures?

Q.3: What do you mean by Decidability?

Q.4: What are different types of statements referred to in Truth-table method?

B) Short Questions: (Answer each question within 150 words)

Q.1: What is a Proof?

Q.2: Is Proof a decision procedure?

Q.3: Explain Direct Truth-table method with the help of an example.

Q.4: Explain Indirect Truth-table method or the method of Reductio AD Absurdum.

Q.5: Construct Truth-tables, Direct and Indirect to test the validity of the given expressions.

i) $(p \supset q) \vee (\sim p \vee \sim r)$

ii) $[(p.q) \supset r] \vee (\sim p \supset (q \supset r)$

iii) $[\sim(p \supset q)] \supset (\sim P.\sim q)$

iv) $[P \supset (q \supset r)] \supset [\ (P \supset q) \supset (P \supset r)$

Q.6: Reduce the following to Conjunctive Normal Form

i) $\sim(F \supset G) \supset (E \supset \sim I)$

ii) $(p \vee q) \supset P$

iii) $(p \supset q).\sim q) \supset \sim P$

Q.7: What do you mean by Reduction to Conjunctive Normal Form? State the Reference Formulae.

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