GEC S3 02 (M)

Quantitative Methods for Economic Analysis - I

SEMESTER - III

ECONOMICS

BLOCK - 2

KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

Subject Experts

1. Professor Madhurjya Prasad Bezbaruah, Dept. of Economics, G.U.
2. Professor Nissar Ahmed Barua, Dept. of Economics, G.U.
3. Dr. Gautam Mazumdar, Dept. of Economics, Cotton University, Guwahati.

Course Coordinator : Bhaskar Sarmah, Ph. D, KKHSOU

SLM Preparation Team

Units	Contributor
7	**Dr. Dipu Nath Sarma,** Handique Girls College, Guwahati.
8, 9, 14	**S. K. Jha,** Lumding College.
10, 12, 13	**Bhaskar Sarmah** & **Utpal Deka,** KKHSOU.
11	**Dr. Parag Dutta,** Dispur College
	Dr. Dibyajyoti Mahanta, KKHSOU
15	**Mr. Sudarshan Narzary,** Department of Economics, Guwahati College.

Editorial Team

Content	:	**Prof. N.R. Das,** Gauhati University
		Dr. Sanjay Dutta, PETC, Gauhati University
		Dr. Chandrama Goswami, KKHSOU.
Structure Format & Graphics :		**Harekrishna Deka,** KKHSOU
		Bhaskar Sarmah, KKHSOU.

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BLOCK INTRODUCTION

This is the second and last block of the course entitled *'Quantitative Methods for Economic Analysis - I'*. This block consists of nine units, from Unit numbers 7 to 15. Unit 7 discusses the concepts of limit and continuity. Thus, through this unit, you will be able to gether preliminary knowlege of limit and continuity. In the next unit, the concept of Differentiation has been discussed. Differentiation is one of the most frequently used mathematical concepts in Economics. Through this Unit, you will be able to derive the meaning of derivative, its economic interpretation and the bassic rules of differentiation. The next unit discussed the related concepts of partial and total derivatives. Thus, these two units will help you gather knowledge about the concept of differentation. In the tenth unit, we shall examine some of the applications of derivatives in the field of Economics, particularly relating to total revenue, average revenue, marginal revenue and elasticities among others. Unit 11 discusses the cancept of maxima and minima. As we know, in Economics, we often confront the problems of maximisation and minimisation. For example, we know that a firm often tries to maximinise his profit, but wants to minimise the cost of production. There are many such problems. The theoretical dicussion of maxima and minima has been applied in the next two units to discuss some of the problems of maximisation and minimisation. The fourteenth unit of the course disucsses the concept of integration, i.e., integral calculus. Probably, you have known that integration is the reverse process of differentiation. In this unit, we shall discuss the basic concept and methods of integration. The next unit, which is also the last unit of the course examines some of the applications of integral calculus in Economic problems. Thus, after going through all the units of this course, it is expected that you would gain the basic knowledge of mathematics and will also be able to solve a few economic problems throuhg the use of mathematical principles. This course would also considerably help you in further study of the subject Economics.

While going through a unit, you will notice some along-sode boxes, which have been included to help you know some of the difficult, unseen terms. Some 'ACTIVITY' (s) has been included to help you apply your own thoughts. Again, we have included some relevant concepts in 'LET US KNOW' along with the text. And, at the end of each section, you will get 'CHECK YOUR PROGRESS' questions. These have been designed to self-check your progress of study. It will be better if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with 'ANSWERS TO CHECK YOUR PROGRESS' given at the end of each unit.

CONTENTS

	Page Nos.
Unit 7: Limit and Continuity	125 – 136
Limits: Finding Limits Analytically, Examples of Evaluations of Limits Using Various Rules, Limits by the Method of Substitution: A Special Limit, Some Other Special Limits, One Sided Limits; Continuity: Basic Definitions and Example	
Unit 8: Differentiation	137 – 148
Concept of Derivative of a Function; Geometrical Interpretation of Derivative of a Function; Economic Interpretation of Derivative of a Function; Rules of Differentiation: The Rules for Sums and Differences, Product Rule of Differentiation, Derivative of composite functions or Chain Rule of Differentiation; Second Order Derivative; Derivative of Various Standard Functions; Examples of One Variable Derivative	
Unit 9: Partial Derivatives and Total Derivatives	149 – 161
Function of Several Variables; Partial Derivatives; Chain Rule of Partial Differentiation or Total Differentiation	
Unit 10: Economic Applications of Derivatives	162 – 178
Economic Applications of Derivatives: Elasticity of Demand (ed), Deriving the Relationship between AR, MR and ed, Deriving the Relationship between AC and MC, Utility Function, Indifference Curve (IC) Function, Iso-Quant (IQ) Function, Relationship between MR, MP and MPR, Consumption Function, Saving Function, Import Function, Elasticity of Supply (e_s); Numerical Problems	
Unit 11: Concepts of Maxima and Minima	179 – 190
Maximum and Minimum of a Function: Methods for Finding Maximum and Minimum	
Unit 12: Economic Applications of Maxima	191 – 205
Economic Applications of Maxima	
Unit 13: Economic Applications of Minima	206 – 216
Economic Applications of Minima	
Unit 14: Integration and Methods of Integration	217 – 233
Integration of Simple Functions; Properties of Indefinite Integrals; Methods of Integration: Integration by Substitution, Integration by Parts, Integration by Partial Fractions; Definite Integral	
Unit 15: Applications of Integral Calculus to Economic Problems	234 – 259
Derivation of Total Functions from Marginal Functions; Estimation of Consumer's Surplus; Estimation of Producer's Surplus	

UNIT 7: LIMIT AND CONTINUITY

UNIT STRUCTURE

7.1 Learning Objectives

7.2 Introduction

7.3 Limits

- 7.3.1 Finding Limits Analytically
- 7.3.2 Examples of Evaluations of Limits Using Various Rules
- 7.3.3 Limits by the Method of Substitution
- 7.3.4 A Special Limit
- 7.3.5 Some Other Special Limits
- 7.3.6 One Sided Limits

7.4 Continuity

- 7.4.1 Basic Definitions and Example

7.5 Let Us Sum Up

7.6 Further Reading

7.7 Answer To Check Your Progress

7.8 Model Questions

7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define the concept of limit.
- evaluate limits by algebraic rules.
- explain the concept of continuity of a function.
- show that some well known functions are continuous.

7.2 INTRODUCTION

So far, we have discussed the Set theory and functions.

In this unit, we shall discuss one of the fundamental concepts in Mathematics, which is of paramount importance to the study of differential calculus. This concept is known as 'Limit'.

The concept of limit is an abstract one. We will learn various algebraic

rules to find limits at various points of well-known functions. We will also learn about left and right hand limits. We will also discuss about the continuity and differentiation of functions. Finally, we will discuss about the concept of integration. We will also discuss about two powerful methods of integration, namely, methods of substitution and the method of integration by parts.

7.3 LIMITS

We are already familiar with the concept of a real function and a real variable. The concept of limit is an abstract one. Consider the function $f(x) = \frac{x^2 - 9}{x - 3}$, which is defined at all points except $x = 3$. In the first column of following table, some values of x close to 3 are written and in the second column the corresponding values of $f(x)$ are written.

The value of the variable	The corresponding value of $f(x) = \frac{x^2 - 9}{x - 3}$
2.97	5.97
2.98	5.98
2.99	5.99
3.01	6.01
3.02	6.02
3.03	6.03

From the above table, we observe that if x is very close to 3, then $\frac{x^2 - 9}{x - 3}$ is very close to 6. It is also true that for all values of x close to 3, $\frac{x^2 - 9}{x - 3}$ is close to 6. We then say:

As x tends to 3, the function $\frac{x^2 - 9}{x - 3}$ tends to 6 or equivalently, we say: The function $\frac{x^2 - 9}{x - 3}$ has the limit 6 at the point 3.

In symbols, we write

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=6.$$

In general, by $\underset{\substack{x\to a\\ |x-a|x\to a}}{\text{Lim}}\ f(x)=l$ we mean that as gets close to *a* then $f(x)$ gets close to a limit of l. This can be put in a more precise way as: $|f(x)-l|$ can be made arbitrarily small, by taking $|x-a|$ to be sufficiently small.

> The value of the function *f* at *a* is immaterial for finding the limit of *f* at *a*. Only the values that are close to *a* but not at *a* matter.

The following two results are simple consequences of the definition.

1. **Constant Rule:** $\lim_{x\to a} c=c$, that is, the limit of the constant function $f(x)=c$ as x tends to a is the value *c*.

2. **Identity Rule:** $\lim_{x\to a} x=a$, that is, the limit of the identity function $f(x)=x$ as x tends to *a* is the value a.

Uniqueness of Limit

> If $\lim_{x\to a} f(x)$ exists, it is unique. There cannot be two distinct numbers l_1 and l_2 such that when x tends to *a* the function $f(x)$ tends to both l_1 and l_2.

LET US KNOW



(1) The symbol $x\to a$ (read as x tends to a) means that 'x' assumes successive values one after another either from the right or from the left and approaches very near to a, so that $|x-a|$ becomes almost zero.

(2) Distinction between f(a) and $\lim_{x\to a} f(x)$:

(i) f(a) stands for the value of f(x) when x is exactly equal to a ... f (a) may be otbained either by definition of the function f(x) at x=a or by putting x=a in f(x) when it exists.

(ii) f(a) and $\lim_{x\to a} f(x)$ may be equal or unequal when both exist.

(iii) both of f(a) and $\lim_{x\to a} f(x)$ may exist or one of them may exist and other may not exist.

7.3.1 Finding Limits Analytically

It is possible to find limits by using algebraic techniques. The limit behaves well with respect to the operations of addition, subtraction, multiplication by a constant, multiplication, division, etc. The most basic theorem governing analysis of limits is the Principal Limit Theorem.

Principal Limit Theorem:

Let *a* and *k* be real numbers, $n \geq 0$, and let f and g be functions with limits at *a* such that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$. Then

1. **Sum Rule:** $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = l + m.$

2. **Difference Rule:**

$\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = l - m.$

3. **Product Rule:** $\lim_{x \to a}[f(x) \bullet g(x)] = \lim_{x \to a} f(x) \bullet \lim_{x \to a} g(x) = l \bullet m.$

4. **Quotient Rule:** $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l}{m}$ provided $\lim_{x \to a} g(x) = m \neq 0.$

5. **Coefficient Rule:** $\lim_{x \to a} k f(x) = k \lim_{x \to a} f(x) = kl.$

6. **Power Rule:** $\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n = l^n.$

We can state the above rules in language. For example, Sum Rule could be expressed as **"The limit of a sum is the sum of the limits."**

7.3.2 Examples of Evaluation of Limits Using Various Rules

Example 7.1: Find $\lim_{x \to 2} x^2$.

Solution: $\lim_{x \to 2} x^2 = \left(\lim_{x \to 2} x\right)^2$ (by Power Rule)

$= 2^2$ (by Identity Rule)

$= 4$

Example 7.2: Find $\lim_{x\to a} 2x^3+3x^2+4x+5.$

Solution:

$$\lim_{x\to a} 2x^3+3x^2+4x+5$$

$= 2\lim_{x\to a} x^3+3\lim_{x\to a} x^2+4\lim_{x\to a} x+\lim_{x\to a} 5$ (by Sum and Coefficient Rules)

$= 2\left(\lim_{x\to a} x\right)^3+3\left(\lim_{x\to a} x\right)^2+4\lim_{x\to a} x+\lim_{x\to a} 5$ (by Power Rule)

$= 2a^3+3a^2+4a+5.$ (by Identity and Constant Rules)

Remark: It is clear from the above example that for all polynomial functions, the limit at a point can be obtained by substituting that value for the variable *x*. But, for non-polynomial functions, this may not be true always. The next two examples show this fact.

Example 7.3: Find $\lim_{x\to 1}\frac{2x-1}{x^3-2}$.

Solution:

$$\lim_{x\to 1}\frac{2x-1}{x^3-2} = \frac{\lim_{x\to 1}(2x-1)}{\lim_{x\to 1}(x^3-2)}$$ (by Quotient Rule)

$$= \frac{\lim_{x\to 1}2x-\lim_{x\to 1}1}{\lim_{x\to 1}x^3-\lim_{x\to 1}2}$$ (by Difference Rule)

$$= \frac{2\lim_{x\to 1}x-\lim_{x\to 1}1}{(\lim_{x\to 1}x)^3-\lim_{x\to 1}2}$$ (by Coefficient and Power Rules)

$$= \frac{2(1)-1}{(1)^3-2}$$ (by Constant and Identity Rules)

$= -1.$

Example 7.4: Find $\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2}$.

Solution: Since the denominator function x^2-3x+2 has the limit 0 at the point 1, that is, $\lim_{x\to 2} x^2-3x+2=0$, so Quotient Rule cannot be applied here. Therefore, we have to resort to another method. Factorizing both the numerator and the denominator, we find that

$$\frac{x^2-4}{x^2-3x+2}=\frac{(x-2)(x+2)}{(x-1)(x-2)}.$$

When $x \neq 2$, we can cancel the common factor $x-2$.

Thus,

$$\frac{x^2-4}{x^2-3x+2}=\frac{x+2}{x-1}.$$

Therefore,

$$\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2}=\lim_{x\to 2}\frac{x+2}{x-1}=\frac{\lim_{x\to 2}x+\lim_{x\to 2}2}{\lim_{x\to 2}x-\lim_{x\to 2}1}=\frac{2+2}{2-1}=3$$



CHECK YOUR PROGRESS

Q 1: Write true (T) or false (F).

(a) $\lim_{x\to 1}(x-1)^2+5=5.$ (T / F)

(b) $\lim_{x\to 2}(x-2)(x+4)-0.$ (T/F)

(c) $\lim_{x\to 3}\frac{x^2-9}{x-3}=0.$ (T / F)

(d) $\lim_{x\to -1}\frac{x^2-1}{x+1}=2.$ (T/F)

7.3.3 Limitis by the Method of Substitution

Consider the example $\lim_{x\to 1}\frac{x^2-1}{x-1}$. We find that

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}=\lim_{x\to 1}(x+1)=2$$

The same limit can be evaluated with the help of substitution. If we substitute $y=x-1$, i.e. $x=y+1$, in the expression $\frac{x^2-1}{x-1}$, then

$\frac{x^2-1}{x-1}=\frac{(y+1)^2-1}{y}=y+2$. Also, note that when x tends 1, then y

tends 0. Therefore,

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{y\to 0}(y+2)=2.$$

In general,

$$\lim_{x\to a} f(x)=\lim_{h\to 0} f(a+h).$$

Equivalently,

$$\lim_{x\to a} f(x)=\lim_{y\to a+h} f(y), \quad where\ y=x+h.$$

7.3.4 A Special Limit

Let us evaluate $\lim_{x\to a}\frac{x^n-a^n}{x-a}$, where n is a positive integer.

By the method of substitution, we find that

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{h\to 0}\frac{(a+h)^n-a^n}{a+h-a}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[a^n+\binom{n}{1}a^{n-1}h+\binom{n}{2}a^{n-2}h^2+\ldots+h^n-a^n\right]$$

[By Binomial Theorem]

$$=\lim_{h\to 0}\left[\binom{n}{1}a^{n-1}+\binom{n}{2}a^{n-2}h+\ldots+h^{n-1}\right]=\binom{n}{1}a^{n-1}=na^{n-1}.$$

Thus,

$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$, where n is a positive integer.

The above formula remains true, when n is a rational number, provided a is positive.

Example 7.5: Find $\lim_{x\to 3}\frac{x^3-27}{x^2-9}$.

Solution:

$$\lim_{x\to 3}\frac{x^3-27}{x^2-9}=\lim_{x\to 3}\left[\frac{x^3-27}{x-3}\div\frac{x^2-9}{x-3}\right]=\lim_{x\to 3}\left[\frac{x^3-27}{x-3}\right]\div\lim_{x\to 3}\left[\frac{x^2-9}{x-3}\right]$$

$$= \lim_{x\to 3}\left[\frac{x^3-3^3}{x-3}\right] \div \lim_{x\to 3}\left[\frac{x^2-3^2}{x-3}\right] = 3(3)^2 \div 2(3) = 27 \div 6 = \frac{9}{2}.$$

Example 7.6: If $\lim_{x\to 2}\frac{x^n-2^n}{x-2} = 12$ and if n is a positive integer, find n.

Solution: We have $\lim_{x\to 2}\frac{x^n-2^n}{x-2} = n2^{n-1}$. According to the given question, $n2^{n-1} = 12 = 3.2^2$. Therefore, n = 3.

7.3.5 Some Other Special Limits

The following limit rule exponential functions are very useful.

Example 7.7: Find $\lim_{x\to 0}\frac{e^{4x}-1}{x}$.

Solution: We have

$$\lim_{x\to 0}\frac{e^{4x}-1}{x} = \lim_{x\to 0}\frac{e^{4x}-1}{4x}\cdot 4 = 4\lim_{y\to 0}\frac{e^{y}-1}{y} = 4.$$

7.3.6 One Sided Limits

In this section we will discuss one sided limits.

The notation $\lim_{x\to a+} f(x) = l$ means that when is close x to a and greater than a then $f(x)$ is close to l.. Similarly, $\lim_{x\to a-} f(x) = l$ means that when is close x to a and less than a, then $f(x)$ is close to l.

$\lim_{x\to a+} f(x)$ is called the **right limit** of f at a and $\lim_{x\to a-} f(x)$ is called the **left limit** of f at a. Both these are called **one-sided limits** of f at a

Example 7.8: Find $\lim_{x\to 0+}\frac{x}{|x|}$ and $\lim_{x\to 0-}\frac{x}{|x|}$.

Solutions: When x takes positive values, then $\frac{x}{|x|} = \frac{x}{x} = 1$.

Therefore, we have $\lim_{x\to 0+}\frac{x}{|x|} = 1$.

On the other hand, when takes negative values, then

$\frac{x}{|x|} = \frac{x}{-x} = -1$. So, $\lim_{x \to 0-} \frac{x}{|x|} = -1$.

If $\lim_{x \to a+} f(x) \neq \lim_{x \to a-} f(x)$ then $\lim_{x \to a} f(x)$ does not exist.

If $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = l$ then $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = l$.

Example 7.9: Decide whether $\lim_{x \to 0} \frac{x}{|x|}$ exists or not.

Solution: From the solutions of Example 11, we note that $\lim_{x \to 0+} \frac{x}{|x|} = 1$ and $\lim_{x \to 0-} \frac{x}{|x|} = -1$. Since the right limit and left limit are not equal, therefore $\lim_{x \to 0} \frac{x}{|x|}$ does not exist.



CHECK YOUR PROGRESS

Q 2: Write true (T) or false (F).

(a) If f is a real function such that $f(x) = f(-x)$ for all x, then $\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x)$. (T /F)

(b) $\lim_{x \to 1} [x] = 1$. (T / F)

(c) $\lim_{x \to 1+} [x] = 1$. (T / F)

Q 3: Find the following limits:

(i) $\lim_{x \to 2} \frac{x-2}{x+2}$ (ii) $\lim_{x \to 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$ (iii) $\lim_{x \to 3} \frac{e^x - e^3}{x-3}$

7.4 CONTINUITY

In this section, we will learn about the concept of continuity of a function. We will show that a large class of well known functions is continuous.

7.4.1 Basic Definition and Examples

Let f be a real function and let a be in the domain of f If the limit of f at a is same as the value of f at a, then f is said to be continuous at a.

Thus, if *f* is to be continuous at *a* the following three conditions must be satisfied.

a) *f (a)* is defined,

b) $\lim_{x\to a} f(x)$ exists,

c) $\lim_{x\to a} f(x) = f(a)$.

Example 7.10: Show that every constant function is continuous at all points.

Solution: Consider a constant function $f(x) = c$ and an arbitrary point a.

Then $f(a) = c$ and $\lim_{x\to a} f(x) = \lim_{x\to a} c = c$, and so, $\lim_{x\to a} f(x) = f(a)$.

Therefore, f is continuous at a

Example 7.11: Show that the identity function is continuous at all points.

Solution: Consider the identity function $f(x) = x$ and an arbitrary point *a*.

Then, we have $f(a) = a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} x = a$.

Thus, $\lim_{x\to a} f(x) = f(a)$, which shows that f is continuous at 'a'.

Example 7.12: Show that the greatest integer function [x] is not continuous at 0.

Solution: Let $f(x) = [x]$.

Now, $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} [x] = 0$ and $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} [x] = -1$.

Thus, the right hand and left hand limits of f at 0 are not equal. Which means that $\lim_{x\to 0} f(x) = \lim_{x\to 0} [x]$ does not exist. Therefore, though $f(0) = [0] = 0$, *f* is not continuous at 0.

Some more results are listed in the following table.

(a) The function x^2 is continuous at 3.

(b) The function $\dfrac{x-4}{x-1}$ is continuous at 2.

(c) The function e^x is continuous at 0.

(d) The function $\frac{x}{|x|}$ is not continuous at 0.

Definition:

A real function is said to be continuous in an open or closed interval if it is continuous at every point of the interval.

Remark: When a function f is considered on a closed interval $[a,b]$, then f is said to be continuous at the end point a if $\lim_{x\to a+} f(x) = f(a)$. Similarly, f is said to be continuous at the end point b if $\lim_{x\to b-} f(x) = f(b)$.



7.5 LET US SUM UP

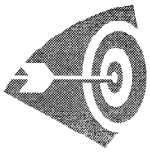
- In this unit, we have learnt about the concept of limits and various algebraic rules to find limits at various points of well-known functions. We have also learnt about the left and right hand limits, and found that they need not be equal always.
- We have learnt about the continuity of functions. In summary, we can say that if the limit exists at a point and it is equal to the value of the function at that point then the function is continuous at that point.



7.6 FURTHER READING

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3) Bose, D. (2004). *Mathematical Economics*; New Delhi: Himalaya Publishing House.
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7.7 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (a) T, (b) T, (c) F, (d) F

Ans to Q No 2: (a) T, (b) F, (c) T.

Ans to Q No 3: (i) 0 (ii) 3, (iii) e^3.



7.8 MODEL QUESTIONS

Q 1: Evaluate the following limits:

(a) $\lim_{x\to 3}\frac{x^2-7x+12}{x^2-4x+3}$ (b) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$

(c) $\lim_{x\to 0}\frac{\sin 10x}{\sin 11x}$ (d) $\lim_{x\to k}\frac{x^{10}-k^{10}}{x^5-k^5}$

Q 2: Suppose a function *f* is defined as follows.

$$f(x)=\begin{cases} x, & x\geq 0 \\ -x, & x>0. \end{cases}$$

By evaluating the left and right hand limits, show that $\lim_{x\to 0} f(x)=0$.

Q 3: If *f* is an odd function (i.e., $f(-x)=-f(x), \quad \forall x.$) and $\lim_{x\to 0} f(x)$ exists, then show that the value of this limit is zero.

Q 4: Decide whether the function given in Question 2 is continuous at $x=0$ or not.

Q 5: A real function *f* is defined as follows.

$$f(x)=\begin{cases} \frac{\tan x}{x}, & x\neq 0 \\ \frac{3}{4}, & x=0. \end{cases}$$

Show that *f* is continuous at 0.

*** ***** ***

UNIT 8: DIFFERENTIATION

UNIT STRUCTURE

8.1 Learning objectives.

8.2 Introduction

8.3 Concept of Derivative of a Function

8.4 Geometrical Interpretation of Derivative

8.5 Economic Interpretation of Derivative

8.6 Rules of Differentiation

- 8.6.1 The Rules for Sums and Differences
- 8.6.2 Product Rule of Differentiation
- 8.6.3 Derivative of composite functions or Chain Rule of Differentiation

8.7 Second Order Derivative

8.8 Derivative of Various Standard Functions

8.9 Examples of One Variable Derivative

8.10 Let Us Sum Up

8.11 Further Reading

8.12 Answers to Check Your Progress

8.13 Model Questions

8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- understand the geometrical meaning of $f'(x)$ or $\frac{dy}{dx}$
- know the different rules of differentiation

8.2 INTRODUCTION

This unit introduces you to the concept of derivative. After going through this unit, you will be able to know the different rules of differentation.

8.3 CONCEPT OF DERIVATIVE OF A FUNCTION

Many phenomena involve changing quantities and their rates of change (increase or decrease). Differential calculus deals with the rate of change of a dependent variable with respect to an independent variable.

Let f be a real valued function and a is a point in its domain of definition.

Then the derivative of f at a is defined by

$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ provided this limit exists and is donoted by f'(a).

If y = f(x) be a function of f, the derivative of y with respect to x is $\frac{dy}{dx}$ or y_1.

When both x and y increase or decrease simulteneously, the derivative $\frac{dy}{dx}$ is positive. But when y increases with a decrease in x, or decreases with an increase in x, then $\frac{dy}{dx}$ is negative.

Remark : The derivative of a function of x is also called as the differential coefficient of x. The process of finding the differential coefficient is called differentiation.

Example 8.1 : Find the derivative of f(x) = 3x at x = 2

Solution : We have

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

$$\therefore f'(2) = \lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h \to 0} \frac{3(2+h)-3(2)}{h}$$

$$= \lim_{h \to 0} \frac{6+3h-6}{h}$$

$$= \lim_{h \to 0} 3$$

$$= 3$$

$\therefore$ Derivative of the function f(x) = 3x at x = 2 is 3.

Example 8.2 : If $f(x) = 2x^2 + 3x - 5$, then using definition show that $f'(0) + 3f'(-1) = 0$.

Solution : Given $f(x) = 2x^2 + 3x - 5$

We have

$$f'(0) = \lim_{h \to 0} \frac{f(o+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(3h^2 + 3h - 5) - (0 + 0 - 5)}{h}$$

$$= \lim_{h \to 0} \frac{h(3h+3)}{h}$$

$$= \lim_{h \to 0} (3h + 3) \qquad [\because h \neq 0 \text{ as } h \to 0]$$

$$= 3$$

Also $f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$

$$= \lim_{h \to 0} \frac{\{2(-1+h)^2 + 3(-1+h) - 5\} - \{2(-1)^2 + 3(-1) - 5\}}{h}$$

$$= \lim_{h \to 0} \frac{2 - 4h + 2h^2 - 3 + 3h - 5 - (2 - 3 - 5)}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 - h}{h}$$

$$= \lim_{h \to 0} \frac{h(2h-1)}{h}$$

$$= \lim_{h \to 0} (2h - 1) = -1$$

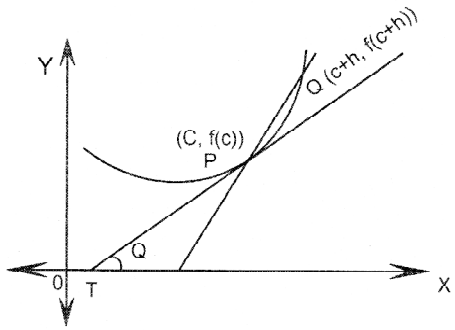
$\therefore f'(0) + 3f'(-1) = 0$

8.4 GEOMETRICAL INTERPRETATION OF DERIVATIVE

Let y=f(x) be a function. Let P (c, f(c)) and Q (c+h, f(c+h)) be two neighbouring points on the graph of y=f(x).

Slope of Chord PQ $= \dfrac{f(c+h)-f(c)}{(c+h)-c}$

$= \dfrac{f(c+h)-f(c)}{h}$



When Q $\rightarrow$ P along the curve, the chord PQ becomes tangent PT at P. Thus, the slope of the chord PQ becomes slope of the tangent at P when Q $\rightarrow$ P, i.e. h $\rightarrow$ 0

$\therefore$ Slope of the tangent at P= $\lim_{h \to 0} \dfrac{f(c+h)-f(c)}{h}$

= f'(c) (by definition)

Note :- If tangent is parallel to x-axis, then f'(c)=0.

8.5 ECONOMIC INTERPRETATION OF DERIVATIVE

Derivative normally represents the marginal function. In case of a simple cost function y = f(x) where y represents the total cost of production and x denotes the output, an infinitely small increase in output (Δx) leads to a corresponding infinitely small increase in cost (Δy). The derivative of a cost function $\lim_{x \to 0} \dfrac{\Delta y}{\Delta x}$ at point x is the marginal cost at the output level x.

8.6 RULES OF DIFFERENTIATION

In this section, we shall discuss the rules of differentiation.

8.6.1 The Rules for Sums and Differences

Theorem 1: If f_1 and f_2 are two differentiable functions on the interval I, and f be their sum function so that $f = f_1 + f_2$, Then f is differentiable and

$$\frac{d}{dx}f = \frac{d}{dx}f_1 + \frac{d}{dx}f_2$$

Theorem 2 If f_1 and f_2 are two differentiable functions on the interval I, and f be their difference function so that $f=f_1-f_2$ then f is differentiable and $\frac{d}{dx}f = \frac{d}{dx}f_1 - \frac{d}{dx}f_2$

Illustrative Examples:

Example 8.3: Find the derivatives of the following with respect to x.

a) $x(1+x)^2$ b) $b\log_e x+x^3+\sqrt{x}$

c) $4\log_{10}x$

Solution :

a) $\frac{d}{dx}\{x(1+x)^2\}=\frac{d}{dx}\{x(1+2x+x^2)\}$

$= \frac{d}{dx}(x+2x^2+x^3)$

$= \frac{d}{dx}(x)+2\frac{d}{dx}(x^2)+\frac{d}{dx}(x^3)$

$= 1 + 2(2x) + 3x^2$

$= 1 + 4x + 3x^2$

b) $\frac{d}{dx}(b\log_e x+x^3+\sqrt{x})= b\frac{d}{dx}(\log_e x)+\frac{d}{dx}(x^3)+\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$

$= b.\frac{1}{x}+3x^2+\frac{1}{2}x^{\frac{1}{2}-1}$

$= \frac{b}{x}+3x^2+\frac{1}{2\sqrt{x}}$

c) $\frac{d}{dx}(4\log_{10}x)=\frac{d}{dx}\left(\frac{4\log_e x}{\log_e 10}\right)$

$= \frac{4}{\log_e 10}\frac{d}{dx}(\log_e x)$

$= 4\log_{10}e\frac{1}{x}$

$= \frac{4\log_{10}e}{x}$

8.6.2 Product Rule of Differentiation

Theorem 3 : Let u(x) and v(x) be two differentiable functions. Then the product u(x) v(x) is also differentiable and

$$\frac{d}{dx}(u(x)v(x)) = u(x)v'(x) + v(x)u'(x)$$

Corollary : If u, v, w are three differentable functions , then

$$\frac{d}{dx}(u(x)v(x)w(x)) = u'(x)v(x)w(x) + v'(x)u(x)w(x) + w'(x)u(x)v(x)$$

Theorem 4: Let u(x) and v(x) be two differentiable functions in the same interval, then the quotient function $\frac{u(x)}{v(x)}$ is also differentiable and

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v(x)\frac{d}{dx}(u(x)) - u(x)\frac{d}{dx}(v(x))}{(v(x))^2}$$

$$= \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$$

Example 8.4 : Find the derivatives of the following functions with respect to x.

a) x^2e^x b) $e^x \log x$

Solution : a) $\frac{d}{dx}(x^2e^x) = e^x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(e^x)$

$= e^x.2x + x^2.e^x$

$= xe^x(2+x)$

b) $\frac{d}{dx}(e^x \log x) = \log x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(\log x)$

$= \log x.e^x + e^x.\frac{1}{x}$

$= e^x\left(\log x + \frac{1}{x}\right)$

8.6.3 Derivative of Composite Functions or Chain Rule of Differentiation

If $y = \phi(t)$ and $t = f(x)$, then $y = \phi(f(x))$ is a composite function.

Theorem 5: Let $y = \phi(t)$ when $t=f(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Generalisation of above theorem :

Let $y=f(u)$, $u=g(t)$, $t=h(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dt} \cdot \frac{dt}{dx}$$

Thus, if the composite function y is a function of many functions, then

$$\frac{dy}{dx} = \frac{d(1^{st} \text{ function})}{d(2^{nd} \text{ function})} \cdot \frac{d(2^{nd} \text{ function})}{d(3^{rd} \text{ function})} \cdots \frac{d(\text{last function})}{dx}$$

Above is also known as **Chain Rule of Differentiation**.

Example 8.5 : Find the derivatives of the following functions with respect to x.

$$y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$$

Solution : Let $t = \sqrt{x} + \frac{1}{\sqrt{x}}$

$\therefore y = \log t$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{d}{dt}(\log t) \cdot \frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$$

$$= \frac{1}{t}\left(\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}\right)$$

$$= \frac{1}{\sqrt{x} + \frac{1}{\sqrt{x}}} \cdot \frac{x-1}{2x\sqrt{x}}$$

$$= \frac{\sqrt{x}}{x+1} \cdot \frac{x-1}{2x\sqrt{x}}$$

$$= \frac{x-1}{2x(x+1)}$$

8.7 SECOND ORDER DERIVATIVE

As we have already said earlier, the deriviative $\frac{dy}{dx}$ (or, $f'(x)$) of a function y = f(x) is also a function of x. If the funciton $f(x)$ is differentiable, the function $f'(x)$ can be differentiated with respect to x. The derivative obtained by differentiating $f'(x)$ is called the second order derivative of a function y = (x) andis written as:

$$\frac{d}{dx}[f'(x)] = f''(x) - \frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2}.$$

Example 8.6:

If $y = f(x) = 5x^3 - 3x^2 + 2x + 10$

$$\frac{dy}{dx} = \frac{d}{dx}(5x^3 - 3x^2 + 2x + 10)$$
$$= 5 \times 3x^{3-1} - 3 \times 2x^{2-1} + 2 \times 1 + 0$$
$$= 15x^2 - 6x + 2$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}(15x^2 - 6x + 2)$$
$$= 15 \times 2x^{2-1} - 6 \times 1 + 0$$
$$= 30x - 6$$

8.8 DERIVATIVES OF VARIOUS STANDARD FUNCTIONS

1. $\frac{d}{dx}(k) = 0$, k is any constant
2. $\frac{d}{dx}(x) = 1$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(e^x) = e^x$
5. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
6. $\frac{d}{dx}(a^x) = a^x \log_e a$

8.9 EXAMPLES OF ONE VARIABLE DERIVATIVE

Find $\frac{dy}{dx}$ of the following functions:

1) $y = 5x^4$

$$\frac{dy}{dx} = 4 \times 5.x^{4-1} = 20x^3$$

2) $y = -6x^5$

$$\frac{dy}{dx} = -6 \times 5.x^{5-1} = -30x^4$$

3) $y = 5x^{-2}$

$$\frac{dy}{dx} = 5 \times (-2)x^{-2-1} = -10x^{3} = -\frac{10}{x^3}$$

4) $y = 5e^{3x}$

$$\frac{dy}{dx} = 5 \times 3.e^{3x} = 15e^{3x}$$

5) $y = \frac{1}{x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-2}) = -2.x^{-2-1} = -2^{x-3} = -\frac{2}{x^3}$$

6) $y = \frac{9}{10}x^{-5}$

$$\frac{dy}{dx} = \frac{9}{10}(-5)x^{-5-1} = \frac{9}{2}x^6$$

7) $y = (\sqrt[3]{x})^4$

$$\therefore y = x^{\frac{4}{3}}$$

$$\frac{dy}{dx} = \frac{4}{3}x^{\frac{4}{3}-1} = \frac{3}{4}x^{\frac{1}{3}}$$

8) $y = x^2(x-3)$

$$\therefore y = x^3 - 3x^2$$

$$\frac{dy}{dx} = (x^3 - 3x^2) = \frac{d}{dx}(x^3) - \frac{d}{dx}(3x^2)$$

$$= 3x^{3-1} - 3x^{2-1} = 3x^2 - 3x$$



CHECK YOUR PROGRESS

Find the first order derivative of the following.

(a) $\frac{x^3 + x^2 - x + 2}{x}$ (b) $\frac{2x+3}{3x+4}$

(c) $\frac{x}{a^2 + x^2}$

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8.10 LET US SUM UP

- In this unit, we have discussed the concept of differentiation. We found several rules to find the derivatives of various functions.
- We have also discussed the application of differentiation, primarily to one variable case.



8.11 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
2) Baruah, S. (2011). *Basic Mathematics and Its Application in Economics*, New Delhi: Trinity Press Pvt. Ltd.
3) Bose, D. (2004). *Mathematical Economics*; New Delhi: Himalaya Publishing House.
4) Chiang, A.C. & (2010). *Fundamental Methods of Economics Analysis*, MC Graw Hill Education India.

5) Kandoi, Balwant (2011). *Mathematics for Business and Economics with Application*; New Delhi: Himalaya Publishing House.

8.12 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: $\frac{d}{dx}\left(\frac{x^3+x^2+x+2}{x}\right) = \frac{d}{dx}\left(\frac{x^3}{x}+\frac{x^2}{x}+\frac{x}{x}+\frac{2}{x}\right)$

$$= \frac{d}{dx}(x^2)+\frac{d}{dx}(x)-\frac{d}{dx}(1)+2\frac{d}{dx}(x^{-1})$$

$$= 2x + 1 - 0 + 2(-1)x^{-2}$$

$$= 2x + 1 - \frac{2}{x^2}$$

Ans to Q No 2: $\frac{d}{dx}\left(\frac{2x+3}{3x+4}\right) = \frac{(3x+4)\frac{d}{dx}(2x+3)-(2x+3)\frac{d}{dx}(3x+4)}{(3x+4)^2}$

$$= \frac{(3x+4)(2+0)-(2x+3)(3+0)}{(3x+4)^2}$$

$$= \frac{(6x+8)-(6x+9)}{(3x+4)^2}$$

$$= \frac{-1}{(3x+4)^2}$$

Ans to Q No 3: $\frac{d}{dx}\left(\frac{x}{a^2+x^2}\right) = \frac{(a^2+x^2)\frac{d}{dx}(x)-x\frac{d}{dx}(a^2+x^2)}{(a^2+x^2)^2}$

$$= \frac{(a^2+x^2)1-x(2x)}{(a^2+x^2)^2}$$

$$= \frac{a^2+x^2-2x^2}{(a^2+x^2)^2}$$

$$= \frac{a^2-x^2}{(a^2+x^2)^2}$$



8.13 MODEL QUESTIONS

Q 1: Determine dy/dx for the following functions:

(a) $y = (x^2+5)(2x^3 - 4x + 3)$ (b) $y = 3x^2(3x + 5x^2)$

$(c) y = x^2 e^x$ $(d) y = \frac{x^2+1}{x-1}$ $(e) y = \frac{e^x}{1+x}$

$(f) y = (x+2)(x+1)^2$ $(g) y = \frac{3x}{x-5}$

$(h) y = 16x^2 + 15x$ $(i) y = \frac{1 \pm \sqrt{x}}{1-\sqrt{x}}$

Q 2: Determine the second order derivative for the following functions:

$(a) y = (3x-2)(x^2+10x)$ $(b) y = \frac{5x^2-2}{x^2-5x}$

$(c) y = x^3 - 6x^2 + 9x$ $(d) y = 10x^5 - 4x^3 + 5x - 2$

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UNIT 9: PARTIAL DERIVATIVES AND TOTAL DERIVATIVES

UNIT STRUCTURE

9.1 Learning objectives.

9.2 Introduction

9.3 Function of Several Variables

9.4 Partial Derivatives

9.5 Total Differentiation

9.6 Let Us Sum Up

9.7 Further Reading

9.8 Answers to Check Your Progress

9.9 Model Questions

9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- acquire knowlege about the partial derivative, total derivative and total differentials
- solve the problems of partial and total differentiation.

9.2 INTRODUCTION

In the previous unit, we have discussed the concept of derivative and also has come across derivative of one variable cases. In this unit, we shall dicuss the concepts of partial and total differentiation. At the end of the unit, you will be able to solve the problems of partial and total differentiation. In the next unit, we shall utilise these concepts to solving a few problems in Economics.

9.3 FUNCTION OF SEVERAL VARIABLES

We are familiar with function of a single variable. But in nature there arise situations where we have to deal with functions of more than one variable.

Suppose we want to forecast the weather this weekend in your town. We construct a formula for the temperature as a function of several environmental variables, each of which is not predictable. Now, we would like to see how our weather forecast would change as one particular environmental factor changes, holding all the other factor constant.To do this investigation, we would use the concept of partial derivatives.

For example, the area of a rectangle A=xy which is a function of the variable x(length) and variable y (breadth). Also, the volume of cuboid V=xyz which is a function of the variables x(length), y(breadth) and z(height).

If three variables u, y, z are so related that the value of u depends upon the values of x and y,then u is called a function of two variables x and y and this is denoted by $u = f(x, y)$. Here u is called the dependent variable while x and y are called independent variables.

Hence, $u = f(x_1, x_2)$ is a function of two variables where u is dependent variable and x_1, x_2 are indepedent variables.

Also, $u = f(x_1, x_2, x_3)$ is a function of three variables where u is dependent variable and x_1, x_2, x_3 are independent variables and $u = f(x_1, x_2, x_3, x_4)$ is a function of four variables where u is dependent variable and x_1, x_2, x_3, x_4 are indepedent variables.

In general, a function of the form $u = f(x_1, x_2, x_3, \cdots, x_n)$ is a function of n variables where u is dependent variable and $x_1, x_2, x_3, \cdots, x_n$ are independent variables.

Examples :

(i) $u = f(x, y) = x^2 + 2xy + y^2$ is a function of two variables x, y.

(ii) $u = f(x, y, z) = x^2 + 3xy + yz^2 + z^3$ is a function of three variables x,y,z.

9.4 PARTIAL DERIVATIVES

If a derivative of a function of several independent variables is found with respect to any of them, keeping the others as constants, it is said to be a **Partial Derivative**. The process of finding the partial derivatives of a function of more than one independent variables is called **Partial differentiation.**

Let $z = f(x, y)$ be a function of two variables.

The partial derivative of z with respect to x , denoted by

$\frac{\partial z}{\partial x}$ or $\frac{f(x,y)}{\partial x}$ or f_x ,is defined by

$\frac{\partial z}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$, provided the limit exists.

Also,the partial derivative of z with respect to y, denoted by

$\frac{\partial z}{\partial y}$ or $\frac{\partial f(x,y)}{\partial y}$ or f_y is defined by

$\frac{\partial z}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$,provided the limit exists.

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called **first order partial derivatives of z.**

Example 9.1: Given Let $f(x,y)=xy^2$. Find the partial derivatives using the limit theorems.

Solution: Given, $f(x,y)=xy^2$

Then , $f_x(x,y)=\lim_{h\to 0}\frac{(x+h)y^2-xy^2}{h}$, $f_y(x,y)=\lim_{h\to 0}\frac{x(y+h)^2-xy^2}{h}$

$= \lim_{h\to 0}\frac{hy^2}{h}$ $= \lim_{h\to 0}\frac{2xyh+xh^2}{h}$

$= y^2$ $= \lim_{h\to 0}(2xy+xh)$

$= 2xy$

Example 9.2: Given $U=5x-6y+8$. Find partial derivatives of fx and fy.

Solution: Given, $U=5x-6y+8$.

$$\therefore f_x = \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(5x-6y+8)$$
$$= 5-0+0=5.$$
$$Similarly, f_y = \frac{\partial U}{\partial y} = \frac{\partial}{\partial y}(5x-6y+8)$$
$$= 0-6+0=-6.$$

Example 9.3: Find partial derivatives of $U=4x^2+4xy+y^2$.

Solution: Given, $U=4x^2+4xy+y^2$.

$$\therefore f_x = \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(4x^2+4xy+y^2)$$

$$= 4 \times 2x^{2-1} + 4 \times 1 \times y + 0 = 8x + 4y$$

$$Similarly, f_y = \frac{\partial U}{\partial y} = \frac{\partial}{\partial y}(4x^2 + 4xy + y^2)$$

$$= 0 + 4 \times x \times 1 + 2 \times y^{2-1} = 4x + 2y$$

Example 9.4: Find the partial derivatives of $y = x_1^{\,3} + 2x_1x_2^{\,2} + 3x_2^{\,2}$

Solution: Given, $y = x_1^{\,3} + 2x_1x_2^{\,2} + 3x_2^{\,2}$

$$\therefore fx_1 = \frac{\partial y}{\partial x_1} = \frac{\partial}{\partial x_1}(x_1^{\,3} + 2x_1x_2^{\,2} + 3x_2^{\,2})$$

$$= 3x_1^{\,3-1} + 2x_2^{\,2} \times 1 + 0 = 3x_1^{\,2} + 2x_2^{\,2}$$

$$Similarly, fx_2 = \frac{\partial y}{\partial x_2} = \frac{\partial}{\partial x_2}(x_1^{\,3} + 2x_1x_2^{\,2} + 3x_2^{\,2})$$

$$= 0 + 2x_1 \times 2x_2^{\,2-1} + 3 \times 2x_2^{\,2-1} = 4x_1x_2 + 6x_2$$

Example 9.5: Find the partial derivatives of $Z = \frac{x+y}{2x+5y}$

Solution: Given, $Z = \frac{x+y}{2x+5y}$

$$\therefore f_x = \frac{(2x+5y)\frac{\partial}{\partial x}(x+y) - (x+y)\frac{\partial}{\partial x}(2x+5y)}{(2x+5y)^2}$$

$$= \frac{(2x+5y) - (x+y) \times 2}{(2x+5y)^2} = \frac{2x+5y-2x-2y}{(2x+5y)^2} = \frac{3y}{(2x+5y)^2}$$

$$Similarly, f_y = \frac{(2x+5y)\frac{\partial}{\partial y}(x+y) - (x+y)\frac{\partial}{\partial y}(2x+5y)}{(2x+5y)^2}$$

$$= \frac{(2x+5y) - (x+y) \times 5}{(2x+5y)^2} = \frac{2x+5y-5x-5y}{(2x+5y)^2} = -\frac{3x}{(2x+5y)^2}$$

Functions of more than two variables : Partial derivatives can also be defined for functions of three or more variables.

Thus if $u = f(x, y, z)$, then the partial differential coefficient of u w.r.t x i.e. $\frac{\partial u}{\partial x}$ is obtained by differentiating u w.r.t x keeping y and z as constants.

i.e. $\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}=f_x(x,y,z)=\lim_{h\to 0}\frac{f(x+h,y,z)-f(x,y,z)}{h}$, provided the limit exists.

Similarly,the partial differential coefficient of u w.r.t y i.e. $\frac{\partial u}{\partial y}$ is obtained by differentiating u w.r.t y keeping x and z as constants

i.e., $\frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}=f_y(x,y,z)=\lim_{h\to 0}\frac{f(x,y+h,z)-f(x,y,z)}{h}$,provided the limit exists.

Again, the partial differential coefficient of u w.r.t zi.e. $\frac{\partial u}{\partial z}$ is obtained by differentiating u w.r.t z keeping x and y as constants.

i.e., $\frac{\partial u}{\partial z}=\frac{\partial f}{\partial z}=f_z(x,y,z)=\lim_{h\to 0}\frac{f(x,y,z+h)-f(x,y,z)}{h}$

In general, if u is a function of n variables, $u=f(x_1,x_2,.....,x_n)$, its partial derivative w.r.t the variable x_i is given by

$$\frac{\partial u}{\partial x_i}=f_{x_i}(x_1,x_2,.....,x_n)$$

$$=\lim_{h\to 0}\frac{f(x_1,x_2,....,x_i+h,x_{i+1},....,x_n)-f(x_1,x_2,.....,x_n)}{h}$$, provided the limit exists.

Second order partial differential coefficients

If $z=f(x,y)$, then $\frac{\partial z}{\partial x}$ or f_x and $\frac{\partial z}{\partial y}$ or f_y are themselves functions of x and y and can be again differentiated partially.

We define $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right),\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right),\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$ as the second order partial derivatives of u and these are as denoted by

$\frac{\partial^2 u}{\partial x\partial y},\frac{\partial^2 u}{\partial y\partial x},\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ respectively.

Higher order partial derivatives : Partial derivatives of higher order of a function are calculated by successive differentiation.

We denote the third order partial derivatives by

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right)=\frac{\partial^3 f}{\partial x^3}=f_{xxx},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right)=\frac{\partial^3 f}{\partial y \partial x^2}=f_{yxx},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right)=\frac{\partial^3 f}{\partial x^2 \partial y}=f_{xxy},\text{etc.}$$

and, in general, the n-th partial derivatives are

$$\frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right)=\frac{\partial^n f}{\partial x^n}-f_{x^n}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right)=\frac{\partial^n f}{\partial y \partial x^{n-1}}=f_{yx^{n-1}}\ \text{etc.}$$

Illustrative Examples :

Example 9.6 : If $z-x^2y+xy^3$, find out the first order partial derivatives.

Solution : We have $z=x^2y+xy^3$

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left(x^2y+xy^3\right)$$

$$=\frac{\partial}{\partial x}\left(x^2y\right)+\frac{\partial}{\partial x}\left(xy^3\right)$$

$$=2xy+y^3$$

Also,

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left(x^2y+xy^3\right)$$

$$=\frac{\partial}{\partial y}\left(x^2y\right)+\frac{\partial}{\partial y}\left(xy^3\right)$$

$$=x^2+3xy^2$$

Example 9.7 : If $z=\log\left(x^2+y^2\right)^{1/2}$, then find out $\frac{\partial z}{\partial x}$ and $\frac{\partial x}{\partial y}$.

Solution : Given $z=\log\left(x^2+y^2\right)^{1/2}$

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left(\log\left(x^2+y^2\right)^{1/2}\right)$$

$$=\frac{1}{2\left(x^2+y^2\right)}.2x$$

$$=\frac{x}{x^2+y^2}$$

Again, $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\log\left(x^2+y^2\right)^{1/2}\right)$

$$= \frac{1}{2\left(x^2+y^2\right)}.2y$$

$$= \frac{y}{x^2+y^2}.$$

Example 9.8 : If $f(x,y) = x^3 + x^2y^3 - 2y^2$,find $f_x(2,1)$ and $f_y(2,1)$.

Solution : Holding y constant and differentiating w.r.t x, we get

$$f_x(x,y) = 3x^2 + 2xy^3$$

and so $$f_x(2,1) = 3.2^2 + 2.2.1^3 = 16$$

Similarly, holding x constant and differentiating w.r.t y, we get

$$f_y(x,y) = 3x^2y^2 - 4y$$

$$f_y(2,1) = 3.2^2.1^2 - 4.1 = 8$$

Example 9.9 : If $u = \log\left(x^2+y^2+z^2\right)$, show that $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} + z.\frac{\partial u}{\partial z} = 2$

Solution : We have $u = \log\left(x^2+y^2+z^2\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2+z^2} \times (2x)$$

$$x.\frac{\partial u}{\partial x} = x.\frac{2x}{x^2+y^2+z^2} = \frac{2x^2}{x^2+y^2+z^2} \quad \ldots (1)$$

Also, $\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2+z^2} \times (2y)$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2+z^2}$$

$$y.\frac{\partial u}{\partial y} = \frac{2y^2}{x^2+y^2+z^2} \quad \ldots (2)$$

and $\frac{\partial u}{\partial z} = \frac{1}{x^2+y^2+z^2} \times (2z)$

$$\frac{\partial u}{\partial z} = \frac{2z}{x^2+y^2+z^2}$$

$$z.\frac{\partial u}{\partial z} = \frac{2z^2}{x^2+y^2+z^2} \quad \ldots (3)$$

Adding (1), (2) and (3), we get

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} + z.\frac{\partial u}{\partial z} = \frac{2\left(x^2+y^2+z^2\right)}{x^2+y^2+z^2} = 2.$$

9.5 TOTAL DIFFERENTIATION

Total Derivatives : If $u = f(x,y)$ where $x = \phi_1(t)$ and $y = \phi_2(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}.$$

$\frac{du}{dt}$ is called the total differential coefficient of u w.r.t 't'

while $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are partial derivatives of u w.r.t x and y respectively.

The technique of partial derivative gives the rate of change in $U = \{U = f(x,y)\}$ when small changes are made in x {or in y} while y {or x} remains unchanged. But if both x and y change simultaneously, the problem is dealt with the technique of total differentials.

Proof : Let there be a small increment δt in t and let the corresponding increments in u, x and y be $\delta u, \delta x$ and δy respectively. Then we have

$u = f(x,y) \ldots (1)$

and $u + \delta u = f(x+\delta x, y+\delta y) \quad \ldots (2)$

$$\therefore \quad \delta u = f(x+\delta x, y+\delta y) - f(x,y)$$
$$= [f(x+\delta x, y+\delta y) - f(x, y+\delta y)] + [f(x, y+\delta y) - f(x,y)]$$

Dividing by δt, we get

$$\frac{\delta u}{\delta t} = \left\{\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta t}\right\} + \left\{\frac{f(x, y+\delta y) - f(x,y)}{\delta t}\right\} \quad \ldots (3)$$

Let $\delta t \to 0$ so that $\delta x \to 0$ and $\delta y \to 0$.

Now, from (3), we get

$$\lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \lim_{\partial t \to 0} \frac{\partial x}{\partial t}$$
$$+ \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y} \lim_{\partial t \to 0} \frac{\delta y}{\delta t} \quad \ldots (4)$$

since δx and δy tend to zero with δt, therefore (4) becomes

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}.$$

Note: (1) If $u = f(x, y, z)$, where $x = \phi_1(t), y = \phi_2(t), z = \phi_3(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{\partial z}{\partial t}$$

(2) The above result can be extended to any finite number of variables.

If $u = f(x_1, x_2, \cdots, x_n)$ where $x_1 = \phi_1(t), x_2 = \phi_2(t), \ldots, x_n = \phi_n(t)$. then

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}.\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}.\frac{dx_2}{dt} + \cdots + \frac{\partial u}{\partial x_n}.\frac{dx_n}{dt}$$

(3) If u is a function of x and y ,i.e., $u = f(x, y)$ where y is a function of x ,then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}$.

Deductions :(1) If $u = f(x, y)$ where $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then we have

$$dZ = \frac{(x^2 + y^2)d(x^2 - y^2) - (x^2 - y^2)d(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2)(2xdx - 2ydy) - (x^2 - y^2)(2xdx + 2ydy)}{(x^2 + y^2)^2}$$

$$= \frac{2x^3dx + 2xy^{22}dx - 2x^2ydy - 2y^3dy - 2x^3dx - 2x^2ydy + 2xy^2dx + 2y^3dy}{(x^2 + y^2)^2}$$

$$- \frac{-4x^2ydy + 4xy^2dx}{(x^2 + y^2)^2}$$

and $\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_2}$

ILLUSTRATIVE EXAMPLES :

Example 9.10: Find total differentials of $Z = 3x^2 + xy - 2y^3$.

Solution : Given, $Z = 3x^2 + xy - 2y^3$.

$$\begin{aligned} dZ &= d(3x^2) + d(xy) - d(2y^3) \\ &= 3 \times 2xdx + xdy + ydx - 2 \times 3y^{3-1}dy \\ &= 6xdx + xdy + ydx - 6y^2dy \end{aligned}$$

Example 9.11: Find total differentials of $y = 10 + 2x + 3x^2$

Solution : Given, $y = 10 + 2x + 3x^2$

$$\begin{aligned} dy &= d(10) + d(2x) + d(3x^2) \\ &= 0 + 2dx + 6xdx \\ &= 2dx + 6xdx \end{aligned}$$

Example 9.12: Find total differentials of $Z = (x^2 + y)(2x - y^2)$

Solution : Given, $Z = (x^2 + y)(2x - y^2)$

$$\begin{aligned} dZ &= (x^2 + y)d(2x - y^2) + (2x - y^2)d(x^2 + y) \\ &= (x^2 + y)(2dx - 2ydy) + (2x - y^2)(2xdx + dy) \\ &= 2x^2dx + 2ydx - 2x^2ydy - 2y^2dy + 4x^2dx - 2xy^2dx + 2xdy - y^2dy) \\ &= (6x^2 + 2y - 2xy^2)dx + (-2x^2 - 3y^2 + 2x)dy \end{aligned}$$

Example 9.13: Find total differentials of $y = \dfrac{2x^2}{10 + x}$

Solution : Given, $y = \dfrac{2x^2}{10 + x}$

$$\begin{aligned} dy &= \frac{(10 + x) \times d(2x^2) - 2x^2 \times d(10 + x)}{(10 + x)^2} \\ &= \frac{(10 + x)(2 \times 2 \times x) - 2x^2(1)}{(10 + x)^2} \\ &= \frac{40x + 4x^2 - 2x^2}{(10 + x)^2} \\ &= \frac{40x + 2x^2}{(10 + x)^2} \end{aligned}$$

Example 9.14: Find total differentials of $Z = \dfrac{x^2 - y^2}{x^2 + y^2}$

Solution : Given, $Z = \dfrac{x^2 - y^2}{x^2 + y^2}$

$$dZ = \frac{(x^2 + y^2)d(x^2 - y^2) - (x^2 - y^2)d(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2)(2xdx - 2ydy) - (x^2 - y^2)(2xdx + 2ydy)}{(x^2 + y^2)^2}$$

$$= \frac{2x^3dx + 2xy^2dx - 2x^2ydy - 2y^3dy - 2x^3dx - 2x^2ydy + 2xy^2dx + 2y^3dy}{(x^2 + y^2)^2}$$

$$= \frac{-4x^2ydy + 4xy^2dx}{(x^2 + y^2)^2}$$

Example 9.15: If $u = f(x - y, y - z, z - x)$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

Solution : Let $x - y = X$, $y - z = Y$, $z - x = Z$

Then $u = f(X, Y, Z)$ where X, Y, Z are functions of x, y, z.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial x}$$

$$= \frac{\partial u}{\partial X}.(1) + \frac{\partial u}{\partial Y}.(0) + \frac{\partial u}{\partial Z}.(-1)$$

$$= \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} \quad \ldots (1)$$

Similarly, $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial X}.\dfrac{\partial X}{\partial y} + \dfrac{\partial u}{\partial Y}\dfrac{\partial Y}{\partial y} + \dfrac{\partial u}{\partial Z}.\dfrac{\partial Z}{\partial y}$

$$= \frac{\partial u}{\partial X}.(-1) + \frac{\partial u}{\partial Y}(1) + \frac{\partial u}{\partial Z}.(0)$$

$$= -\frac{\partial u}{\partial X}.+\frac{\partial u}{\partial Y} \quad \ldots (2)$$

and $\dfrac{\partial u}{\partial z} = -\dfrac{\partial u}{\partial Y} + \dfrac{\partial u}{\partial Z} \quad \ldots (3)$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

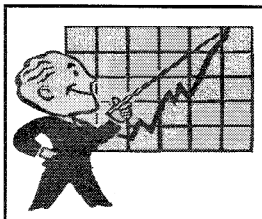
Example 9.16: Find $\frac{dz}{dt}$ when $z=xy^2+x^2y$, $x=at^2$, $y=2at$.

Solution : $\frac{\partial z}{\partial x}=y^2+2xy$, $\frac{\partial z}{\partial y}=2xy+x^2$

$$\frac{dx}{dt}=2at, \frac{dy}{dt}=2a$$

Now, we have $\frac{dz}{dt}=\frac{\partial z}{\partial x}.\frac{dx}{dt}+\frac{\partial z}{\partial y}.\frac{dy}{dt}$

$$=\left(y^2+2xy\right)2at+\left(2xy+x^2\right)2a$$
$$=\left(4a^2t^2+4a^2t^3\right)2at+\left(4a^2t^3+a^2t^4\right)2a$$
$$=a^3\left(16t^3+10t^4\right)$$



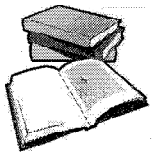
CHECK YOUR PROGRESS

Q 1: If $u=x^5y^4$ where $x=t^2$ and $y=t^3$, find $\frac{du}{dt}$.



9.6 LET US SUM UP

- In this unit, we have discussed the concept of partial differentiation.
- We have also learn to solve the problems of partial differentiation.



9.7 FURTHER READING

1) Baruah, S. *Basic Mathematics and Its Applications in Economics*, Trinity Press Pvt Ltd.

2) Chiang, A.C. & Wainright, K. *Fundamental Methods of Economic Analysis*, McGrawHill Education (India)

3) Madnani, B C, Mehata & G. M. *Mathematics for Economists*, Sultan Chand & Sons



9.8 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No. 1 : Here u is a function of x and y where x and y are the functions of t.
Again, u is also a function of single variable t.

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$$= \left(5x^4y^4\right)2t + \left(4x^5y^3\right)3t^2$$

$$= 10.x^4y^4t + 12x^5y^3t^2$$

$$= 10.t^8t^{12}t + 12t^{10}t^9t^2$$

$$= 22t^{21}$$



9.9 MODEL QUESTIONS

Q 1: Find the total differentiation of $u = x^2 + 2y^2$.

Q 2: Find the total differentiation of $z = x^2 + y^2 + 3xy$.

Q 3: Find the total differentiation of $z = (x^2 + y)(2x-y^2)$

Q 4: Determine dy/dx for the following functions:

(a) $y = (x^2+5)(2x^3 - 4x+ 3)$

(b) $y = 3x^2(3x + 5x^2)$

Q 5: For the production function $Q = 40F + 3F^2 + F^{3/3}$, calculate marginal and average production functions.

Q 6: Find marginal productivity for the total productivity $Y = (x^2+2)(x+3)$.

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UNIT 10: ECONOMIC APPLICATIONS OF DERIVATIVES

UNIT STRUCTURE

10.1 Learning Objectives

10.2 Introduction

10.3 Economic Applications of Derivatives

- 10.3.1 Elasticity of Demand (ed)
- 10.3.2 Deriving the Relationship between AR, MR and ed
- 10.3.3 Deriving the Relationship between AC and MC
- 10.3.4 Utility Function
- 10.3.5 Indifference Curve (IC) Function
- 10.3.6 Production Function
- 10.3.7 Iso-Quant (IQ) Function
- 10.3.8 Relationship between MR, MP and MPR
- 10.3.9 Consumption Function
- 10.3.10 Saving Function
- 10.3.11 Import Function
- 10.3.12 Elasticity of Supply (e_s)

10.4 Numerical Problems

10.5 Let Us Sum Up

10.6 Further Reading

10.7 Answers to Check Your Progress

10.8 Model Questions

10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to -

- acquire knowledge of the different functions in Economics where differential calculus is used
- apply differential calculus to economic problems.

10.2 INTRODUCTION

In the earlier two units, we discussed the concept of differential calculus. In this unit, we shall look into a few areas of its applications in

Economics. In the field of Economics, differential calculus is frequently used to find marginals from totals; e.g, in finding marginal case-or revenu from total cast or revenue. The concept is also frequently used in finding elasticities, among others.

10.3 ECONOMIC APPLICATIONS OF DERIVATIVES

In this section, we shall look into a few economic applications of derivatives.

10.3.1 Elasticity of Demand (Ed)

Elasticity of demand can be expressed as;

$$Ed = \frac{\text{Proportionate change in quantity demand}}{\text{Proportinate change in price}}$$

$$= \frac{\frac{dQ}{Q}}{\frac{dP}{P}}$$

$$Ed = \frac{dQ}{dP} \cdot \frac{P}{Q} \quad (\therefore \text{ P and Q are – vely related})$$

$$\text{or } |Ed| = \left|\frac{dQ}{dP} . P/Q\right|$$

We can express ed in following way also.

$$Ed = \frac{d(\log Q)}{d(\log P)}$$

$$\text{or } Ed = \frac{AR}{AR - MR}$$

If Ed>1, then the commodity has elastic demand

Ed<1, inelastic demand

Ed=1, then the commodity has Unitary elasticity of demand.

10.3.2 Deriving the Relationship between AR, MR and ed

We know;

$$TR = AR \times Q \text{ ———————— (i)}$$

Deriving (i) w.r. to 'Q' we get,

$$\frac{dTR}{dQ} = AR\frac{dQ}{dQ} + Q\frac{dAR}{dQ}$$

$$\Rightarrow MR = AR + Q.\frac{dAR}{dQ}$$

$$\therefore MR = P + Q\frac{dP}{dQ} (\because AR = P = Price)$$

$$\Rightarrow MR = P\left(1 + \frac{Q}{P}\frac{dP}{dQ}\right)$$

$$\Rightarrow MR = P\left(1 + \frac{1}{P/Q\frac{dQ}{dP}}\right)$$

$$\Rightarrow MR = P\left(1 - \frac{1}{ed}\right)\left(\because Ed = -\frac{dQ}{dP}P/Q\right)$$

$$\Rightarrow MR = P - P/Ed$$

$$\Rightarrow AR/Ed = AR - MR \ (\because \text{ AR=P as mentioned earlier})$$

$$\Rightarrow \frac{AR}{AR - MR} = Ed$$

$$\therefore Ed\frac{AR}{AR - MR}$$

10.3.3 Deriving the Relationship between AC and MC

We know;

$$TC = AC \times Q \text{ ——————— (i)}$$

Deriving (i) w.r. to 'Q' we get;

$$\frac{d}{dQ}TC = \frac{d}{dQ}(AC \times Q)$$

$$\Rightarrow MC = AC\frac{dQ}{dQ} + Q\frac{dAC}{dQ}$$

$$\Rightarrow MC = AC + Q.\text{Slope of } AC$$

$$\Rightarrow Q.\text{Slope of } AC = MC - AC$$

$$\therefore \text{Slope of AC} = \frac{1}{Q}(MC - AC) \quad \text{——— (ii)}$$

From (ii) we derive the relationship between AC and MC as follows:

(i) When MC>AC then slope of AC>0

(ii) When MC<AC then slope of AC = 0

(iii) When MC < AC then slope of AC<0

10.3.4 Utility Function

A utility (U) function for two commodities can be expressed as:

U = f(x,y) where u = utility

x and y are two goods

Marginal utility :

$MU_x = \frac{\partial u}{\partial x}$, here y is constant

where, MU_x = marginal utility of x

$MU_y = \frac{\partial u}{\partial y}$, here x is constant

where, MU_y = Marginal utility of y

10.3.5 Indifference Curve (IC) Function

An IC function can be expressed as:

U = f(x,y) = C where c is constant.

Slope of IC = $-\frac{dy}{dx} - \frac{MU_x}{MU_y}$

Note : IC possesses two characteristics which are :

(i) IC is downward sloping is $\frac{dy}{dx} < 0$.

(ii) IC is convex to the origin is $\frac{d^2y}{dx^2} > 0$

10.3.6 Production function :

A Production function for two factors of poduction can be expressed as:

Q=f (L, K) where; Q = Output, L and K are inputs Labour and capital respectively.

Marginal product :

$MP_L = \frac{\partial Q}{\partial L}$, here K is constant

where; MP_L = Marginal Product of labour

$MP_K = \frac{dQ}{\partial K}$, here L is constant.

where; MP_k = Marginal product of capital.

10.3.7 Iso-Quant (IQ) Function

An Iso-Quant (IQ) function can be expressed as:

Q = f(L,K) = C where C is constant.

Slope of $IQ = -\frac{dK}{dL} = \frac{MP_L}{MP_K}$

Note : IC possesses two characteristics which are :

(i) IQ is downward sloping is $\frac{dK}{dL} < 0$.

(ii) IQ is convex to the origin is $\frac{d^2K}{dL^2} > 0$.

10.3.8 Relationship between MR, MP and MPR

Let us consider the two- input production function as :

Q = f (L, K) where; L = Labour input

K = Capital input.

Here, $MP_L = \frac{\partial Q}{\partial L}$, here K is constant.

where = Marginal physical product of labour.

$MP_K = \frac{\partial Q}{\partial K}$, here L is constant

where MP_K = Marginal physical product of capital.

Note : Marginal product (MP) and Marginal Physical Product (MPP) are similar.

Again; Total Revenue (TR) is a function of quantity (Q) which can be written as;

TR = f (Q)

Here, $MR = \frac{dTR}{dQ}$

Now; $MRP_L = MP_L \times MR$

and $MRP_K = MP_K \times MR$

10.3.9 Consumption Function

Consumption (C) is a function of disposable income (yd), which can be written as;

c = f (yd) where Yd = Y – T

Y = Income

T = Tax

Marginal Propensity to consume (MPC) :

$$MPC = \frac{dc}{dy}$$

Note : 0<MPC<1

10.3.10 Saving Function

Saving (s) is a function of income (y) which can be written as;

s = f (y)

Marginal Propensity to Save (MPS) :

$$MPS = \frac{ds}{dy}$$

It is important to note that MPC+MPS=1.

10.3.11 Import Function

Import (M) is a function of income (y) which can be written as;

M = f (y)

Margainal Proensity to Import (MPM)

$$MPM = \frac{dM}{dy}$$

10.3.12 Elasticity of Supply (E_s)

Elasticity of supply can be expressed as:

$$E_s = \frac{\text{Proportionate change in quantity Supply}}{\text{Proportionate change in Price}}$$

$$= \frac{dQs}{Qs} / \frac{dp}{p} \quad \text{where;} \quad Qs = \text{Quantity Supply}$$

P = price of the commodity

$$\therefore E_s = \frac{dQs}{dp} \cdot \frac{P}{Qs}$$

We can express e_s in the following way also;

$$\therefore E_s = \frac{d(\log Qs)}{d(\log P)}$$

10.4 NUMERICAL PROBLEMS

In this section, we shall discuss a few numerical examples.

Example 10.1 : The demand function of a firm is given as $Q = 30–5P–P^2$. Find the elasticity of demand and MR when P = 2

Solution: Given, demand function $Q = 30 - 5P - P^2$.

Now when P = 2, we get

$$Q = 30 - 5(2) - (2)^2$$

$$= 30 - 10 - 4$$

$$= 16.$$

Now,

$$Q = 30 - 5P - P^2$$

$$\therefore \frac{dQ}{dP} = -5 - 2P$$

Putting P = 2, we get,

$$\frac{dQ}{dP} = -5 - 2(2)$$

$$= -9.$$

(i) $$E_d = -\frac{dQ}{dP} \times \frac{P}{Q}$$

$$= -(-9) \times \frac{P}{Q}$$

$$= 9 \times \frac{2}{16}$$

$$= \frac{18}{16} - 1.125.$$

(ii) We know, $MR = AR\frac{(Ed-1)}{(Ed)}$

$$MR = P\frac{(1.125-1)}{(1.125)}$$

$$= 2\left(\frac{125}{1125}\right)$$

$$= 2\ (0.11)$$

$$= 0.22.$$

Example 10.2 : The deamand fucntion of a firm is given as $Q = 100 - 4P - 2P^2$. Calculate price elasticity of demand when values of P are 2, 5 and 10 respectively and interpret.

Solution: Given demand function $Q = 100 - 4P - 2P^2$ ——— (i)

$$\therefore \frac{dQ}{dP} = \frac{d}{dP}(100 - 4P - 2P^2)$$

$$= -4 - 4P \text{ ———— (ii)}$$

We know that $Ed = (-)\left(\frac{P}{Q}\right) \times \left(\frac{dQ}{dP}\right)$

(i) Now putting P = 2 in (i) we get,

$$Q = 100 - 4(2) - 2(2)^2$$

$$= 100 - 8 - 8$$

$$= 84$$

$$\therefore \frac{dQ}{dP} = -4 - 4(2) \qquad \text{[From equation (ii)]}$$

$$= -4 - 8$$

$$= -12.$$

$$\therefore Ed = (-)\left(\frac{P}{Q}\right) \times \left(\frac{dQ}{dP}\right)$$

$$= (-)\frac{2}{84}(-12)$$

$$= \frac{24}{84} = \frac{2}{7} = 0.286$$

As |Ed| = 0.29 > 1. So, we can say that at P = 2, the demand is relatively inelastic. Thus, at P = 2, corresponding to a unit charge is price, the quantity demanded (Q) changes by 0.29 or by 29%.

(ii) Putting P = 5 in equation (i), we get

$Q = 100 - 4(5) - 2(5)^2$

$= 100 - 20 - 50$

$= 30.$

And, $\frac{dQ}{dP} = -4 - 4(5)$ [From equation (ii)]

$= -4 - 20$

$= -24.$

$$\therefore Ed = (-)\left(\frac{P}{Q}\right)\left(\frac{dQ}{dP}\right)$$

$$= (-)\left(\frac{5}{30}\right)(-24)$$

$$= \frac{5 \times 24}{30}$$

$= 4.$

As, Ed = 4>1, we can say that at P = 5, the demand is relatively elastic. Thus at P = 5, corresponding to a unit change in price, the quantity demanded of Q changes 4 times a by 400%.

(iii) Putting P = 10 in equation (i), we get

$Q = 100 - 4(10) - 2(10)^2$

$= 100 - 40 - 200$

$= -140.$

$\therefore \frac{dQ}{dP} = (-4) - 4(10)$ [From equation (ii)]

$= -4 - 40$

$= -44.$

$$\therefore Ed = (-)\left(\frac{P}{Q}\right)\left(\frac{dQ}{dP}\right)$$

$$= (-)\left(\frac{10}{-140}\right)(-44)$$

$$= -\frac{10 \times 44}{140}$$

$$= -\frac{22}{7} = -3.14$$

Or |Ed| = 3.14

As |Ed| = 3.14>1, we can say that at P = 10, the demand is relatively elastic. Thus, at P = 10, corresponding to a unit change in price, quantity demanded 3.14 times or 314 percent.

Example 10.3 : Supply function of a firm is given as $P = 4 + 5Q^2$. Find elasticity of supply at P = 9, P = 6, P = 4 and P = 3 and interpret.

Solution: Given the supply function

$$P = 4 + 5Q^2 \text{ ———————— (i)}$$

$$\Rightarrow 5Q^2 = P - 4$$

$$\Rightarrow Q^2 = \frac{1}{5}(P-4) - \text{(ii)}$$

$$\Rightarrow Q = \pm\sqrt{\frac{1}{5}(P-4)}$$

[– sign is ignored as it does not make any sense in Economics.]

Now, $\frac{dP}{dQ} = \frac{d}{dQ}(4+5Q^2)$

$= 10Q$ ——— (iii)

and $\frac{dQ}{dP} = \frac{1}{10Q}$

We know, $Es = \frac{P}{Q}\frac{dQ}{dP}$

$$= \left(\frac{4+5Q^2}{Q}\right)\left(\frac{1}{10Q}\right)$$

$$= \frac{1}{10}\left(\frac{4+5Q^2}{Q^2}\right) \text{ ———— (iv)}$$

(i) When P = Q

$$Q^2 = \frac{1}{5}(9-4) \text{ ———— [From equation (ii)]}$$

$= \frac{1}{5} \times 5$

= 1.

$\therefore Q = 1$. (– sign is ignored).

$$\therefore Es = \frac{1}{10}\left\{\frac{4+5(1)^2}{(1)^2}\right\} \qquad \text{[From (iv)]}$$

$$= \frac{1}{10} \times 9 = 0.9. < 1.$$

As Es is 0.9<1, we can say that at P = 9, elasticity of supply is relatively in elastic. Thus, at P = 9, corresponding to a unit change in price, the quantity supplied changed by 0.9 or 90%.

(ii) When P = 6,

$$Q^2 = \frac{1}{5}(6-4)$$

$$= \frac{2}{5}$$

$$\therefore Q = \pm\sqrt{\frac{2}{5}}$$

$$= \pm\sqrt{0.40}$$

$$= \pm 0.63 = 0.63 \qquad \text{[Negative sing is ignored].}$$

$$\therefore Es = \frac{1}{10}\left\{\frac{4+5\left(\frac{2}{5}\right)}{\frac{2}{5}}\right\} \qquad \text{[From equation (iv]}$$

$$\frac{1}{10}\left(6 \times \frac{5}{2}\right)$$

$$= \frac{15}{10} = 1.5 > 1.$$

As Es is 1.5>1, we can say that at P = 6, elasticity of supply is relatively elastic. Thus at P = 6, corresponding to a unit change in price, the quantity supplied changes by 1.5 times, or by 150%.

(iii) When P = 4

$$Q^2 = \frac{1}{5}(4-4)$$

$$= 0$$

$$Q = 0$$

$$\therefore \frac{dP}{dQ} = 10(0) = 0$$

$$\frac{dQ}{dP} = \frac{1}{0} \text{ or } \infty$$

and $$\therefore Es = \frac{1}{0}\left[\frac{4+5(10)^2}{(0)^2}\right]$$

Thus, at P = 4, Es is perfectly (or infinitely) elastic)

(iv) At P = 3,

$$Q^2 = \frac{1}{5}(3-4)$$

$$= \frac{-1}{5}$$

Or, $$Q = \sqrt{-\frac{1}{5}}$$

$Q = \sqrt{-\frac{1}{5}}$ is a non-real quantity.

$\therefore$ At p = 3, Es is not defined in real terms.

Example 10.4 : The total utility function of two commodities x and y givemn as $u = 2x^3y + 3xy^2 + 3x + 3y$. Find marginal utility of y at x = 2 and y = 3.

Solution: Given, $u - 2x^3y + 3xy^2 + 3x + 3y$

$\therefore$ Marginal utility of $y = \frac{\partial u}{dy}$

$$= \frac{\partial}{dy}(2x^3y + 3xy^2 + 3x + 3y)$$

$$= 2x^3 + 6xy + 3$$

For x = 2 and y = 3

Marginal utility of $y = 2(2)^3 + 6(2)(3) + 3$

$$= 16 + 36 + 3$$

$$= 55.$$

Example 10.5: The total utility function of two goods x and y is given as $U = 5x^2y + 2xy^3 + 3x + 9y$. Find marginal utilities of x and y at x = –3 and y=2.

Solution: Given total utility function $U = 5x^2y + 2xy^3 + 3x + 9y$.

$\therefore$ Marginal utility of x $(MU_x) = \frac{\partial u}{\partial x}$

$$= \frac{\partial}{\partial x}\left(5x^2y + 2xy^3 + 3x + 9y\right)$$

$= 10xy + 2y^3 + 3$ ——————— (i)

Marginal utility of y $(MU_y) = \frac{\partial u}{\partial y}$

$$= \frac{\partial}{dy}(5x^2 + 2xy^3 + 3x + 9y)$$

$= 5x^2 + 6xy^2 + 9$ ——————— (ii)

Putting x = 3 and y = 2 in (i), we get

$MU_x = 10(3)(2) + 2(2)^3 + 3$

$= 60 + 16 + 3$

$= 79$

Similarly, putting x = 3 and y = 2 in (ii), we get

$MU_y = 5(3)^2 + 6(3)(2)^2 + 9$

$= 45 + 72 + 9$

$= 126.$

Example 10.6: The production function of a firm is given as $Q = K^2 + 2KL + L^3$, where Q stands for Qunatity, L for labour and K for capital. Compute the marginal product of capital at K = 3 and L = 2

Solution: Given the production function $Q = K^2 + 2KL + L^3$.

Now, marginal product of capital $(MP_K) = \frac{\partial Q}{\partial K}$

$$= \frac{\partial}{\partial x}\left(K^2 + 2KL + L^3\right)$$

$= 2K + 2L.$

$\therefore$ Marginal product of capital at K = 3 and L = 2

$= 2\ (3) + 2\ (L)$

$= 6 + 4$

$= 10.$

Example 10.7: The production function of a firm is given as $Q = 40F + 3F^2 - F^{3}/_{3}$. Find out marginal and average production functions.

Solution: Given, the production function $Q = 40F + 3F^2 - F^{3}/_{3}$

(i) Marginal production function $= \frac{d}{dF}(Q)$

$$= \frac{d}{dF}\left(40F + 3F^2 - F^3/3\right)$$

$$= 40 + 6F - \frac{1}{3}3F^2$$

$$= 40 + 6F - F^2$$

(ii) Average production function $= \frac{Q}{F}$

$$= \frac{40F + 3F^2 - F^3/3}{F}$$

$$= 40 + 3F - F^2/3$$

Example 10.8 : The consumption function is given by $C = 1000 - \frac{5000}{3+y}$.

Find the following :

(i) Marginal propensity to consume when y = 97.

(ii) Marginal propensity to save when y = 97.

Solution: Given consumption function $C = 1000 - \frac{5000}{3+y}$.

$= 1000 - 5000\ (3+y)^{-1}$

(i) $MPC = \frac{dc}{dy}$

$$= \frac{d}{dy}\left\{1000 - 5000(3+y)^{-1}\right\}$$

$$= 0 - (-1)5000(3+y)^{-}$$

$$= \frac{5000}{(3+y)^2}$$

Now when y = 97, $MPC = \frac{5000}{(3+97)^2}$

$$MPC = \frac{5000}{(3+97)^2}$$

$$= \frac{5000}{10{,}000}$$

$$= 1/2$$

$$= 0.5$$

(ii) The saving function is defined as S = Y – C

or $S = Y - 1000 + 5000\,(3+y)^{-1}$

$$\therefore \text{MPS} = \frac{ds}{dy}$$

$$= \frac{d}{dy}\left\{y - 1000 + 5000(3+y)^{-1}\right\}$$

$$-1 - 0 + (-1)5000(3+y)^2$$

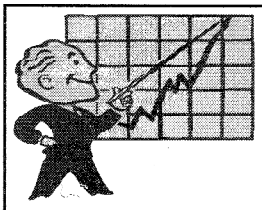
$$= 1 - \frac{5000}{(3+y)^2}$$

Now when y = 97, $\text{MPS} = 1 - \frac{5000}{(3+97)^2}$

$$= 1 - \frac{5000}{10000}$$

$$= 1 - 0.5$$

$$= 0.5$$



CHECK YOUR PROGRESS

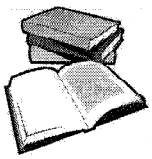
Q 1: Given demand function Q = 200 – 4P. Find price elasticity of demand.

Q 2 : Given MR = 15 and elasticity of demand is 4. Find AR.



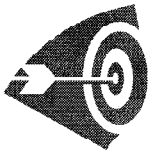
10.5 LET US SUM UP

- In this unit, we have discussed some of the applications of differenttial calculus in Economics.



10.6 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
2) Baruah, S. (2011). *Basic Mathematics and Its Application in Economics*, New Delhi: Trinity Press Pvt. Ltd.
3) Bose, D. (2004). *Mathematical Economics*; New Delhi: Himalaya Publishing House.
4) Chiang, A.C. & (2010). *Fundamental Methods of Economics Analysis*, MC Graw Hill Education India.
5) Kandoi, Balwant (2011). *Mathematics for Business and Economics with Application*; New Delhi: Himalaya Publishing House.



10.7 ANSWER TO CHECK YOUR PROGRESS

Ans to Q.No.1: Try yourself.

Ans to Q.No.2: We know

$$AR = MR\left[\frac{Ed}{Ed-1}\right]$$

$$= 15\left[\frac{4}{4-1}\right]$$

$$= 15 \times \frac{4}{3}$$

= 20.



10.8 MODEL QUESTIONS

Q 1: Find the elasticity of demand and AR, at P = 3, if the demand function $q = 32 - 4p - p^2$.

Q 2: Find the elasticity of demand and MR for the demand function $P = -2q^2 + 18$ for q = 1.

Q 3: Given the demand curve $P = (10 - 2D)(20 - D)^2$. Find the elasticity of demand for D = 1 and D = 4.

In jthis case Find AR, if MR is Rs 18 and the elasticity is 9.

Q 4: Find the Marginal Utilities for the following functions :

i) $U = 9x^3 + 2x^2 + x - 7$

ii) $U = 3x^3 - 3x + 4$

iii) $U = x^2 + y^2 + 2xy^2$

iv) $U = 2x^2 + 3x^2 - 2x^2y^2$

v) $U = 2x^2y + 3xy - 3x$.

Q 5: The total cost function is : $C = 2Q - 2Q^2 + Q^3$

(C – total cost and Q – total quanity produced)

Find (i) Average cost function.

(ii) Martginal cost function.

Also verify that at the minimum of average cost, average cost is equal to marginal cost.

Q 6: Given the following demand curves, demonstrate the relationship between marginal revenue and elasticity of demand

(i) $p = 1.47 - 0.1x$ (ii) $p = 86 - 25x$

Q 7: The total cost functions are :

(i) $C = 2Q - 2Q^2 + Q^3$

(ii) $C = 4Q - Q^2 + 2Q^3$

Find the (i) *AC* function

(ii) *MC* function

(iii) At what level of output *AC* is minimum?

(iv) Verify that at minimum average cost *AC* = *MC*.

Q 8: The total cost of producing a product is C = 3x + 200 and 50 units are produced. Find (i) Fixed Cost (ii) Variable Cost (iii) Total Cost (iv) Variable Cost per unit (v) Average Cost per unit.

Q 9: The Average Cost function for a commodity is given by $AC = \frac{2}{3}x + 15.2$. Find the Total and Marginal Cost.

Q 10: Given the Revenue Function $R = 40x - 2x^2 - 20$, find the Average Revenue and Marginal Revenue.

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UNIT 11: CONCEPTS OF MAXIMA AND MINIMA

UNIT STRUCTURE

11.1 Learning Objectives

11.2 Introduction

11.3 Maximum and Minimum of a Function

11.3.1 Methods for Finding Maximum and Minimum

11.4 Let Us Sum Up

11.5 Further Reading

11.6 Answers to Check Your Progress.

11.7 Model Questions

11.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- find maximum and minimum value of a function
- determine the points on a given curve where extremum exist, if any

11.2 INTRODUCTION

In this unit, we shall discuss maximum and minimum of a function. Also, we also discuss the methods of finding maximum and minimum values of a function. Lastly, we will discuss application of maxima and minima.

11.3 MAXIMUM AND MINIMUM OF A FUNCTION

A function f(x) is said to have a maximum at x=a if f(a) is greater than every other value assumed by f(x) in the immediate neighbourhood of x=a.

This means that f(a) > f(a+h) and f(a) > f(a–h) for a sufficiently small positive h. (See Figure 11.1)

Similarly, a function f(x) is said to have a minimum at x=b if f(b) is smaller than every other value assumed by f(x) in the immediate neighbourhood of x=b.

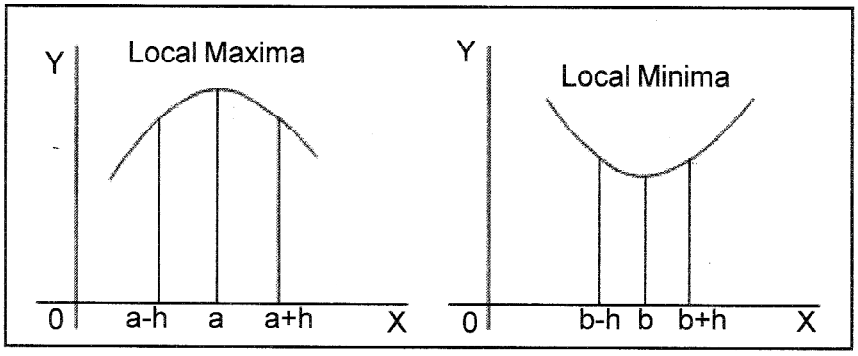
This means that

f(b) < f(b+h)

and $f(b) < f(b-h)$

for a sufficiently small positive h (See Figure 11.1).

Figure 11.1: Local Maxima and Local Minima



Geometrically, a function f continuous at a point has a maximum or a minimum at that point according as the graph of f has a peak or a cavity at that point.

Note : i) The maximum and minimum values of a function are also known as local (relative) maxima or local (relative) minma as these are the greatest and least values of the function relative to some neighbourhood of the point in the question. The term extremum is also used for both a maximum and minimum value.

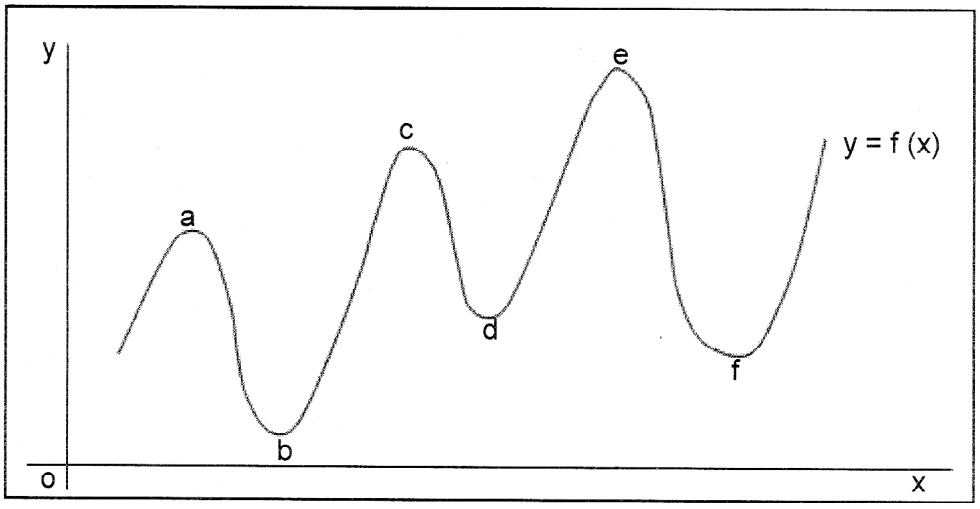
ii) A local maximum value at a point may be less than a local minimum value at another point.

iii) A function can have several maximum and minimum values. The maximum and minimum values of a continuous function occur alternatively.

iv) At maximum, the function f(x) changes from increasing to decreasing state and at minimum the function f(x)changes from decreasing to increasing state.

Concept of Global Maxima and Global Minima: So far, we have discussed the concept of relative or local maxima and minima. Let us consider the function y = f(x) shown with the help of Figure 11.2. From the figure it can be seen that in the entire fuction y = f (x), 'b' is the lowest point, while 'e' is the highest point. The other minimum points, e.g., 'd' and 'f' are minimum points in their immediate neighbourhood. Similarly, point 'a' and 'c' are maximum points in their immediate neighbourhood. Hence, these

Figure 11.2: Local and Global Maxima and Minima



points are called local minima and maxima respectively. On the other hand, 'b' being the lowest minimum point in the entire function is called the global minimum and 'e' being the highest maximum point in the entire function, is called the global maximum. From the above, we can say that there may be many local minima or maxima, but there can be only one global minimum or global maximum.

11.3.1 Methods of Finding Maximum or Minimum

Method I :

1) Find $\frac{dy}{dx}$

2) Solve the equation $\frac{dy}{dx}=0$ for real values of x. Let these values be a, b, c etc.

3) Consider x=a. Study the sign of $\frac{dy}{dx}$ for values of a slightly less than a and slightly greater than a.

4) If $\frac{dy}{dx}$ changes sign from positive to negative, then f(x) is maximum at x=a.

If $\frac{dy}{dx}$ changes sign from negative to positive, then f(x) is minimum at x=a.

If $\frac{dy}{dx}$ does not change sign, then x=a is a point neither of maximum nor of minimum.

Method II :

Steps I and 2 are same as in method I.

3) Find $\frac{d^2y}{dx^2}$ and put values. Consider x=a.

i) If $\frac{d^2y}{dx^2}<0$ at x=a, then f(x) is maximum at x=a.

ii) If $\frac{d^2y}{dx^2}>0$ at x=a, then f(x) is minimum at x=a.

iii) If $\frac{d^2y}{dx^2}=0$ but $\frac{d^3y}{dx^3}\neq 0$ at x=a, then there is neither maximum nor minimum at x=a.

Illustrative Examples

Example 11.1 : Find x for which f(x)=x(5–x) is the maximum. What is the maximum value of f(x) ?

Solution : Given, f(x)=x(5–x)

$$\Rightarrow f(x) = 5x - x^2$$

$\therefore f'(x)= 5 - 2x$

For an extremum, we have

$f'(x)=0$

$\Rightarrow 5-2x=0$

$\Rightarrow x=\frac{5}{2}$

Also, $f''(x) = \frac{d}{dx}(5-2x)$

$= -2$

$$f''\left(\frac{5}{2}\right) = -2 < 0$$

$\therefore$ f(x) is maximum at $x=\frac{5}{2}$ and the maximum value

$$= f\left(\frac{5}{2}\right)$$

$$= \frac{5}{2}\left(5-\frac{5}{2}\right)$$

$$= \frac{25}{4}$$

Example 11.2: Find the values of Maximum and Minimum of $y = 2x^3 + 3x^2 - 36x + 10$.

Solution : Given $y = 2x^3 + 3x^2 - 36x + 10$

$$\Rightarrow \frac{dy}{dx} = 6x^2 + 6x - 36$$

$$= 6\left(x^2 + x - 6\right)$$

For a maximum or minimum

$$\frac{dy}{dx} = 0$$

$$\Rightarrow 6\left(x^2 + x - 6\right) = 0$$

$$\Rightarrow 6(x+3)(x-2) = 0$$

$$\Rightarrow x = 2 \text{ or } x = -3$$

Now, $\frac{d^2y}{dx^2} = 12x + 6$

At $x = 2$, $\frac{d^2y}{dx^2} = 12(2) + 6 = 30 > 0$

At $x = -3$, $\frac{d^2y}{dx^2} = 12(-3) + 6 = -30 < 0$

Thus y is maximum at the point $x = -3$ and is minimum at the point $x = 2$.

$\therefore$ Minimum value of y $= 2(2)^3 + 3(2)^2 - 36(2) + 10$

$$= 16 + 12 - 72 + 10$$

$$= -34$$

and Maximum value of y $= 2(-3)^3 + 3(-3)^2 - 36(-3) + 10$

$$= -54 + 27 + 108 + 10 \qquad = 91$$

Example 11.3 : Show that the function $y = x^3 - 6x^2 + 9x - 8$ has a maximum value at $x = 1$ and a minimum value at $x = 3$.

Solution : Given $y = x^3 - 6x^2 + 9x - 8$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 12x + 9$$

$$= 3\left(x^2 - 4x + 3\right)$$

For maximum and minima, $\frac{dy}{dx} = 0$

$$\Rightarrow 3\left(x^2 - 4x + 3\right) = 0$$

$$\Rightarrow 3(x-1)(x-3) = 0$$

$$\Rightarrow x = 1,3$$

Now, $\frac{d^2y}{dx^2} = 2x - 4$

At $x = 1$, $\frac{d^2y}{dx^2} = -2 < 0$

At $x = 3$, $\frac{d^2y}{dx^2} = 2 > 0$

Thus y is maximum at $x = 1$ and the maximum value is

$= (1)^3 - 6(1)^2 + 9(1) - 8$

= -4

y is minimum at $x = 3$ and the minimum value is

$= (3)^3 - 6(3)^2 + 9(3) - 8$

$= -8$

Example 11.4 : Find out the relative extrema of the function y=20x - $2x^2$.

Solution : The first order condition is: f'(x) = 0

$$\therefore \frac{dy}{dx} = 20 \times 1 - 2 \times 2x^{2-1}$$

$$= 20 - 4x$$

$$Now, \frac{dy}{dx} = 20 - 4x = 0$$

$$\Rightarrow 4x = 20$$

$$\Rightarrow x = 5$$

Second order condition: Now, to verify whether the function will attain maximum or minimum when x=5, we test the second order whether f''(x) >0 or, f''(x) <0.

$$Now, f''(x) = \frac{d2y}{dx2} = \frac{d}{dx}(20 - 4x)$$

$= -4 < 0.$

$\therefore x = 5$, gives the maximum of the function $y = 20x - 2x^2$.

Therefore, Maximum value is:

$$y = 20 \times 5 - 2 \times (5)^2$$
$$= 100 - 50$$
$$= 50$$

Example 11.5 : Find out the relative extrema of the function: $y=2x^2-16x+50$.

Solution : The first order condition is: $f'(x) = 0$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^2 - 16x + 50)$$
$$= 2 \times 2x^{2-1} - 16 \times 1 + 0$$
$$= 4x - 16$$

$$Now, \frac{dy}{dx} = 4x - 16 = 0$$
$$\Rightarrow 4x = 16$$
$$\rightarrow x = 4$$

To verify whether the function will attain maximum or minimum when x = 4, we have to find the second order derivative $\frac{d^2y}{dx^2}$ and see whether $\frac{d^2y}{dx^2} > 0$ or $\frac{d^2y}{dx^2} < 0$.

$$Now, \frac{d^2y}{dx^2} = \frac{d}{dx}(4x - 16)$$
$$= 4 > 0.$$

Therefore, the function $y=2x^2-16x+50$ will have relative minimum at x = 4, and the minimum value is:

$$y = 2 \times 4^2 - 16 \times 4 + 50$$
$$= 32 - 64 + 50$$
$$= 82 - 64 = 18.$$

Example 11.6 : Find out the maximum and minimum values of the function: $y=x^3-3x+1$.

Solution : The first order condition for maximum or minimum is:

$$\frac{dy}{dx} = 0.$$

$$Now, \frac{dy}{dx} = \frac{d}{dx}(x^3 - 3x + 1)$$
$$= 3 \times x^{3-1} - 3 \times 1 + 0$$
$$= 3x^2 - 3$$
$$Now, \frac{dy}{dx} = 3x^2 - 3 = 0$$
$$\Rightarrow 3x^2 = 3$$
$$\Rightarrow x^2 = \pm 1$$

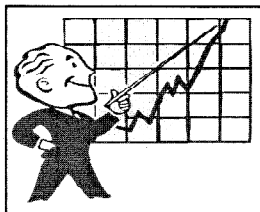
$\therefore x = 1,$ or $x = -1.$

To distinguish between maximum and minimum values, we apply the second order condition.

$$f''(x) = \frac{d}{dx}(3x^2 - 3)$$
$$= 6x$$

(i) At x = 1, $f''(x) = 6 \times 1 = 6 > 0$, i.e., minimum value.

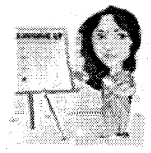
(ii) At x = -1, $f''(x) = 6 \times (-1) = -6 < 0$, i.e., maximum value.



CHECK YOUR PROGRESS

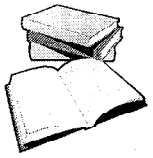
Q 1: Find the maximum or minimum values of the following functions.

a) $2x^2-6x+3$ b) $x+\frac{1}{x}$ c) $4x^3-15x^2+12x-1$



11.4 LET US SUM UP

- $\frac{dy}{dx}$ represents the rate of change of y with respect to x.
- The values of x obtained on solving $f'(x)=0$ are called stationary or critical points. A function have local maximum or minimum at these points according at $f''(x)<0$ or $f''(x)>0$ respectively.



11.5 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
2) Baruah, S. (2011). *Basic Mathematics and Its Application in Economics*, New Delhi: Trinity Press Pvt. Ltd.
3) Bose, D. (2004). *Mathematical Economics*; New Delhi: Himalaya Publishing House.
4) Chiang, A.C. & (2010). *Fundamental Methods of Economics Analysis*, MC Graw Hill Education India.
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11.6 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (a) Let $f(x) = 2x^2-6x+3$

$\therefore f'(x)=4x-6$

For maximum or minimum, we have

$f'(x)=0$

$\Rightarrow 4x-6=0$

$\Rightarrow x = \frac{3}{2}$

Now, $f''(x) = \frac{d}{dx}(4x-6)$

$\Rightarrow f''(x) = 4$

$\Rightarrow f''\left(\frac{3}{2}\right) = 4 > 0$

$\therefore$ f(x) is minimum when $x=\frac{3}{2}$

and the minimum value is

$$= f\left(\frac{3}{2}\right)$$

$$= 2\times\frac{9}{4} - 6\times\frac{3}{2} + 3$$

$$= \frac{9-18+6}{2}$$

$$= -\frac{3}{2}$$

(b) Let $f(x) = x + \frac{1}{x}$

$$\therefore f'(x) = 1 - \frac{1}{x^2}$$

For maximum or minimum, we have

$f'(x) = 0$

$$\Rightarrow 1 - \frac{1}{x^2} = 0$$

$\Rightarrow x^2 - 1 = 0$

$\Rightarrow (x+1)(x-1) = 0$

$\Rightarrow x = -1, x = 1$

Now, $f''(x) = \frac{d}{dx}\left(1 - \frac{1}{x^2}\right)$

$$= 0 - \frac{-2}{x^3}$$

$$= \frac{2}{x^3}$$

$$\therefore f''(-1) = -2 < 0$$

$\Rightarrow$ f(x) is maximum at x=–1, and the maximum value is

$f(-1) = -1 + \frac{1}{-1} = -2$.

Again, $f''(1) = 2 > 0$

$\therefore$ f(x) is minimum at x=1 and the minimum value is f(1)=1+1=2

(c) Let $f(x) = 4x^3 - 15x^2 + 12x - 1$

$\Rightarrow f'(x) = 12x^2 - 30x + 12$

For maximum or minimum, we have $f'(x) = 0$

$\Rightarrow 12x^2 - 30x + 12 = 0$

$\Rightarrow 6(2x^2 - 5x + 2) = 0$

$\Rightarrow (2x-1)(x-2) = 0$

$\Rightarrow x = \frac{1}{2},\quad x = 2$

Now, $f''(x) = \frac{d}{dx}\left(12x^2 - 30x + 12\right)$

$= 24x - 30$

$\therefore f''\left(\frac{1}{2}\right) = 12 - 30$

$= -18 < 0$

$\Rightarrow$ f(x) is maximum when $x = \frac{1}{2}$ and the maximum value is

$= f\left(\frac{1}{2}\right)$

$= 4 \times \frac{1}{8} - 15 \times \frac{1}{4} + 12 \times \frac{1}{2} - 1$

$= \frac{1}{2} - \frac{15}{4} + 5$

$= \frac{2 - 15 + 20}{4}$

$= \frac{7}{4}$

Also, $f''(2) = 24 \times 2 - 30$

$= 18 > 0$

$\Rightarrow$ f(x) is minimum when x=2 and the minimum value is

$= f(2)$

$= 4 \times 8 - 15 \times 4 + 12 \times 2 - 1$

$= 32 - 60 + 24 - 1$

$= 56 - 61$

$= -5$



11.7 MODEL QUESTIONS

Q 1: Find the points at which the following functions have maximum and minimum values :

(a) $y = 9x^3 - 45x^2 + 48x + 11$ (b) $y = 11 - 12x + 6x^2 - x^3$

Q 2: Find the maximum and minimum values of :

(a) $y = \left(\dfrac{x^2 - 3x - 4}{x - 8} \right)$ (b) $y = 2x^3 - 15x^2 + 36x + 10$

Q 3: For what values of x are the following functions minimum or maximum?

(a) $y = x^3 - 9x^2 + 15x - 3$

(b) $y = \left(\dfrac{2x + 1}{x^2 - 8x - 2} \right)$

(c) $y = 2x^3 - 15x^2 + 36x + 10$

Q 4: Find the extreme values of the function $y = x^3 - 9x^2 + 15x + 20$.

Q 5: Find the extreme values of the following functions:

(a) $y = 3x^2 + 18x + 12$

(b) $y = 5x^2 - 30x + 15$

(c) $y = x^3 - 48x + 10$

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UNIT 12: ECONOMIC APPLICATIONS OF MAXIMA

UNIT STRUCTURE

12.1 Learning Objectives

12.2 Introduction

12.3 Economic Applications of Maxima

12.4 Let Us Sum Up

12.5 Further Reading

12.6 Answers to Check Your Progress

12.7 Model Questions

12.1 LEARNING OBJECTIVES

After going through this unit, you will be able to -

- solve profit maximisation problems
- solve revenue maximisation problems

12.2 INTRODUCTION

In the previous unit, we discussed the concept of maximisation and minimisation. In this unit, we shall look into a few maximisation problems in Economics, particularly relating to maximisation of profit and revenue. From our theoretical knowledge, we have known that an entrepreneur always try to maximise his/her revenue and profits. In the next unit, we shall discuss a few minimisation problems.

12.3 ECONOMIC APPLICATIONS OF MAXIMA

In this section, we shall basically look into a few maximisation problems, particularly relating to profit and revenue maximisation.

We have already discussed that profit (π) is defined as the difference between total revenue (TR) and total cost (TC). Let us consider a few examples of profit maximisation.

Example 12.1 : The demand function of a business firm is given as $p = 12 - 3q$, where p is price and q is output. Find the values of p and q at which the revenue is maximum. Also find maximum revenue.

Solution : Given $p = 12 - 3q$ ——— (i)

We know, Total Revenue (TR) = Price x output

$\therefore$ TR = p.q

$= (12 - 3q).q$

$= 12q - 3q^2$.

The first order condition for maximisation of revenue is $\frac{dTR}{dq} = 0$

$$\Rightarrow \frac{dTR}{dq} = \frac{d}{dq}\left(12q - 3q^2\right)$$

$= 12 - 6q$

Thus, at the point of maximum $\frac{dTR}{dq} = 0$

$\Rightarrow 12 - 6q = 0$

$\Rightarrow 6q = 12$

$\Rightarrow q = 2$.

Now, the demand function will be maximum at q = 2 if the second order condition of teh function is satisfied.

The second order condition for maximisation is $\frac{d^2TR}{dq^2} < 0$.

Now $\frac{d^2TR}{dq^2} = \frac{d}{dq}(12 - 6q)$

$= -6 < 0$.

As $\frac{d^2TR}{dq^2}$ is negative, the given function (i) is maximum at q = 2.

Now putting q = 2, in equation (i) we get

$p = 12 - 3.2$

$= 12 - 6 = 6$

$\therefore$ Maximum revenue at q = 2 and p = 6 is :

$TR = p.q$

$= 6 \times 2$

$= 12$

Example 12.2 : The demand and cost function of a firm are given as $p = 46 - 3q$ and $c = 110 + 6q + \frac{q^2}{2}$ respectively. Find the no. of units and price at which profit is maximum. Find out maximum profit.

Solution : Given,

demand function P = 46 – 3q ——— (i)

and cost function $C = 110 + 6q + \frac{q^2}{2}$.

We know that, total revenue (TR) = p.q

= (46 – 3q) q

= 46q – 3q²

∴ Profit (π) = TR – TC

$$-\left(46q - Sq^2\right) - \left(110 + 6q + \frac{q^2}{2}\right)$$

$$= 46q - 3q^2 - 110 - 6q - \frac{q^2}{2}$$

$$= 40q - 3q^2 - \frac{q^2}{2} - 110$$

$$= 40q - \frac{\left(6q^2q^2\right)}{52} - 110$$

$$= 40q - \frac{5q^2}{2} - 110 \quad \text{——— (ii)}$$

The first order condition for maximisation of profit is $\frac{d\pi}{dq} = 0$

$$= \frac{d}{dq} = \left(40q - \frac{5q^2}{2} \quad 110\right) = 0$$

$$\Rightarrow 40 - \frac{10q}{2} = 0$$

$$\Rightarrow \frac{10q}{2} = 40$$

$\Rightarrow$ 10q = 80

$\Rightarrow$ q = 8.

Now, the given demand function will be maximum at q = 8, if the second order condition of maximisation is satisfied.

The second order condition for maximisation is $\frac{d^2\pi < 0}{dq^2}$

Now, $\frac{d^2\pi}{dq^2} = \frac{d}{dq}\left(40 - \frac{10q}{2}\right)$

$= -5 < 0.$

As $\frac{d^2\pi}{dq^2}$ is negative, the given demand function (i) is maximum at q = 8.

Putting q = 8 in equation (ii) we get,

Profit (π) $= 40.8 - \frac{5.8^2}{2} - 110$

= 320 – 160 – 110

= 50. Ans

Example 12.3 : The demand function of a firm is given by P = 500 – 0.2q and its cost function is C = 25q + 10000 (p = price, q = output and c=cost). Find the output at which the profits of the firm are maximum. Also find the price the firm will change.

Solution : Given, demand function P = 500 – 0.2q ——— (i)

and cost function (TC) = 25q + 10000

We know that, total revenue (TR) = p.q

= (500 – 0.2q) q

= 500q – 0.2q²

∴ Profit (π) = TR – TC

= 500q – 0.2q² – 25q – 10000

= – 0.2q² + 475q – 10000 ——— (ii)

The first order condition for maximisation of profit is $\frac{d\pi}{dq} = 0$

$$\Rightarrow \frac{d}{dq}\left(-0.2q^2 + 475q - 10000\right) = 0$$

$$\Rightarrow -0.4q + 475 = 0$$

$$\Rightarrow 0.4q = 475$$

$$\Rightarrow q = 1187.5$$

Now, the given demand function will be maximum at q = 1187.5 if the second order condition is satisfied.

The second order condition for maximisation is $\frac{d^2\pi}{dq^2} < 0$

Now, $\frac{d^2\pi}{dq^2} = \frac{d}{dq}(-0.4q + 475) < 0$

$= -0.4. < 0.$

As $\frac{d^2\pi}{dq^2}$ is negative, the given demand function (i) is maximum at q = 1187.5

$\therefore$ Putting q = 1187.5 we get
Price = 500 – 237.5
= 262.5

Ans : (i) Profit maximising output (q) = 1187.5
(ii) Price = 262.5.

Example 12.4 : A firm under non-perfect competition has the following total cost and demand functions :

$C = c(q) = 20 + 2q + 3q^2$

$P = f(p) = 50 - q.$

(i) find the profit maximising levels of output and price.

(ii) find the profit maximising levels of q and p of GST (Goods and Service Tax) Rs. 5 per unit is imposed.

(iii) find the rate of GST that would maximise the tax revenue to the govt.

Solution : Given, cost function (TC) $= 20 + 2q + 3q^2$

demand function (P) $= 50 - q$ ——— (i)

$\therefore TR = P.q = (50 - q) . q$

$= 50q - q^2$

(i) Profit (π) = TR – TC

$= (50q - q^2) - (20 + 2q + 3q^2)$

$= 50q - q^2 - 20 - 2q - 3q^2$

$= -4q^2 + 48q - 20$ ——— (ii)

For maximisation of profit (π), the first order condition is $\frac{d\pi}{dq} = 0$

$\Rightarrow \frac{d}{dq}(-4q^2 + 48q - 20) - 0$

$\Rightarrow -8q + 48 = 0$

$\Rightarrow q = 48/8 = 6.$

Now, the given profit function (ii) will be maximum at q = 6, if the second order condition of maximisation is satisfied.

The second order condition for maximisation is :

$$\frac{d^2\pi}{dq^2} < 0 \qquad \Rightarrow \frac{d}{dq}(-8q + 48) < 0$$
$$\Rightarrow -8 < 0$$

As the second order condition of maximisation is satisfied, the π function is maximum at q = 6.

$\therefore$ q = 6

p = 50 – 6 = 44.

(ii) In the event of inposition of GST (i) Rs. 5 per unit, the new profit function becomes :

Profit (π) = TR – (TC + t.q) (Here t= Rate of GST)

= $(50q - q^2) - (20 + 2q + 3q^2 + 5.q)$

= $50q - q^2 - 20 - 2q - 3q2 - 5q$

= $-4q^2 + 43q - 20$ ———————— (iii)

Now, the first order condition for maximisation of (π) (iii) $\frac{d\pi}{dq} = 0$

$$\Rightarrow \frac{d}{dq}\left(-4q^2 + 43q - 20\right) = 0$$

$$\Rightarrow -8q + 43 = 0$$

$$\Rightarrow q = 43/8 = 5.375$$

Now, the given profit function (iii) will be maximum at q = 5.375 of the second order condition is fulfilled.

The second order condition for maximisation is ——

$$\frac{d^2\pi}{dq^2} < 0 \qquad \Rightarrow \frac{d}{dq}(-8q + 43) < 0$$
$$\Rightarrow -8 < 0$$

$\therefore$ As the second order condition is fulfilled, the given π function is maximum at q = 5.375.

$\therefore$ q = 5.375

p = 50 – 5.375 = 44.625

Thus, it is clear that with the imposition of GST, price has increased, while

profit has come down.

(iii) Let us suppose, the govt. impose GST @ Rs 't', per unit. Now the profit function becomes :

$$\begin{aligned}\pi &= TR - (TC + t.q)\\ &= (50q - q^2) - (20 + 2q + 3q^2 + tq)\\ &= 50q - q^2 - 20 - 2q - 3q^2 - tq\\ &= -4q^2 + 48q - tq - 20\\ &= -4q^2 + q(48 - t) - 20. \qquad \text{——————} \text{ (iv)}\end{aligned}$$

To find the rate of tax that maximises the govt. revenue, we have to find the profit maximising quantity (q) of the firm first.

Now, the first order condition for maximisation of profit (π) is $\frac{d\pi}{dq} = 0$

$$\Rightarrow \frac{d}{dq}\left\{-4q^2 + q(48 - t) - 20\right\} = 0$$

$$\Rightarrow -8q + (48 - t) = 0$$

$$\Rightarrow q = \frac{(48 - t)}{8}$$

The given π function (iv) will be maximum at q = (48 – t)/8 if the second order condition of maximisation is fulfilled. The second order condition for maximisation is $\frac{d\pi}{dq} =< 0$

$$\Rightarrow \frac{d}{dq}(-8q + 48 - t) < 0$$

$$\Rightarrow -8 < 0$$

As the second order condition is fulfilled, the profit maximising output of the firm is

$$q = \left(\frac{48 - t}{8}\right)$$

Now, total tax revenue of the govt is

$$T = t.q = t.\frac{(48 - t)}{8}$$

$$= 6t - \frac{t^2}{8} \qquad \text{——————} \text{ (v)}$$

For maximisation of (T), with respect to rate of GST, 't', the first order condition is $\frac{dT}{dt} = 0$

$$\Rightarrow \frac{d}{dt}\left(6t - \frac{t^2}{8}\right) = 0$$

$$\Rightarrow 6 - \frac{2t}{8} = 0$$

$$\Rightarrow \frac{2t}{8} = 6$$

$$\Rightarrow t = 6 \times 4 = 24.$$

The taxation function (v) will be maximum at t = 24, if the second order condition of maximisation is fulfilled. The second order condition for maximisation is –

$$\frac{d^2T}{dt^2} < 0$$

$$\Rightarrow \frac{d}{dt}\left(6 - \frac{2t}{8}\right) < 0$$

$$\Rightarrow -\frac{2}{8} < 0.$$

As the second order condition is fulfilled at given function (v) is maximum at t = 24.

$\therefore$ Rate of tax for maximising tax revenue = Rs. 24 per unit.

Example 12.5 : A firm has the following cost and demand functions.

$P = 150 - 0.5q$ $C = 100 + 3q + 7q^2$.

(i) Find the profit maximising levels of output and price, if the firm is paid a subsidy of Rs. 3 per unit by the govt.

(ii) Would anything be produced in the absence of subsidy?

(iii) What would be the change of equilibrium price and quantity with respect to subsidy?

Solution : Give demand function P $= 150 - 0.5q$

Total revenue (TR) $= P.q$

$= 150q - 0.5q^2$

TC $= 100 + 3q + 7q^2$

$\therefore$ Profit (π) $= TR - TC$

$= (150q - 0.5q^2) - (100 + 3q + 7q^2)$

$= 150q - 0.5q^2 - 100 - 3q - 7q^2$

$= -7.5q^2 + 147q - 100$ ———— (i)

(i) When the government offers Rs 3 per unit it will reduce cost of the firm. Thus, subsidy it will reduce cost of the firm. Thus, the profit function becomes–

$\pi = TR - (TC - tq)$

$= (150q - 0.5q^2) - (100 + 3q + 7q^2 - 3.q)$

$= 150q - 0.5q^2 - 100 - 3q - 7q^2 + 3q$

$= -7.5q^2 + 150q - 100.$ ———— (ii)

Now, for the maximisation of profit of the given function (iii), the first order condition is

$$\Rightarrow \frac{d\pi}{dq} = 0$$

$$\Rightarrow \frac{d}{dq}(-7.5q^2 + 150q - 100) = 0$$

$\Rightarrow (-15q + 150) = 0$

$\Rightarrow q = 150/15 = 10$

The given profit function (ii) will be maximum at q = 10, if the second order condition is satisfied. The second order condition for maximisation is

$$\Rightarrow \frac{d^2\pi}{dq^2} < 0$$

$$\Rightarrow \frac{d}{dq}(-15q + 150) < 0$$

$$\Rightarrow -15 < 0$$

Thus, second order condition is fulfilled at q = 10.

$\therefore q = 10$

$P = 150 - 0.5.\ 10$

$= 150 - 5 = 145.$

$\pi = -7.5.10^2 + 150.10 - 100$

$= 1500 - 850 = 650.$

(ii) If subsidy is withdrawn, we get the profit function as shown in equation (i). the first order condition for maximisation of the profit function (i) is –

$$\frac{d\pi}{dq} = 0$$

$$\Rightarrow \frac{d}{dq}\left(-7.5q^2 + 147q - 100\right) = 0$$

$$\Rightarrow -15q + 147 = 0$$

$$\Rightarrow q = \frac{147}{15} = 9.8$$

Now, the profit function (i) will be maximum at q = 9.8, if the second order condition for maximisation is satisfied. The second order condition for maximisation is :

$$\frac{d^2\pi}{dq^2} < 0$$

$$\Rightarrow \frac{d}{dq}(-15q + 147) < 0$$

$$\Rightarrow -15 < 0$$

As the second order condition is satisfied, the $\overline{\Lambda}$ function (i) is maximum at q=9.8.

$\therefore$ price p = 150 – 0.5 (9.8)

= 150 – 4.9

= 145.1

π = 7.5 $(9.8)^2$ + 147 (9.8) – 100

= –720.3 + 1440.6 – 100

= 620.3

(iii) When subsidy is withdrawn,

(a) price increases from 145 to 145.1.

(b) quantity decreases from 10 to 9.8.

(c) profit decreases from 650 to 620.3.

Example 12.6 : The following utility function is given U = 8Q – Q^2. Maximise U. Find the value of u when Q = 3.

Solution (i) Given, u = 8Q – Q^2.

The first order condition for maximisation is $\frac{dU}{dQ} = 0$

$\Rightarrow \frac{d}{dQ}(8Q - Q^2) = 0$

$\Rightarrow 8 - 2Q = 0$

$\Rightarrow Q = 4$

The given function U will be maximum at Q = 4, if the second order condition is satisfied. The second order condition for maximisation is $\frac{d^2U}{dQ_2} < 0$

$\Rightarrow \frac{d}{dQ}(8 - 2Q) < 0$

$\Rightarrow -2 < 0$

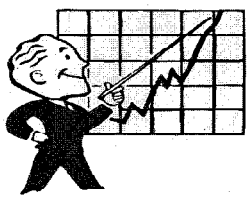
As the second order condition is satisfied, the function U is maximum at Q = 4.

(ii) Now when Q = 3,

U = $8.3 - 3^2$

= 24 – 9

= 15 utils.



CHECK YOUR PROGRESS

Q 1: The total cost function of a firm is $C - 5x^2 + 4x + 10$ where C is the total and x is output. A tax at the tate of Rs. 3 per unit of output is imposed and the producer adds it to his cost. If the market demand function is given by p = 100 – 3x, where Rs. p is the price per unit of output, find the profit maximizing output and price.

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12.4 LET US SUM UP

In this unit, we have seen the solution of a few:

- profit maximisation problems
- revenue maximisation problems



12.5 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
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12.6 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q No. 1 : We know, Total Revenue = Price x Qunatity

i.e., $R = (100 - 3x) \times x = 100 - 3x^2$

After imposing the tax Rs. 3 per unit, the total cost becomes

$$C = 5x^2 + 4x + 10 + 3x = 5x^2 + 7x + 10$$

$$\text{profit} = R - C$$

Or $$\pi = (100x - 3x^2) - (5x^2 + 7x + 10)$$

$$= -8x^2 + 93x - 10$$

Differentiating both sides *w.r.* to x, we get

$$\frac{d\pi}{dx} = -16x + 93$$

Now $\frac{d\pi}{dx} = 0 \Rightarrow x = \frac{93}{16} = 5.8125$

Also $\Rightarrow \frac{d^2\pi}{dx^2} = -16 = -ve$

$\therefore$ At $x = 5.8125$ Profit id maximised

Hence the profit optimizing output is 5.8125 units.

Price p = 100 – 3 x 5.8125 = 82.56



12.7 MODEL QUESTIONS

Short Questions

Q 1: Given the demand function $q = 25 - \frac{1}{4}p$, Find the value of q and p at which revenue is maximum.

Q 2: The Total Revenue (TR) curve for a commodity is given by

$$TR = 12x + \frac{1}{2}x^2 - \frac{1}{3}x^3$$

Find the level of output at which AR function is highest.

Q 3: If $Q = \sqrt{300 - \frac{p}{2}}$ is the demand function, find the level of output at which revenue is maximum.

Q 4: If the market demand function of a firm is 4ptx – 16 = 0 and average cost (AC) function takes the form $AC = \frac{4}{x} + 2 - 0.3x + 0.05x^2$. Find the level of output x which gives the maximum profit.

Long Questions

Q 1: Given the $TR = 83x - 4x^2 - 21$ and $TC = x^3 - 12x^2 + 48x + 11$ where x=output, TR = Total Revenue, TC = Total Cost. Obtain the output (x) for which profit is maximum and also find the maximum profit.

Q 2: A monopolist has the following demand and Cost functions :

$$p = f(x) = 100 - 5x$$

$$c = c(x) = x^2 + 4x + 5$$

Find the level of profit maximising output and the maximum profit.

Q 3: The demand function and average cost (AC) function of an industry are both linear. Show that the output for maximum profit will be half the output for zero profit.

Q 4 to 6: If the demand and cost functions of the monopolist are as under and an aedditive tax of *t* per unit output is imposed, determine the change in price, the maximum profit and revenue received by the government as function of *t*. Also, determine the maximum revenue that can be obtained by the government.

Q 4: $x_d = p/4 - 3$ $\quad TC = x^2 + 2x$

Q 5: $p = 12 - 4x_d$ $\quad TC = 8x + x^2$

Q 6: $x_d = 75 - 3p$ $\quad TC = 100 + 3x$

Q 7: If the demand law is $p = \beta + \alpha x$ of a monopolist with total cost function $C = ax^2 + bx + c$, determine the quantity and price that maximise the profit of monopolist if tax of 2 per unit of output is levied by the government.

Q 8: If the demand law is p = 12 – 5Q and total cost is $C = -Q^3 + 3Q^3$, determine the change in the price due to imposition of sales tax of 20%. Find the corresponding profit. Determine the price if government grants subsidy (instead of imposing tax) of 1% per unit of output.

What will be the profit if a sales tax of 10% is imposed?

Q 9: Given the demand function $p = (10 - x)^2$ and the cost function $C = (55 - 8x^2)$; find the maximum profit. What would be the effect of an imposition of a tax of Rs. 9 per unit qunatity on price?

Q 10: Given the demand function $p = 100 - 0.5x^2$ and the cost function $C = \frac{1}{3}x^3 - 6x^2 + 108x$, determine the equilibrium output. Determine the change in price, output and profit after the imposition of a tax of

Re. 1 per unit of output.

Q 11: Given the deamnd function : $p = 12 - x^2$ and the cost function : $C = -\frac{4}{3}x^3 + 4x^2 + 10$, where *p* denotes price and x the number of units of the product determine the profit maximising output of the firm. Whatwill be the impact of a tax of Rs. 12 per unit of product on price and profit? Interpret the results.

Determine the tax per unit that maximises the tax revenue if $C = x^3 + 4x^2 + 10$

Q 12: Given the demand function $p = 90 - x^2$ and the cost function $C = 10 + 2x^2 + 3x^3$, determine the profit maximising output of a monopolist firm. What would be the impact of a tax of Rs. 10 per unit of output on price and profit?

Q 13: The total cost function of a firm is given by $C = 5x^2 + 4x + 10$, where C is total cost and x is output. A tax rate of Rs. 3 per unit is imposed and producer adds it to his cost. If the market demand function is given by $p = 100 - 3x$, where p is the price per unit of output, find the profit maximising output and price.

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UNIT 13 : ECONOMIC APPLICATIONS OF MINIMA

UNIT STRUCTURE

13.1 Learning Objectives

13.2 Introduction

13.3 Economic Applications of Minima

13.4 Let Us Sum Up

13.5 Further Reading

13.6 Answers to Check Your Progress

13.7 Model Questions

13.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- Acquire knowledge about minimization problems in Economics.
- Solve minimisation problems.

13.2 INTRODUCTION

In the earlier unit, we discussed a few applications of minimisation problems. In this unit, we shall look into a few minimisation problems and discuss how to solve them.

13.3 ECONOMIC APPLICATIONS OF MINIMA

In general, the minimisation problems in Economics relates to cost minimsation. Let us consider the following examples.

Example 13.1 The total cost function of a business firm is given as $c(q)=q^3-6q^2+15q$. Find the number of units for which AC is minimum. Here q stands for quantity, AC for average cost.

Solution : Given Total cost $c(q) = q^3-6q^2+15q$

$$\text{Average cost (AC)} = \frac{q^3 - 6q^2 + 15q}{q}$$

$= q^2-6q+15$ ———(i)

The first order condition for minimisation is

$$\frac{d}{dq}(AC) = 0$$

$$\Rightarrow \frac{d}{dq}\left(q^2 - 6q + 15\right) = 0$$

$\Rightarrow 2q-6 = 0$

$\Rightarrow q = {}^{6}\!/_{2} = 3$

The average cost function (i) will be minimum if the second order condition is fulfilled. The second order condition for minimisation is

$$\frac{d^2(AC)}{dq^2} > 0$$

$$\Rightarrow \frac{d}{dq}(2q-6) > 0$$

$\Rightarrow 2 > 0$

As the second order condition is satisfied, the AC function (i) is minimum at q=3

$\therefore$ AC is minimum at q=3

From (i), Minimum $AC=3^2-6.3+15=6$

Example 13.2 The total cost function of a firm is given by $c=300q-10q^2+\frac{1}{3}q^3$. Find the output levels at which the marginal cost and average cost of the firm are minimum.

Solution : Given total cost $= 300q-10q^2+\frac{1}{3}q^3$

Now $\therefore MC = \frac{dc}{dq}$

$$= \frac{d}{dq}\left(300q - 10q^2 + \frac{1}{3}q^3\right)$$

$$= 300 - 20q + q^2 \quad \text{——————(i)}$$

For minimisation of MC function (i), the first order condition is

$$\frac{d(MC)}{dq} = 0$$

$$-\frac{d}{dq}(300 - 20q + q^2) = 0$$

$$\Rightarrow -20 + 2q = 0$$

$$\Rightarrow q = {}^{20}\!/_{q} = 10$$

The marginal cost function (i) will be minimum at q=10, if the second order condition is fulfilled. The second order condition for minimisation is

$$\frac{d^2(MC)}{d^2q} > 0$$

$$\Rightarrow \frac{d(-20 + 2q)}{dq} > 0$$

$$\Rightarrow 2 > 0$$

As the second order condition is fulfilled, the MC function is minimum at q=10.

$$\text{Now, AC} = \frac{TC}{q}$$

$$= \frac{\left(300q - 10q^2 + \frac{1}{3}q^3\right)}{q}$$

$$= 300 - 10q + \frac{1}{3}q^2 \quad \text{——————(ii)}$$

For minimisation of the AC function (ii), the first order condition is

$$\frac{d(AC)}{dq} = 0$$

$$\Rightarrow \frac{d\left(300 - 10q + \frac{1}{3}q^2\right)}{dq} - 0$$

$$\Rightarrow \frac{d}{dq}(300) - \frac{d}{dq}(10q) + \frac{1}{3}\frac{d}{dq}(q^2) = 0$$

$$\Rightarrow -10 + \frac{2}{3}q - 0$$

$$\Rightarrow q = 10 \times \frac{3}{2} = 15$$

The average cost function (ii) will be maximum at q=15, if the second order condition is fulfilled. The second order condition for minimisation is

$$\frac{d^2(AC)}{dq^2} > 0$$

$$\Rightarrow \frac{d}{dq}\left(-10 + \frac{2}{3}q\right) > 0$$

$$\Rightarrow \frac{2}{3} > 0$$

As the second order condition is fulfilled, the AC function is minimum at q=15

$\therefore$ MC is minimum at 10 units.

AC is minimum at 15 units.

Example 13.3

Prove that MC=AC at the minimum point of AC

Proof : let us consider;

TC= f(Q)

$\therefore$ $AC = \frac{TC}{Q}$

$$MC=\frac{d}{dQ}TC$$

$$\therefore \quad TC = AC\times Q \quad \text{————(i)}$$

Differentiating (i) w.r. to Q we get,

$$\frac{d}{dQ}TC=\frac{d}{dQ}(AC\times Q)$$

$$\Rightarrow MC = AC\frac{dQ}{dQ}+Q\frac{dAC}{dQ}$$

$$\Rightarrow MC=AC.1+Q \text{ Slope of AC}$$

$$\Rightarrow MC-AC=Q \text{ Slope of AC}$$

$$\therefore \quad \text{Slope of AC}=\frac{1}{Q}(MC-AC), Q>0$$

From the above it is clear that when MC=AC then slope of AC=0 (is the minimum point of AC)

$\therefore$ MC=AC at the minimum point of AC

Example 13.4 The manufacturing cost of a business firm is as follows :

Fixed cost : Rs. 1000/- per unit

Material cost : Rs. 2/- per unit

Labour cost : $q^2/90$ for q units produced.

Find how many items the firm would produce to minimise average cost of production.

Solution : Given,

Fixed cost : Rs. 1000/- per unit

Material cost : Rs. 2/- per unit

Labour cost : $q^2/90$ for q units produced.

Now, as fixed cost remains the same respective of the volume of production, the total cost function of the business firm for producing q volume of production can be devined as–

$$C=1000+2.q+\frac{q^2}{90}$$

$\therefore$ Average cost (AC) $= \dfrac{1000 + 2.q + \dfrac{q^2}{90}}{q}$

$$= \frac{1000}{q} + 2 + \frac{q}{90} \quad \text{——————(i)}$$

The first order condition for minimisation of the AC function (i) is $\dfrac{d(AC)}{dq} = 0$

$$\Rightarrow \frac{d\left(\frac{1000}{q} + 2 + \frac{q}{90}\right)}{dq} = 0$$

$$\Rightarrow d\frac{\left(\frac{1000}{q}\right)}{dq} + \frac{d(2)}{dq} + \frac{d\left(\frac{q}{90}\right)}{dq} = 0$$

$$\Rightarrow 1000\frac{d}{dq}\left(-q^{-1}\right) + \frac{d\left(\frac{q}{90}\right)}{dq} = 0$$

$$\Rightarrow 1000(-1)q^{-1-1} + \frac{1}{90} = 0$$

$$\Rightarrow -1000\frac{1}{q^2} + \frac{1}{90} = 0$$

$$\Rightarrow q^2 = 1000 \times 90$$

$$= 90000$$

$$\Rightarrow q = \pm\sqrt{900000}$$

$$= 300 \text{ (output cannot be negative)}$$

The average cost function (i) will be minimum if the second order condition is satisfied. The second order condition for minimisation is-

$$\frac{d^2(AC)}{dq^2} > 0$$

$$\Rightarrow \frac{d\left(-1000\frac{1}{q^2}+\frac{1}{90}\right)}{dq} > 0$$

$$\Rightarrow -1000\frac{d}{dq}\left(\frac{1}{q_2}\right) > 0$$

$$\Rightarrow -1000(-2)q^{-2-1} > 0$$

$$\Rightarrow 2000\frac{1}{q^3} > 0$$

As the second order condition is satisfied, the AC function (i) is minimum at q=300

$\therefore$ AC is minimum at q=300

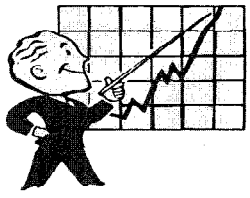
From (i), when q=300

$$AC = \frac{1000}{300}+2+\frac{300}{90}$$

$$= \frac{10}{3}+2+\frac{30}{9}$$

$$= \frac{30+18+30}{9} = \frac{78}{9} = 8.67$$

Thus, the company has to produce 300 units for achieving at minimum AC of 8.67 per unit.



CHECK YOUR PROGRESS

Q 1: If c be total cost, x be the total output and the Total Cost Function be $C = (2+3x^2)^2$, find the Marginal cost.

.....

.....

Q 2: Compute Marginal Productivity of labour at K = 1 and L =2 for the Production Function $X=3KL^2+4K^2L+2L+2K$.

.....

.....

Q 3: Find out $\frac{\partial Q}{\partial L}$ and $\frac{\partial Q}{\partial K}$ for Production Function

$Q=24KL - 10L^2 - 8K^2$

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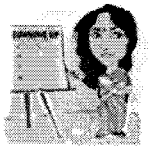
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Q 4: Compute Marginal Utilities of x and y for the Utility Function

$u=5xy-y^2$

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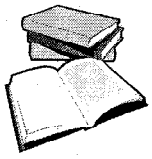
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13.4 LET US SUM UP

In this unit, we have seen the solution of a few:

- cost minisation problems



13.5 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
2) Baruah, S. (2011). *Basic Mathematics and Its Application in Economics*, New Delhi: Trinity Press Pvt. Ltd.
3) Bose, D. (2004). *Mathematical Economics*; New Delhi: Himalaya Publishing House.
4) Chiang, A.C. & (2010). *Fundamental Methods of Economics Analysis*, MC Graw Hill Education India.
5) Kandoi, Balwant (2011). *Mathematics for Business and Economics with Application*; New Delhi: Himalaya Publishing House.

13.6 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q No 1: We find the Mariginal cost $\left(\frac{dC}{dx}\right)$ be the chain Rule.

$c= (2+3x^2)^2$

Let $y= (2+3x^2)$ and $C=y^2$

Therefore, Marginal cost (i.e., $\frac{dC}{dx}$)$=\frac{dC}{dy}\times\frac{dy}{dx}$

$=2y\ (6x)$

$=2(2+3x^2)(6x)$

$=(4+6x^2)\ (6x)$

$=24x+36x^3$

Alternatively,

Total cost (i.e.,C) $= (2+3x^2)^2$

$= 4+12x^2+9x^4$ ..... [$(a+b)^2$ formula]

Therefore, MC (i.e., $\frac{dC}{dx}$)$=24x+36x^3$

Answer to Q No 2: MP_L i.e., $\frac{\partial X}{\partial L}=\frac{\partial}{\partial L}[3KL^2+4K^2L+2L+2K]$

$= 6LK + 4K^2+2$ (K is Constant)

MP_L at K = 1 and L = 2 $= 6\ (2)\ (1) + 4(1)^2+2$

$= 12+4+2$

$= 18$

Answer to Q No 3: We are given the production function as :

$Q=24KL - 10\ L^2 - 8K^2$

Now, $\frac{\partial Q}{\partial L} = \frac{\partial}{\partial L}[24KL-10L^2-8K^2]$

$= 24K - 20L$

Similarly, $\frac{\partial Q}{\partial K} = \frac{\partial}{\partial K}[24KL-10L^2-8K^2]$

$= 24L - 16K$

Answer to Q No 4: We are given the production function as :

$u=5xy-y^2$

Now, marginal utility of x is :

$$=\frac{\partial u}{\partial x}=\frac{\partial}{\partial y}\left[5xy-y^2\right]=5y$$

Similarly, marginal utility of y is :

$$=\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left[5xy-y^2\right]$$

$= 5x-2y$



13.7 MODEL QUESTIONS

Q 1: A cheap gramophone sells for Rs. p and is known that the demand, x hundred machines per year is given by x = [90/(p+5)] – 6. Determine at what price and output the total revenue is greatest.

Q 2: The utility function of the consumer is given by $U = x_1x_2 - 10x_1$, where x_1 and x_2 are the quantities of two commoditeis consumed. Find the optimal value of x_1 and x_2 if his income is 116, and product prices are 2 and 8 respectively.

Q 3: A tour operator charges Rs. 136 per passenger for 100 passengers up to a discuount of Rs. 4 for each 10 passengers in excess of 100. Determine the number of passengers that will maximize the amount of money the tour operator receives.

Q 4: A company has examined its cost structure and revenue structure and has determined that C the total cost, R the total revenue and X the numbers of units produced are related as $C = 100 + 0.015X^3$ and R = 3X. Find the production rate X that will maximise profits of the company. Find that profits when X=120.

Q 5: The demand function of a commodity is $p = 100e^{-x/400}$, whose x unis are demanded when market price is Rs. p. Determine the price and quantity which maximize total revenue.

Q 6: The demand function a commodity is p = 1000 – 2x with the total cost function :

$$C = (x) = x^3 - 59x^2 + 1315x + 2000$$

Determine the equilibrium price and quantity.

Q 7: The cost C of manufacturing a certain product is $C=q^2-6q+38$, where q is the quantity manufactured. Find minimum value of C.

Q 8 : Suppose the TC of producing X units of a product is $C=x^3-6x^2+15x$. Find the number of units for which the average cost is minimum.

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UNIT 14: INTEGRATION AND METHODS OF INTEGRATION

UNIT STRUCTURE

14.1 Learning Objectives

14.2 Introduction

14.3 Intregration of Simple Functions

14.4 Definite Integral

14.5 Let Us Sum Up

14.6 Further Reading

14.7 Answers to Check Your Progress

14.8 Model Questions

14.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand about integration of simple functions from the view point of inverse of differentiation
- know about definite basic integrate and learn to solve the problems of indefinite and definite integrals.

14.2 INTRODUCTION

The development of integral calculus can be viewed from two different directions which are related to solving of problems of the following two types.

(i) to find the function when its derivative is known

(ii) to find the area bounded by the curve (or graph) of a function under certain given conditions.

The process related to the first category leads to the development of "indefinite integration" and the other leads to "definite integration".

The word "integration" literally means "summation". It is, in fact, the process of finding the "limit of a sum" of a certain number of elements, as the number of elements tends to infinity and each of the elements

becomes infinitesimally small.

In this unit, we shall confine ourselves to the study of indefinite integral including some process of integration.

14.3 INTEGRATION OF SIMPLE FUNCTIONS

We have already learnt how to find the derivative *of* a functional of defined in an interval. If $y = f(x)$ is a function then its derivative is denoted by $f'(x)$ or $\frac{dy}{dx}$. Now, here in this unit, we will learn how to find f whenever f' is given. Integration is the inverse process of differentiation, i.e., to find a function when the differential coefficient is given.

Let us take this example :

$$\frac{d}{dx}\left(\frac{x^4}{4} + C\right) = x^3 \quad \text{Here C is any real number}$$

It is evident from above that $\frac{x^4}{4} + C$ is also an antiderivative of x^3. Therefore, we have understood that the antiderivative of a function is not unique.

Definition : The set of functions that could possibly have the given function as a derivative are called "antiderivatives" (or integrals or primitives) of the function.

More generally, if there is a function $\phi(x)$ defined in an interval I such that $\frac{d}{dx}[\phi(x)] = f(x),\ x \in I$, then for any real number C $\frac{d}{dx}[\phi(x) + C] = f(x),\ x \in I$. Thus $\phi(x) + C$ denotes the family of anti-derivatives of $f(x)$. We introduce the symbol $\int f(x)dx$ which will represent the whole class of anti-derivatives, i.e. $\int f(x)dx = \phi(x) + C$.

The symbol $\int f(x)dx$ is read as the indefinite indegral of $f(x)$ with respect to x. The process of finding the anti-derivatives is known as "integration". The sign '$\int$' is called the integral sign. The symbol '$\int$' is an

elongated S which is taken from the first letter of 'sum'.

Some Basic Results

Based upon the above definition and also on some standard derivatives, we obtain the following integration formulae.

	Derivatives	**Integrals (or anti derivatives)**
1.	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$	$\int x^n dx=\frac{x^{n+1}}{n+1}+C,\ (n\neq -1)$
2.	$\frac{d}{dx}(e^x)=e^x$	$\int e^x dx=e^x+C$
3.	$\frac{d}{dx}(\log^x)=\frac{1}{x}$	$\int\frac{1}{x}dx=\log x+c$
4.	$\frac{d}{dx}[f(x)+g(x)]$	$\int[f(x)+g(x)]dx=\int f(x)dx+\int g(x)dx$
	$=f'(x)+g'(x)$	$=F(x)+c_1+G(x)+c_2 \quad =F(x)+G(x)+C$
		Where $C=C_1+C_2$

NOTE:

1) The rule of integration in Sl. no. 1 is known as the Power rule of integration.
2) The rule of integration in Sl. no. 2 is known as the Exponential rule of integration.
3) The rule of integration in Sl. no. 1 is known as the Logarithimic rule of integration.
4) The rule of integration in Sl. no. 1 is known as the Sum rule of integration.
5) The product rule of integration

$$\int kf(x)dx=k\int f(x)dx$$

Examples :

Example 14.1 : Evaluate (i) $\int x^4 dx$ (ii) $\int\frac{1}{\sqrt{x}}dx$ (iii) $\int 3^x dx$

Solutions : (i) $\int x^4 dx=\frac{x^{4+1}}{4+1}+C=\frac{x^5}{5}+C$ (using formula 1)

(ii) $\int \frac{1}{\sqrt{x}} dx = \int x^{-½} dx = \frac{x^{-½+1}}{-½+1} + C = 2\sqrt{x} + C$

(iii) $\int 3^x dx = \frac{3^x}{\log_e 3} + C$

Example 14.2 : Evaluate $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

Solution : $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

$= \int \left(x + 2 + \frac{1}{x}\right) dx$

$= \int x dx + \int 2 dx + \int \frac{1}{x} dx$

$= \frac{x^{1+1}}{1+1} + C_1 + 2\int dx + \log x + C_2$

$= \frac{x^2}{2} + C_1 + 2x + C_3 + \log x + C_2$

$= \frac{x^2}{2} + 2x + \log x + C$ (Here $C_1 + C_2 + C_3 = C$)

Example 14.3 : Evaluate (i) $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$

(ii) $\int \frac{3x^4 + 7x + 5}{2\sqrt{x}} dx$

Solutions : (i) $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$

$= \int \frac{(x-1)(x^2+1)}{(x-1)} dx$

$= \int (x^2 + 1) dx$

$= \frac{x^3}{3} + x + C$

(ii) $\int \frac{3x^4 + 7x + 5}{2\sqrt{x}} dx$

$$= \int\left(\frac{3}{2}x^{7/2}dx + \frac{7}{2}x^{1/2} + \frac{5}{2}x^{-1/2}\right)dx$$

$$= \frac{3}{2}.\frac{2}{9}x^{9/2} + \frac{7}{2}.\frac{2}{3}x^{3/2} + \frac{5}{2}.\frac{2}{1}x^{1/2} + C$$

$$= \frac{1}{3}x^{9/2} + \frac{7}{3}x^{3/2} + 5\sqrt{x} + C$$

Example 14.4: Find $\int x^5 dx$

Solution : $\int x^5 dx = \frac{1}{1+5}x^{5+1} + c = \frac{1}{6}x^6 + c$

Example 14.5: Find $\int dx$

Solution : $\int dx = \frac{1}{0+1}x^{0+1} + c = x + c$

Example 14.6: Find $\int \frac{1}{x^2}dx$

Solution : $\int \frac{1}{x^2}dx = \int x^{-2}dx$

$$= \frac{1}{-2+1}x^{-2+1} + c = -x^{-1} + c$$

$$= -\frac{1}{x} + c$$

Example 14.7: Find $\int\left(2x^2 + x^3\right)dx$

Solution : $\int\left(2x^2 + x^3\right)dx$

$$= \int 2x^2 dx + \int x^3 dx$$

$$= 2 \times \frac{x^{2+1}}{2+1} + \frac{x^{3+1}}{3+1} + c$$

$$= \frac{2}{3}x^3 + \frac{1}{4}x^4 + c$$

Example 14.8: Find $\int\left(2x^2 + \sqrt{x} + \frac{x}{4}\right)dx$

Solution : $\int\left(2x^2 + \sqrt{x} + \frac{x}{4}\right)dx$

$$= \int 2x^2 dx + \int \sqrt{x} dx + \int x/4$$

$$= 2 \times \frac{x^{2+1}}{2+1} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{1}{4}\frac{x^2}{2} + C$$

$$= \frac{2}{3}x^3 + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^2}{8} + C$$

$$= \frac{2}{3}x^3 + \frac{2}{3}x^{\frac{3}{2}} + \frac{x^2}{8} + C$$

Example 14.9: Find $\int 5\sqrt{x} dx$

Solution : $\int 5\sqrt{x} dx$

$$= 5\int x^{\frac{1}{2}} dx$$

$$= 5 \times \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= 5 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{10}{3}x^{\frac{3}{2}} + c$$

Example 14.10: Find $\int (3x+2)^2 dx$

Solution : $\int (3x+2)^2 dx$

$$= \int (9x^2 + 12x + 4) dx$$

$$= \int 9x^2 dx + \int 12x dx + \int 4 dx$$

$$= 9 \times \frac{x^{2+1}}{2+1} + 12 \times \frac{x^{1+1}}{1+1} + 4x + c$$

$$= \frac{9}{3}x^3 + \frac{12}{2}x^2 + 4x + c = 3x^3 + 6x^2 + 4x + c$$

Example 14.11: Find $\int\left(6e^x-2x^3+\frac{5}{x}\right)dx$

Solution : $\int\left(6e^x-2x^3+\frac{5}{x}\right)dx$

$$=\int 6e^x dx-\int 2x^3 dx+\int\frac{5}{x}dx$$

$$=6e^x-2\times\frac{x^{3+1}}{3+1}+5\log x+c$$

$$=6e^x-\frac{1}{2}x^4+5\log x+c$$

Example 14.12: Find $\int\left(e^x+\frac{1}{x^2}\right)dx$

Solution : $\int\left(e^x+\frac{1}{x^2}\right)dx$

$$=\int e^x dx+\int\frac{1}{x^2}dx$$

$$=e^x+\frac{x^{-2+1}}{-2+1}+c$$

$$=e^x-\frac{1}{x}+c$$

Example 14.13: Find $\int\left(5e^x-x^{-2}+\frac{4}{x}\right)dx$

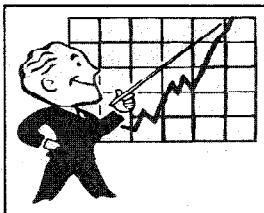
Solution : $\int\left(5e^x-x^{-2}+\frac{4}{x}\right)dx$

$$=\int 5e^x dx-\int x^{-2}dx+\int\frac{4}{x}dx$$

$$=5e^x-\frac{x^{-2+1}}{-2+1}+4\log x+c$$

$$=5e^x+x^{-1}+4\log x+c$$

$$=5e^x+\frac{1}{x}+4\log x+c$$



CHECK YOUR PROGRESS

Q 1: Find the following integrals

(i) $\int x^3\left(x+\frac{1}{x}\right)^2 dx$ (ii) $\int \frac{(1+x)^2}{\sqrt{x}} dx$

Q 2: Evaluate (i) $\int \frac{x^4+x^2+1}{x^2-x+1} dx$ (ii) $\int \frac{x^{-1/3}+\sqrt{x}+2}{\sqrt[3]{x}} dx$

14.4 DEFINITE INTEGRAL : CONCEPT AND CALCULATION

The integrals we have so far discussed in this unit are basically indefinite integrals. We have defined intergration as : $\int f(x)dx = \phi(x)+c$. The value of c is unknown, and hence, we call it an indefinite integral. Let us now move on the discuss definite integral. Suppose we have a function f(x). The indefinite integration of this f(x) is given by F(x) + K. Now if we have two values of x which lie in the domain of the function, we can evaluate the primite function at the higher value of x (say, xh) and then the lower value of x (say, xe), and finally we can calculate the difference between these two values. The final value arrived at, is thus, independent if the value of the 'constant' of integration. Symbolically, we can say,

$$\int_{x_l}^{x_h} F(x)dx = \{F(x_h)+K\} - \{F(x_l)+K\}$$
$$= F(x_h)+K-F(x_l)-K$$
$$= F(x_h)-K-F(x_l)$$

Example 14.14: Evaluate $\int_2^4 2x^2 dx$.

Solution : The integral of $2x^2dx$ is given by –

$$\int 2x^2 dx = 2\frac{x^{2+1}}{2+1} = \frac{2}{3}x^3$$

$$\therefore \int_2^4 2x^2 dx = \frac{2}{3}(4)^3 = \frac{2}{3}(2)^3$$

$$-\frac{2}{3}(64-8) = \frac{112}{3}$$

Example 14.15:

Evaluate (i) $\int_0^{10} 2x dx$ (ii) $\int_2^3 x^3 dx$ (iii) $\int_2^3 (x^2+6x+7)dx$ (iv) $\int_{10}^{25} 100x^{-2} dx$

Solutions :

(i) The integral of $\int 2x dx$ is given by –

$$\int 2x dx = 2\int x dx$$

$$= 2\frac{x^1+1}{1+1}$$

$$= x^2$$

$$\therefore \int_0^{10} 2x dx = 10^2 - 0^2 = 100$$

(ii) The intergral of $x^3 dx$ is given by –

$$\int x^3 dx = \frac{x^{3+1}}{3+1} = \frac{x^4}{4}$$

$$\therefore \int_2^3 x^3 dx = \left(\frac{3^4}{4}\right) - \left(\frac{2^4}{4}\right)$$

$$= \frac{81}{4} - \frac{16}{4} = \frac{1}{4}(81-16) = \frac{65}{4} = 16\frac{1}{4}$$

(iii) The integral of $(x^2+6x+7)dx$ is given by –

$$\int (x^2+6x+7)dx = \left(\frac{x^{2+1}}{2+1} + \frac{6x^{1+1}}{1+1} + 7x\right)$$

$$= \left(\frac{x^3}{3} + \frac{6x^2}{2} + 7x\right)$$

$$- \left(\frac{x^3}{3} + 3x^2 + 7x\right)$$

$$\therefore \int_2^3 (x^2+6x+7)dx = \left\{\frac{3^3}{3}+3(3)^2+7(3)\right\} - \left\{\frac{2^3}{3}+3(2)^2+7(2)\right\}$$

$$= \left(\frac{27+3^4+7(3)^2}{3}\right) - \left(\frac{8+9(2)^2+7(2)(3)}{3}\right)$$

$$= \left(\frac{27+81+63}{3}\right) - \left(\frac{8+36+42}{3}\right)$$

$$= \frac{171}{3} - \frac{86}{3}$$

$$= \frac{1}{3}(85) = 28\frac{1}{3}$$

(iv) The integral of $100x^{-2}dx$ is given by –

$$\int 100x^{-2}dx = 100\frac{x-2+1}{-2+1} = -100x^{-1}$$

$$\therefore \int_{10}^{25} 100x^{-2}dx = -100(25)^{-1} - \left(-100(10)^{-1}\right)$$

$$= -\frac{100}{25} - \left(-\frac{100}{10}\right) \qquad \left[\therefore (25)^{-1} = \frac{1}{25} \text{ and } (10)^{-1} = \frac{1}{10}\right]$$

$$= -\frac{100}{25} + \frac{100}{10}$$

$$= -4 + 10 = 6$$

Example 14.16: Find the value of $\int_2^5 2xdx$

Solution: $\int_2^5 2xdx$

$$= 2\times\left[\frac{x^2}{2}\right]_2^5$$

$$= 2\times\left(\frac{5^2}{2}-\frac{2^2}{2}\right)$$

$$= 2 \times \left(\frac{25-4}{2} \right) = 21$$

Example 14.17: Find the value of $\int_1^5 \frac{12}{x} dx$

Solution: $\int_1^5 \frac{12}{x} dx$

$$= 12[\log x]_1^5$$
$$= 12(\log 5 - \log 1)$$

Example 14.18: Find the value of $\int_2^4 3x^2(x^3+1)dx$

Solution: $\int_2^4 3x^2(x^3+1)dx$

$$= \int_2^4 (3x^5 + 3x^2)dx$$

$$= 3\left[\frac{x^6}{6}\right]_2^4 + 3\left[\frac{x^3}{3}\right]_2^4$$

$$= \frac{1}{2}\left(4^6 - 2^6\right) + (4^3 - 2^3)$$

$$= \frac{1}{2} \times (4032) + (64 - 8)$$

$$= 2016 + 56 = 2072$$

Example 14.19: Find the value of $\int_2^4 2x^2 dx$

Solution: $\int_2^4 2x^2 dx$

$$= 2\left[\frac{x^3}{3}\right]_2^4$$

$$= \frac{2}{3}(4^3 - 2^3)$$

$$= \frac{2}{3}(64-8)$$

$$= \frac{2}{3} \times 56 = \frac{112}{3}$$

Example 14.20: Evaluate: (1) $\int_0^{10} 2x dx$ (2) $\int_2^3 x^3 dx$

Solutions: (1) $\int_0^{10} 2x dx$

$$= 2\left[\frac{x2}{2}\right]_0^{10}$$

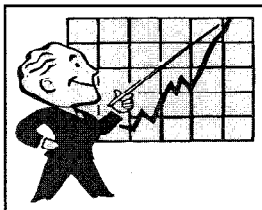
$$= 2 \times \frac{1}{2}(10^2 - 0) = 100$$

(2) $\int_2^3 x^3 dx$

$$= \left[\frac{x^4}{4}\right]_2^3$$

$$= \frac{1}{4}\left[3^4 - 2^4\right]$$

$$= \frac{1}{4}(81-16) \quad = \frac{1}{4} \times 65 = \frac{65}{4}$$



CHECK YOUR PROGRESS

Q 3: Evaluate (i) $\int_2^3 (x^2 + 6x + 7) dx$

(ii) $\int_{10}^{25} (100x^{-2}) dx$

Q 4: Prove that : (i) $\int_0^2 (2x+7)dx = 18$ (ii) $\int_1^2 (x^2 + 4x + 3)dx = \frac{34}{3}$



14.5 LET US SUM UP

- Differentiation and integration are inverse processes of one another.
- We may also define integration as a process of finding the limit of a sum.
- The set of functions that could possibly have in given functon as a derivative are called anti derivatives (or integrats or primitives) of a function.
- Based on the results of some standard derivatives, we obtain the formula of some standard simple Integration.
- A few important forms of integrals

(i) $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$

(ii) $\int \{f(x)\}^n f'(x) dx = \frac{1}{n+1}[f(x)]^{n+1} + C, \quad n \neq -1$

(iii) $\int f'(ax+b) dx = \frac{1}{a} f(ax+b) + C$

- An important formula for product of two functions is

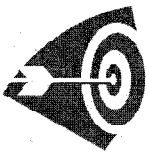
$$\int uv\, dx = u \int v dx - \int \left\{ \frac{d}{dx}(u) . \int v dx \right\} dx$$



14.6 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
2) Baruah, S. (2011). *Basic Mathematics and Its Application in Economics*, New Delhi: Trinity Press Pvt. Ltd.
3) Bose, D. (2004). *Mathematical Economics*; New Delhi: Himalaya Publishing House.
4) Chiang, A.C. & (2010). *Fundamental Methods of Economics Analysis*, MC Graw Hill Education India.

5) Kandoi, Balwant (2011). *Mathematics for Business and Economics with Application*; New Delhi: Himalaya Publishing House.



14.7 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q No 1:

(i) $\int x^3\left(x+\frac{1}{x}\right)^2 dx$

$$=\int x^3\left(x^2+\frac{1}{x^2}+2x.\frac{1}{x}\right)dx=\int\left(x^5+x+2x^3\right)dx$$

$$=\frac{x^6}{6}+\frac{x^2}{2}+2\frac{x^4}{4}+C$$

$$=\frac{x^6}{6}+\frac{x^2}{2}+\frac{x^4}{2}+C.$$

(ii) $\int\frac{(1+x)^2}{\sqrt{x}}dx=\int\frac{1+2x+x^2}{\sqrt{x}}dx$

$$=\int\left(\frac{1}{\sqrt{x}}+2\sqrt{x}+x^{\frac{3}{2}}\right)dx$$

$$=\int x^{-\frac{1}{2}}dx+2\int x^{\frac{1}{2}}dx+\int x^{\frac{3}{2}}dx$$

$$=\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+2.\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}+C$$

$$=2x^{\frac{1}{2}}+\frac{4}{3}x^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}+C$$

Answer to Q No 2: (i) $\int\frac{x^4+x^2+1}{x^2-x+1}dx=\int\frac{\left(x^2+1\right)^2-x^2}{x^2-x+1}dx$

$$=\int\frac{\left(x^2+1+x\right)\left(x^2+1-x\right)}{x^2-x+1}dx$$

$$= \int \left(x^2 + x + 1\right) dx = \frac{x^3}{3} + \frac{x^2}{2} + x + C .$$

(ii) $$\int \frac{x^{-1/3} + \sqrt{x} + 2}{\sqrt[3]{x}}\, dx = \int \frac{x^{-\frac{1}{3}} + x^{\frac{1}{2}} + 2}{x^{\frac{1}{3}}} dx$$

$$= \int \left(\frac{x^{-\frac{1}{3}}}{x^{\frac{1}{3}}} + \frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}} + \frac{2}{x^{\frac{1}{3}}} \right) dx$$

$$= \int \left(x^{-\frac{2}{3}} + x^{\frac{1}{2}-\frac{1}{3}} + 2x^{-\frac{1}{3}} \right) dx$$

$$= \int \left(x^{-\frac{2}{3}} + x^{\frac{1}{6}} + 2x^{-\frac{1}{3}} \right) dx$$

$$= \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + \frac{x^{\frac{1}{6}+1}}{\frac{1}{6}+1} + 2\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C$$

$$= \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + \frac{x^{\frac{7}{6}}}{\frac{7}{6}} + 2\frac{x^{\frac{2}{3}}}{\frac{2}{3}} + C$$

$$= 3x^{\frac{1}{3}} + \frac{6}{7}x^{\frac{7}{6}} + 3x^{\frac{2}{3}} + C .$$

Answer to Q No. 3 (i) $$\int_2^3 (x^2 + 6x + 7)dx$$

$$= \int_2^3 x^2 dx + \int_2^3 6x dx + \int_2^3 7 dx$$

$$= \left[\frac{x^3}{3}\right]_2^3 + 6\left[\frac{x^2}{2}\right]_2^3 + 7\left[x\right]_2^3$$

$$= \frac{1}{3}(3^3 - 2^3) + 3(3^2 - 2^2) + 7(3-2)$$

$$= \frac{1}{3} \times 21 + 3 \times 5 + 7$$

$$= \frac{1}{3} \times 19 + 22$$

$$= \frac{19+66}{3} = \frac{85}{3}$$

(ii) $\int_{10}^{25} (100x^{-2})dx$

$$= 100\left[\frac{x^{-2+1}}{-2+1}\right]_{10}^{25}$$

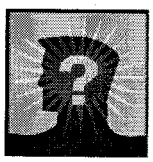
$$= 100\left[-\frac{1}{x}\right]_{10}^{25}$$

$$= -100\left[\frac{1}{25} - \frac{1}{10}\right]$$

$$= -100\left[\frac{2-5}{50}\right]$$

$$= -100\left(\frac{-3}{50}\right) = 6$$

Ans to Q No. 4 (i) and (ii): Try yourself.



14.8 MODEL QUESTIONS

A. Evaluate

1. $\int \frac{x^4}{x^2+1} dx$ 2. $\int \left(1 + \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} + \frac{5}{x\sqrt{x^2-1}} + 3^x\right) dx$

3. $\int \frac{4^x + 3^x}{5^x} dx$

B. Integrate the following–

1. $\int \frac{5}{x^2 - 8x + 25} dx$
2. $\int \frac{1}{3 - 2x - x^2} dx$
3. $\int \frac{1}{x(x^n + 1)} dx$
4. $\int \frac{1}{\sqrt{x^2 + 4x + 2}} dx$
5. $\int \frac{1}{\sqrt{6 + x - x}} dx$
6. $\int \frac{x + 2}{2x^2 + 6x + 5} dx$
7. $\int \frac{x^2 + 2}{x^2 - 5x + 6} dx$

C. Evaluate

1. $\int \sqrt{5x^2 + 8x + 4}\, dx$
2. $\int \frac{3x + 2}{x^3 - 6x^2 + 11x - 6} dx$
3. $\int \frac{x^2 + x - 1}{x^2 + x - 6} dx$
4. $\int \frac{3x}{(x^2 - 4)(x + 1)} dx$
5. $\int \frac{x}{(1 + x)(1 + x^2)} dx$
6. $\int \frac{2x}{(x^2 + 1)(x^2 + 3)} dx$
7. $\int \frac{2x + 1}{(x + 2)(x - 3)^2} dx$

*** ***** ***

UNIT 15: APPLICATIONS OF INTEGRAL CALCULUS TO ECONOMIC PROBLEMS

UNIT STRUCTURE

15.1 Learning Objectives

15.2 Introduction

15.3 Derivation of Total Functions from Marginal Functions

15.4 Estimation of Consumer's Surplus

15.5 Estimation of Producer's Surplus

15.6 Let Us Sum Up

15.7 Further Reading

15.8 Answers to Check Your Progress

15.9 Model Questions

15.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- derive the total functions (like Total Revenus, Total Cost, Saving function, Consumption, etc.) from marginal functions (of Marginal Revenue, Marginal Cost, Marginal Propensity to Save, Marginal Propensity to Consume, etc.) by integrating the marginal function with respect to the variable involved;
- obtain Consumer's Surplus and Producer's Surplus with the help of definite integral.

15.2 INTRODUCTION

Integral calculus is widely used in the field of Economics. Integration is exactly opposite of differentiation. As the differention of any total function gives us marginal function, therefore, in order to get the total function from the marginal function we need to find the integral of the marginal function w.r.t the variable involved. For example, if TR = R (Q), where Q is the number of quantity sold, then MR can be obtained by differentiating the TR function w.r.t. Q

i.e. $MR = \frac{d}{dQ}TR = \frac{d}{dQ}R(Q) = R'(Q)$

As the integration is the inverse of differentiation, therefore, the TR function can be obtain from MR function by simply integrating the MR function w.r.t. Q

i.e. $TR = \int MR dQ = \int R'(Q) dQ$

Similarly, integration is also used to derive Total Cost function, Consumption function, saving function from Marginal Cost function, Marginal propensity to consume function, Marginal propensity to save function respectively.

Further, definite integral is used to obtain consumer's surplus and producer's surplus. This unit will deals with some of the important applications of integration in the field of Economics.

15.3 DERIVATION OF TOTAL FUNCTIONS FROM MARGINAL FUNCTIONS

(a) Derivation of Total Revenue (TR) function from Marginal Revenue (MR) function :

Since, marginal revenue function is the derivative of total revenue function w.r.t. output, the total revenue function can be obtained from marginal revenue function by simply integrating the marginal revenue function w.r.t. output. If the marginal revenue is given by

MR = a - bQ, where Q is quantity, a and b are parameters, then the TR will be

$TR = \int MR dQ$

$= \int (a - bQ) dQ$

$= \int a dQ - \int bQ dQ$

$= aQ - \frac{bQ^2}{2} + c$

Since, TR is always zero when Q is zero

$$\therefore TR = aQ - \frac{bQ^2}{2} + c$$

$$\rightarrow O = a.o - \frac{b.o^2}{2} + c$$

$$\Rightarrow C = O$$

$$\therefore TR = aQ - \frac{bQ^2}{2}$$

Example 15.1: Obtain the total revenue function from the following marginal revenue function

MR = 100 – 0.5Q, where Q is quantity of output.

Solution: Given MR = 100 – 0.5 Q

$$\therefore TR = \int MR dQ$$

$$= \int (100 - 0.5Q) dQ$$

$$= \int 100 dQ - \int 0.5Q dQ$$

$$= 100Q - \frac{0.5Q^2}{2} + c$$

TR= 100Q – 0.25Q² [$\because$ c = 0, because TR = 0, when Q = O]

Example 15.2: If the marginal revenue function is given by

MR = 50 – 4Q, where Q is quantity, find average revenue (AR) when quanntity is 10.

Solution: Given, MR = 50 – 4Q

To find $AR = \frac{TR}{Q}$, Let us first find out TR function

TR = $\int$MRdQ

$$= \int (50 - 4Q)\, dQ$$

$$= \int 50 dQ - \int 4Q dQ$$

$$= 50Q - 4.\frac{Q^2}{2} + c$$

TR = 50Q – 2Q² [$\because$ C = 0]

$$\therefore AR - \frac{TR}{Q} = \frac{50Q - 2Q^2}{Q}$$

= 50 – 2Q

= 50 – 2 x 10 [$\because$ Q = 10]

= 50 – 20

AR = 30

Example 15.3: Given the marginal revenue function

MR = 60 – 5Q

Find the point elasticity of demand when Q = 10

Solution: Given, MR = 60 – 5Q

We know, that point elasticity of demand

$$e_d = \frac{AR}{AR - MR}$$

To find AR = $\frac{TR}{Q}$, Let us first find out TR.

TR = $\int$MRdQ

= $\int$(60 – 5Q) dQ

$$= 60Q - \frac{5}{2}Q^2 + c$$

$$TR = 60Q - \frac{5}{2}Q^2 \qquad [\because C = 0]$$

$$\therefore AR = \frac{TR}{Q} = \frac{60Q - \frac{5}{2}Q^2}{Q}$$

$$= 60 - \frac{5}{2}Q$$

$$= 60 - \frac{5}{2}.10 \qquad [\because Q = 10]$$

= 60 – 25

AR = 35

Now, MR = 60 – 5Q

= 60 – 5.10 [$\because$ Q = 10]

= 60 – 50

MR = 10

$$\therefore e_d = \frac{AR}{AR - MR}$$

$$= \frac{35}{35 - 10}$$

$$= \frac{35}{25}$$

$$= \frac{7}{5}$$

$$= 1.4$$

Example 15.4: If MR = 30 – 4Q, where Q is quantity, Find out the increases in TR if sales increases from 8 to 12

Solution: Given,

$$MR = 30 - 4Q$$

$$\therefore TR = \int MR\,dQ$$

$$= \int (30 - 4Q)\,dQ$$

$$= \int 30\,dQ - \int 4Q\,dQ$$

$$= 30Q - 4\frac{Q^2}{2} + C$$

$$TR = 30Q - 2Q^2 \quad [\because C = 0]$$

Now, at Q = 8, $TR = 30Q - 2Q^2$

$= 30(8) - 2(8)^2$

$= 240 - 128$

$= 112$

at Q = 12, $TR = 30Q - 2Q^2$

$= 30(12) - 2(12)^2$

$= 360 - 288$

$= 72$

$\therefore$ When sales increases from 8 to 12, TR falls from Rs. 112 to Rs. 72, i.e. TR falls by (112–72) = Rs. 40.

(b) Derivation of Total Cost (TC) function from Marginal Cost (MC) function:

Since derivative of TC function gives MC function, thus the integration, which is the inverse of differentiation, of the marginal cost function will give the total cost function. Thus if

MC = 2aQ + b, where Q is the output, then

$$TC = \int MC\,dQ$$

$$= \int (2aQ + b)\,dQ$$

$= \int 2aQdQ + \int bdQ$

$= 2a.\frac{Q^2}{2} + bQ + C$

$= aQ^2 + bQ + C$, where C is an arbitrary constant.

We know that,

TC = Total variable cost (TVC) + Total Fixed Cost (TFC)

and TC = TFC, when Q = 0, because in this case the TVC is zero.

$\therefore$ $TC = aQ^2 + bQ + c$

$= a.\ 0^2 + b.0 + c$ [when Q = 0]

$= c$

Thus, TC = C = TFC

Here, the arbitrary constant C represents the TFC and $[aQ^2 + bQ]$ represents the TVC.

Example 15.5: If the marginal cost (MC) function of a firm is given by

$MC = C'(Q) = 5Q + 2$ and the total fixed cost is 10, find the total fixed cost is 10, find the total fixed cost and the average cost functions.

Solution: Given,

$MC = C'(Q) = 5Q + 2$

We have,

$TC = \int MCdQ$

$= \int (5Q + 2)dQ$

$= \frac{5}{2}Q^2 + 2Q + C$

When Q = 0, TC = C = 10

$\Rightarrow C = 10$

$\therefore TC = \frac{5}{2}Q^2 + 2Q + 10$

Againg, $AC = \frac{TC}{Q} = \frac{\frac{5}{2}Q^2 + 2Q + 10}{Q}$

$\therefore AC = \frac{5}{2}Q + 2 + \frac{10}{Q}$

Example 15.6: The marginal cost function is given by

MC = $5Q^2 - 4Q + 1$, find the average cost and average variable cost functions if the total fixed cost is 100.

Solution: Given,

$MC = 5Q^2 - 4Q + 1$

We have,

$$TC = \int MC\,dQ$$

$$= \int (5Q^2 - 4Q + 1)\,dQ$$

$$= 5\frac{Q^3}{3} - 4\frac{Q^2}{2} + Q + C$$

$$= \frac{5}{3}Q^3 - 2Q^2 + Q + C$$

When, Q = 0, TC = C = 100

$$\therefore TC = \frac{5}{3}Q^3 - 2Q^2 + Q + 100$$

Now, $AC = \frac{TC}{Q} = \frac{\frac{5}{3}Q^3 - 2Q^2 + Q + 100}{Q}$

$$\therefore AC = \frac{5}{3}Q^2 - 2Q + 1 + \frac{100}{Q}$$

Again, TC = TVC + TFC

$$\therefore TVC = \frac{5}{3}Q^3 - 2Q^2 + Q$$

Since, $AVC = \frac{TVC}{Q}$

$$= \frac{\frac{5}{3}Q^3 - 2Q^2 + Q}{Q}$$

$$= \frac{5}{3}Q^2 - 2Q + 1$$

Example 15.7: The marginal revenue and marginal cost functions of a firm are given by

$MR = 4$

$MC = 2Q - 6$

and the total fixed cost is 10. Find out total profit when the firm produces and sells 5 units of output.

Solution: Given,

$MR = 4$

$MC = 2Q - 6$

We have, the profit function of a firm is

$\pi = TR - TC$

To find TR and TC :

$TR = \int MR\,dQ$

$= \int 4\,dQ$

$= 4Q + C$

$\therefore TR = 4Q \qquad [\because C = 0]$

$TC = \int MC\,dQ$

$= \int (2Q-6)\,dQ$

$= 2.\frac{Q^2}{2} - 6Q + C$

$= Q^2 - 6Q + C$

When $Q = 0$, $TC = C = 10$

$\therefore TC = Q^2 - 6Q + 10$

$\therefore \pi = TR - TC$

$= 4Q - (Q^2 - 6Q + 10)$

$= 4Q - Q^2 + 6Q - 10$

$= 10Q - Q^2 - 10$

$= 10.5 - 5^2 - 10 \qquad$ [given $Q = 5$]

$= 50 - 25 - 10$

$= 50 - 35$

$\pi = 15.$

Example 15.8: Given $MC = 4Q + 5$ and the total fixed cost is 18. Find the output at which AC is minimum.

Solution: Given,

$MC = 4Q + 5$

$\therefore TC = \int MC\,dQ$

$= \int (4Q + 5)\, dQ$

$= \int 4Q dQ + \int 5 dQ$

$= 4.\frac{Q^2}{2} + 5Q + C$

$= 2Q^2 + 5Q + C$

$TC = 2Q^2 + 5Q + 18 \quad [\because C = 18]$

Now, $AC = \frac{TC}{Q} = 2Q + 5 + \frac{18}{Q}$

Minimisation of AC requires that

1st order condition:

$$\frac{dAC}{dQ} = 0$$

Here, $\frac{dAC}{dQ} = \frac{d}{dQ}\left(2Q + 5 + \frac{18}{Q}\right)$

$$= \frac{d}{dQ}2Q + \frac{d}{dQ}5 + \frac{d}{dQ}18Q^{-1}$$

$$= 2 + 0 + 18(-1)Q^{-1-1} = 2 - 18Q^{-2}$$

$$-2 - \frac{18}{Q^2}$$

According to 1st order condition

$$\frac{dAC}{dQ} = 2 - \frac{18}{Q^2} = 0$$

$$\Rightarrow \frac{18}{Q^2} = 2$$

$$\Rightarrow 2Q^2 = 18$$

$$\Rightarrow Q^2 = 9$$

$$\Rightarrow Q = \sqrt{9}$$

$$\therefore Q = \pm 3$$

$\therefore Q = 3$ ($\because$ Q can never be negative)

2nd order condition : $\frac{d^2AC}{dQ^2} > 0$

Here, $\frac{d^2AC}{dQ^2} = \frac{d}{dQ}\left(\frac{dAC}{dQ}\right)$

$$= \frac{d}{dQ}\left(2 - \frac{18}{Q^2}\right)$$

$$= \frac{d}{dQ}.2 - \frac{d}{dQ}18Q^{-2}$$

$$= 0 - 18(-2)Q^{-2-1}$$

$$= 36Q^{3}$$

$$= \frac{36}{Q^3}$$

Now, when Q = 3,

$$\frac{d^2AC}{dQ^2} = \frac{36}{Q^3} = \frac{36}{3^3} = \frac{36}{27} > 0$$

$\therefore$ 2nd order condition of AC minimisation is satisfied when Q = 3.

(c) Derivation of Consumption Function from Marginal Propensity to Consume.

The derivative of consumption function with respect to (w.r.t.) income gives marginal propensity to consume (MPC) and hence the integration of the MPC function w.r.t. income will give consumption function.

If the consumption function is

C = C(Y)

then, $MPC = \frac{d}{dy}C = \frac{d}{dy}C(y) = C'(y)$

The consumption function can be obtain from MPC by integrating the MPC function w.r.t. y as

C = C(Y) = $\int$MPC dY = $\int$c'(y) dy.

Example 15.9: Derive the consumption function from the given MPC function $MPC = C'(y) = 0.8 + 0.1y^{-\frac{1}{2}}$ with aggregate consumption 100 when income is zero.

Solution: Given,

$$MPC = C'(Y) = 0.8 + 0.1y^{-\frac{1}{2}}$$

$$\therefore C(Y) = \int MPC\, dY$$

$$= \int \left(0.8 + 0.1y^{-\frac{1}{2}}\right) dy$$

$$= \int 0.8dy + \int 0.1y^{-\frac{1}{2}} dy$$

$$= 0.8y + 0.1\frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$-0.8y + 0.1\frac{y^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 0.8y + 0.2y^{1}{}^{2} + c$$

Given,

$C(Y) = 100$, when $y = 0$

$\therefore C(0) = C = 100$

$\Rightarrow C = 100$

$\therefore C(y) = 0.8y + 0.2y^{\frac{1}{2}} + 100.$

Example 15.10: Derive consumption function if

$$MPC = \frac{0.4}{\sqrt{y}}$$

Given that aggregate consumption is 50 when income is zero.

Solution: Given,

$$MPC = \frac{0.4}{\sqrt{y}}$$

$$= \frac{0.4}{y^{\frac{1}{2}}}$$

$$= 0.4y^{-\frac{1}{2}}$$

We have,

$$C(y) = \int MPC\,dy$$

$$= \int 0.4y^{-1/2}dy$$

$$= 0.4\frac{y^{-1/2+1}}{-1/2+1} + c$$

$$= 0.4\frac{y^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 0.8y^{1/2} + c$$

Given, aggregate consumption c(y) = 50, when y = 0

$\therefore$ c(o) = c = 50

$\Rightarrow$ c = 50

$\therefore$ c(y) = $0.8y^{1/2}$ + 50

(d) Derivation of saving function from Marginal propensity to save :

Saving function can be obtained from marginal propensity to save by integrating the marginal propensity to save function w.r.t. income. Say

S = S(Y)

then, $MPS = \frac{ds}{dy} = \frac{d}{dy}s(y) = s'(y)$

Saving function can be obtained from MPS as

S = S(Y) = $\int$MPSdy

= $\int$S'(y)dy.

Example 15.11: Given the marginal propensity to save

$MPS = 1 - \frac{0.2}{\sqrt{y}}$, find aggregate saving function

S(Y) with the aggregate saving is zero when income is 144.

Solution: Given,

$$MPS = 1 - \frac{0.2}{\sqrt{y}}$$

$$-1 - 0.2/y^{1/2}$$

$$= 1 - 0.2y^{-1/2}$$

$\therefore$ S(Y) = $\int$MPSdy

$$= \int\left(1 - 0.2y^{-1/2}\right)dy$$

$$= \int 1dy - \int 0.2y^{-1/2}dy$$

$$= y - 0.2\frac{y^{-1/2+1}}{-\frac{1}{2}+1} + c$$

$$= y - 0.2\frac{y^{1/2}}{\frac{1}{2}} + c$$

$$= y - 0.4y^{1/2} + c$$

Given, S(y) = 0, when y = 144

$$\therefore 0 = 144 - 0.4(144)^{1/2} + c$$

= 144 – 0.4 x 12 + c

= 144 – 4.8 + c

= 139.2 + c

$\Rightarrow$ C = –139.2

$\therefore$ S(y) = 144 – 0.4$y^{1/2}$ – 139.2

Example 15.12: Derive the aggregate saving function s(y) if the marginal propensity to consume function is given by MPC = 0.2$y^{-1/3}$ with zero saving when y = 125.

Solution: Given, MPC = 0.2$y^{-1/3}$

Since, aggregate saving function is the integral of MPS function, first of all we have to derive MPS function from MPC function as

We have

MPC + MPS = 1

$\therefore$ MPS = 1 – MPC

$\Rightarrow$ MPS = 1 – 0.2$y^{-1/3}$

$\therefore$ S(y) = $\int$MPSdy

= $\int$(1– 0.2$y^{-1/3}$)dy

$$= y - 0.2\frac{y^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c$$

$$= y - 0.2\frac{y^{\frac{2}{3}}}{\frac{2}{3}} + c$$

$$= y - 0.2 \times \frac{3}{2} y^{\frac{2}{3}} + c$$

$$= y - 0.3 y^{\frac{2}{3}} + c$$

Given, S(y) = 0, when y = 125

$\therefore$ $0 = 125 - 0.3\,(125)^{\frac{2}{3}} + c$

$= 125 - 0.3\,(5^3)^{\frac{2}{3}} + c$

$= 125 - 0.3 \times 5^2 + c$

$= 125 - 0.3 \times 25 + c$

$= 125 - 7.5 + c$

$= 117.5 + c$

$\Rightarrow C = -117.5$

$\therefore S(y) = y - 0.3y^{\frac{2}{3}} - 117.5$

15.4 ESTIMATION OF CONSUMER'S SURPLUS

The consumer's surplus is a concept introduced by Alfred Marshall. Consumer's surplus is equal to the difference between the price that a consumer is willing to pay for a commodity rather than go withour it and the actual price he pays for the commodity, consumer's surplus can derive using the concept of definite integral.

Consumer's surplus (c.s) = $\int_0^Q$ (Demand function expressed in terms of Q) dQ – P x Q

or, C. S. = $\int_0^Q f(Q)\,dQ - P \times Q$, where P is price and Q is quantity.

Example 15.13: If the demand function is $P = 19 - 2Q - 3Q^2$ and the quantity

demanded (Q) is 2, what will be the consumer's surplus.

Solution: Given,

$P = 19 - 2Q - 3Q^2$ and Q = 2

Putting Q = 2, $P = 19 - 2.2\ 3.2^2$

$= 19 - 4 - 12$

$= 3$

Now,

Consumer's surplus $= \int_0^Q f(Q)dQ - P \times Q$

$$= \int_0^2 \left(19 - 2Q - 3Q^2\right)dQ - 3 \times 2$$

$$= \left[19Q - Q^2 - Q^3\right]_0^2 - 6$$

$= (19.2 - 2^2 - 2^3) - 6$

$= (38 - 4 - 8) - 6$

$= 26 - 6$

$= 20$

Example 15.14: Given the consumer's demand function

$$Q = \sqrt{60 - 2p}$$

Find the consumer's surplus when market price P = 12

Solution: Given,

$$Q = \sqrt{60 - 2p}$$

The demand function can be rewritten in terms of quantity as

$Q^2 = 60 - 2P$

$\Rightarrow 2P = 60 - Q^2$

$$\Rightarrow p = 30 - \frac{Q^2}{2}$$

Now,

Putting P = 12 in the demand function, we have

$$Q = \sqrt{60 - 2.12}$$

$$= \sqrt{60 - 24}$$

$$= \sqrt{36}$$

$$= \pm 6$$

Since, output cannot be negative, therefore, Q = 6

$\therefore$ Consumer;s surplus = $\int_0^Q f(Q)\,dQ - P \times Q$

$$= \int_0^6 \left(30 - \frac{Q^3}{2}\right) dQ = 12 \times 6$$

$$= \left[30 - \frac{Q^3}{6}\right]_0^6 - 72$$

$$= 30.6 - \frac{6^3}{6} - 72$$

$$= 180 - 36 - 72$$

$$= 72$$

Example 15.15: Given the demand function P = 20 – 5Q and the supply function P = 4 + 3Q, where P is price and Q is quantity. Find consumer's surplus.

Solution: Given, the demand function

$$P = 20 - 5Q$$

and the supply function

$$P = 4 + 3Q$$

The equilibrim price and quantity can be determined by equating the demand and supply functions as

$$20 - 5Q = 4 + 3Q$$

$$\Rightarrow -5Q - 3Q = 4 - 20$$

$$\Rightarrow -8Q = -16$$

$$\Rightarrow Q = 2$$

Putting Q = 2 in the demand function, we have

$$P = 20 - 5Q$$

$$= 20 - 5.2$$

$$= 20 - 10$$

$$= 10$$

$\therefore$ Consumer's surplus $= \int_0^Q f(Q)dQ - PQ$

$$= \int_0^6 (20 - 5Q)dQ - 10 \times 2$$

$$= \left[20Q - \frac{5}{2}Q^2\right]_0^2 - 20$$

$$= 20 \times 2 - \frac{5}{2}2^2 - 20$$

$$= 40 - 10 - 20$$

$$= 10.$$

Example 15.16: Given the demand function P = 16 – Q and supply function P = 4 + Q, find consumer's surplus if

(i) the demand anc price are determined under perfect competition.

(ii) the demand and price are determined under monopoly so as to maximise profit and the supply function is identified with the marginal cost function.

Solutions: (i) Determination of consumer's surplus under perfect competition market :

Given the demand function

P = 16 – Q

and the supply function

P = 4 + Q

At equilibrium,

Demand = Supply

$\Rightarrow$ 16 – Q = 4 + Q

$\Rightarrow$ 2Q = 12

$\Rightarrow$ Q = 6

Putting Q = 6 in the demand function, we get

P = 16 – Q

= 16 – 6

= 10

Now,

$$\text{Consumer's surplus} = \int_0^Q f(Q)dQ - PQ$$

$$= \int_0^6 f(16-Q)dQ - 10 \times 6$$

$$= \left[16Q - \frac{Q^2}{2}\right]_0^6 - 60$$

$$= 16.6 - \frac{6^2}{2} - 60$$

$$= 96 - 18 - 60$$

$$= 18$$

(ii) Derivation of consumer's surplus where demand and price are determined under monopoly market :

In monopoly market, the equilibrium price and quantity are determined by the interaction of marginal revenue and marginal cost curves.

Marginal revenue can be derived as

$TR = P \times Q$

$= (16 - Q)Q$

$= 16Q - Q^2$

$$\therefore MR = \frac{d}{dQ}TR = \frac{d}{dQ}\left(16Q - Q^2\right)$$

$= 16 - 2Q$

$\because$ Supply function is identified with MC function, hence

$MC = 4 + Q$

At equilibrium,

$MR = MC$

$\Rightarrow 16 - 2Q = 4 + Q$

$\Rightarrow 3Q = 12$

$\Rightarrow Q = 4$

Putting Q = 4, in demand function, we get

$P = 16 - Q$

$= 16 - 4$

$= 12$

Now,

$$\text{Consumer's surplus} = \int_0^Q f(Q)dQ - PQ$$

$$= \int_0^4 f(16 - Q)dQ - 12 \times 4$$

$$= \left[16Q - \frac{Q^2}{2}\right]_0^4 - 48$$

$$= 16.4 - \frac{4^2}{2} - 48$$

$$= 64 - 8 - 48$$

$$= 8$$

15.5 ESTIMATION OF PRODUCER'S SURPLUS

Producer's surplus is the excess of what a produceer actually receives and the revenue he would be willing to recieved from the supply of a given amount of goods. In other words, producer's surplus = actual revenue receive– expected revenue receive. Producer's surplus can be derive by using definite integral as

Producer's surplus (PS) = PQ – ∫(supply function expressed in terms of Q)dQ

or, PS = PQ– $\int_0^Q f$ (Q)dQ, where P is price and Q is quantity.

Example 15.17: Given the producer's supply function

$$Q = \sqrt{-4 + 4P}$$

and market price is 10. Find the producer's surplus.

Solution: Given,

$$Q = \sqrt{-4 + 4P}$$

The supply function can be rewritten in terms of quantity as

$Q^2 = -4 + 4P$

$\Rightarrow 4P = Q^2 + 4$

$$\Rightarrow P = \frac{Q^2}{4} + 1$$

Now,

Putting P = 0 in the supply function, we have

$$Q = \sqrt{-4 + 4.10}$$

$$= \sqrt{36}$$

$$= \pm 6$$

Since, output cannot be negative, therefore, Q = 6

Now,

$$\text{Producer's surplus} = PQ - \int_0^Q f(Q)dQ$$

$$= 10 \times 6 - \int_0^6 f\left(\frac{Q^2}{4} + 1\right)dQ$$

$$= 60 - \left[\frac{1}{4} \cdot \frac{Q^3}{3} + Q\right]_0^6$$

$$= 60 - \frac{1}{12}6^3 - 6$$

$$= 60 - \frac{216}{12} - 6$$

$$= 60 - 18 - 6$$

$$= 36$$

Example 15.18: Given the supply function

$P = 0.5Q + 1.5$

Find producer's surplus when price = 6

Solution: Given, $P = 0.5\,Q + 1.5$

At P = 6,

$$P = 0.5Q + 1.5$$

$$\Rightarrow 6 = 0.5Q + 1.5$$

$$\Rightarrow 0.5Q = 4.5$$

$$\Rightarrow Q = 9$$

Now,

$$\text{Producer's surplus} = PQ - \int_0^Q f(Q)dQ$$

$$= 6 \times 9 - \int_0^9 (0.5Q + 1.5)dQ$$

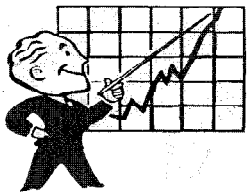
$$= 54 - \left[0.5\frac{Q^2}{2} + 1.5Q\right]dQ$$

$$= 54 - \left[0.25Q^2 + 1.5Q\right]_0^9$$

$$= 54 - [0.25 \times 9^2 + 1.5 \times 9]$$

$= 54 - [20.25 + 13.5]$

$= 20.25$



CHECK YOUR PROGRESS

Q 1: Obtain the total revenue function given the marginal revenue function

$MR = R'(Q) = 25 - 0.5Q$

Q 2: Given the marginal revenue function

$MR = R'(Q) = 40 - 8Q$

Where Q is output. What will happen to average revenue when sales (Q) increases from 6 to 8

Q 3: Given the marginal cost function

$MC = C'(Q) = 3Q^2 - 12Q + 2$

Find the level of output at which the average variable cost (AVC) will be minimum.

Q 4: Derive the total consumption function if the marginal propensity to save function is given by

$MPS = 0.5y^{-0.5}$

and if total consumption is 100 when income is zero.

Q 5: Given the demand function $P = 8 - 2Q$ and the supply function $P = 2 + Q$, find consumer's surplus and producer's surplus.



15.6 LET US SUM UP

- Derivation of total function from marginal function is one of the most important application of integral calculas in economic analysis.

  For example, $TC = \int MC dQ$ $TR = \int MR dQ$

  $C(Y) = \int MPC dy$ $S(Y) = \int MPS dy$

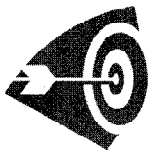
- The concept of definite integral is used to derive consumer's surplus and producer's surplus.

- Consumer's surplus = $\int_0^Q$ (Demand function expressed in terms of Q)dQ – PQ
- Procucer's surplus = PQ – $\int_0^Q$ (Supply function expressed in terms of Q) dQ where P is price and Q is quantity



15.8 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
2) Baruah, S. (2011). *Basic Mathematics and Its Application in Economics*, New Delhi: Trinity Press Pvt. Ltd.
3) Bose, D. (2004). *Mathematical Economics*; New Delhi: Himalaya Publishing House.
4) Chiang, A.C. & (2010). *Fundamental Methods of Economics Analysis*, MC Graw Hill Education India.
5) Kandoi, Balwant (2011). *Mathematics for Business and Economics with Application*; New Delhi: Himalaya Publishing House.



15.8 ANSWERS TO CHECK YOUR PROGRESS

Answer to Q No 1: Given–

$MR = R'(Q) = 25 - 0.5Q$

$\therefore TR = \int MRdQ$

$= \int(25 - 0.5Q)dQ$

$= 25Q - 0.5\frac{Q^2}{2} + C$

$TR = 25Q - 0.25Q^2 \quad [\because C = 0]$

Answer to Q No 2: Given,

$MR = R'(Q) = 40 - 8Q$

$\therefore TR = \int MRdQ$

$= \int(40 - 8Q)dQ$

$$= 40Q - 8.\frac{Q^2}{2} + C$$

$$TR = 40Q - 4Q^2 \quad [\because C = 0]$$

Again,

$$AR = \frac{TR}{Q} = \frac{40Q - 4Q^2}{Q}$$

$$= 40 - 4Q$$

Now, At Q = 6, AR = 40 – 4Q

= 40 – 4.6 = 40 – 24 = 16

At Q = 8, AR = 40 – 4Q

40 – 4.8 = 40 – 32 = 8

$\therefore$ When sales increases from 6 to 8, average revenue decreases from 16 to 8 i.e by 8.

Answer to Q No 3: Given,

$MC = C'(Q) = 3Q^2 - 12Q + 2$

$\therefore TC = \int MC dQ$

$= \int (3Q^2 - 12Q + 2)dQ$

$= Q^3 - 6Q^2 + 2Q + C$

Here, $TVC = Q^3 - 6Q^2 + 2Q$

$\therefore AVC = \frac{TVC}{Q} = Q^2 - 6Q + 2$

Minimisation of AVC requires that

1st order condition :

$$\frac{dAVC}{dQ} = 0$$

$$\frac{dAVC}{dQ} = \frac{d}{dQ}\left(Q^2 - 6Q + 2\right)$$

$= 2Q - 6$

According to 1st order condition

$$\frac{dAVC}{dQ} = 2Q - 6 = 0$$

$\rightarrow 2Q = 6$

$\Rightarrow$ Q = 3

2nd order condition

$$\frac{d^2AVC}{dQ^2} > 0$$

Here, $\frac{d^2AVC}{dQ^2} = \frac{d}{dQ}\left(\frac{dAVC}{dQ}\right) = \frac{d}{dQ}(2Q-6) = 2 > 0$

$\therefore$ 2nd order condition for AVC minimisation is satisfied when Q = 3.

Answer to Q No 4: Given, MPS = $0.5y^{-0.5}$

We have, MPC + MPS = 1

$\therefore$ MPC = 1 – MPS

$\Rightarrow$ MPC = $1 - 0.5y^{-0.5}$

$\therefore$ C(y) = $\int$MPCdy

= $\int(1 - 0.5y^{-0.5})dy$

$$= y - 0.5.\frac{y^{-0.5+1}}{-0.5+1} + c$$

$$= y - 0.5\frac{y^{0.5}}{0.5} + c$$

= $y - y^{0.5} + c$

Given C(y) = 100, when y = 0

$\therefore$ C(y) = $y - y^{0.5} + c$

$\Rightarrow$ $100 = 0 - 0^{0.5} + c$

$\Rightarrow$ 100 = C

$\therefore$ C(y) = $y - y^{0.5} + 100$

Answer to Q No 5: Given, Demand function

P = 8 – 2Q

and supply function

P = 2 + Q

At equilibrium,

Demand = Supply

$\Rightarrow$ 8 – 2Q = 2 + Q

$\Rightarrow$ 3Q = 6

$\Rightarrow$ Q = 2

Putting Q = 2 in the demand function, we get

$P = 8 - 2Q$

$= 8 - 2.2$

$P = 4$

Now, Consumer's surplus $= \int_0^Q f(Q)dQ - PQ$

$= \int_0^2 (8 - 2Q)dQ - 4.2$

$= \left[8Q - Q^2\right]_0^2 - 8$

$= 8.2 - 2^2 - 8$

$= 16 - 12$

$= 4$

And,

Producer's surplus $= PQ - \int_0^Q f(Q)dQ$

$= 4.2 - \int_0^2 (2 + Q)dQ$

$= 8 - \left[2Q + \frac{Q^2}{2}\right]_0^2$

$= 8 - \left(2.2 + \frac{2^2}{2}\right)$

$= 8 - (4 + 2)$

$= 2$



15.9 MODEL QUESTIONS

Q 1: The marginal cost of a firm is given by $MC = 3Q^2 - 10Q + 6$

With total fixed cost equal to 100, find the average cost when output is 5.

Q 2: If the marginal revenue function

MR = 50 – 3Q

Find the total revenue function.

Q 3: Given the marginal propensity to consume

MPC = $C'(y) = 0.8 + 0.1y^{-\frac{1}{2}}$

and the information C = Y, when Y = 100, Find the consumption function C(y).

Q 4: Find out the saving function given

MPS = $S'(y) = 1 - 0.2y^{-\frac{1}{2}}$, with zero saving when y = 81

Q 5: The demand and supplyu Laws are $P_d = (6 - Q)^2$ and $P_s = 14 + Q$ respectively. Find the consumer's surplus if–

(a) The demand and price are determined under pure competition.

(b) The demand and price are determined under monopoly so as to maximise profit and the supply function is identified with the manginal cost function.

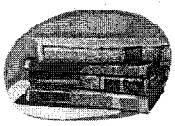
Q 6: If the demand and supply laws are given by

$P_d = 3x^2 - 20x + 5$

$P_s = 15 + 9x$

Find producer's surplus.

*** ***** ***



REFERENCES

Books:

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.
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