GMA S3-01

Differential Calculus

SEMESTER - III

MATHEMATICS

Block 1



KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

Subject Experts

Professor N. R. Das, Dept. of Mathematics, Gauhati University

Dr. Dilip Kumar Sarma, Former Head, Dept. of Mathematics, Cotton College

Course Co-ordinator : Harekrishna Deka, KKHSOU

SLM Preparation Team

UNITS	CONTRIBUTORS
1, 2, 3, 5, 6 & 7	S. K. Jha, Lumding College, Lumding

Editorial Team

Content : Dr. Dilip Kumar Sarma, Dept. of Mathematics, Cotton College

Structure, Format : Harekrishna Deka, KKHSOU

First Edition: June, 2018

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Printed and published by Registrar on behalf of the Krishna Kanta Handiqui State Open University.

Head Office : Patgaon, Rani Gate, Guwahati-781 017; Web: www.kkhsou.in

Sub-Head Office : Housefed Complex, Dispur, Guwahati-781 006

The university acknowledges with thanks the financial support provided by the **Distance Education Bureau, UGC**, for the preparation of this study material.

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COURSE INTRODUCTION

'Differential Calculus' is the third course of the programme B. A Mathematics (pass). The course is divided into two blocks.The first block consists of seven units and second block consists of seven units.

Block 1 deals with the concept of functions and limits of a function.Continuity and Differentiability of a function are discussed in this block.Concept of derivatives and indeterminate form of functions are also been introduced in this block.

Block 2 deals with successive differentiation of various functions,Mean value theorems and partial differentiation.Concept of Jocabian is also introduced in this block. Maximum and Minimum values of function of two variables will also covered in this block.

BLOCK INTRODUCTION

This is the first block of the course " Differential Calculus". The block consists of seven units.

In Unit 1, We will discuss about function and various types of function.

In Unit 2, We will introduce you to the concept of limit of a function.

In Unit 3, The concept of continuity of a function will be introduced.

In Unit 4, We will discuss the differentiability of a function.

In Unit 5, deals with the concept of derivatives.

In Unit 6, We will discuss the derivatives of various types functions.

In Unit 7, deals with limit of functions of indeterminate forms.

While going through a unit, you will notice some along-side boxes, which have been included to help you know some of the difficult, unseen terms. Again, we have included some relevant concepts in "LET US KNOW'' along with the text. And, at the end of each section, you will get "CHECK YOUR PROGRESS" questions. These have been designed to self-check your progress of study. It will be better if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with "ANSWERS TO CHECK YOUR PPROGRESS" given at the end of each unit.

UNIT 1: FUNCTION

UNIT STUCTURE

1.1 Learning Objectives

1.2 Introduction

1.3 Function

1.3.1 Domain and Range of a Function

1.4 Classification of a Function

1.4.1 Some Important Functions

1.5 Let us Sum up

1.6 Answers to check Your Progress

1.7 Further Reading

1.8 Model Questions

1.1 LEARNING OBJECTIVES

After going through this unit, you will able to

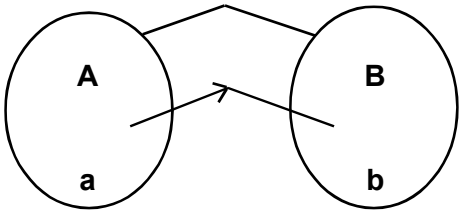
- find domain and range of a function
- understand different types of functions.

1.2 INTRODUCTION

The concept of a function plays an important role in mathematics. A function is a special type of relation which expresses how one quantity depends on another quantity.

1.3 FUNCTION

Let A and B be any two non-empty sets and let there be correspondence between the elements of A and B such that for each element $a \in A$, $\exists$ a unique $b \in B$, written as b = f (a).Then we say that f is a function from A to B and is written as $f : A \to B$.

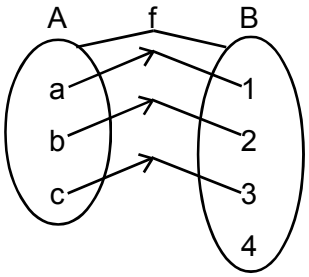


Remarks:

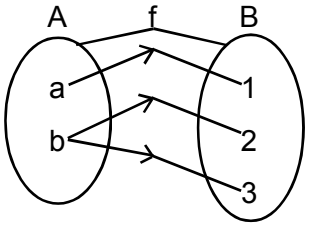
1) The element $b \in B$ is called the image of a under f.
2) If there is an element in A which is related to more than one element in B, then f is not a function.
3) If there is an element in A which is not related to any element in B, then f is not a function.

Illustrations:

1) If A = {a, b, c}, B = {1, 2, 3, 4,} and f(a) = 1, f(b) = 2, f(c) = 3
Then f is a function.



2) If A = {a, b}, B = {1, 2, 3} and f(a) = 1, f(b)=2, f(b) = 3
Then f is not a function.



1.3.1 Domain and Range of a Function

If $f : X \to Y$ be a function, then the set X is said to be the domain of f.

Range of f is the set of all image points in Y. The set Y is called the codomain of f.

Cleasly, Range $\subseteq$ Codomain.

Illustrations: Let $X = \{0, \pm 1, \pm 2\}$

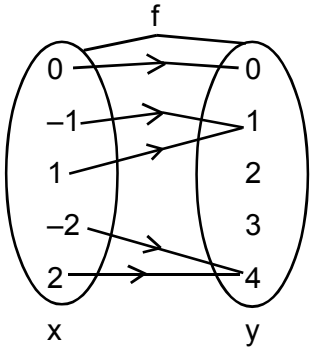
$Y = \{0, 1, 2, 3, 4\}$

Let $f : X \to Y$ be defined by $f(x) = x^2$

Here, Domain of f = X

Range of f = {0, 1, 4)

Codomain of f = Y.



Interval:

Let a,b be two real numbers such that $a < b$.

Closed Interval: The set $\{x : a \le x \le b\}$ is called a closed interval.It is denoted by $[a,b]$.

Open Interval: The set $\{x : a < x < b\}$ is called an open interval.It is denoted by $]a,b[$ or (a,b).

Semi Open or Semi Closed Interval: The sets $\{a \le x < b\}$ and $\{a < x \le b\}$ are called semi-open or semi-closed intervals.These are denoted by $[a,b[$ and $]a,b]$ respectively.

Note: The set of real numbers R is the open interval $(-\infty,-\infty)$.

Illustrative Examples:

Example 1: If f : x $\to$ x+3 and the domain of = {x : –2 $\le$ x $\le$ 2, x is an integer}, find the function f and its range.

Solution: The domain of the function

= {x :–2 $\le$ x $\le$ 2, x is an integer}

= {–2, –1, 0, 1, 2}

$\because$ f(x) = x+3

$\therefore$ f(–2) = –2 + 3 = 1

f(–1) = –1 + 3 = 2

f(0) = 0 + 3 = 3

f(1) = 1 + 3 = 4

f(2) = 2 + 3 = 5

$\therefore$ The function is

f = {(–2, 1), (–1, 2), (0, 3), (1, 4), (2, 5)}

and its range = {1, 2, 3, 4, 5)

Example 2: Find the domain and range of the function f(x) = $\frac{x}{1+x^2}$

Solution: Given f(x) = $\frac{x}{1+x^2}$

Clearly, f is defined for all real numbers.

$\therefore$ Domain of f is R

Let $y = f(x) \Rightarrow y = \dfrac{x}{1+x^2}$

$\Rightarrow y + yx^2 = x$

$\Rightarrow yx^2 - x + y = 0$

$\therefore (-1)^2 - 4.y.y \geq 0$ ($\because x \in R$)

and $y \neq 0$

$\Rightarrow 1 - 4y^2 \geq 0$ and $y \neq 0$

$\Rightarrow (1+2y)(1-2y) \geq 0$ and $y \neq 0$

$\Rightarrow -\dfrac{1}{2} \leq y \leq \dfrac{1}{2}$ and $y \neq 0$

But $y = 0$ gives $x = 0$

So, $y = 0$ is also in range

Hence, range of $f = \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

Example 3: Find the domain and range of $f(x) = \dfrac{1}{2-\sin 3x}$

Solution: Given $f(x) = \dfrac{1}{2-\sin 3x}$

$\because -1 \leq \sin x \leq 1, \forall x \in R$

$\therefore 2 - \sin x > 0, \forall x \in R$

$\therefore$ f is defined for all $x \in R$

$\therefore$ Domain is R.

Let $y = f(x) \Rightarrow y = \dfrac{1}{2-\sin 3x}$

$\Rightarrow \dfrac{1}{y} = 2 - \sin 3x \quad [\because y > 0]$

$\Rightarrow \sin 3x = 2 - \dfrac{1}{y}$

$\Rightarrow x = \dfrac{1}{3}\sin^{-1}\left(\dfrac{2y-1}{y}\right)$

$\Rightarrow -1 \leq \dfrac{2y-1}{y} \leq 1 \quad (\because x \in R)$

$\Rightarrow -y \leq 2y - 1 \leq y \quad (\because y > 0)$

$\Rightarrow 2y-1 \geq -y$ and $2y-1 \leq y$

$\Rightarrow 3y \geq 1$ and $y \leq 1$

$\Rightarrow y \geq \frac{1}{3}$ and $y \leq 1$

$\therefore$ Range is $\left[\frac{1}{3}, 1\right]$

Example 4: Find the domain and range of the function $f(x) = \frac{x}{|x|}$

Solution: Given $f(x) = \frac{x}{|x|}$

Clearly, f is defined for all real values of x except at x = 0

$\therefore$ Domain is R – {0}

Also, $f(x) = \frac{x}{|x|}$

$$= \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases}$$

$\therefore$ Range of f is {–1, 1}

Example 5: Find the domain and range of the function

$$f(x) = \frac{x^2 - 3x + 2}{x^2 + x - 6}$$

Solution: Given, $f(x) = \frac{x^2 - 3x + 2}{x^2 + x - 6}$

Clearly, f is defined if $x^2+x-6 \neq 0$

$\Rightarrow (x+3)(x-2) \neq 0$

$\Rightarrow x \neq -3, 2$

$\therefore$ Domain is R– {–3, 2}

Let $y = \frac{x^2 - 3x + 2}{x^2 + x - 6}$

$\Rightarrow x^2y + xy - 6y = x^2 - 3x + 2$

$\Rightarrow (y-1)x^2 + (y+3)x - (6y+2) = 0$

$\because$ x is real

$\therefore (y+3)^2 - 4(y-1)\{-(6y+2)\} \geq 0$

$\Rightarrow y^2 + 6y + 9 + 4(6y^2 - 4y - 2) \geq 0$

$\Rightarrow 25y^2 - 10y + 1 \geq 0$

$\Rightarrow (5y - 1)^2 \geq 0$, which is true for all real y

$\therefore$ Range of f is R.



CHECK YOUR PROGRESS

Q.1: Find the domain of the function

f(x) = $\sqrt{2 - 2x - x^2}$

Q.2: Find the domain of the function f(x) = $\frac{1}{\sqrt{x^2 - 3x + 2}}$

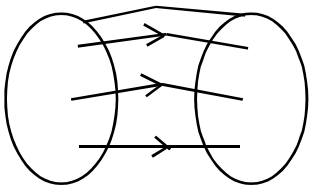
Q.3: Find the domain and range of the function $y = \frac{x^2}{1 + x^2}$

Q.4: For the function f(x) = sinx – cosx, find domain and range.

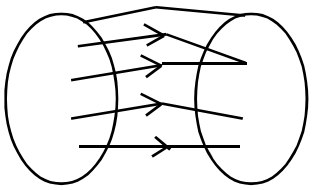
Q.5: Find domain and range of f(x) = $\frac{x + 2}{|x + 2|}$

1.4 CLASSIFICATION OF A FUNCTION

Injective (one-one) Function: A function f : X $\rightarrow$ Y is said to be one-one or injective if distinct elements of X have distinct images in Y.



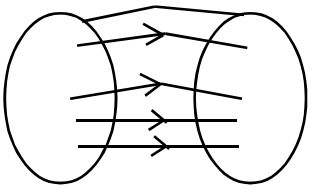
Many One Function: A function f : X $\rightarrow$ Y is said to be many one if there exists at least two distinct elements in X whose image are same.



Surjective (onto) Function: A function f: X $\rightarrow$ Y is said to be onto if every element of Y is the image of some element of X.

Note: A function which is not onto is called an into function.

Bijective Function: A function $f : X \to Y$ is said to be bijective, if f is both one-one and onto.



Note: A function is invertible if it is both one-one and onto i.e, bijective.

Method to check whether a function is one-one:

For the function $f : X \to Y$, if $f(x) = f(y) \Rightarrow x = y$, then f is one-one.

Method to check whether a function is onto:

Find the range of the function $f : X \to Y$,

If Range = Y, then the function is onto.

Illustrative Examples:

Example 1: Examine if the function $f : R \to R$ defined by $f(x) = x + \sqrt{x^2}$ is one-one and onto.

Solution: Given, $f(x) = x + \sqrt{x^2}$

$= x + |x|$

We have $f(-1) = -1 + |-1| = 0$

$f(-2) = -2 + |-2| = 0$

$\therefore f(-1) = f(-2)$ but $-1 \neq -2$

$\therefore$ f is not one - one

Also, f is not onto as $f(x) \geq 0, \ \forall x \in R$

And, Range $\neq$ codomain

Example 2: Show that the function $f : R - \{1\} \to R - \{1\}$ given by $f(x) = \frac{x-3}{x+1}$ is a bijective function.

Solution: Let x and y are two elements of $R - \{1\}$ such that $f(x) = f(y)$.

Now, $f(x) = f(y) \Rightarrow \frac{x-3}{x+1} = \frac{y-3}{y+1}$

$\Rightarrow xy + x - 3y - 3 = xy - 3x + y - 3$

$\Rightarrow x - 3y = -3x + y$

$\Rightarrow 4x = 4y$

$\Rightarrow x = y$

$\therefore$ f is one-one

Next, let $y \in R - \{1\}$ be arbitrary

$$\therefore f(x) = y \Rightarrow \frac{x-3}{x+1} = y$$

$$\Rightarrow xy + y = x - 3$$

$$\Rightarrow x(y-1) = -(y+3)$$

$$\Rightarrow x = \frac{y+3}{1-y} \in R, \forall y \in R - \{1\}$$

$\therefore$ f is onto.

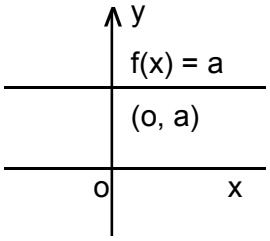
$\therefore$ f is both one-one and onto

Hence f is a bijective function.

Note: Since f is bijective function, so $f^{-1}: R - \{1\} \to R - \{1\}$ exists and is given by $f^{-1}(x) = \frac{x+3}{1-x}$

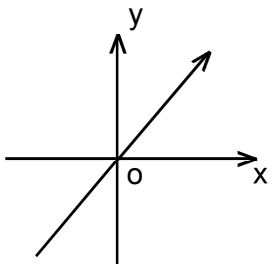
1.4.1 Some Important Functions

Constant Function: A function $f: R \to R$ defined as $f(x) = a$, $\forall x \in R$, where a is a constant, is called a constant function.



Note: The graph of a constant function is a straight line paralled to x-axis. The range of a constant function is a singletion set.

Identity Function: The function $f: R \to R$ defined as $f(x) = x$, $\forall x \in R$ is called the identity function. Its domain is R and range is also R.

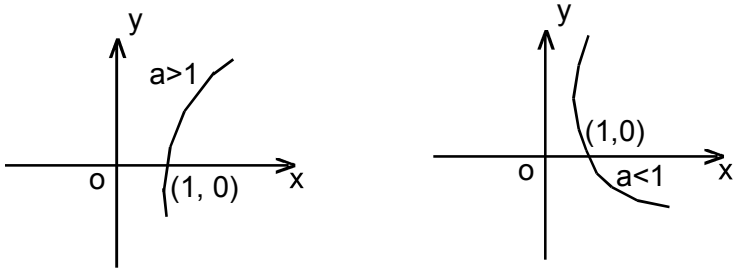


Note: The graph of the identity function is a straight line passing through origin and inclined at an angle of 45^0 with x–axis.

Polynomial Function : A function f : R $\rightarrow$ R, defined by f(x) = a_0 + $a_1x + a_2x^2 + ... + a_nx^n$, where $n \in N$ and $a_0, a_1, a_2, ... , a_n \in$ R is called a polynomial function ($a_n \neq 0$). n is called the degree of the polynomial. The domain of a polynomial function is R.

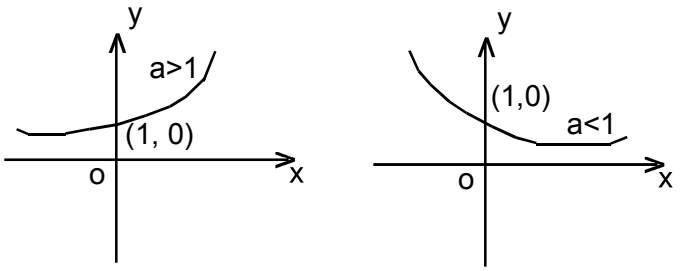
Logarithmic Function:

Let x be a positive real number other than 1. Then the function f : $]0, \infty[\rightarrow$ R defined by $f(x) = \log_a x$ is called the logarithmic function.



Remark: $\log_e x$ is also denoted by lnx.

Exponential Function: Let a be a positive real number other than 1. then the function f : R $\rightarrow$ R,defined by $f(x) = a^x$ is called the exponential function.



Explicit and Implicit Function: A function y is said to be an explicit function of x, if the dependent variable y can be expressed totally in terms of the independent variable x.

e.g. $y = \cos^2 x$, $x \in R$, $y = \log x + 2e^{3x} - x^3$; x > 0. When the variables x and y occur together in an equation f(x, y) = 0, in which y cannot be expressed explicitly in terms of x, then y is said to be an implicit function of x.

e.g. $xy = \sin(x+y)$, $e^{xy} + \cos x + x^2y - 1 = 0$.

Odd Function: A function f is said to be an odd function if f(–x) = –f(x), for all x in the domain of f. e. g. f(x) = sinx, $f(x) = x^3 + x$.

Even Function: A function f is said to be an even function if f(–x) = f(x), for all x in the domain of f. e.g. $f(x) = \cos x$, $f(x) = x^4 - 2x^2 + 2$.

Absolute Value (or Modulus) Function:

The function f is defined by

f(x) = x, if $x \geq 0$ = –x if x < 0

is called absolute value function. It is denoted by $|x|$,i.e.

$|x| = x$, if $x \geq 0$ = –x, if x < 0

We state the following theorems without proofs.

Theorems: Let a,b be two real numbers

i) $|a+b| \leq |a| + |b|$

ii) $||a| - |b|| \leq |a-b|$

iii) $|a-b| \leq k \Leftrightarrow b-k \leq a \leq b+k$

iv) $|ab| = |a||b|$

v) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0$

Note: Absolute value function is an even function, for $f(x) = |x| = |-x| = f(-x)$.



1.5 LET US SUM UP

- A function is a special type of relation that expresses how one quantity depends on another quantity.
- If range of a function is same as its codomain then it is onto (surjective), otherwise it is into.
- If $f : X \to Y$ is a one-one function such that the number of elements in X and Y are equal(X and Y are finite), then f is also onto.
- The graph of an even function is symmetrical about the y-axis.
- The graph of an odd function is symmetrical about the origin and it is placed either in the first and third quadrant or in the second and fourth quadrant.
- f(x) = 0 is the only function which is both even and odd.

1.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: Given $f(x) = \sqrt{2 - 2x - x^2}$

$\therefore 2 - 2x - x^2 \geq 0$

$\Rightarrow x^2 + 2x - 2 \leq 0$

$\Rightarrow (x+1)^2 - 3 \leq 0$

$\Rightarrow \left(x + 1 + \sqrt{3}\right)\left(x + 1 - \sqrt{3}\right) \leq 0$

$\Rightarrow -1 - \sqrt{3} \leq x \leq -1 + \sqrt{3}$

$\therefore$ Domain of f is $\left[-1 - \sqrt{3}, -1 + \sqrt{3}\right]$

Ans. to Q. No. 2: Given $f(x) = \dfrac{1}{\sqrt{x^2 - 3x + 2}}$

$\therefore x^2 - 3x + 2 > 0$

$\Rightarrow (x-1)(x-2) > 0$

$\Rightarrow x < 1$ or $x > 2$

$\therefore$ Domain is $]-\infty, 1[\cup]2, \infty]$

Ans. to Q. No. 3: Given, $y = \dfrac{x^2}{1 + x^2}$

Clearly, y is defined for all real x

$\therefore$ Domain is R

Also, $y = \dfrac{x^2}{1 + x^2} \Rightarrow y + yx^2 = x^2$

$\Rightarrow x^2 (1 - y) = y$

$\Rightarrow x = \sqrt{\dfrac{y}{1 - y}}$

$\Rightarrow x = \sqrt{\dfrac{y(1-y)}{(1-y)^2}}$

$\Rightarrow x = \dfrac{\sqrt{y(1-y)}}{1 - y}$

For x to be real, $1 - y \neq 0$ and $y(1 - y) \geq 0$

$\Rightarrow y \neq 1$ and $0 \leq y \leq 1$

$\Rightarrow 0 \leq y < 1$

$\therefore$ Range is [0, 1]

Ans. to Q. No. 4: Given, f (x) = sinx – cosx

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4}\sin x - \sin\frac{\pi}{4}\cos x\right)$$

$$\therefore f(x) = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$$

Clearly, f(x) is defined for all real x.

$\therefore$ Domain of f is R

Let y = f(x) $\quad \Rightarrow y = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

$$\Rightarrow \sin^{-1}\frac{y}{\sqrt{2}} = x - \frac{\pi}{4}$$

$$\Rightarrow x = \frac{\pi}{4} + \sin^{-1}\frac{y}{\sqrt{2}}$$

$\because x \in R$ we have $-1 \le \frac{y}{\sqrt{2}} \le 1$

$$\Rightarrow -\sqrt{2} \le y \le \sqrt{2}$$

$\therefore$ Range of f is $\left[-\sqrt{2}, \sqrt{2}\right]$

Ans. to Q. No. 5: Given function is $f(x) = y = \frac{x+2}{|x+2|}$

Clearly f(x) is defined for all real ralues of x except at x = –2

$\therefore$ Domain of f = R – {–2}

Also, $f(x) = \frac{x+2}{|x+2|}$

$$= \begin{cases} 1 & \text{if } x > -2 \\ -1 & \text{if } x < -2 \end{cases}$$

$\therefore$ Range of f is {–1, 1}



1.7 FURTHER READING

1) Gupta, Sudhir (2011) (1st edition): *Basics of Differentential Calculus*; New Delhi: Anmol publication Pvt. Ltd.

2) Habib, Amber (2011) (1st edition): *The Calculus of Finance*; Hyderabad: Universities Press India Pvt. Ltd.

3) Maron, I. A. (1993) (1st edition): *Problems in Calculus of One Variable*; New Delhi: CBS Publishers & Distributors Pvt. Ltd.

4) Sharma, A. K. (2004) (1st edition): *Text Book of Differential Calculus for B.A., B.Sc., B.Com., I.A.S., P.C.S.*; New Delhi: Discovery Publishing House.



1.8 MODEL QUESTIONS

Q.1: Find the domain of each of the following functions:

i) $f(x) = \dfrac{x}{x^2 - 3x + 2}$ ii) $f(x) = \dfrac{x^2 - 5x + 3}{x^2 - 1}$

iii) $f(x) = \dfrac{x + 7}{x^2 - 5x + 4}$ iv) $f(x) = \dfrac{x}{x^2 - 3x + 2}$

v) $f(x) = \dfrac{1}{5 - \cos x}$

Q.2: Find the domain and range of each of the following functions:

i) $f(x) = 1 - |x - 3|$ ii) $f(x) = \dfrac{1}{\sqrt{x + 3\sin x}}$

iii) $f(x) = 1 + 3\cos 2x$ iv) $f(x) = \dfrac{x^2 - 9}{x - 3}$

v) $f(x) = \dfrac{4 - x}{x - 4}$ vi) $f(x) = \dfrac{1}{2 - \cos 3x}$

vi) $f(x) = \dfrac{ax + b}{bx - a}$

Q.3: Find the domain if $f(x) = \sqrt{x^2 - 6x + 5}$

*** ***** ***

UNIT 2: LIMIT OF A FUNCTION

UNIT STUCTURE

2.1 Learning Objectives

2.2 Introduction

2.3 Limit of a Function

- 2.3.1 Methods of Evaluation Limits of Functions
- 2.3.2 Some Important Limits
- 2.3.3 Right Hand and Left Hand Limits

2.4 Let us Sum up

2.5 Answers to check Your Progress

2.6 Further Reading

2.7 Model Questions

2.1 LEARNING OBJECTIVES

After going through this unit, you will able to:

- evaluate limit of a given function
- distinguish between left hand limit and right hand limit.

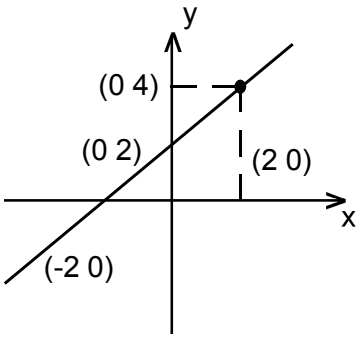
2.2 INTRODUCTION

The concepts of limit is basic to the study of calculus. The concept of limit helps us to describe the nature of a function when the independent variable takes values very-very close to a particular value.

2.3 LIMIT OF A FUNCTION

Let us sketch the graph of the function $f(x) = \dfrac{x^2 - 4}{x - 2}$

x	0	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4
y	2	3	$3\frac{1}{2}$	not defined	$4\frac{1}{2}$	5	6



A glance at the graph of f discloses the following facts :

i) f(2) is undefined.

ii) On either side of x = 2, the graph of f is without break.

iii) The difference of f(x) from 4 can be made as small as we please, by having x in the heighbourhood of 2 small enough for the purpose.

iv) As x approaches 2 from either side, f(x) approaches 4.

We study the behaviour of the function in the neighbourhood of x = 2.

for $x \neq 2$, $f(x) = \dfrac{(x-2)(x+2)}{(x-2)}$

$\Rightarrow$ f(x) = x+2, $x \neq 2$

$\therefore$ f(x) – 4 = x – 2

$\Rightarrow$ |f(x) – 4|= | x –2|

Now, |f (x) – 4| <0.1, if we take x such that |x–2| < 0.1

|f (x) – 4| < 0.1, if we take x such that |x–2| < 0.001

|f (x) – 4| < 0,001, if we take x such that |x–2| < 0.00001

This confirms the statement (iii) made above. From the above result it is always possible to take |x-2| small enough to make |f(x) – 4| as small as we like.

$\therefore$ f(x) approaches 4 as x approaches 2.

Definition of Limit of a Function:

Let f be a function defined in some neighbourhood of a. The function is said to approach a limit A as x approaches a if the difference |f(x) – A| can be made as small as we please by making |x–a| small enough.

Symbolically, $\lim\limits_{x \to a} f(x) = A$

The above statement may be more rigorously presented as below :

The number A is said to be the limit of f(x) as x approaches a, if for an arbitrary small positive number $\in$ another small number $\delta > 0$ can be found such that |f(x)–A|< $\in$ for all those values of x for which $0 < |\, x - a \,| < \delta$.

Note: We state the following theorems without proof:

i) $\lim\limits_{x \to a} f(x)$, if it exists, is unique.

If $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$, then

ii) $\lim_{x\to a}(f(x)\pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = l \pm m$

iii) $\lim_{x\to a}(f(x).g(x)) = \left(\lim_{x\to a} f(x)\right).\left(\lim_{x\to a} g(x)\right) = l.m$

iv) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{l}{m}$, where $m \neq 0$.

2.3.1 Methods of Evaluating Limits of Functions

Case I: Rational Function: A rational function of x is of the type $h(x) = \frac{f(x)}{g(x)}$.

When f(a) = 0 and g(a) = 0, factorize f(x) and g(x) and divide the factor which makes f(x) and g(x) zero.

Then find limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$.

Example 1: Evaluate : $\lim_{x\to 3} \frac{x^2+x-12}{x-3}$

Solution:

$$\lim_{x\to 3} \frac{x^2+x-12}{x-3} = \lim_{x\to 3} = \frac{(x-3)(x+4)}{(x-3)} = \lim_{x\to 3}(x+4) = 7$$

Example 2: Evaluate: $\lim_{x\to 0} \frac{(1+x)^{\frac{1}{n}}-1}{x}$

Solution: $\lim_{x\to 0} \frac{(1+x)^{\frac{1}{n}}-1}{x}$

$$= \lim_{x\to 0} \frac{(1+\frac{1}{n}x + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2!}x^2 + ...)-1}{x}$$

$$= \lim_{x\to 0}\left[\frac{1}{n} + \frac{1}{n}\left(\frac{1}{n}-1\right).\frac{x}{2} + \text{terms containing higher powers of } x\right]$$

$$= \frac{1}{n}$$

Case II: Irrational Functions:

If $h(x) = \frac{f(x)}{g(x)}$, rationalise the denominator or numerator, then find limits.

Example 3 : Evaluate: $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

Solution: $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

$$= \lim_{x \to 0} \frac{(1+x)-(1-x)}{x\left(\sqrt{1+x} + \sqrt{1+x}\right)}$$

$$= \lim_{x \to 0} \frac{2x}{x\left(\sqrt{1+x} + \sqrt{1-x}\right)}$$

$$= \lim_{x \to 0} \frac{2}{\sqrt{1+x} + \sqrt{1-x}}$$

$$= \frac{2}{1+1}$$

$$= 1$$

Example 4: Evaluate: $\lim_{x \to 0} \frac{x}{\sqrt{1+x} - 1}$

Solution : $\lim_{x \to 0} \frac{x}{\sqrt{1+x} - 1}$

$$= \lim_{x \to 0} \frac{x\left(\sqrt{1+x} + 1\right)}{1+x-1}$$

$$= \lim_{x \to 0} \frac{x\left(\sqrt{1+x} + 1\right)}{x}$$

$$= \lim_{x \to 0} \frac{\sqrt{1+x} + 1}{1}$$

$$= 1 + 1$$

$$= 2$$

Case III: When $x \to \infty$: Divide the numerator and denominator by highest power of x. Put $x = \frac{1}{z}$. As $x \to \infty$, we get $z \to 0$. Hence proceed as in case I or case II.

Example 5: Evaluate: $\lim_{x \to \infty} \frac{3x^2+7x+1}{3x^2+2x+7}$

Solution : $\lim_{x \to \infty} \frac{3x^2+7x+1}{3x^2+2x+7}$

$$= \lim_{x \to \infty} \frac{3+\frac{7}{x}+\frac{1}{x^2}}{3+\frac{2}{x}+\frac{7}{x^2}}$$

$$= \lim_{z \to 0} \frac{3+7z+z^2}{3+2z+7z^2} \qquad \left[\begin{array}{l}\text{Let } x = \frac{1}{z} \\ \because x \to \infty \therefore z \to 0\end{array}\right]$$

$$= \frac{3+0+0}{3+0+0}$$

$$= \frac{3}{3} = 1$$

$$= 1$$

Case IV: Many limits can be evaluated with the help of following expansions:

1) $(x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{2!}x^{n-2}a^2 + \ldots\ldots\ldots + a^n [n \in N]$

2) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\infty, \forall x \in R$

3) $a^x = 1 + x\log_e a + \frac{x^2(\log_e a)^2}{2!} + \ldots\ldots\ldots\infty, a > 0, a \neq 1, x \in R$

4) a) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\ldots\infty,\ -1 < x \leq 1$

b) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots\ldots\infty,\ -1 < x \leq 1$

5) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\ldots\infty$

6) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots\infty$

Example 6: Evaluate: $\lim_{x \to 0} \frac{a^x - 1}{x}$

Solution: $\lim_{x \to 0} \frac{a^x - 1}{x}$

$$= \lim_{x \to 0} \frac{\left\{1 + x\log a + \frac{x^2(\log a)^2}{2!} + ...\infty\right\} - 1}{x}$$

$$= \lim_{x \to 0} \frac{1}{x}\left[x\log a + \frac{x^2}{2!}(\log a)^2 + ...\infty\right]$$

$$= \lim_{x \to 0}\left[\log a + \frac{x}{2!}(\log a)^2 + \text{terms containing higher powers of } x\right]$$

= log a.

Example 7: Evaluate: $\lim_{x \to 1} \frac{\log x}{x - 1}$

Solution: $\lim_{x \to 1} \frac{\log x}{x - 1}$

$$= \lim_{h \to 0} \frac{\log(1+h)}{h} \qquad \left[\begin{matrix}\text{Let } x = 1 + h \\ \because x \to 1 \ \therefore h \to 0\end{matrix}\right]$$

$$= \lim_{h \to 0} \frac{h - \frac{h^2}{2} + \frac{h^3}{3} - .....\infty}{h}$$

$$= \lim_{h \to 0}\left[1 - \frac{h}{2} + \text{terms containing higher powers of } h\right]$$

= 1

2.3.2 Some Important Limits

1) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (x is measured in radian)

2) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

3) $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$

4) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$; a > 0 and n is a rational number.

5) $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a,\quad a>0,\ a\neq 1$

6) $\lim_{x\to 0}\frac{e^x-1}{x}=1$

7) $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$

8) $\lim_{x\to 0}\frac{\log(1-x)}{x}=-1.$

Illustrative Examples:

Example 1: Find $\lim_{x\to 0}\frac{\sin^{-1}x}{x}$

Solution: Let $\sin^{-1}x=y \qquad \therefore x=\sin y$

$\because\ x\to 0 \qquad \therefore y\to 0$

$$\therefore \lim_{x\to 0}\frac{\sin^{-1}x}{x}=\lim_{y\to 0}\frac{y}{\sin y}=\frac{1}{\lim_{y\to 0}\frac{\sin y}{y}}=\frac{1}{1}=1$$

Example 2: Evaluate: $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$

Solution:
$$\lim_{x\to 0}\frac{\sin ax}{\sin bx}=\lim_{x\to 0}\frac{\frac{\sin ax}{ax}.ax}{\frac{\sin bx}{bx}.bx}$$

$$=\lim_{x\to 0}\left[\frac{\frac{\sin ax}{ax}}{\frac{\sin bx}{bx}}.\frac{a}{b}\right]$$

$$=\frac{a}{b}.\frac{\lim_{ax\to 0}\frac{\sin ax}{ax}}{\lim_{bx\to 0}\frac{\sin bx}{bx}}\quad\left[\begin{array}{l}x\to 0\\ \Rightarrow ax\to 0 \text{ and } bx\to 0\end{array}\right]$$

$$=\frac{a}{b}.\frac{1}{1}$$

$$=\frac{a}{b}$$

Example 3: Evaluate: $\lim_{x \to 2} \frac{x^{10} - 1024}{x^5 - 32}$

Solution: $\lim_{x \to 2} \frac{x^{10} - 1024}{x^5 - 32}$

$$= \lim_{x \to 2} \frac{x^{10} - 2^{10}}{x^5 - 2^5}$$

$$= \lim_{x \to 2} \left[\frac{\frac{x^{10} - 2^{10}}{x - 2}}{\frac{x^5 - 2^5}{x - 2}} \right] = \frac{\lim_{x \to 2} \frac{x^{10} - 2^{10}}{x - 2}}{\lim_{x \to 2} \frac{x^5 - 2^5}{x - 2}}$$

$$= \frac{10 \times 2^{10-1}}{5 \times 2^{5-1}}$$

$$= \frac{2^{10}}{2^4} = 64$$

Example 4: Evaluate: $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{3x}$

Solution: $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{3x}$

$$= \lim_{x \to \infty} \left(1 + \frac{1}{\frac{x}{2}}\right)^{\frac{x}{2} \cdot 6}$$

$$= \left[\lim_{x \to \infty} \left(1 + \frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}} \right]^6$$

$$= \left[\lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^{y} \right]^6 \qquad \left[\begin{array}{l} \text{Let } \frac{x}{2} = y \\ \because x \to \infty \ \ \therefore y \to \infty \end{array} \right]$$

$= e^6$



CHECK YOUR PROGRESS

Evaluate the followings:

Q.1: $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1+x^2}}{x}$ **Q.2:** $\lim_{x \to 3} \frac{x^5 - 243}{x^3 - 27}$

Q.3: $\lim_{x \to \infty} \frac{4x^3 - 2x^2 + 6}{6x^3 + 5x + 1}$ **Q.4:** $\lim_{x \to 0} \frac{\sin x^0}{x}$

Q.5: $\lim_{x \to 0} \frac{\log(1+x)}{x}$ **Q.6:** $\lim_{x \to 1} \frac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2 + x - 3}$

Q.7: $\lim_{x \to 0} \frac{e^{ax} - e^{bx}}{x}$ **Q.8:** $\lim_{x \to 0} \frac{e^x + e^{-x} - 2}{x^2}$

2.3.3 Right Hand and Left Hand Limits

While defining the limit of a function f(x) as $x \to a$, we consider the behaviour of the function f(x) as x takes values closer to a. If we consider the behaviour of f(x) for those values of x > a or for those values of x < a, we say that x approaches a from the right (from above) or from the left (from below).

It is expressed as $x \to a + 0$ or $x \to a - 0$.

These limits are known as one sided limits.

The Right Hand Limit: A function f(x) is said to tend to a limit l as $x \to a$ from above or from the right, if to each $\varepsilon > 0, \exists\ \delta > 0$, such that $|f(x) - l| < \in$ whenever $a < x < a + \delta$.

Symbolically, we write $\lim_{x \to a+0} f(x) = l$

The Left Hand Limit : A function f(x) is said to tend to a limit l as $x \to a$ from below or from the left, if to each $\varepsilon > 0, \exists\ \delta > 0$, such that $|f(x) - l| < \in$ whenever $a - \delta < x < a$.

Symbolically, we write $\lim_{x \to a-0} f(x) = l$

Note: A function is said to tend to a limit l as $x \to a$ if both R.H.L. and L.H.L. exists and are equal to l.

Illustrative Examples:

Example 1: Let $f(x)=\frac{|x|}{x}$, show that $\lim_{x\to 0} f(x)$ does not exist.

Solution: Given, $f(x)=\frac{|x|}{x}$

$$\text{R.H.L.} = \lim_{x\to 0+0} f(x) = \lim_{x\to 0+0} \frac{|x|}{x}$$

$$= \lim_{x\to 0+0} \frac{x}{x} \qquad [\because |x| = x \text{ if } x > 0]$$

$$= 1$$

$$\text{L.H.L.} = \lim_{x\to 0+0} f(x) = \lim_{x\to 0+0} \frac{|x|}{x}$$

$$= \lim_{x\to 0+0} \frac{-x}{x} \qquad [\because |x| = -x \text{ if } x < 0]$$

$$= -1$$

$\therefore$ R. H. L. $\neq$ L. H. L.

$\therefore \lim_{x\to 0} f(x)$ does not exist.

Example 2: Let f be defined as $f(x)=\begin{cases} x^2 \text{ if } x \le 2 \\ 8-2x \text{ if } x > 2 \end{cases}$

Find a) $\lim_{x\to 2+0} f(x)$ b) $\lim_{x\to 2-0} f(x)$

c) $\lim_{x\to 2} f(x)$, if exists.

Solution: Given, $f(x)=\begin{cases} x^2 \text{ if } x \le 2 \\ 8-2x \text{ if } x > 2 \end{cases}$

a) $\lim_{x\to 2+0} f(x) = \lim_{x\to 2+0} (8-2x) = 8-4 = 4$

b) $\lim_{x\to 2-0} f(x) = \lim_{x\to 2-0} x^2 = 2^2 = 4$

c) $\because \lim_{x\to 2+0} f(x) = \lim_{x\to 2-0} = 4$

$\therefore \lim_{x\to 2} f(x)$ exists and it is equal to 4.



CHECK YOUR PROGRESS

Q.9: If $f(x)=\begin{cases}\dfrac{x-|x|}{x} & , x\neq 0\\ 0 & , x=0\end{cases}$

Examine the existence of $\lim\limits_{x\to 0} f(x)$

Q.10: If $f(x)=\begin{cases}5x-4, & 0<x\leq 1\\ 4x^2-3x, & 1<x<2\\ 3x^2+4, & x\geq 2\end{cases}$

Find $\lim\limits_{x\to 1} f(x)$ and $\lim\limits_{x\to 2} f(x)$, if exist.

Q.11: Evaluate $\lim\limits_{x\to 0}\dfrac{1}{1-e^{\frac{1}{x}}}$



2.4 LET US SUM UP

- Useful standard limits

a) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

b) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=e$

c) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$

d) $\lim\limits_{x\to 0}\dfrac{x^n-a^n}{x-a}=na^{n-1}\ ;\ a>0$

e) $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$

f) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log_e a\ ;\ a>0$

- Limit of a function f(x) as $x \to a$ exists provided $\lim_{x \to a-0} f(x)$ and $\lim_{x \to a+0} f(x)$ exist and are equal.



2.5 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1 : $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1+x^2}}{x}$

$$= \lim_{x \to 0} \frac{(1+x) - \left(1+x^2\right)}{x\left(\sqrt{1+x} + \sqrt{1+x^2}\right)}$$

$$= \lim_{x \to 0} \frac{1+x-1-x^2}{x\left(\sqrt{1+x} + \sqrt{1+x^2}\right)}$$

$$= \lim_{x \to 0} \frac{x(1-x)}{x\left(\sqrt{1+x} + \sqrt{1+x^2}\right)}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

Ans. to Q. No. 2: $\lim_{x \to 3} \frac{x^5 - 243}{x^3 - 27} = \lim_{x \to 3} \frac{x^5 - 3^5}{x^3 - 3^3}$

$$= \lim_{x \to 3} \frac{\frac{x^5 - 3^5}{x-3}}{\frac{x^3 - 3^3}{x-3}} = \frac{\lim_{x \to 3} \frac{x^5 - 3^5}{x-3}}{\lim_{x \to 3} \frac{x^3 - 3^3}{x-3}}$$

$$= \frac{5 \times 3^4}{3 \times 3^2}$$

$$= 15$$

Ans. to Q. No. 3: $\lim_{x \to \infty} \frac{4x^3 - 2x^2 + 6}{6x^3 + 5x + 1}$

$$= \lim_{x \to \infty} \frac{4 - \frac{2}{x} + \frac{6}{x^3}}{6 + \frac{5}{x^2} + \frac{1}{x^3}}$$

$$= \lim_{y \to 0} \frac{4 - 2y + 6y^3}{6 + 5y^2 + y^3} \quad \left[\text{Let } y = \frac{1}{x}, \; \therefore x \to \infty \Rightarrow y \to 0\right]$$

$$= \frac{4 - 0 + 0}{6 + 0 + 0}$$

$$= \frac{2}{3}$$

Ans. to Q. No. 4 : $\lim_{x \to 0} \frac{\sin x^0}{x} = \lim_{x \to 0} \frac{\sin \frac{\pi x}{180}}{x}$

$$= \lim_{x \to 0} \left(\frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}} \times \frac{\pi}{180} \right)$$

$$= \frac{\pi}{180} \lim_{\frac{\pi x}{180} \to 0} \frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}}$$

$$= \frac{\pi}{180} \times 1$$

$$= \frac{\pi}{180}$$

Ans. to Q. No. 5: $\lim_{x \to 0} \frac{\log(1 + x)}{x}$

$$= \lim_{x \to 0} \frac{1}{x} \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty \right]$$

$$= \lim_{x \to 0} \left[1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots \infty \right]$$

$= 1$

Ans. to Q. No. 6: $\lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$

$$= \lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}$$

$$= \lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(\sqrt{x}+1)(\sqrt{x}-1)}$$

$$= \lim_{x \to 1} \frac{2x-3}{(2x+3)(\sqrt{x}+1)}$$

$$= \frac{2-3}{(2+3)(1+1)} = -\frac{1}{10}$$

Ans. to Q. No. 7 : $\lim_{x \to 0} \frac{e^{ax}-e^{bx}}{x}$

$$= \lim_{x \to 0} \frac{(e^{ax}-1)-(e^{bx}-1)}{x}$$

$$= \lim_{x \to 0} \frac{e^{ax}-1}{x} - \lim_{x \to 0} \frac{e^{bx}-1}{x}$$

$$= \lim_{x \to 0} \left(\frac{e^{ax}-1}{ax} \times a \right) - \lim_{x \to 0} \left(\frac{e^{bx}-1}{bx} \times b \right)$$

$$= a \lim_{x \to 0} \frac{e^{ax}-1}{ax} - b \lim_{x \to 0} \frac{e^{bx}-1}{bx}$$

$$= a \times 1 - b \times 1$$

$$= a - b$$

Ans. to Q. No. 8: $\lim_{x \to 0} \frac{e^{x}+e^{-x}-2}{x^2}$

$$= \lim_{x \to 0} \frac{(e^{x}+e^{-x})^2}{x^2}$$

$$= \lim_{x \to 0} \left[\frac{(e^{x}-1)-(e^{-x}-1)}{x} \right]^2$$

$$= \lim_{x \to 0} \left[\frac{e^{x}-1}{x} + \frac{e^{-x}-1}{-x} \right]^2$$

$= (1+1)^2 = 4$

Ans. to Q. No. 9: Given, $f(x)=\begin{cases}\frac{x-|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

$$\text{L.H.L.}=\lim_{x\to 0-0} f(x)=\lim_{x\to 0-0}\frac{x-|x|}{x}$$

$$=\lim_{x\to 0-0}\frac{x+x}{x} \qquad [\because |x|=-x \text{ if } x<0]$$

$$=\lim_{x\to 0-0} 2$$

$$= 2$$

$$\text{R.H.L.}=\lim_{x\to 0-0} f(x)=\lim_{x\to 0+0}\frac{x-|x|}{x}$$

$$=\lim_{x\to 0-0}\frac{x-x}{x} \qquad [\because |x|=x \text{ if } x>0]$$

$$= 0$$

$\therefore$ L.H.L. $\neq$ R.H.L.

$\Rightarrow \lim_{x\to 0} f(x)$ does not exist.

Ans. to Q. No. 10: Given, $f(x)=\begin{cases}5x-4, & 0<x\leq 1\\ 4x^2-3x, & 1<x<2\\ 3x^2+4, & x\geq 2\end{cases}$

For $x\to 1$

$$\text{L.H.L.}=\lim_{x\to 1-0} f(x)=\lim_{x\to 1+0}(5x-4)=5-4=3$$

$$\text{R.H.L.}=\lim_{x\to 1+0} f(x)=\lim_{x\to 1+0}(4x^2-3x)=4-3=1$$

$\therefore$ L.H.L. $\neq$ R.H.L.

$\therefore \lim_{x\to 1} f(x)$ does not exist.

For $x\to 2$

$$\text{L.H.L.}=\lim_{x\to 2-0} f(x)=\lim_{x\to 2-0}(4x^2-3x)=4\times 4-3\times 2=10$$

$$\text{R.H.L.} = \lim_{x \to 2+0} f(x) = \lim_{x \to 2+0} (3x^2 + 4) = 3 \times 2^2 + 4 = 16$$

$\therefore$ L.H.L. $\neq$ R.H.L.

$\therefore \lim_{x \to 2} f(x)$ does not exist.

Ans. to Q. No. 11: $$\text{R.H.L.} = \lim_{x \to 0+0} \frac{1}{1 - e^{\frac{1}{x}}}$$

$$= \lim_{x \to 0+0} \frac{e^{-\frac{1}{x}}}{e^{-\frac{1}{x}} - 1}$$

$$= \frac{0}{0-1}$$

$$= 0$$

$$\text{L.H.L.} = \lim_{x \to 0-0} \frac{1}{1 - e^{\frac{1}{x}}} = \frac{1}{1-0} = 1$$

$\therefore$ L.H.L. $\neq$ R.H.L.

$\lim_{x \to 0} \frac{1}{1 - e^{\frac{1}{x}}}$ does not exist.



2.6 FURTHER READING

1) Gupta, Sudhir (2011) (1st edition): *Basics of Differentential Calculus*; New Delhi: Anmol publication Pvt. Ltd.
2) Habib, Amber (2011) (1st edition): *The Calculus of Finance*; Hyderabad: Universities Press India Pvt. Ltd.
3) Maron, I. A. (1993) (1st edition): *Problems in Calculus of One Variable*; New Delhi: CBS Publishers & Distributors Pvt. Ltd.
4) Sharma, A. K. (2004) (1st edition): *Text Book of Differential Calculus for B.A., B.Sc., B.Com., I.A.S., P.C.S.*; New Delhi: Discovery Publishing House.



2.7 MODEL QUESTIONS

Q.1: Evaluate the followings

a) $\lim\limits_{x\to 2} \dfrac{x^2-5x+6}{x^2+2x-8}$ b) $\lim\limits_{\theta\to 0} \dfrac{\theta}{\theta+\cos\theta}$

c) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+2x}-\sqrt{1-3x}}{x}$ d) $\lim\limits_{x\to 0} \dfrac{a-\sqrt{a^2-x^2}}{x^2}$

e) $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{x+2}-\sqrt{2-x}}$

Q.2: Evaluate the followings:

a) $\lim\limits_{x\to\infty} \dfrac{2x^3+5x+9}{x^3-7x+3}$ b) $\lim\limits_{x\to\infty} \dfrac{(2x-3)(3x+5)}{3x^2+1}$

c) $\lim\limits_{x\to\infty}\left(\sqrt{x+1}-1\right)$

Q.3: Examine if the following limits exists :

a) $\lim\limits_{x\to 3} f(x)$ here $f(x)=x^2$ if $x>0$

= 0 if x = 0

= –x if x < 0

b) $\lim\limits_{x\to 0} \phi(x)$ where $\phi(x)=x^2$ if x < 0

= 25 if x = 1

= x^2 + 2 if x > 1

Q.4: Evaluate: $\lim\limits_{y\to 1} \dfrac{y^2-1}{\sqrt{3y+1}-\sqrt{5y-1}}$

Q.5: Find the values of :

a) $\lim\limits_{x\to 0} \dfrac{(2+x)^4-16}{x}$ b) $\lim\limits_{h\to 1} \dfrac{h^2+2h-3}{h-1}$

c) $\lim\limits_{y\to 1} \dfrac{y^2-1}{\sqrt{3y+1}-\sqrt{5y-1}}$

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UNIT 3: CONTINUITY OF A FUNCTION

UNIT STUCTURE

3.1 Learning Objectives

3.2 Introduction

3.3 Continuity of a Function at a Point

3.4 Let us Sum up

3.5 Answers to Check Your Progress

3.6 Further Reading

3.7 Model Questions

3.1 LEARNING OBJECTIVES

After going through this unit, you will able to:

- determine the continuity of a function at a given point on it.

3.2 INTRODUCTION

The concepts of continuity is basic to the study of calculus. Most of the results in calculus are not true unless we are dealing with functions that are continuous. We may intuitively think of continuous functions as those functions whose graphs we can draw without lifting the pencil.

3.3 CONTINUITY OF A FUNCTION AT A POINT

A function is continuous at a point if the graph of the function has no break at that point. The continuity of a function at a point is closely related to the limiting behaviour of the function in the neighbourhood of that point.

Definition: A function f is said to be continuous at x = a if for any positive smallest possible number $\in$, another positive number δ can be found out such that $|f(x)-f(a)|<\in$ for all those values of x for which $0<|x-a|<\delta$

In simple language, for a function to be continuous at a point x = a,

$$\lim_{x \to a} f(x) = f(a) \text{ or } \lim_{x \to a-0} f(x) = \lim_{x \to a+0} f(x) = f(a)$$

Illustrative Examples:

Example 1: Show that the function f(x) = |x| is continuous at x = 0.

Solution: $f(x) = |x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$

$$\lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} (-x) = 0$$

$$\lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} x = 0$$

And f (0) = 0

$$\therefore \lim_{x \to 0+0} f(x) = f(0) = \lim_{x \to 0-0} f(x)$$

$\therefore$ f is continuous at x = 0

Example 2: Examine the continuity of the following function at x = 0

$$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Solution: We have $\lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$

$$= \lim_{x \to 0+0} \frac{1 - e^{-\frac{1}{x}}}{1 + e^{-\frac{1}{x}}}$$

$$= \frac{1-0}{1+0} \qquad \left[\because x \to 0+0, \ \therefore e^{-\frac{1}{x}} \to 0\right]$$

$$\lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = \frac{0-1}{0+1} = -1$$

$$\therefore \lim_{x \to 0+0} f(x) \neq \lim_{x \to 0-0} f(x)$$

$\therefore$ f is not continuous at x = 0.

Example 3: Find the values of a, b, c for which the function

$$f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin x}{x}, & x<0\\ c, & x=0\\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx\sqrt{x}}, & x>0\end{cases}$$

may be continuous at x = 0.

Solution: We have $\lim\limits_{x\to 0+0} f(x) = \lim\limits_{x\to 0} \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx\sqrt{x}}$

$$= \lim_{x\to 0} \frac{\sqrt{x}\left(\sqrt{1+bx}-1\right)}{bx\sqrt{x}}$$

$$= \lim_{x\to 0} \frac{\left(\sqrt{1+bx}-1\right)\left(\sqrt{1+bx}+1\right)}{bx\left(\sqrt{1+bx}+1\right)}$$

$$= \lim_{x\to 0} \frac{1+bx-1}{bx\left(\sqrt{1+bx}+1\right)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{1+bx}+1} = \frac{1}{2}$$

Also, $\lim\limits_{x\to 0-0} f(x) = \lim\limits_{x\to 0} \dfrac{\sin(a+1)x+\sin x}{x}$

$$= \lim_{x\to 0}\left[\frac{\sin(a+1)x}{(a+1)x}.(a+1)+\frac{\sin x}{x}\right]$$

= 1. (a+1) + 1

= a + 2

And f (0) = c, given

If f is continuous at x = 0, then we must have

$$\lim_{x\to 0+0} f(x) = \lim_{x\to 0-0} f(x) = f(0) \Rightarrow \frac{1}{2} = a+2 = c$$

$$\therefore c = \frac{1}{2} \text{ and } a = -\frac{3}{2}$$

$\because$ The result is independent of b $\quad \therefore b \in R$



CHECK YOUR PROGRESS

Q.1: Prove the following:

a) $f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x, & x\neq 0\\ 1, & x=0\end{cases}$

is not continuous at x = 0

b) f(x)=|x-1|+|x-2| is continuous at x = 1 and x = 2

c) $f(x)=\begin{cases}\dfrac{\log(1+ax)-\log(1-bx)}{x}, & x\neq 0\\ a+b, & x=0\end{cases}$

is continuous at x = 0

Q.2: Find a and b so that f becomes continuous everywhere.

a) $f(x)=\begin{cases}1, & x\leq 3\\ ax+b, & 3<x<5\\ 7, & x\geq 5\end{cases}$

b) $f(x)=\begin{cases}-4 & \text{if } x\leq -1\\ ax^2+b & \text{if } -1<x<0\\ \cos x & \text{if } x\geq 0\end{cases}$

c) $f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}; & x<\dfrac{\pi}{2}\\ 2a; & x=\dfrac{\pi}{2}\\ \dfrac{b(1-\sin x)}{(\pi-2x)^2}; & x>\dfrac{\pi}{2}\end{cases}$



3.4 LET US SUM UP

- A function f is continuous at

  x = a if $\lim_{x\to a-0} f(x)=\lim_{x\to a+0} f(x)=f(a)$

- Every polynomial is continuous at every point of the real line.

3.5 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: a) We have $\lim_{x \to 0} f(x) = \lim_{x \to 0}\left(\frac{\sin x}{x} + \cos x\right)$

$$= \lim_{x \to 0} \frac{\sin x}{x} + \lim_{x \to 0} \cos x$$

$$= 1 + 1$$

$$= 2$$

Also, f(0) = 1 (given)

$$\therefore \lim_{x \to 0} f(x) \neq f(0)$$

$\therefore$ f is not continuous at x = 0

b) f(x) = |x – 1| + |x – 2|

$$= \begin{cases} -(x-1)-(x-2) & \text{when} \quad x < 1 \\ (x-1)-(x-2) & \text{when} \quad 1 \leq x < 2 \\ (x-1)+(x-2) & \text{when} \quad x \geq 2 \end{cases}$$

$$= \begin{cases} -2x+3 \; ; & x < 1 \\ 1 \; ; & 1 \leq x < 2 \\ 2x-3 \; ; & x \geq 2 \end{cases}$$

We have

$$\lim_{x \to 1+0} f(x) = \lim_{x \to 1} 1 = 1$$

$$\lim_{x \to 1-0} f(x) = \lim_{x \to 1} (-2x+3) = -2+3 = 1$$

f(1) = 1

$$\therefore \lim_{x \to 1+0} f(x) = \lim_{x \to 1-0} f(x) = f(1)$$

$\therefore$ f is continuous at x = 1.

Similarly verify continuity at x = 2.

c) We have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\log(1+ax) - \log(1-bx)}{x}$

$= \lim_{x \to 0} \frac{\log(1+ax)}{x} - \lim_{x \to 0} \frac{\log(1-bx)}{x}$

$= a \lim_{x \to 0} \left\{ \frac{1}{ax} . \log(1+ax) \right\} - b \lim_{x \to 0} \left\{ \frac{1}{bx} . \log(1-bx) \right\}$

$= a \times 1 - b \times (-1)$ [Ref 1.5.2]

= a + b

Also, f (0) = a + b, given

$\therefore \lim_{x \to 0} f(x) = f(0)$

$\therefore$ f is continuous at x = 0

Ans. to Q. No. 2: a) $\therefore$ f is continuous everywhere

$\therefore$ f is continuous at x = 3 and x=5

$\therefore$ f is continuous at x = 3

$\therefore \lim_{x \to 3-0} f(x) = \lim_{x \to 3+0} f(x) = f(3)$

$\Rightarrow \lim_{x \to 3} 1 = \lim_{x \to 3} (ax + b) = 1$

$\Rightarrow 1 = 3a + b = 1$

$\Rightarrow 3a + b = 1$ ..... (1)

Also, f is continuous at x = 5

$\therefore \lim_{x \to 5-0} f(x) = \lim_{x \to 5+0} f(x) = f(5)$

$\Rightarrow \lim_{x \to 5} (ax + b) = \lim_{x \to 5} 7 = 7$

$\Rightarrow$ 5a + b = 7

$\Rightarrow$ 5a + 1 – 3a = 7, using (1)

$\Rightarrow$ 2a = 6

$\Rightarrow$ a = 3

$\therefore$ (1) gives b = – 8

Here, a = 3, b – 8.

b) Given, f is continuous everywhere

$\therefore$ f is continuous at x = –1 and x = 0

$\Rightarrow \lim\limits_{x \to 1-0} f(x) = \lim\limits_{x \to 1+0} f(x) = f(-1)$

$\Rightarrow \lim\limits_{x \to -1-4} = \lim\limits_{x \to -1} ax^2 + b = -4$

$\Rightarrow$ –4 = a + b = –4

$\Rightarrow$ a + b = – 4 ..... (i)

And $\lim\limits_{x \to 0+0} f(x) = \lim\limits_{x \to 0-0} f(x) = f(0)$

$\Rightarrow \lim\limits_{x \to 0} \cos x = \lim\limits_{x \to 0} (ax^2 + b) = 1$

$\Rightarrow$ 1 = b = 1

$\therefore$ (i) $\Rightarrow$ a + 1 = – 4 $\Rightarrow$ a = – 5

c) As f is continuous everywhere

$\therefore$ f is also continuous at $x = \frac{\pi}{2}$

Now, $\lim\limits_{x \to \frac{\pi}{2}-0} f(x) = \lim\limits_{x \to \frac{\pi}{2}} \frac{1-\sin^3 x}{3\cos^2 x}$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1-\sin^3 x}{3(1-\sin^2 x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{(1-\sin x)(1+\sin x+\sin^2 x)}{3(1-\sin x)(1+\sin x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1+\sin x+\sin^2 x}{3(1+\sin x)}$$

$$= \frac{1+1+1}{3(1+1)}$$

$$= \frac{1}{2}$$

$\lim\limits_{x \to \frac{\pi}{2}+0} f(x) = \lim\limits_{x \to \frac{\pi}{2}} \frac{b(1-\sin x)}{(\pi-2x)^2}$

$$= \lim_{y \to 0} \frac{b\left\{1-\sin\left(\frac{\pi}{2}-\frac{y}{2}\right)\right\}}{y^2} \qquad \left[\begin{array}{l} \text{Let } y = \pi - 2x \\ \therefore x = \frac{\pi}{2} - \frac{y}{2} \\ \because x \to \frac{\pi}{2} \therefore y \to 0 \end{array}\right]$$

$$= \lim_{y \to 0} \frac{b\left(1-\cos\frac{y}{2}\right)\left(1+\cos\frac{y}{2}\right)}{y^2\left(1+\cos\frac{y}{2}\right)}$$

$$= \lim_{y \to 0} \frac{b\sin^2\frac{y}{2}}{y^2\left(1+\cos\frac{y}{2}\right)}$$

$$= \lim_{y \to 0} \left\{b.\frac{\sin\frac{y}{2}}{\frac{y}{2}}.\frac{\sin\frac{y}{2}}{\frac{y}{2}}.\frac{1}{4\left(1+\cos\frac{y}{2}\right)}\right\}$$

$$= b.1.1.\frac{1}{4(1+1)}$$

$$= \frac{b}{8}$$

Also, $f\left(\frac{\pi}{2}\right) = 2a$, given

But f is continuous at $x = \frac{\pi}{2}$

$$\therefore \lim_{x \to \frac{\pi}{2}-0} f(x) = \lim_{x \to \frac{\pi}{2}+0} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{1}{2} = \frac{b}{8} = 2a$$

$$\therefore \frac{1}{2} = \frac{b}{8} \text{ and } \frac{1}{2} = 2a$$

$$\Rightarrow b = 4 \text{ and } a = \frac{1}{4}$$



3.6 FURTHER READING

1) Gupta, Sudhir (2011) (1st edition): *Basics of Differentential Calculus*; New Delhi: Anmol publication Pvt. Ltd.
2) Habib, Amber (2011) (1st edition): *The Calculus of Finance*; Hyderabad: Universities Press India Pvt. Ltd.
3) Maron, I. A. (1993) (1st edition): *Problems in Calculus of One Variable*; New Delhi: CBS Publishers & Distributors Pvt. Ltd.
4) Sharma, A. K. (2004) (1st edition): *Text Book of Differential Calculus for B.A., B.Sc., B.Com., I.A.S., P.C.S.*; New Delhi: Discovery Publishing House.



3.7 MODEL QUESTIONS

Q.1: Define the continuity of a function x at x = a.

Q.2: Discuss the continuity of
f(x) = |x + 1| + |x – 2| at x = –1 and at x = 2.

Q.3: Examine the continuity of the functions defined below at the indicated points :

a) $f(x) = 2 + x$ if x > 1
= 3 if x = 1
= 4 – x if x < 1

b) $f(x) = \frac{1}{2} - x$ if x < 3
= 1 if x = 3
$= x - \frac{1}{2}$ if x > 3

c) $\psi(x) = x^2 + 1$ if x < 1
= x + 1 if $1 \le x < 3$
$= \frac{x^3}{27} + 3$ if $x \ge 3$

d) $g(x) = \dfrac{\tan 5x}{2x}$ if $x \neq 0$

$= \dfrac{2}{5}$ if x = 0

at x = 0.

Q.4: If $f(x) = \dfrac{1-\cos x}{x \sin x}$, $x \neq 0$

$= \dfrac{a^2}{2}$, x = 0

is continuous at x = 0, find the value of a.

Q.5: If $f(x) = \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}$, $x \neq 2$

= m, x = 2

is continuous for all values of x, find the value of m.

Q.6: A function f defined as follows:

a) $f(x) = \dfrac{x^2}{a} - a$ if 0 < x < a

= 0 if x = a

$= a - \dfrac{a^3}{x^2}$ if x > a

Prove that f is continuous at x = a.

b) $f(x) = \dfrac{e^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}$ if $x \neq 0$

= 0 if x = 0

Prove that f is discontinuous at x = 0.

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UNIT 4: DIFFERENTIABILITY OF A FUNCTION

UNIT STUCTURE

4.1 Learning Objectives

4.2 Introduction

4.3 Concept of Derivative

4.3.1 Derivative of A Function

4.3.2 Left and Right Hand Derivatives

4.4 Let us Sum up

4.5 Answers to Check Your Progress

4.6 Further Reading

4.7 Model Questions

4.1 LEARNING OBJECTIVES

After going through this unit, you will able to:

- differentiate a function using limit
- distinguish between left hand derivative and right hand derivative.

4.2 INTRODUCTION

In unit 3, we have already discussed the continuity of a function.In this unit, we will discuss the differentiability of a function.

4.3 CONCEPT OF DERIVATIVE

Many phenomena involve changing quantities and their rates of change (increase or decrease). Differential calculus deals with the rate of change of a dependent variable with respect to an independent variable.

4.3.1 Derivative of a Function

Let f be a real valued function and a is a point in its domain of definition. Then the derivative of f at a is defined by–

$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ provided this limit exists and is donoted by $f'(a)$.

If y = f(x) be a function of f, the derivative of y with respect to x is $\frac{dy}{dx}$ or y_1.

When both x and y increase or decrease simulteneously, the derivative $\frac{dy}{dx}$ is positive. But when y increases with a decrease in x, or decreases with an increase in x, then $\frac{dy}{dx}$ is negative.

Remark: The derivative of a function of x is also called as the differential coefficient of x. The process of finding the differential coefficient is called differentiation.

Example 1: Find the derivative of f(x) = 3x at x = 2

Solution: We have $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

$$\therefore f'(2) = \lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h \to 0} \frac{3(2+h)-3(2)}{h}$$

$$= \lim_{h \to 0} \frac{6+3h-6}{h}$$

$$= \lim_{h \to 0} 3 = 3$$

$\therefore$ Derivative of the function f(x) = 3x at x = 2 is 3.

Example 2: If $f(x) = 2x^2 + 3x - 5$, then using definition show that $f'(0) + 3f'(-1) = 0$.

Solution: Given $f(x) = 2x^2 + 3x - 5$

We have $f'(0) = \lim_{h \to 0} \frac{f(o+h)-f(0)}{h}$

$$= \lim_{h \to 0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(3h^2+3h-5)-(0+0-5)}{h}$$

$$= \lim_{h \to 0} \frac{h(3h+3)}{h}$$

$$= \lim_{h \to 0} (3h+3) \qquad [\because h \neq 0 \text{ as } h \to 0]$$

$$= 3$$

Also $f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$

$$= \lim_{h \to 0} \frac{\{2(-1+h)^2 + 3(-1+h) - 5\} - \{2(-1)^2 + 3(-1) - 5\}}{h}$$

$$= \lim_{h \to 0} \frac{2 - 4h + 2h^2 - 3 + 3h - 5 - (2 - 3 - 5)}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 - h}{h}$$

$$= \lim_{h \to 0} \frac{h(2h-1)}{h}$$

$$= \lim_{h \to 0} (2h - 1) = -1$$

$\therefore$ $f'(0) + 3f'(-1) = 0$



CHECK YOUR PROGRESS

Q.1: Using definition, evaluate the following:

a) $f'(2)$ if $f(x) = x^2 - 6x + 1$

b) $f'(0)$ if $f(x) = \cos x$

c) $f'(1)$ if $f(x) = e^x$

d) $f'(-1)$ if $f(x) = x + \frac{1}{x}$

e) $f'(x)$ if $f(x) = \sqrt{x+2}$

4.3.2 Left and Right Hand Derivatives

A function f is said to be left derivable at x = a if

$\lim_{h \to 0-0} \frac{f(a+h) - f(a)}{h}$ exists finitely and is denoted by $Lf'(a)$.

A function f is said to be right derivable at x = a if

$\lim_{h \to 0+0} \frac{f(a+h) - f(a)}{h}$ exists finitely and is denoted by $Rf'(a)$.

In case $Lf'(a) = Rf'(a)$, we say f is derivable at x = a and denote by $f'(a)$.

Note: If a function is derivable at a point then it must be continuous at that point. But the converse is not true.

Illustrative Examples:

Example 1: Examine the differentiability of the function

$$f(x)=\begin{cases} -x, & x<0 \\ x^2, & 0\le x\le 1 \\ x^2-x+1, & x>1 \end{cases}$$

at the point x = 0 and x = 1

Solution: Given function is $f(x)=\begin{cases} -x, & x<0 \\ x^2, & 0\le x\le 1 \\ x^2-x+1, & x>1 \end{cases}$

We have $Rf'(0)=\lim_{h\to 0+0}\frac{f(0+h)-f(0)}{h}$

$$=\lim_{h\to 0}\frac{h^2-0}{h}$$

$$=\lim_{h\to 0} h = 0$$

$$Lf'(0)=\lim_{h\to 0-0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{-h-0}{h}$$

$$=\lim_{h\to 0} -1 = -1$$

$\therefore Rf'(0)\neq Lf'(0)$

$\therefore$ f is not differentiable at x = 0

Again, $Rf'(1)=\lim_{h\to 0+0}\frac{f(1+h)-f(1)}{h}$

$$=\lim_{h\to 0}\frac{(1+h)^2-(1+h)+1-1}{h}$$

$$=\lim_{h\to 0}\frac{1+2h+h^2-1-h}{h}$$

$$=\lim_{h\to 0}\frac{h^2+h}{h}$$

$$=\lim_{h\to 0}(h+1) = 1$$

$$Lf'(1) = \lim_{h \to 0-0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{1+2h+h^2-1}{h}$$

$$= \lim_{h \to 0} (2+h)$$

$$= 2$$

$\therefore Rf'(1) \neq Lf'(1)$

$\therefore$ f is not differentiable at x = 1.

Example 2: If $f(x) = \begin{cases} x+a & \text{if } x \leq 1 \\ ax^2+1 & \text{if } x > 1 \end{cases}$

Find 'a' so that f becomes differentiable at x = 1.

Solution: Given, $f(x) = \begin{cases} x+a & ; x \leq 1 \\ ax^2+1 & ; x > 1 \end{cases}$

We have $Rf'(1) = \lim_{h \to 0+0} \frac{f(1+h)-f(1)}{h}$

$$= \lim_{h \to 0} \frac{\{a(1+h)^2+1\} - (1+a)}{h}$$

$$= \lim_{h \to 0} \frac{a+2ah+ah^2+1-1-a}{h}$$

$$= \lim_{h \to 0} \frac{h(2a+ah)}{h}$$

$$= \lim_{h \to 0} (2a+ah) = 2a$$

And $Lf'(1) = \lim_{h \to 0-0} \frac{f(1+h)-f(1)}{h}$

$$= \lim_{h \to 0} \frac{(1+h+a)-(1+a)}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= 1$$

But f (x) is differentiable at x = 1

$\therefore$ $Rf'(1) = Lf'(1)$

$\Rightarrow 2a = 1$

$\Rightarrow a = \frac{1}{2}$

Example 3: If f is differentiable at x = 1,

Prove that $\lim_{x \to a} \frac{xf(a) - af(x)}{x - a} = f(a) - af'(a)$

Solution: $\because$ f is differentiable at x = 1

$\therefore \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$ .....(1)

L.H.S. = $\lim_{x \to a} \frac{xf(a) - af(x)}{x - a}$

$= \lim_{h \to 0} \frac{(a+h)f(a) - af(a+h)}{h}$ $\begin{bmatrix} \text{Let } h = x - a \\ \because x \to a \\ \therefore h \to 0 \end{bmatrix}$

$= \lim_{h \to 0} \frac{af(a) + hf(a) - af(a+h)}{h}$

$= \lim_{h \to 0} \frac{hf(a) - a\{f(a+h) - f(a)\}}{h}$

$= \lim_{h \to 0} \left[f(a) - a.\frac{f(a+h) - f(a)}{h} \right]$

$= f(a) - a \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$= f(a) - af'(a)$, using (1)

= R.H.S.

Example 4: Let $f(x) = \begin{cases} \frac{x\left(e^{\frac{1}{x}} - e^{-\frac{1}{x}}\right)}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}} ; x \neq 0 \\ 0 \quad ; x = 0 \end{cases}$

Show that f is continuous but not derivable at x = 0

Solution: Given $f(x)=\begin{cases}\dfrac{x\left(e^{\frac{1}{x}}-e^{-\frac{1}{x}}\right)}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}} ; x\neq 0\\ 0 \quad ; x=0\end{cases}$

Continuity at x = 0

$$\lim_{x\to 0+0} f(x)=\lim_{x\to 0+0}\frac{x\left(e^{\frac{1}{x}}-e^{-\frac{1}{x}}\right)}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}$$

$$=\lim_{x\to 0+0}\frac{x\left(1-e^{-\frac{2}{x}}\right)}{1+e^{-\frac{2}{x}}}$$

$$=\frac{0(1-0)}{1+0}=0 \qquad \left[\because x\to 0+0,\ \therefore e^{-\frac{2}{x}}\to 0\right]$$

And $\lim_{x\to 0-0} f(x)=\lim_{x\to 0-0}\frac{x\left(e^{\frac{1}{x}}-e^{-\frac{1}{x}}\right)}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}$

$$=\lim_{x\to 0-0}\frac{x\left(e^{\frac{2}{x}}-1\right)}{e^{\frac{2}{x}}+1}$$

$$=\frac{0(0-1)}{0+1}$$

$$=0$$

Also, f (0) = 0 (given)

$$\therefore \lim_{x\to 0+0} f(x)=\lim_{x\to 0-0} f(x)=f(0)$$

$\therefore$ f is continuous at x = 0

Differentiability at x = 0

$$Rf'(0)=\lim_{h\to 0+0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{h\left(e^{\frac{1}{h}} - e^{-\frac{1}{h}} \right)}{e^{\frac{1}{h}} + e^{-\frac{1}{h}}} - 0 \right]$$

$$= \lim_{h \to 0} \frac{1 - e^{-\frac{2}{h}}}{1 + e^{-\frac{2}{h}}}$$

$$= \frac{1-0}{1+0}$$

$$= 1$$

$$Lf'(0) = \lim_{h \to 0-0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0-0} \frac{1}{h} \left[\frac{h\left(e^{\frac{1}{h}} - e^{-\frac{1}{h}} \right)}{e^{\frac{1}{h}} + e^{-\frac{1}{h}}} - 0 \right]$$

$$= \lim_{h \to 0-0} \frac{\left(e^{\frac{2}{h}} - 1 \right)}{\left(e^{\frac{2}{h}} + 1 \right)}$$

$$= \frac{0-1}{0+1}$$

$$= -1$$

$\therefore Rf'(0) \neq Lf'(0)$

$\therefore$ f is not differentiable at x = 0



CHECK YOUR PROGRESS

Q.2: Show that $f(x) = \begin{cases} x e^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)}; & x \neq 0 \\ 0 \quad\quad\quad\

Q.3: Find the value of a and b such that the function

$$f(x)=\begin{cases} x^2+3x+a; & x\le 0 \\ bx+c & ; x>0 \end{cases}$$ is differentiable everywhere.

Q.4: Show that the function $f(x)=\begin{cases}(x-a)\cos\left(\dfrac{1}{x-a}\right); & x\neq a \\ 0 & ; x=a\end{cases}$

is not differentiable at x = a.



4.4 LET US SUM UP

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- If a function is differentiable at a point x = a, then it is continuous at x = a.
- If f is not continuous at x = a, then it is not differentiable at x = a.
- $f'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$
- $Rf'(a)=\lim\limits_{h\to 0+0}\dfrac{f(a+h)-f(a)}{h}$

  $Lf'(a)=\lim\limits_{h\to 0-0}\dfrac{f(a+h)-f(a)}{h}$

  $f'(a)$ exists provided $Rf'(a)$, $Lf'(a)$ exist and are equal.



4.5 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: a) Given $f(x) = x^2 - 6x + 1$

We have $f'(2)=\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$

$$=\lim_{h\to 0}\frac{\{(2+h)^2-6(2+h)+1\}-(2^2-6\times 2+1)}{h}$$

$$=\lim_{h\to 0}\frac{4+4h+h^2-12-6h+1+7}{h}$$

$$=\lim_{h\to 0}\frac{h^2-2h}{h}$$

$$= \lim_{h\to 0} \frac{h(h-2)}{h}$$

$$= \lim_{h\to 0} (h-2)$$

$$= -2$$

b) Given, f(x) = cosx.

We have $f'(0) = \lim_{h \to 0} \frac{f(h)-f(0)}{h}$

$$= \lim_{h \to 0} \frac{\cos h - \cos 0}{h}$$

$$= \lim_{h \to 0} \frac{-(1-\cos h)}{h}$$

$$= \lim_{h \to 0} \frac{-(1-\cos h)(1+\cos h)}{h(1+\cos h)}$$

$$= \lim_{h \to 0} \frac{-\sin^2 h}{h(1+\cos h)}$$

$$= \lim_{h \to 0} \left[-\frac{\sin h}{h} . \frac{\sin h}{h} . \frac{h}{1+\cos h} \right]$$

$$= -1.1.\frac{0}{1+1}$$

$$= 0$$

c) Given $f(x) = e^x$

We have $f'(1) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$

$$= \lim_{h \to 0} \frac{e^{1+h} - e}{h}$$

$$= \lim_{h \to 0} \frac{e(e^h - 1)}{h}$$

$$= e. \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e.1$$

$$= e$$

d) Given, $f(x) = x + \frac{1}{x}$

We have $f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$

$$= \lim_{h \to 0} \frac{\left(-1+h+\frac{1}{-1+h}\right) - \left(-1+\frac{1}{-1}\right)}{h}$$

$$= \lim_{h \to 0} \frac{(h-1)^2 + 1 + 2(h-1)}{h(h-1)}$$

$$= \lim_{h \to 0} \frac{h^2 - 2h + 1 + 1 + 2h - 2}{h(h-1)}$$

$$= \lim_{h \to 0} \frac{h^2}{h(h-1)}$$

$$= \lim_{h \to 0} \frac{h}{h-1}$$

$$= 0$$

e) Given $f(x) = \sqrt{x+2}$

We have $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}$$

$$= \lim_{h \to 0} \frac{x+h+2-x-2}{h\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\sqrt{x+h+2} + \sqrt{x+2}}$$

$$= \frac{1}{\sqrt{x+2} + \sqrt{x+2}}$$

$$= \frac{1}{2\sqrt{x+2}}$$

Ans. to Q. No. 2: a) Given, $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

$$=\begin{cases} xe^{-\left(\frac{1}{x}+\frac{1}{x}\right)} & ; x > 0 \\ xe^{-\left(-\frac{1}{x}+\frac{1}{x}\right)} & ; x < 0 \\ 0 & ; x = 0 \end{cases}$$

$$=\begin{cases} xe^{-\frac{2}{x}} & ; x > 0 \\ x & ; x < 0 \\ 0 & ; x = 0 \end{cases}$$

Now, $Rf'(0)=\lim_{h \to 0+0} \frac{f(0+h)-f(0)}{h}$

$$=\lim_{h \to 0+0} \frac{he^{-\frac{2}{h}}-0}{h}$$

$$=\lim_{h \to 0+0} e^{-\frac{2}{h}}$$

$= 0$

And, $Lf'(0)=\lim_{h \to 0-0} \frac{f(0+h)-f(0)}{h}$

$$=\lim_{h \to 0-0} \frac{h-0}{h} = 1$$

$\therefore Rf'(0) \neq Lf'(0)$

$\therefore$ f is not differentiable at x = 0.

Ans. to Q. No. 3: Given function is $f(x)=\begin{cases} x^2+3x+a & ; x \leq 0 \\ bx+2 & ; x > 0 \end{cases}$

$\therefore$ f is differentable everywhere

$\therefore$ f is continuous as well as differentiable at x = 1

$\therefore \lim_{x \to 1-0} f(x) = \lim_{x \to 1+0} f(x) = f(1)$

$$\Rightarrow \lim_{x \to 1}(x^2 + 3x + a) = \lim_{x \to 1}(bx + 2) = 1^2 + 3 \times 1 + a$$

$$\Rightarrow 1 + 3 + a = b + 2 = 1 + 3 + a$$

$$\Rightarrow 4 + a = b + 2$$

$$\Rightarrow a = b - 2 \qquad \text{.....(1)}$$

Also, f is differentiable at x = 1

$$\therefore Rf'(1) = Lf'(1)$$

$$\Rightarrow \lim_{h \to 0+0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0-0} \frac{f(1+h) - f(1)}{h}$$

$$\Rightarrow \lim_{h \to 0+0} \frac{\{b(1+h)+2\} - \{1+3+a\}}{h} = \lim_{h \to 0-0} \frac{(1+h)^2 - 3(1+h) + a - \{1+3+a\}}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{b + bh + 2 - 4 - b + 2}{h} = \lim_{h \to 0} \frac{1 + 2h - h^2 + 3 + 3h + a - 4 - a}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{bh}{h} = \lim_{h \to 0} \frac{h_2 + 5h}{h}$$

$$\Rightarrow \lim_{h \to 0} b = \lim_{h \to 0}(h + 5)$$

$$\therefore b = 5 \qquad \therefore a = 3$$

Ans. to Q. No. 4: Given, $f(x) = \begin{cases} (x-a)\cos\left(\frac{1}{x-a}\right) ; x \neq a \\ 0 ; x = a \end{cases}$

We have, $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$$= \lim_{h \to 0} \frac{(a+h-a)\cos\left(\frac{1}{a+h-a}\right) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h\cos\frac{1}{h}}{h}$$

$$= \lim_{h \to 0} \cos\frac{1}{h}, \text{ which does not exist.}$$

Hence, f is not differentiable at x = a.



4.6 FURTHER READING

1) Gupta, Sudhir (2011) (1st edition): *Basics of Differentential Calculus*; New Delhi: Anmol publication Pvt. Ltd.
2) Habib, Amber (2011) (1st edition): *The Calculus of Finance*; Hyderabad: Universities Press India Pvt. Ltd.
3) Maron, I. A. (1993) (1st edition): *Problems in Calculus of One Variable*; New Delhi: CBS Publishers & Distributors Pvt. Ltd.
4) Sharma, A. K. (2004) (1st edition): *Text Book of Differential Calculus for B.A., B.Sc., B.Com., I.A.S., P.C.S.*; New Delhi: Discovery Publishing House.



4.7 MODEL QUESTIONS

Q.1: If a function is differentiable at a point, prove that it is also continuous at that point.

Q.2: If $f(x) = 5x - 4$ if $0 < x \leq 1$

$= 4x^3 - 3$ if $1 < x < 2$

$= 3x + 4$ if $x \geq 2$

Examine the continuity and differentiability of the function at x = 1 and x = 2.

Q.3: If $f(x) = \frac{x^2}{a} - a$ if 0 < x < a

$= a$ if x = a

$= a - \frac{x^2}{a}$ if x > a

then, examine its continuity and differentiability at x = 0.

Q.4: If $\psi(x) = x^2 \cos\frac{1}{x}$, $x \neq 0 = 0$, $x = 0$, then prove that the function ψ is continuous and differentiable at x = 0.

Q.5: Show that f(x) = |x|+|x–1| is continuous but not differentiable at x = 0 and x = 1.

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UNIT 5: DERIVATIVE OF FUNCTIONS

UNIT STRUCTURE

5.1 Learning Objectives

5.2 Introduction

5.3 Geometrical Interpretation of Derivative of a Function

5.4 Derivative of Various Standard Functions

5.5 Derivative of Sum and Difference of Functions

5.6 Derivative of Product of Functions and Quotient Rule

5.7 Let us Sum up

5.8 Answers to Check Your Progress

5.9 Further Reading

5.10 Model Questions

5.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- understand the geometrical meaning of $f'(x)$ or $\frac{dy}{dx}$
- evaluate derivative of algebraic, trigonometrical, exponential and logarithm functions.

5.2 INTRODUCTION

We have already introduced the concept of derivative in unit 4. In this unit, we discuss algebra of derivative including derivative of certain standard function. The notion of differentiation is essentially concerned with the rate of charge of a dependent variable with respect to an independent variable.

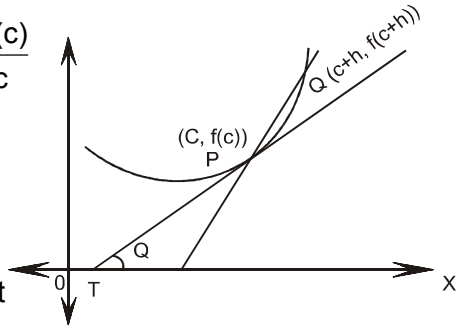
5.3 GEOMETRICAL INTERPRETATION OF DERIVATIVE

Let y = f(x) be a function. Let P (c, f(c)) and Q (c+h, f(c+h)) be two neighbouring points on the graph of y = f(x).

Slope of Chord PQ = $\frac{f(c+h)-f(c)}{(c+h)-c}$

= $\frac{f(c+h)-f(c)}{h}$

When Q $\to$ P along the curve, the chord PQ becomes tangent PT at P.



Thus, the slope of the chord PQ becomes slope of the tangent at P when Q $\to$ P, i.e. h $\to$ 0

$\therefore$ Slope of the tangent at P = $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$

= $f'(c)$ (by definition)

Also, if θ is the angle made by tangent at P with x-axis, then slope at the tangent at P = $\tan\theta$.

Thus, $f'(c) = \tan\theta = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$

Note: If tangent is parallel to x-axis, then $f'(c) = 0$.

5.4 DERIVATIVES OF VARIOUS STANDARD FUNCTIONS

1) $\frac{d}{dx}(k) = 0$, k is any constant

2) $\frac{d}{dx}(x) = 1$

3) $\frac{d}{dx}(x^n) = nx^{n-1}$

4) $\frac{d}{dx}(e^x) = e^x$

5) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

6) $\frac{d}{dx}(a^x) = a^x \log_e a$

10) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

11) $\frac{d}{dx}(\sec x) = \sec x \tan x$

12) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x . \cot x$

13) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$; $-1<x<1$

14) $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$; $-1<x<1$

15) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$; $x \in R$

7) $\frac{d}{dx}(\sin x) = \cos x$

8) $\frac{d}{dx}(\cos x) = -\sin x$

9) $\frac{d}{dx}(\tan x) = \sec^2 x$

16) $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$; $x \in R$

17) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}; |x| > 1$

18) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}; |x| > 1$

Proof:

1) Let $y = f(x) = k$

$\therefore$ $f(x+h) = k$

By definition, $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$\Rightarrow \frac{d}{dx}(k) = \lim_{h \to 0} \frac{k-k}{h} = 0$

2) Let $y = f(x) = x$

$\therefore$ $f(x+h) = x + h$

By definition, $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$\Rightarrow \frac{d}{dx}(x) = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} 1 = 1$

3) Let $y = f(x) = x^n$

$\therefore$ $f(x+h) = (x+h)^n$

By definition, $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$\Rightarrow \frac{d}{dx}(x^n) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$

Case I: Let n be a positive integer

$$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{\left(x^n + {}^nc_1 x^{n-1}h + {}^nc_2 x^{n-2}h^2 + \ldots + h^n\right) - x^n}{h}$$

$$= \lim_{h \to 0} \left({}^nc_1 x^{n-1} + {}^nc_2 x^{n-2}h + \ldots + h^{n-1}\right)$$

$= {}^nc_1\, x^{n-1}$

$= nx^{n-1}$

Case II: Let n be a negative integer or a fraction

$$\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{x^n\left(1+\frac{h}{x}\right)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[x^n\left\{1+n.\frac{h}{x}+\frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2+...\infty\right\}-x^n\right]$$

$$= \lim_{h\to 0}\left[x^n\left\{n.\frac{1}{x}+\frac{n(n-1)}{2!}.\frac{h}{x^2}+...\infty\right\}\right]$$

$$= x^n.n.\frac{1}{x}$$

$$= nx^{n-1}$$

$\therefore$ for all rational n, $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

4) Let $y = f(x) = e^x$

$\therefore$ $f(x+h) = e^{x+h}$

By definition, $\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\Rightarrow \frac{d}{dx}\left(e^x\right) = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x.e^h - e^x}{h}$$

$$= e^x \lim_{h\to 0}\frac{e^h-1}{h}$$

$$= e^x.1$$

$$= e^x$$

5) Let $y = f(x) = \log_e x$

$\therefore$ $f(x+h) = \log_e(x+h)$

By definition, $\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\Rightarrow \frac{d}{dx}\left(\log_e x\right) = \lim_{h\to 0}\frac{\log_e(x+h)-\log_e x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\log_e\left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\log\left(1+\frac{h}{x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}....\infty\right]$$

$$= \lim_{h\to 0} \left[\frac{1}{x}-\frac{h}{2x^2}+\frac{h^2}{3x^3}...\infty\right]$$

$$= \frac{1}{x}$$

6) Let $y = f(x) = a^x$

$\therefore$ $f(x+h)=a^{x+h}$

By definition, $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\Rightarrow \frac{d}{dx}\left(a^x\right) = \lim_{h\to 0} \frac{a^x.a^h - a^x}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^h-1}{h}$$

$$= a^x \log_e a$$

7) Let $y = f(x) = \sin x$

$\therefore$ $f(x+h) = \sin (x+h)$

By definition, $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\Rightarrow \frac{d}{dx}\left(\sin x\right) = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\ \frac{x+h+x}{2}\ \sin\frac{x+h-x}{2}}{h}$$

$$= \lim_{h\to 0} \left[2\cos\left(x+\frac{h}{2}\right).\frac{\sin\frac{h}{2}}{h}\right]$$

$$= \lim_{h\to 0}.2\cos\left(x+\frac{x}{2}\right) \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= \cos x.1 = \cos x$$

8) Let $y = f(x) = \cos x$

$\therefore\ f(x+h) = \cos(x+h)$

By definition, $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\Rightarrow \frac{d}{dx}(\cos x) = \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0} \frac{-2\sin\frac{x+h+x}{2}\sin\frac{x+h-x}{2}}{h}$$

$$= \lim_{h\to 0} \left[-2\sin\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{h}\right]$$

$$= \left(\lim_{h\to 0} -\sin\left(x+\frac{h}{2}\right)\right).\left(\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)$$

$$= -\sin x.$$

9) Let $y = f(x) = \tan x$

$\therefore\ f(x+h) = \tan(x+h)$

By definition, $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\Rightarrow \frac{d}{dx}(\tan x) = \lim_{h\to 0} \frac{\tan(x+h)-\tan x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0} \frac{\sin(x+h-x)}{h\cos(x+h)\cos x}$$

$$= \left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$= 1.\frac{1}{\cos x.\cos x}$$

$$= \sec^2 x$$

10) Let y = f(x) = cotx

$\therefore$ f(x+h) = cot (x+h)

By definition, $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\Rightarrow \frac{d}{dx}(\cot x) = \lim_{h\to 0} \frac{\cot(x+h)-\cot x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right]$$

$$= \lim_{h\to 0} \frac{\sin x\cos(x+h)-\cos x\sin(x+h)}{h\sin(x+h)\sin x}$$

$$= \lim_{h\to 0} \frac{\sin\{x-(x+h)\}}{h\sin(x+h)\sin x}$$

$$= \lim_{h\to 0} \frac{\sin(-h)}{h\sin(x+h)\sin x}$$

$$= -\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\sin(x+h).\sin x}\right)$$

$$= -1.\frac{1}{\sin x.\sin x} = -\operatorname{cosec}^2 x$$

11) Let y = f(x) = sec x

$\therefore$ f(x+h) = sec (x+h)

By definition, $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\Rightarrow \frac{d}{dx}(\sec x) = \lim_{h\to 0} \frac{\sec(x+h)-\sec x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}.\frac{\cos x-\cos(x+h)}{\cos(x+h)\cos x}$$

$$= \lim_{h\to 0} \frac{1}{h}.\frac{2\sin\frac{x+x+h}{2}\ \sin\frac{x+h-x}{2}}{\cos(x+h)\cos x}$$

$$= \lim_{h\to 0}\left[\sin\left(x+\frac{h}{2}\right).\frac{\sin\frac{h}{2}}{\frac{h}{2}}.\frac{1}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0}\sin\left(x+\frac{h}{2}\right).\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\cdot\frac{1}{\lim\{\cos(x+h).\cos x\}}$$

$$= \sin x.1.\frac{1}{\cos x.\cos x}$$

$$= \frac{1}{\cos x}.\frac{\sin x}{\cos x}$$

$$= \sec x.\tan x.$$

12) Let $y = f(x) = \operatorname{cosec} x$

$\therefore$ $f(x+h) = \operatorname{cosec}(x+h)$

By definition, $\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\Rightarrow \frac{d}{dx}(\operatorname{cosec} x) = \lim_{h\to 0}\frac{\operatorname{cosec}(x+h)-\operatorname{cosec} x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$= \lim_{h\to 0}\frac{\sin x-\sin(x+h)}{h.\sin(x+h)\sin x}$$

$$= \lim_{h\to 0}\frac{2\cos\frac{x+x+h}{2}\sin\frac{x-x-h}{2}}{h\sin(x+h)\sin x}$$

$$= \lim_{h\to 0}\left[\frac{2.\cos\left(x+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{h\sin(x+h)\sin x}\right]$$

$$= \lim_{h\to 0}\left[-\frac{\cos\left(x+\frac{h}{2}\right)}{\sin(x+h).\sin x}.\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]$$

$$= -\lim_{h\to 0}\frac{\cos\left(x+\frac{h}{2}\right)}{\sin(x+h).\sin x}.\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= -\frac{\cos x}{\sin x.\sin x}.1 = -\operatorname{cosec} x\cot x$$

13) Let $y = \sin^{-1}x$

$\therefore \sin y = x$

$\therefore \frac{d}{dy}(\sin y) = \frac{d}{dy}(x)$

$\Rightarrow \cos y = \frac{dx}{dy}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$ where $\cos y \neq 0$

$\Rightarrow \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-\sin^2 y}}$ $\left[\because \cos y > 0\right]$

$= \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$

14) Let $y = \cos^{-1}x$

$\therefore \cos y = x$

$\Rightarrow \frac{d}{dy}(\cos y) = \frac{d}{dy}(x)$

$\Rightarrow -\sin y = \frac{dx}{dy}$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y}$, where $\sin y \neq 0$

$\Rightarrow \frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-\cos^2 y}}$

$= \frac{-1}{\sqrt{1-x^2}}$, $-1 < x < 1$

15) Let $y = \tan^{-1}x$

$\Rightarrow \tan y = x$

$\therefore \frac{d}{dy}(\tan y) = \frac{dx}{dy}$

$\Rightarrow \sec^2 y = \frac{dx}{dy}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$ $\left[\sec y \neq 0\right]$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\tan^2 y}$$

$$\Rightarrow \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2};\ \ x \in R$$

16) Let $y = \cot^{-1}x$

$$\Rightarrow \cot y = x$$

$$\Rightarrow \frac{d}{dy}(\cot y) = \frac{d}{dy}(x)$$

$$\Rightarrow -\operatorname{cosec}^2 y = \frac{dx}{dy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\operatorname{cosec}^2 y} \ \ [\operatorname{cosec} y \neq 0]$$

$$\Rightarrow \frac{d}{dx}\left(\cot^{-1}x\right) = \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2},\ x \in R$$

17) Let $y = \sec^{-1}x$

$$\Rightarrow \sec y = x$$

$$\therefore \frac{d}{dy}(\sec y) = \frac{d}{dy}(x)$$

$$\Rightarrow \sec y \tan y = \frac{dx}{dy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y} \ \ , \sec y \neq 0, \tan y \neq 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y\left(\pm\sqrt{\sec^2 y - 1}\right)}$$

$$\Rightarrow \frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{\pm x\sqrt{x^2-1}}$$

$$= \begin{cases} \dfrac{1}{x\sqrt{x^2-1}} & \text{for } x > 1, \\ \dfrac{1}{-x\sqrt{x^2-1}} & \text{for } x < -1, \end{cases}$$

$$= \frac{1}{|x|\sqrt{x^2-1}} \ ; |x| > 1$$

18) Let $y = \text{cosec}^{-1}x$

$$\Rightarrow \text{cosec}\, y = x$$

$$\therefore \frac{d}{dy}(\text{cosec}\, y) = \frac{d}{dy}(x)$$

$$\Rightarrow -\text{cosec}\, y \cot y = \frac{dx}{dy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\text{cosec}\, y . \cot y}, \text{cosec}\, y \neq 0, \cot y \neq 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\text{cosec}\, y\left(\pm\sqrt{\text{cosec}^2 y - 1}\right)}$$

$$\Rightarrow \frac{d}{dx}\left(\text{cosec}^{-1}x\right) = \frac{-1}{\pm x\sqrt{x^2-1}}$$

$$= \begin{cases} \dfrac{-1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ \dfrac{-1}{-x\sqrt{x^2-1}} & \text{if } x < -1 \end{cases}$$

$$= \frac{-1}{|x|\sqrt{x^2-1}};\ |x| > 1$$

5.5 THE DERIVATIVE OF SUM AND DIFFERENCE OF FUNCTIONS

Theorem 5.5.1: If f_1 and f_2 are two differentiable functions on the interval I, and f be their sum function so that $f = f_1 + f_2$, Then f is differentiable and $\frac{d}{dx}f = \frac{d}{dx}f_1 + \frac{d}{dx}f_2$

Proof: Let x be any point of I and x+h be a neighbouring point.

Then, $f(x) = f_1(x) + f_2(x)$

and $f(x+h) = f_1(x+h) + f_2(x+h)$

$\therefore f(x+h)-f(x) = f_1(x+h) + f_2(x+h) - f_1(x) - f_2(x)$

$$\Rightarrow \frac{f(x+h)-f(x)}{h} = \frac{f_1(x+h)-f_1(x)}{h} + \frac{f_2(x+h)-f_2(x)}{h}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f_1(x+h)-f_1(x)}{h} \lim_{h\to 0}\frac{f_2(x+h)-f_2(x)}{h} \quad \text{.....(1)}$$

$\because f_1$ and f_2 are differential in I

$$\therefore \frac{df_1}{dx} = \lim_{h\to 0} \frac{f_1(x+h)-f_1(x)}{h}$$

and $\frac{df_2}{dx} = \lim_{h\to 0} \frac{f_2(x+h)-f_2(x)}{h}$ exist.

$$\therefore (1) \Rightarrow \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x)$$

This shows that f is differentiable

and $\frac{d}{dx} f(x) = \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x)$

i.e., $\frac{d}{dx} f = \frac{d}{dx} f_1 + \frac{d}{dx} f_2$

Theorem 5.5.2: If f_1 and f_2 are two differentiable functions on the interval I, and f be their difference function so that $f=f_1-f_2$ then f is differentiable and $\frac{d}{dx} f = \frac{d}{dx} f_1 - \frac{d}{dx} f_2$

Proof: Try yourself.

Illustrative Example:

Example 1: Find the derivatives of the following with respect to x.

a) $x(1+x)^2$ b) $b\log_e x + x^3 + \sqrt{x}$

c) $4\log_{10} x$ d) $\sqrt[3]{1+3\log_a x+3(\log_a x)^2+(\log_a x)^3}$

Solution:

a) $$\frac{d}{dx}\left\{x(1+x)^2\right\} = \frac{d}{dx}\left\{x\left(1+2x+x^2\right)\right\}$$

$$= \frac{d}{dx}\left(x+2x^2+x^3\right)$$

$$= \frac{d}{dx}(x) + 2\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(x^3\right)$$

$$= 1 + 2(2x) + 3x^2$$

$$= 1 + 4x + 3x^2$$

b) $$\frac{d}{dx}\left(b\log_e x + x^3 + \sqrt{x}\right) = b\frac{d}{dx}(\log_e x) + \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= b.\frac{1}{x} + 3x^2 + \frac{1}{2}x^{\frac{1}{2}-1}$$

$$= \frac{b}{x} + 3x^2 + \frac{1}{2\sqrt{x}}$$

c) $\frac{d}{dx}(4\log_{10} x) = \frac{d}{dx}\left(\frac{4\log_e x}{\log_e 10}\right)$

$= \frac{4}{\log_e 10}\frac{d}{dx}(\log_e x)$

$= 4\log_{10} e\,\frac{1}{x}$

$= \frac{4\log_{10} e}{x}$

d) $\frac{d}{dx}\sqrt[3]{1+3\log_a x+3\left(\log_a x\right)^2+\left(\log_a x\right)^3} = \frac{d}{dx}\left\{\left(1+\log_a x\right)^3\right\}^{\frac{1}{3}}$

$= \frac{d}{dx}\left\{1+\frac{\log_e x}{\log_e a}\right\} = \frac{d}{dx}(1)+\frac{1}{\log_e a}\frac{d}{dx}\left(\log_e x\right)$

$= 0+\frac{1}{\log_e a}.\frac{1}{x} = \frac{1}{x}\log_a e$

Example 2: Find the derivatives of y with respect to x for the following

a) $y = \frac{9}{\sin x} - 3\cot x + a^x$ b) $y = \tan x - e^x + 4$

c) $y = 3\log x + x^7 - 4\sec x$

Solution:

a) $y = \frac{9}{\sin x} - 3\cot x + a^x$

$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{9}{\sin x} - 3\cot x + a^x\right)$

$= 9\frac{d}{dx}(\operatorname{cosec} x) - 3\frac{d}{dx}(\cot x) + \frac{d}{dx}(a^x)$

$= 9(-\operatorname{cosec} x.\cot x) - 3(-\operatorname{cosec}^2 x) + a^x \log_e a$

$= -9\operatorname{cosec} x.\cot x + 3\operatorname{cosec}^2 x + a^x \log_e a$

b) $y = \tan x - e^x + 4$

$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\tan x - e^x + 4\right)$

$= \frac{d}{dx}(\tan x) - \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}(4)$

$= \sec^2 x - e^x$

c) $y = 3\log x + x^7 - 4\sec x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(3\log x + x^7 - 4\sec x\right)$$

$$= 3\frac{d}{dx}(\log x) + \frac{d}{dx}\left(x^7\right) - 4\frac{d}{dx}(\sec x)$$

$$= \frac{3}{x} + 7x^6 - 4\sec x \tan x$$

Example 3: Find the point on the curve $y = x^2 - 3x$ at which the tangent is parallel to the x–axis.

Solution: Given $y = x^2 - 3x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^3 + 2x\right) = 2x - 3$$

For tangents parallel to x – axis, we have

$$\frac{dy}{dx} = 0 \Rightarrow 2x - 3 = 0 \Rightarrow x = \frac{3}{2}$$

$$\therefore y = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) = \frac{9}{4} - \frac{9}{2} = \frac{-9}{4}$$

$\therefore$ The point is $\left(\frac{3}{2}, \frac{-9}{4}\right)$



CHECK YOUR PROGRESS

Q.1: Find the derivative of the following functions with respect to x.

a) $\dfrac{x^3 + x^2 - x + 2}{x}$

b) $\sin x + e^x - \dfrac{1}{x^4}$

c) $\operatorname{cosec} x + a^x$

d) $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2 + 2$

e) $\log(a^2 x^3 e^x)$

f) $3\sin^{-1}x + \cos x - a^x + 1$

Q.2: Prove that no tangent to the curve $y = x^3 + 2x$ is parallel to the x-axis.

5.6 DERIVATIVE OF PRODUCT OF FUNCTIONS AND QUOTIENT RULE

Theorem 5.6.1: Let u(x) and v(x) be two differentiable functions. Then the product u(x)v(x) is also differentiable and

$$\frac{d}{dx}\big(u(x)v(x)\big) = u(x)v'(x) + v(x)u'(x)$$

Proof: Let x be any point in the domain, and x+h be a neighbouring point.

f(x)=u(x)v(x)

$\therefore$ f(x+h) = u(x+h)v(x+h)

$\therefore$ f(x+h) – f(x) = u(x+h)v(x+h) – u(x)v(x)

= u(x+h)v(x+h) – u(x+h)v(x) + u(x+h)v(x) – u(x)v(x)

= u(x+h){v(x+h) – v(x)} + v(x){u(x+h) – u(x)}

$$\therefore \frac{f(x+h)-f(x)}{h} = u(x+h).\frac{v(x+h)-v(x)}{h} + v(x)\frac{u(x+h)-u(x)}{h}$$

$$\therefore \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} u(x+h).\lim_{h\to 0} \frac{v(x+h)-v(x)}{h} + v(x)\lim_{h\to 0} \frac{u(x+h)-u(x)}{h}$$

$$= u(x).\lim_{h\to 0} \frac{v(x+h)-v(x)}{h} + v(x).\lim_{h\to 0} \frac{u(x+h)-u(x)}{h} \quad .....(1)$$

$\because$ u and v are differentiable.

$$\therefore u'(x) = \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} \text{ and } v'(x) = \lim_{h\to 0} \frac{v(x+h)-v(x)}{h}$$

$$\therefore (1) \Rightarrow \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = u(x)v'(x) + v(x)u'(x)$$

Thus f is differentable at x and

$f'(x) = u(x)v'(x) + v(x)u'(x)$

i.e., $\frac{d}{dx}(f(x)) = u(x)v'(x) + v(x)u'(x)$

i.e., $\frac{d}{dx}u(x)v(x) = u(x)v'(x) + v(x)u'(x)$.

Corollary: If u, v, w are three differentable functions, then

$$\frac{d}{dx}(u(x)v(x)w(x)) = u'(x)v(x)w(x) + v'(x)u(x)w(x) + w'(x)u(x)v(x)$$

Theorem 5.6.2: Let u(x) and v(x) be two differentiable functions in the same interval, then the quotient function $\frac{u(x)}{v(x)}$ is also differentiable and

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right)=\frac{v(x)\frac{d}{dx}(u(x))-u(x)\frac{d}{dx}(v(x))}{(v(x))^2}$$

$$= \frac{v(x)u'(x)-u(x)v'(x)}{(v(x))^2}$$

Proof: Let $f(x) = \frac{u(x)}{v(x)}$

$$\therefore f(x+h)=\frac{u(x+h)}{v(x+h)}$$

$$\therefore f(x+h)-f(x)=\frac{u(x+h)}{v(x+h)}-\frac{u(x)}{v(x)}$$

$$= \frac{v(x)u(x+h)-u(x)v(x+h)}{v(x+h)v(x)}$$

$$\Rightarrow \frac{f(x+h)-f(x)}{h}=\frac{v(x)u(x+h)-v(x)u(x)+v(x)u(x)-u(x)v(x+h)}{h.v(x+h)v(x)}$$

$$= \frac{v(x)[u(x+h)-u(x)]+u(x)[v(x)-v(x+h)]}{h.v(x+h)v(x)}$$

$$= \frac{v(x).\frac{[u(x+h)-u(x)]}{h}-u(x)\frac{[v(x+h)-v(x)]}{h}}{v(x+h)v(x)}$$

$$\therefore \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \frac{v(x).\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}-u(x).\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}}{v(x).v(x)} \quad .....(1)$$

$\because$ u(x) and v(x) are differentiable

$$\therefore u'(x) = \lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

and $$v'(x) = \lim_{h\to 0}\frac{v(x+h)-v(x)}{h}$$

$$\therefore (1) \Rightarrow \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{v(x)u'(x)-u(x)v'(x)}{\{v(x)\}^2}$$

Thus, f(x) is differentiable at x and $f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{\{v(x)\}^2}$

i.e. $\frac{d}{dx}(f(x)) = \frac{v(x)u'(x) - u(x)v'(x)}{\{v(x)\}^2}$

i.e. $\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$

Illustrative Examples:

Example 1: Find the derivatives of the following functions with respect to x.

a) x^2e^x b) $e^x\log x$ c) $x^3e^x\sin x$

Solution:

a) $\frac{d}{dx}(x^2e^x) = e^x\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(e^x)$

$= e^x.2x + x^2.e^x$

$= xe^x(2+x)$

b) $\frac{d}{dx}(e^x\log x) = \log x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(\log x)$

$= \log x.e^x + e^x.\frac{1}{x}$

$= e^x\left(\log x + \frac{1}{x}\right)$

c) $\frac{d}{dx}(x^3e^x\sin x) = e^x\sin x\frac{d}{dx}(x^3) + x^3\sin x\frac{d}{dx}(e^x) + x^3e^x\frac{d}{dx}(\sin x)$

$= e^x\sin x.3x^2 + x^3\sin x.e^x + x^3e^x\cos x$

$= x^2e^x(3\sin x + x\sin x + x\cos x)$

Example 2: Find the derivatives of the following with respect to x.

a) $\frac{\cos x}{\log_a x}$ b) $\frac{x^n}{\log_e x}$ c) $\frac{x^2}{e^x}$

Solution: a) $\frac{d}{dx}\left(\frac{\cos x}{\log_a x}\right) = \frac{\log_a x\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(\log_a x)}{(\log_a x)^2}$

$= \frac{\log_a x(-\sin x) - \cos x.\frac{d}{dx}\left(\frac{\log_e x}{\log_e a}\right)}{(\log_a x)^2}$

$$= \frac{-\sin x \log_a x - \cos x . \frac{1}{\log_e a} . \frac{1}{x}}{(\log_a x)^2}$$

$$= \frac{-\sin x \log_a x - \cos x \log_a e}{x(\log_a x)^2}$$

b) $$\frac{d}{dx}\left(\frac{x^n}{\log_e x}\right) = \frac{\log_e x \frac{d}{dx}(x^n) - x^n \frac{d}{dx}(\log_e x)}{(\log_e x)^2}$$

$$= \frac{\log_e x . n x^{n-1} - x^n . \frac{1}{x}}{(\log_e x)^2}$$

$$= \frac{n x^{n-1} \log_e x - x^{n-1}}{(\log_e x)^2}$$

c) $$\frac{d}{dx}\left(\frac{x^2}{e^x}\right) = \frac{e^x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(e^x)}{(e^x)^2}$$

$$= \frac{e^x . 2x - x^2 . e^x}{(e^x)^2}$$

$$= \frac{x e^x (2 - x)}{(e^x)^2} = \frac{x(2-x)}{e^x}$$

Example 3: Find the derivatives of the following functions with respect to x.

a) $3x^2+x^3\log x$ b) $\frac{x^2 + x - 1}{x^2 - x + 1}$

c) $\frac{\sec x}{x + e^x}$ d) $x^3 \sin x + \frac{e^x + 1}{\log x}$

Solution: a) $\frac{d}{dx}(3x^2+x^3\log x) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(x^3 \log x)$

$$= 3\frac{d}{dx}(x^2) + \log x \frac{d}{dx}(x^3) + \frac{d}{dx} x^3 (\log x)$$

$$= 3(2x) + \log x\,(3x^2) + x^3 . \frac{1}{x}$$

$$= 6x + 3x^2 \log x + x^2$$

b) $\frac{d}{dx}\left(\frac{x^2+x-1}{x^2-x+1}\right)$

$$= \frac{(x^2-x+1)\frac{d}{dx}(x^2+x-1)-(x^2+x-1)\frac{d}{dx}(x^2-x+1)}{(x^2-x+1)^2}$$

$$= \frac{(x^2-x+1)(2x+1)-(x^2+x-1)(2x-1)}{(x^2-x+1)^2}$$

$$= \frac{2x^3-2x^2+2x+x^2-x+1-2x^3-2x^2+2x-x^2-x+1}{(x^2-x+1)^2}$$

$$= \frac{-4x^2+2x+2}{(x^2-x+1)^2}$$

c) $\frac{d}{dx}\left(\frac{\sec x}{x+e^x}\right)=\frac{(x+e^x)\frac{d}{dx}(\sec x)-\sec x\frac{d}{dx}(x+e^x)}{(x+e^x)^2}$

$$= \frac{(x+e^x).\sec x\tan x-\sec x(1+e^x)}{(x+e^x)^2}$$

$$= \frac{\sec x\{(x+e^x)\tan x-1-e^x\}}{(x+e^x)^2}$$

d) $\frac{d}{dx}\left(x^3\sin x+\frac{e^x+1}{\log x}\right)$

$$= \frac{d}{dx}(x^3\sin x)+\frac{d}{dx}\left(\frac{e^x+1}{\log x}\right)$$

$$= \sin x\frac{d}{dx}(x^3)+x^3\frac{d}{dx}\sin x+\frac{\log x\frac{d}{dx}(e^x+1)-(e^x+1)\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \sin x.3x^2 + x^3.\cos x + \frac{\log x.(e^x+0)-(e^x+1).\frac{1}{x}}{(\log x)^2}$$

$$= x^2(3\sin x + x\cos x) + \frac{e^x(x\log x)-e^x-1}{x(\log x)^2}$$

Example 4: If $y = \frac{x}{x+5}$, prove that $x\frac{dy}{dx} = y(1-y)$

Solution: We have $y = \frac{x}{x+5}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{x+5}\right) = \frac{(x+5)\frac{d}{dx}(x) - x\frac{d}{dx}(x+5)}{(x+5)^2}$$

$$= \frac{(x+5).1 - x(1+0)}{(x+5)^2}$$

$$= \frac{x+5-x}{(x+5)^2}$$

$$\therefore x\frac{dy}{dx} = \frac{5x}{(x+5)^2} \qquad .....(1)$$

Also, $y\,(1-y) = \frac{x}{x+5}\left(1 - \frac{x}{x+5}\right)$

$$= \frac{x}{x+5}.\frac{x+5-x}{x+5}$$

$$= \frac{5x}{(x+5)^2} \qquad .....(2)$$

From (1) and (2), we get $x\frac{dy}{dx} = y(1-y)$



CHECK YOUR PROGRESS

Q.3: Find the derivatives of the following functions with respect to x.

a) $\frac{2x+3}{3x+4}$ b) $\frac{1+\tan x}{1-\tan x}$ c) $\frac{x}{a^2+x^2}$

d) $7xa^x + \log x^x$ e) $3\log x \tan x - x^3e^x + 2$

f) $\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$ g) $\log (x^3\, e^{4x}\, a^x)$

Q.4: If $y = \frac{\sin x}{1+\cos x}$ Prove that $\frac{dy}{dx} = \frac{1}{1+\cos x}$

Q.5: If $f(x) = \frac{x-2}{2\sqrt{x}}$, find $f'(4)$.



5.7 LET US SUM UP

- $\frac{dy}{dx}$ at a point on a given curve y = f(x) represents the gradient of the tangent line at that point.
- For tangents parallel to x–axis, $\frac{dy}{dx} = 0$,
- $\frac{d}{dx}(k) = 0 \ \ (k = \text{cons tan t})$

  $\frac{d}{dx}(x^n) = nx^{n-1}$

  $\frac{d}{dx}(e^x) = e^x$

  $\frac{d}{dx}(\log x) = \frac{1}{x}$

  $\frac{d}{dx}(a^x) = a^x \log_e a$

  $\frac{d}{dx}(\sin x) = \cos x$

  $\frac{d}{dx}(\cos x) = -\sin x$

  $\frac{d}{dx}(\tan x) = \sec^2 x$

  $\frac{d}{dx}(\cot x) = -\text{cos ec}^2 x$

  $\frac{d}{dx}(\sec x) = \sec x.\tan x$

  $\frac{d}{dx}(\text{cos ec}x) = -\text{cos ec}x.\cot x$

  $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}; -1 < x < 1$

  $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}; -1 < x < 1$

  $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}; x \in \Re$

$$\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2};\ x \in \Re$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}};\ |x| > 1$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}};\ |x| > 1$$

- $\frac{d}{dx}(f_1 \pm f_2) = \frac{df_1}{dx} \pm \frac{df_2}{dx}$
- $\frac{d}{dx}(uv) = v.\frac{du}{dx} + u.\frac{dv}{dx}$
- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} + u.\frac{dv}{dx}}{v^2}, v \neq 0$

5.8 ANSWERS TO CHECK YOU PROGRESS

Ans. to Q. No. 1: a) $\frac{d}{dx}\left(\frac{x^3+x^2+x+2}{x}\right)$

$$= \frac{d}{dx}\left(\frac{x^3}{x} + \frac{x^2}{x} + \frac{x}{x} + \frac{2}{x}\right)$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(x) - \frac{d}{dx}(1) + 2\frac{d}{dx}(x^{-1})$$

$$= 2x + 1 - 0 + 2(-1)x^{-2}$$

$$= 2x + 1 - \frac{2}{x^2}$$

b) $\frac{d}{dx}\left(\sin x + e^x - \frac{1}{x^4}\right) = \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x) - \frac{d}{dx}(x^{-4})$

$$= \cos x + e^x - (-4)x^{-4-1}$$

$$= \cos x + e^x + \frac{4}{x^5}$$

c) $\frac{d}{dx}(\operatorname{cosec} x + a^x) = \frac{d}{dx}(\operatorname{cosec} x) + \frac{d}{dx}(a^x)$

$$= -\operatorname{cosec} x \cot x + a^x \log ea$$

d) $\frac{d}{dx}\left\{\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2+2\right\}=\frac{d}{dx}\left(x+\frac{1}{x}+2+2\right)$

$= \frac{d}{dx}(x)+\frac{d}{dx}\left(x^{-1}\right)+\frac{d}{dx}(4)$

$= 1 + (-1)x^{-1-1} + 0$

$= 1-\frac{1}{x^2}$

e) $\frac{d}{dx}\left\{\log\left(a^2x^3e^x\right)\right\}=\frac{d}{dx}\left(\log a^2+\log x^3+\log e^x\right)$

$= \frac{d}{dx}(2\log a+3\log x+x\log e)$

$= \frac{d}{dx}(2\log a)+3\frac{d}{dx}(\log x)+\frac{d}{dx}(x) \quad (\because \log e = 1)$

$= 0 + \frac{3}{x} + 1$

$= 1 + \frac{3}{x}$

f) $\frac{d}{dx}\left(3\sin^{-1}x+\cos x-a^x+1\right)$

$= 3\frac{d}{dx}\left(\sin^{-1}x\right)+\frac{d}{dx}(\cos x)-\frac{d}{dx}\left(a^x\right)+\frac{d}{dx}(1)$

$= \frac{3}{\sqrt{1-x^2}} + (-\sin x) - a^x\log a + 0$

$= \frac{3}{\sqrt{1-x^2}} - \sin x - ax\log a$

Ans. to Q. No. 2: Given, $y = x^3 + 2x$

$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(x^3+2x\right)$

$= \frac{d}{dx}\left(x^3\right)+2\frac{d}{dx}(x)$

$= 3x^2 + 2$

$\neq 0$ for $x \in \Re$

$\therefore$ There does not exist any value of x for which $\frac{dy}{dx}=0$

$\therefore$ No tangent to the curve $y = x^3 + 2x$ will be parallel to x-asis.

Ans. to Q. No. 3: a) $\frac{d}{dx}\left(\frac{2x+3}{3x+4}\right)=\frac{(3x+4)\frac{d}{dx}(2x+3)-(2x+3)\frac{d}{dx}(3x+4)}{(3x+4)^2}$

$$= \frac{(3x+4)(2+0)-(2x+3)(3+0)}{(3x+4)^2}$$

$$= \frac{(6x+8)-(6x+9)}{(3x+4)^2}$$

$$= \frac{-1}{(3x+4)^2}$$

b) $\frac{d}{dx}\left(\frac{1+\tan x}{1-\tan x}\right)=\frac{(1-\tan x)\frac{d}{dx}(1+\tan x)-(1+\tan x)\frac{d}{dx}(1-\tan x)}{(1-\tan x)^2}$

$$= \frac{(1-\tan x)(\sec^2 x)-(1+\tan x)(-\sec^2 x)}{(1-\tan x)^2}$$

$$= \frac{\sec^2 x(1-\tan x+1+\tan x)}{(1-\tan x)^2}$$

$$= \frac{2\sec^2 x}{(1-\tan x)^2}$$

c) $\frac{d}{dx}\left(\frac{x}{a^2+x^2}\right)=\frac{(a^2+x^2)\frac{d}{dx}(x)-x\frac{d}{dx}(a^2+x^2)}{(a^2+x^2)^2}$

$$= \frac{(a^2+x^2).1-x(2x)}{(a^2+x^2)^2}$$

$$= \frac{a^2+x^2-2x^2}{(a^2+x^2)^2}$$

$$= \frac{a^2-x^2}{(a^2+x^2)^2}$$

d) $\frac{d}{dx}(7xa^x+\log x^x) = \frac{d}{dx}(7xa^x+x\log x)$

$$= 7\frac{d}{dx}(xa^x)+\frac{d}{dx}(x\log x)$$

$$= 7\left[a^x\frac{d}{dx}(x)+x\frac{d}{dx}(a^x)\right]+\left[\log x\frac{d}{dx}(x)+x\frac{d}{dx}(\log x)\right]$$

$$= 7\left[a^x.1 + x.a^x \log a_e\right] + \left[\log x.1 + x.\frac{1}{x}\right]$$

$$= 7a^x(1 + x \log a_e) + \log x + 1$$

e) $\frac{d}{dx}\left[3\log x \tan x - x^3 e^x + 2\right]$

$$= 3\frac{d}{dx}(\log x \tan x) - \frac{d}{dx}\left(x^3 e^x\right) + \frac{d}{dx}(2)$$

$$= 3\left[\tan x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\tan x)\right] - \left[e^x \frac{d}{dx}\left(x^3\right) + x^3 \frac{d}{dx}\left(e^x\right)\right] + 0$$

$$= 3\left[\tan x.\frac{1}{x} + \log x.\sec^2 x\right] - \left[e^x.3x^2 + x^3.e^x\right]$$

$$= 3\left[\frac{\tan x}{x} + \log x \sec^2 x\right] - x^2 e^x[3 + x]$$

f) $\frac{d}{dx}\left[\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}\right] = \frac{\left(\sqrt{a}-\sqrt{x}\right)\frac{d}{dx}\left(\sqrt{a}+\sqrt{x}\right) - \left(\sqrt{a}+\sqrt{x}\right)\frac{d}{dx}\left(\sqrt{a}-\sqrt{x}\right)}{\left(\sqrt{a}-\sqrt{x}\right)^2}$

$$= \frac{\left(\sqrt{a}-\sqrt{x}\right)\left(0 + \frac{1}{2\sqrt{x}}\right) - \left(\sqrt{a}+\sqrt{x}\right)\left(0 - \frac{1}{2\sqrt{x}}\right)}{\left(\sqrt{a}-\sqrt{x}\right)^2}$$

$$= \frac{\frac{1}{2\sqrt{x}}\left(\sqrt{a}-\sqrt{x}+\sqrt{a}+\sqrt{x}\right)}{\left(\sqrt{a}-\sqrt{x}\right)^2}$$

$$= \frac{2\sqrt{a}}{2\sqrt{x}\left(\sqrt{a}-\sqrt{x}\right)^2}$$

$$= \frac{\sqrt{a}}{\sqrt{x}\left(\sqrt{a}-\sqrt{x}\right)^2}$$

g) $\frac{d}{dx}\left[\log\left(x^3 e^{4x} a^x\right)\right] = \frac{d}{dx}\left[\log x^3 + \log e^{4x} + \log a^x\right]$

$$= \frac{d}{dx}\left[3\log x + 4x \log e + x \log a\right]$$

$$= 3\frac{d}{dx}(\log x) + 4\frac{d}{dx}(x) + \log a \frac{d}{dx}(x) \left[\because \log e = 1\right]$$

$$= \frac{3}{x} + 4 + \log a$$

Ans. to Q. No. 4: $y = \dfrac{\sin x}{1+\cos x} \quad \therefore \dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\sin x}{1+\cos x}\right)$

$$= \frac{(1+\cos x)\frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x + 1}{(1+\cos x)^2}$$

$$= \frac{1}{1+\cos x}$$

Ans. to Q. No. 5: Given, $f(x) = \dfrac{x-2}{2\sqrt{x}}$

$$= \frac{\sqrt{x}}{2} - \frac{1}{\sqrt{x}}$$

$$\therefore f'(x) = \frac{d}{dx}\left(\frac{\sqrt{x}}{2} - \frac{1}{\sqrt{x}}\right)$$

$$= \frac{1}{2}\frac{d}{dx}\left(x^{\frac{1}{2}}\right) - \frac{d}{dx}\left(x^{-\frac{1}{2}}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2}x^{\frac{1}{2}-1} - \left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right)$$

$$= \frac{1}{4\sqrt{x}} + \frac{1}{2x\sqrt{x}}$$

$$\therefore f'(4) = \frac{1}{4(2)} + \frac{1}{2(4)(2)}$$

$$= \frac{1}{8} + \frac{1}{16}$$

$$= \frac{3}{16}$$



5.9 FURTHER READING

1) Gupta, Sudhir (2011) (1st edition): *Basics of Differentential Calculus*; New Delhi: Anmol publication Pvt. Ltd.
2) Habib, Amber (2011) (1st edition): *The Calculus of Finance*; Hyderabad: Universities Press India Pvt. Ltd.
3) Maron, I. A. (1993) (1st edition): *Problems in Calculus of One Variable*; New Delhi: CBS Publishers & Distributors Pvt. Ltd.
4) Sharma, A. K. (2004) (1st edition): *Text Book of Differential Calculus for B.A., B.Sc., B.Com., I.A.S., P.C.S.*; New Delhi: Discovery Publishing House.



5.10 MODEL QUESTIONS

Q.1: Differentiate the following with respect to x

a) $\frac{1}{3x+5}$ b) $\sqrt{ax+b}$ c) $\sqrt{\sec x}$

d) $\sqrt{\log x}$ (e) $\frac{x^4+8x+5}{x}$

f) $2e^{3x}+\tan x-\cos 2x+9\sin^{-1}x$ g) $\left(\sqrt{x}+1\right)\left(\frac{1}{x}+\frac{1}{\sqrt{x}}\right)$

Q.2: Find the differential coefficient of the following with respect to x

a) $\frac{1-x^2}{1+x^2}$ b) $\frac{1-\cos x}{1+\cos x}$

c) $\frac{\sin x+\cos x}{\sin x-\cos x}$ d) $\frac{a+b\cos x}{a-b\cos x}$

Q.3: If $y=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$, then Prove that $\frac{dy}{dx}=y$.

Q.4: Find the differential coefficient of $y=\cos x\log x+e^x\operatorname{cosec} x$ with respect to x.

Q.5: Find the differential coefficient of $y=\frac{e^x\log x}{\sin x+\cos x}$ with respect to x.

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UNIT 6: DERIVATIVE OF SOME SPECIAL FUNCTIONS

UNIT STRUCTURE

6.1 Learning Objectives

6.2 Introduction

6.3 Derivative of Composite Functions

6.4 Derivative of Logarithm Functions and Implicit Functions

6.5 Let us Sum up

6.6 Answers to Check Your Progress

6.7 Further Reading

6.8 Model Questions

6.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- determine derivatives of composite functions (function of functions)
- evaluate derivative of implicit functions.

6.2 INTRODUCTION

We have already introduced the concept of derivative of functions in unit 5. In this unit, we will discuss derivative of composite ,logarithmic and implicit functions.

6.3 DERIVATIVE OF COMPOSITE FUNCTIONS

If $y = \phi(t)$ and $t = f(x)$, then $y = \phi(f(x))$ is a composite function. Examples of composite functions are $\cos 2x$, $\sin x^2$, $\log \cos x$, $e^{\sin x}$ etc.

Theorem 6.3.1: Let $y = \phi(t)$ when $t = f(x)$. Then $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

Proof: Let k be the increment in t corresponding to the increment h in x.

$\because t = f(x)$ .....(1)

$\therefore t + k = f(x + h)$ .....(2)

(2) – (1) $\Rightarrow$ k = f(x+h) – f(x) .....(3)

Also, $h \to 0$ gives $t+k \to t$ or $k \to 0$

Now, $\phi(f(x)) = \phi(t) \Rightarrow \phi(f(x+h)) = \phi(t+k)$

$\therefore \phi(f(x+h)) - \phi(f(x)) = \phi(t+k) - \phi(t)$

$$\Rightarrow \frac{\phi(f(x+h)) - \phi(f(x))}{h} = \frac{\phi(t+k) - \phi(t)}{k} \cdot \frac{k}{h}$$

$$= \frac{\phi(t+k) - \phi(t)}{k} \cdot \frac{f(x+h) - f(x)}{h}, \text{ by (3)}$$

Taking limit on both sides, we get

$$\lim_{h\to 0} \frac{\phi(f(x+h)) - \phi(f(x))}{h} = \lim_{h\to 0} \frac{\phi(t+k) - \phi(t)}{k} \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(\phi(f(x))) = \frac{d}{dt}(\phi(t)) \cdot \frac{d}{dx}(f(x))$$

$$\Rightarrow \frac{d}{dx}(\phi(t)) = \frac{d}{dt}(y) \cdot \frac{d}{dx}(t)$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Generalisation of above theorem:

Let y = f(u), u = g(t), t = h(x) then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dt} \cdot \frac{dt}{dx}$

Thus, if the composite function y is a function of many functions,

then $\frac{dy}{dx} = \frac{d(1^{st} \text{ function})}{d(2^{nd} \text{ function})} \cdot \frac{d(2^{nd} \text{ function})}{d(3^{rd} \text{ function})} \cdots \frac{d(\text{last function})}{dx}$

Above is also known as **chain Rule**.

Illustrative Examples:

Example 1: Find the derivatives of the following functions with respect to x.

a) $\cos x^3$ b) $e^{\sin x^2}$ c) $\sec^3 3x$.

Solution: a) Let $t = x^3$ $\therefore \frac{dt}{dx} = 3x^2$

$$\therefore \frac{d}{dx}(\cos x^3) = \frac{d}{dx}(\cos t)$$

$$= \frac{d}{dt}(\cos t) \cdot \frac{dt}{dx}$$

$$= -\sin t . 3x^2 = -3x^2 \sin x^3$$

b) Let $y = e^{\sin x^2}$

$= e^{\sin t}$, where $t = x^2$

$= e^u$, where $u = \sin t$

By chain rule, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dt}.\frac{dt}{dx} = \frac{d}{du}(e^u)\frac{d}{dt}(\sin t).\frac{d}{dx}(x^2)$

$= e^u \cos t.2x$

$= 2xe^{\sin t}.\cos t$

$= 2xe^{\sin x^2}\cos x^2$

c) Let $y = \sec^3 3x$

$= \sec^3 u$, where $u = 3x$

$= t^3$, where $t = \sec u$

By chain rule, $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{du}.\frac{du}{dx} = \frac{d}{dt}(t^3).\frac{d}{du}(\sec u).\frac{d}{dx}(3x)$

$= 3t^2 \sec u \tan u.3$

$= 9\sec^2 u.\ \sec u \tan u$

$= 9\sec^3 3x \tan 3x.$

Example 2: Find the derivatives of the following functions with respect to x.

a) $y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ b) $y = \log\log\log x^2$

c) $y = \cos^{-1} x^2$ d) $y = a\cot^{-1}(m\tan^{-1} bx)$

Solution: a) Let $t = \sqrt{x} + \frac{1}{\sqrt{x}}$ $\therefore$ $y = \log t$

By chain rule, $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{d}{dt}(\log t).\frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

$= \frac{1}{t}\left(\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}\right)$

$= \frac{1}{\sqrt{x} + \frac{1}{\sqrt{x}}}.\frac{x-1}{2x\sqrt{x}}$

$= \frac{\sqrt{x}}{x+1}.\frac{x-1}{2x\sqrt{x}}$

$= \frac{x-1}{2x(x+1)}$

b) $y = \log \log \log x^2$

Let $v = \log \log x^2$

$w = \log x^2$

$t = x^2$

$$\therefore \frac{dy}{dx} = \frac{dy}{dv}.\frac{dv}{dw}.\frac{dw}{dt}.\frac{dt}{dx}$$

$$= \frac{d}{dv}(\log v).\frac{d}{dw}(\log w).\frac{d}{dt}(\log t).\frac{d}{dx}(x^2)$$

$$= \frac{1}{v}.\frac{1}{w}.\frac{1}{t}.2x$$

$$= \frac{2x}{\log\log x^2.\log x^2.x^2}$$

$$= \frac{2}{x(\log\log x^2)(\log x^2)}$$

c) $y = \cos^{-1}x^2$

$= \cos^{-1}u$,where $u = x^2$

By chain rule, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{d}{du}\left(\cos^{-1}u\right)\frac{d}{dx}\left(x^2\right)$

$$= \frac{-1}{\sqrt{1-u^2}}.2x$$

$$= \frac{-2x}{\sqrt{1-x^4}}$$

d) $y = a\cot^{-1}(m\tan^{-1}bx)$

$= a\cot^{-1}(m \tan^{-1} u)$, where $u = bx$,

$= a \cot^{-1} v$, where $v = m\tan^{-1}u$,

$= aw$, where $w = \cot^{-1} v$.

By chain rule, we get, $\frac{dy}{dx} = \frac{dy}{dw}.\frac{dw}{dv}.\frac{dv}{du}.\frac{du}{dx}$

$$= \frac{d}{dw}(aw).\frac{d}{dv}\left(\cot^{-1}v\right).\frac{d}{du}\left(m\tan^{-1}u\right).\frac{d}{dx}(bx)$$

$$= a.1.\frac{-1}{1+v^2}.\frac{m}{1+u^2}.b.1$$

$$= \frac{-mab}{\left\{1+\left(m\tan^{-1}bx\right)^2\right\}\left(1+b^2x^2\right)}$$

Example 3: Prove that the derivative of an even function is always an odd function.

Solution: Let f be an even function.

Then f(x) = f(–x)

$$\therefore \frac{d}{dx}f(x) = \frac{d}{dx}f(-x)$$

$$\Rightarrow f'(x) = f'(-x)\frac{d}{dx}(-x)$$

$$= -f'(-x).(-1)$$

$$= -f'(-x)$$

$$\therefore f'(-x) = -f'(x)$$

$\Rightarrow$ f′ is an odd function.

Example 4: If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$, Prove that $(1-x^2)\frac{dy}{dx} = xy + 1$

Solution: $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$ .....(1)

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-x^2}.\frac{1}{\sqrt{1-x^2}} - \sin^{-1}x.\frac{1}{2\sqrt{1-x^2}}(-2x)}{1-x^2}$$

$$= \frac{1 + x.\frac{\sin^{-1}x}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{1+xy}{1-x^2}$$

$$\therefore (1-x^2)\frac{dy}{dx} = xy + 1$$



CHECK YOUR PROGRESS

Q.1: Find the derivatives of the following functions with respect with respect to x.

a) y = log(sec x + tan x) b) y = sin(cos x)

c) $y = \log\left(x + \frac{1}{x}\right)$

Q.2: Find $\frac{dy}{dx}$ of the following:

a) $y = \log\left(x + \sqrt{a^2 + x^2}\right)$ b) $y = e^{ax^2+bx+c}$

c) $y = \log(e^{3x}.x^3)$ d) $y = x^3 \sin 2x.\log x$

Q.3: Prove that the derivative of an odd function is always an even function.

6.4 DERIVATIVE OF LOGARITHMIC FUNCTIONS AND IMPLICIT FUNCTIONS

If a function contains a variable as its exponent or if it is the product or quotient of two or more functions, we first take the logarithm of the function and then differentiate. The process is known as logarithmic differentiation.

Illustrative Examples:

Example 1: Find the derivatives of the following functions with respect to x.

a) x^x b) $a^{\tan x}$ c) $(\log_e x)^{\sin x}$

Solution: a) Let $y = x^x$

$\therefore \log_e y = \log_e x^x \Rightarrow \log_e y = x \log_e x$

Differentiating both sides with respect to x, we get–

$$\frac{1}{y}\frac{dy}{dx} = 1.\log_e x + x.\frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = y(\log_e x + 1)$$

$$\Rightarrow \frac{d}{dx}(x^x) = x^x(1 + \log_e x)$$

b) Let $y = a^{\tan x}$

$\therefore \log_e y = \log_e a^{\tan x} \Rightarrow \log_e y = \tan x \log_e a$

Differentating both sides with respect to x, we get.

$$\frac{1}{y}.\frac{dy}{dx} = \sec^2 x.\log_e a$$

$$\Rightarrow \frac{dy}{dx} = y \log_e a.\sec^2 x$$

$$\Rightarrow \frac{d}{dx}\left(a^{\tan x}\right) = a^{\tan x}.\log_e a.\sec^2 x$$

c) Let $y = (\log_e x)^{\sin x}$

$\therefore \log_e y = \log_e (\log_e x)^{\sin x} \Rightarrow \log_e y = \sin x . \log_e (\log_e x)$

Differentiating both side with respect to x, we get,

$$\frac{1}{y}.\frac{dy}{dx} = \cos x . \log_e (\log_e x) + \sin x . \frac{1}{\log_e x}.\frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log_e (\log_e x)\cos x + \frac{\sin x}{x \log_e x}\right]$$

$$\Rightarrow \frac{d}{dx}\left\{(\log_e x)^{\sin x}\right\} = (\log_e x)^{\sin x}\left[\log_e (\log_e x)\cos x + \frac{\sin x}{x \log_e x}\right]$$

Example 2: If $y = x^{x^{x^{\ldots\ldots\infty}}}$, Prove that $x\frac{dy}{dx} = \frac{y^2}{1 - y \log x}$

Solution: $y = x^{x^{x^{\ldots\ldots\infty}}} \Rightarrow y = x^y$

$\therefore \log y = \log x^y \Rightarrow \log y = y \log x$

Differentiating bothsides with respect to x, we get,

$$\frac{1}{y}\frac{dy}{dx} = y.\frac{1}{x} + \frac{dy}{dx}.\log x$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1 - y \log x}{y}\right) = \frac{y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y^2}{x(1 - y \log x)}$$

Example 3: Find the differential coefficient of $\frac{(x+1)^2\sqrt{x-1}}{(x+4)^3 e^x}$ with respect to x.

Solution: Let $y = \frac{(x+1)^2\sqrt{x-1}}{(x+4)^3 e^x}$

$$\therefore \log_e y = \log_e \left\{\frac{(x+1)^2\sqrt{x-1}}{(x+4)^3 e^x}\right\}$$

$$\Rightarrow \log_e y = 2\log_e (x+1) + \frac{1}{2}\log_e (x-1) - 3\log_e (x+4) - x \log_e e$$

$$\therefore \frac{d}{dx}(\log_e y) = \frac{d}{dx}\left[2\log_e(x+1) + \frac{1}{2}\log_e(x-1) - 3\log_e(x+4) - x\right]$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = 2.\frac{1}{x+1} + \frac{1}{2}.\frac{1}{x-1} - 3.\frac{1}{x+4} - 1$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{2}{x+1} + \frac{1}{2(x-1)} - \frac{3}{x+4} - 1\right)$$

$$\therefore \frac{d}{dx}\left[\frac{(x+1)^2\sqrt{x-1}}{(x+4)^3 e^x}\right] = \frac{(x+1)^2\sqrt{x-1}}{(x+4)^3 e^x}\left(\frac{2}{x+1} + \frac{1}{2(x-1)} - \frac{3}{x+4} - 1\right)$$

Example 4: If $x^y = e^{x-y}$ Prove that $\frac{dy}{dx} = \frac{\log_e x}{(\log_e(ex))^2}$

Solution: Given, $x^y = e^{x-y}$

$$\therefore \log_e x^y = \log_e e^{x-y}$$

$$\Rightarrow y\log_e x = (x-y)\log_e e \qquad \text{.....(1)}$$

$$\therefore \frac{d}{dx}(y\log_e x) = \frac{d}{dx}(x-y)$$

$$\Rightarrow y.\frac{1}{x} + \frac{dy}{dx}.\log_e x = 1 - \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(1 + \log_e x\right) = 1 - \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x\left(1+\log_e x\right)} = \frac{x-y}{x(\log_e e + \log_e x)}$$

$$= \frac{x-y}{x\log(ex)}$$

$$= \frac{y\log_e x}{x\log(ex)}, \text{ using (1) .....(2)}$$

Again, (1) $\Rightarrow y\log_e x = x - y$

$$\Rightarrow y\left(1+\log_e x\right) = x$$

$$\Rightarrow \frac{y}{x} = \frac{1}{\log_e e + \log_e x} = \frac{1}{\log_e(ex)}$$

$$\therefore (2) \Rightarrow \frac{dy}{dx} = \frac{1}{\log_e(ex)}.\frac{\log_e x}{\log_e(ex)} = \frac{\log_e x}{(\log_e(ex))^2}$$

Derivatives of Implicit Functions: Sometimes x and y are so related that it is not possible to find y in terms of x. For example consider the following relations.

1) $x^3 + y^2 - 3xy = 0$ 2) $\cos y = x\sin(x+y)$

The above examples y cannot be explicity obtained as a function of x.such functions are called implicit functions. In the following examples we illustrate the way to find $\frac{dy}{dx}$ in such cases.

Illustrative Examples:

Example 1: If $y= \sin(x+y) + x^2y$, find $\frac{dy}{dx}$

Solution: Given, $y = \sin(x+y) + x^2y$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left\{\sin(x+y) + x^2y\right\}$$

$$\Rightarrow \frac{dy}{dx} = \cos(x+y).\frac{d}{dx}(x+y) + 2xy + x^2.\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \cos(x+y).\left(1+\frac{dy}{dx}\right) + 2xy + x^2.\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \cos(x+y) + \cos(x+y).\frac{dy}{dx} + 2xy + x^2.\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left[1 - x^2 - \cos(x+y)\right] = \cos(x+y) + 2xy$$

$$\therefore \frac{dy}{dx} = \frac{\cos(x+y) + 2xy}{1 - x^2 - \cos(x+y)}$$

Example 2: If $x^y.y^x = 1$ find $\frac{dy}{dx}$.

Solution: $x^y.y^x = 1$

$$\therefore \log\left(x^y.y^x\right) = \log 1$$

$$\Rightarrow y\log x + x\log y = 0$$

$$\therefore \frac{d}{dx}(y\log x + x\log y) = 0$$

$$\Rightarrow y.\frac{1}{x} + \frac{dy}{dx}.\log x + x.\frac{1}{y}\frac{dy}{dx} + 1.\log y = 0$$

$$\Rightarrow \frac{dy}{dx}\left(\log x + \frac{x}{y}\right) = -\left(\frac{y}{x} + \log y\right)$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{y\log x + x}{y}\right) = -\frac{y + x\log y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{-y(y + x\log y)}{x(y\log x + x)}$$

Example 3: If siny = xsin(a+y), Show that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

Solution: Given siny = xsin(a+y)

$$\Rightarrow x = \frac{\sin y}{\sin(a+y)}$$

$$\therefore \frac{d}{dy}(x) = \frac{d}{dy}\left(\frac{\sin y}{\sin(a+y)}\right)$$

$$\Rightarrow \frac{dx}{dy} = \frac{\sin(a+y)\frac{d}{dy}(\sin y) - \sin y \frac{d}{dy}\sin(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y)\cos y - \sin y \cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$= \frac{\sin a}{\sin^2(a+y)}$$

$$\therefore \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Example 4: If, $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + ...\infty}}}$

Show that $\frac{dy}{dx} = \frac{\sin x}{1-2y}$

Solution: Given, $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + ...\infty}}}$

$$\Rightarrow y = \sqrt{\cos x + y}$$

$$\Rightarrow y^2 = \cos x + y$$

$$\therefore \frac{d}{dx}(y^2) = \frac{d}{dx}(\cos x + y)$$

$$\Rightarrow 2y\frac{dy}{dx} = -\sin x + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(2y-1) = -\sin x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x}{1-2y}$$



CHECK YOUR PROGRESS

Q.4: Find the differential coefficient of $\frac{1}{(x+p)(x+q)(x+t)}$ with respect to x.

Q.5: Find $\frac{dy}{dx}$ if

a) $y = (\log x)^{\cos x}$ b) $y = (1+\cos x)^x$

c) $y = \sqrt{\frac{x^2+x+1}{x^2-x+1}}$ d) $y = \sin x \sin 2x \sin 3x$

Q.6: If $y = e^{x+e^{x+e^{x+\ldots\ldots}}}$, show that $\frac{dy}{dx} = \frac{y}{1-y}$

Q.7: If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, prove that $\frac{dy}{dx} = -(1+x)^{-2}$, $x \neq y$

Q.8: Find $\frac{dy}{dx}$ if

a) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

b) $x^n + y^n = a^x$ c) $\sqrt{x^2+y^2} + \log(x^2-y^2) = 0$

The derivative of parametric function and derivative of a function with respect to another function: Sometimes x and y both are expressed as function of another variable. This third variable is known as parameter.

For example $x = a\cos\theta$, $y = a\sin\theta$

In such case, we find $\frac{dy}{dx}$, without first eliminating the parameter,

by the result $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$, if θ is the parameter.

If y_1 and y_2 are two functions of x such that $y_1 = f_1(x)$ and $y_2 = f_2(x)$ then we obtain–

$$\frac{dy_1}{dy_2} = \frac{\text{derivative of } y_1 \text{ with respect to } x}{\text{derivative of } y_2 \text{ with respect to } x}$$

Illustrative Examples:

Example 1: Find $\frac{dy}{dx}$, if x = acosθ, y = asinθ

Solution: $\because x = a\cos\theta \qquad \therefore \frac{dx}{d\theta} = -a\sin\theta$

and y = asinθ $\qquad \therefore \frac{dy}{d\theta} = a\cos\theta$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$$

Example 2: Find $\frac{dy}{dx}$ if $x = a\log\left(\tan\frac{t}{2} + \cos t\right)$ and y = asint

Solution: $\because x = a\left(\log\tan\frac{t}{2} + \cos t\right)$

$$\therefore \frac{dx}{dt} = a\left[\frac{1}{\tan\frac{t}{2}}.\sec\frac{2t}{2}.\frac{1}{2} + (-\sin t)\right]$$

$$= a\left[\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}.\frac{1}{\cos^2\frac{t}{2}}.\frac{1}{2} - \sin t\right]$$

$$= a\left(\frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}} - \sin t\right)$$

$$= a\left(\frac{1}{\sin t} - \sin t\right)$$

$$= \frac{a(1-\sin^2 t)}{\sin t} = \frac{a\cos^2 t}{\sin t}$$

Also, $\frac{dy}{dt} = \frac{d}{dt}(a\sin t) = a\cos t$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}} = \frac{\sin t}{\cos t} = \tan t.$$

Example 3: Find the differential coefficient of $\log(x^2+2x+1)$ with respect to (x^2+2x).

Solution: Let $u = \log(x^2+2x+1)$

$$\therefore \frac{du}{dx} = \frac{1}{x^2+2x+1}(2x+2) = \frac{2(x+1)}{x^2+2x+1}$$

Also, let $v = x^2 + 2x$

$$\therefore \frac{dv}{dx} = 2x+2 = 2(x+1)$$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{2(x+1)}{x^2+2x+1} \times \frac{1}{2(x+1)} = \frac{1}{x^2+2x+1}$$

Example 4: Find the differential coefficient of $e^{\sin x}$ with respect to $\cos x$.

Solution: Let $u = e^{\sin x}$ $\quad \therefore \frac{du}{dx} = e^{\sin x}.\cos x$

Again, let $v = \cos x$ $\quad \therefore \frac{dv}{dx} = -\sin x$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{e^{\sin x}.\cos x}{-\sin x} = -\cot x.e^{\sin x}$$

Example 5: Find the derivative of $u = x^{\sin x}$ with respect to $v = (\sin x)^x$.

Solution: Given, $u = x^{\sin x}$

$\therefore \log u = \log x^{\sin x}$

$\Rightarrow \log u = \sin x.\log x$

$$\therefore \frac{d}{dx}(\log u) = \frac{d}{dx}(\sin x.\log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x.\log x + \sin x.\frac{1}{x}$$

$$\Rightarrow \frac{du}{dx} = u\left(\cos x \log x + \frac{\sin x}{x}\right)$$

$$= x^{\sin x}.\frac{x\cos x\log x + \sin x}{x}$$

Also, $v = (\sin x)^x$

$\therefore$ logv = log$(\sin x)^x$

$\Rightarrow \log v = x\log(\sin x)$

$\therefore \frac{d}{dx}(\log v) = \frac{d}{dx}\{x\log(\sin x)\}$

$\Rightarrow \frac{1}{v}.\frac{dv}{dx} = 1.\log(\sin x) + x.\frac{1}{\sin x}.\cos x$

$\Rightarrow \frac{dv}{dx} = v\{\log(\sin x) + x\cot x\}$

$= (\sin x)^x [\log(\sin x)+x\cot x]$

$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{x^{\sin x}.(x\cos x\log x + \sin x)}{x(\sin x)^x\ [\log(\sin x) + x\cot x]}$



CHECK YOUR PROGRESS

Q.9: Find $\frac{dy}{dx}$ of the following parametric equations.

a) $x = a\cos^3\theta$, $y = a\sin^3\theta$

b) $x = \sqrt{1+t}$, $y = \sqrt{1-t}$

c) $x = e^t(\cos t - \sin t)$, $y = e^t(\cos t + \sin t)$

Q.10: Find the differential coefficient of $e^{\sin^{-1}x}$ with respect to $\sin^{-1}x$.

Q.11: Find the derivative of $\frac{x^2}{1+x^2}$ with respect to x^2.

Q.12: Find the derivative of logx + sinx with respect to e^x + cosx.



6.5 LET US SUM UP

- If y = f(u), u = g(t), t = h(x), then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dt}.\frac{dt}{dx}$
- If u and v are two functions of x such that u=f(x) and v = g(x), then

$$\frac{du}{dv} = \frac{\text{Derivative of u with respect to x}}{\text{Derivative of v with respect to x}}$$

6.6 ANSWERS TO CHECK YOU PROGRESS

Ans. to Q. No. 1: a) y = log(secx + tanx) = logt, where t = secx + tanx

By chain rule, $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{d}{dt}(\log t)\frac{d}{dx}(\sec x + \tan x)$

$= \frac{1}{t}(\sec x \tan x + \sec^2 x)$

$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$

= secx.

b) y = sin(cosx) = sint, where t = cosx

By chain rule, $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{d}{dt}(\sin t)\frac{d}{dx}(\cos x)$

= cost (–sinx)

= – cos(cosx). sinx

c) $y = \log\left(x + \frac{1}{x}\right)$ = logt ,where $t = x + \frac{1}{x}$

By chain rule, $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{d}{dt}(\log t)\frac{d}{dx}\left(x + \frac{1}{x}\right)$

$= \frac{1}{t}\left(1 - \frac{1}{x^2}\right)$

$= \frac{1}{x + \frac{1}{x}}\left(1 - \frac{1}{x^2}\right)$

$= \frac{x}{x^2 + 1}.\frac{x^2 - 1}{x^2}$

$= \frac{x^2 - 1}{x(x^2 + 1)}$

Ans. to Q. No. 2: a) $y = \log\left(x + \sqrt{a^2 + x^2}\right)$

$\Rightarrow$ y = log t, where $t = x + \sqrt{a^2 + x^2}$

By chain rule, $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{d}{dt}(\log t).\frac{d}{dx}\left(x + \sqrt{a^2 + x^2}\right)$

$= \frac{1}{t}\left[1 + \frac{1}{2\sqrt{a^2 + x^2}}.2x\right]$

$= \frac{\sqrt{a^2 + x^2} + x}{\sqrt{a^2 + x^2}\left(x + \sqrt{a^2 + x^2}\right)}$

$= \frac{1}{\sqrt{a^2 + x^2}}$

b) $y = e^{ax^2+bx+c}$

$\Rightarrow y = e^t$, where $t = ax^2 + bx + c$

By chain rule, $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$= \frac{d}{dt}\left(e^t\right).\frac{d}{dx}\left(ax^2 + bx + c\right)$

$= e^t(2ax+b)$

$= e^{\left(ax^2+bx+c\right)}.(2ax + b)$

c) $y = \log(e^{3x}.x^3)$

$= 3x\log e + 3\log x$

$= 3x+3\log x$

$\therefore \frac{dy}{dx} = \frac{d}{dx}(3x + 3\log x) = 3 + \frac{3}{x}$

d) $y = x^3 \sin 2x \log x$

$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^3 \sin 2x \log x\right)$

$= \frac{d}{dx}\left(x^3\right)\sin 2x \log x + x^3 \log x \frac{d}{dx}(\sin 2x) + x^3 \sin 2x \frac{d}{dx}(\log x)$

$= 3x^2 \sin 2x \log x + x^3 \log x.\cos 2x.2 + x^3 \sin 2x.\frac{1}{x}$

$= 3x^2 \sin 2x \log x + 2x^3 \log x.\cos 2x + x^2 \sin 2x$

Ans. to Q. No. 3: Let f be an odd function.

Then $f(-x) = -f(x)$

$$\therefore \frac{d}{dx}f(-x) = \frac{d}{dx}\{-f(x)\}$$

$$\Rightarrow f'(-x)\frac{d}{dx}(-x) = -f'(x)$$

$$\Rightarrow f'(-x).(-1) = -f'(x)$$

$$\Rightarrow -f'(-x) = -f'(x)$$

$$\Rightarrow f'(-x) = f'(x)$$

$\therefore$ f is an even function.

Ans. to Q. No. 4: Let $y = \frac{1}{(x+p)(x+q)(x+t)}$

$$\therefore \log y = \log\frac{1}{(x+p)(x+q)(x+t)}$$

$$\Rightarrow \log y = \log 1 - \log(x+p) - \log(x+q) - \log(x+t)$$

$$\therefore \frac{d}{dx}(\log y) = \frac{d}{dx}\left[-\log(x+p) - \log(x+q) - \log(x+t)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{1}{x+p} - \frac{1}{x+q} - \frac{1}{x+t}$$

$$\Rightarrow \frac{dy}{dx} = -y\left(\frac{1}{x+p} + \frac{1}{x+q} + \frac{1}{x+t}\right)$$

$$\therefore \frac{d}{dx}\left[\frac{1}{(x+p)(x+q)(x+t)}\right] = -\frac{1}{(x+p)(x+q)(x+t)}\left(\frac{1}{x+p} + \frac{1}{x+q} + \frac{1}{x+r}\right)$$

Ans. to Q. No. 5: a) $y = (\log x)^{\cos x}$

$$\therefore \log y = \log (\log x)^{\cos x}$$

$$\Rightarrow \log y = \cos x.\log(\log x)$$

$$\therefore \frac{d}{dx}(\log y) = \frac{d}{dx}\left(\cos x.\log(\log x)\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\sin x.\log(\log x) + \cos x.\frac{1}{\log x}.\frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{\cos x}{x\log x} - \sin x\log(\log x)\right]$$

$$= (\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x\log(\log x)\right]$$

b) $y = (1+\cos x)^x$

$$\therefore \log y = \log(1+\cos x)^x$$

$$\Rightarrow \log y = x\log(1+\cos x)$$

$$\therefore \frac{d}{dx}(\log y) = \frac{d}{dx}\{x\log(1+\cos x)\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 1.\log(1+\cos x) + x.\frac{1}{1+\cos x}.(-\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(1+\cos x) - \frac{x\sin x}{1+\cos x}\right]$$

$$= (1+\cos x)^x\left[\log(1+\cos x) - \frac{x\sin x}{1+\cos x}\right]$$

c) $y = \sqrt{\frac{x^2+x+1}{x^2-x+1}}$

$$\Rightarrow \log y = \log\left(\frac{x^2+x+1}{x^2-x+1}\right)^{\frac{1}{2}}$$

$$\Rightarrow \log y = \frac{1}{2}\left[\log(x^2+x+1) - \log(x^2-x+1)\right]$$

Differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x^2+x+1}(2x+1) - \frac{1}{x^2-x+1}(2x-1)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2}\left[\frac{(2x+1)(x^2-x+1) - (2x-1)(x^2+x+1)}{(x^2+x+1)(x^2-x+1)}\right]$$

$$= \frac{y}{2}\left[\frac{2x^3+x^2-2x^2-x+2x+1-2x^3+x^2-2x^2+x-2x+1}{(x^2+x+1)(x^2-x+1)}\right]$$

$$= \frac{y}{2}\frac{-2x^2+2}{(x^2+x+1)(x^2-x+1)}$$

$$= y\frac{-x^2+1}{(x^2+x+1)(x^2-x+1)}$$

$$= \sqrt{\frac{x^2+x+1}{x^2-x+1}}.\frac{-x^2+1}{(x^2+x+1)(x^2-x+1)}$$

d) $y = \sin x \sin 2x \sin 3x$

$\therefore \log y = \log(\sin x \sin 2x \sin 3x)$

$\Rightarrow \log y = \log \sin x + \log \sin 2x + \log \sin 3x$

Differentrating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\sin x}(\cos x) + \frac{1}{\sin 2x}(\cos 2x).2 + \frac{1}{\sin 3x}(\cos 3x).3$$

$$\Rightarrow \frac{dy}{dx} = y[\cot x + 2\cot 2x + 3\cot 3x]$$

$= \sin x \sin 2x \sin 3x (\cot x + 2\cot 2x + 3\cot 3x).$

Ans. to Q. No. 6: Given $y = e^{x+e^{x+e^{x\ldots\ldots\infty}}}$

$\Rightarrow y = e^{x+y}$

$\therefore \log y = \log e^{x+y}$

$\Rightarrow \log y = x + y$

Differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y} - 1\right) = 1$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1-y}{y}\right) = 1$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{1-y}\right)$$

Ans. to Q. No. 7: Given $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

$\therefore x^2(1+y) = y^2(1+x)$

$\Rightarrow x^2 - y^2 + x^2y - xy^2 = 0$

$\Rightarrow (x+y)(x-y) + xy(x-y) = 0$

$\Rightarrow (x-y)(x+y+xy) = 0$

$\Rightarrow x + y + xy = 0 \quad [\because x \neq y]$

$\Rightarrow y(1+x) = -x$

$$\Rightarrow y = -\frac{x}{1+x}$$

$$\therefore \frac{dy}{dx} = -\frac{d}{dx}\left(\frac{x}{(1+x)}\right)$$

$$= \frac{-[(1+x).1-x.1]}{(1+x)^2}$$

$$= \frac{-1}{(1+x)^2}$$

$$= -(1+x)^{-2}$$

Ans. to Q. No. 8: a) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$$\therefore \frac{d}{dx}\left(ax^2 + 2hxy + by^2 + 2gx + 2fy + c\right) = 0$$

$$\Rightarrow a(2x) + 2h\left(1.y + x\frac{dy}{dx}\right) + b\left(2y\frac{dy}{dx}\right) + 2g(1) + 2f\frac{dy}{dx} + 0 = 0$$

$$\Rightarrow 2ax + 2hy + 2hx\frac{dy}{dx} + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(hx + by + f) = -(ax + hy + g)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}$$

b) $x^n + y^n = a^n$

$$\therefore \frac{d}{dx}\left(x^n + y^n\right) = \frac{d}{dx}\left(a^n\right)$$

$$\Rightarrow nx^{n-1} + ny^{n-1}\frac{dy}{dx} = 0$$

$$\Rightarrow ny^{n-1}\frac{dy}{dx} = -nx^{n-1}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{n-1}}{y^{n-1}}$$

c) $\sqrt{x^2 + y^2} + \log\left(x^2 - y^2\right) = 0$

$$\therefore \frac{d}{dx}\left\{\sqrt{x^2 + y^2} + \log\left(x^2 - y^2\right)\right\} = 0$$

$$\Rightarrow \frac{1}{2\sqrt{x^2 + y^2}}\left(2x + 2y\frac{dy}{dx}\right) + \frac{1}{x^2 - y^2}\left(2x - 2y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}}\frac{dy}{dx} + \frac{2x}{x^2 - y^2} - \frac{2y}{x^2 - y^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{y}{\sqrt{x^2+y^2}}-\frac{2y}{x^2-y^2}\right)=-\left(\frac{x}{\sqrt{x^2+y^2}}+\frac{2x}{x^2-y^2}\right)$$

$$\Rightarrow \frac{dy}{dx}\left\{\frac{y\left(x^2-y^2-2\sqrt{x^2+y^2}\right)}{\left(x^2-y^2\right)\sqrt{x^2+y^2}}\right\}=-x\left\{\frac{x^2-y^2+2\sqrt{x^2+y^2}}{\left(x^2-y^2\right)\sqrt{x^2-y^2}}\right\}$$

$$\therefore \frac{dy}{dx}=\frac{-x\left(x^2-y^2+2\sqrt{x^2+y^2}\right)}{y\left(x^2-y^2-2\sqrt{x^2+y^2}\right)}$$

Ans. to Q. No. 9: a) $\because x=a\cos^3\theta$

$$\therefore \frac{dx}{d\theta}=\frac{d}{d\theta}\left(a\cos^3\theta\right)=a.3\cos^2\theta\left(-\sin\theta\right)=-3a\cos^2\theta\sin\theta$$

Also, $y = a\sin^3\theta$

$$\therefore \frac{dy}{d\theta}=\frac{d}{d\theta}\left(a\sin^3\theta\right)=a.3\sin^2\theta\left(\cos\theta\right)=3a\sin^2\theta\cos\theta$$

$$\therefore \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}=-\tan\theta$$

b) $\because x=\sqrt{1+t}$

$$\therefore \frac{dx}{dt}=\frac{1}{2\sqrt{1+t}}$$

Also, $y = \sqrt{1-t}$

$$\therefore \frac{dy}{dt}=\frac{1}{2\sqrt{1-t}}(-1)=\frac{-1}{2\sqrt{1-t}}$$

$$\therefore \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{-1}{2\sqrt{1-t}}.2\sqrt{1+t}=-\sqrt{\frac{1+t}{1-t}}$$

c) $\because x=e^t\left(\cos t-\sin t\right)$

$$\therefore \frac{dx}{dt}=e^t\left(\cos t-\sin t\right)+e^t\left(-\sin t-\cos t\right)$$

$$= e^t\left(\cos t-\sin t-\sin t-\cos t\right)$$

$$= -2e^t\sin t$$

Also, $y = e^t(\cos t + \sin t)$

$$\therefore \frac{dy}{dt} = e^t(\cos t + \sin t) + e^t(-\sin t + \cos t)$$

$= e^t(\cos t + \sin t - \sin t + \cos t)$

$= 2e^t \cos t$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2e^t \cos t}{-2e^t \sin t} = -\cot t.$$

Ans. to Q. No. 10: Let $u = e^{\sin^{-1}x}$ and $v = \sin^{-1}x$

$$\therefore \frac{du}{dx} = e^{\sin^{-1}x}.\frac{1}{\sqrt{1-x^2}} \text{ and } \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = e^{\sin^{-1}x}.\frac{1}{\sqrt{1-x^2}} \times \sqrt{1-x^2} = e^{\sin^{-1}x}$$

Ans to Q No 11: Let $u = \frac{x^2}{1+x^2}$ and $v = x^2$

$$\therefore \frac{du}{dx} = \frac{(1+x^2).2x - x^2.2x}{(1+x^2)^2} = \frac{2x + 2x^3 - 2x^3}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$$

And $\frac{dv}{dx} = 2x$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{2x}{(1+x^2)^2} \times \frac{1}{2x} = \frac{1}{(1+x^2)^2}$$

Ans. to Q. No. 12: Let $u = \log x + \sin x$ and $v = e^x + \cos x$

$$\therefore \frac{du}{dx} = \frac{1}{x} + \cos x = \frac{1 + x\cos x}{x}$$

and $\frac{dv}{dx} = e^x - \sin x$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{1 + x\cos x}{x(e^x - \sin x)}$$



6.7 FURTHER READINGS

1) Gupta, Sudhir (2011) (1st edition): *Basics of Differentential Calculus*; New Delhi: Anmol publication Pvt. Ltd.
2) Habib, Amber (2011) (1st edition): *The Calculus of Finance*; Hyderabad: Universities Press India Pvt. Ltd.
3) Maron, I. A. (1993) (1st edition): *Problems in Calculus of One Variable*; New Delhi: CBS Publishers & Distributors Pvt. Ltd.
4) Sharma, A. K. (2004) (1st edition): *Text Book of Differential Calculus for B.A., B.Sc., B.Com., I.A.S., P.C.S.*; New Delhi: Discovery Publishing House.



6.8 MODEL QUESTIONS

Q.1: Differentiate the following with respect to x

a) $(\sin x)^{\cos x}$ b) $(\tan x)^{\log x}$ c) $(\log x)^x + (\sin x)^{\sin x}$ d) $x^{\sin x} + (\sin x)^x$

Q.2: Find $\frac{dy}{dx}$ if

a) $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$ b) $ax^2 + 2hxy + by^2 = 0$

c) $y = \log y^x$ d) $x^y - y = 0$

Q.3: Differentiate with respect to x

a) $\sin^{-1}\frac{2x}{1+x^2}$ b) $\frac{1-\cos x}{1+\cos x}$ c) $\cos^{-1}\frac{1-x^2}{1+x^2}$

d) $\tan^{-1}\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}}$ e) $\tan^{-1}\frac{\cos x}{1+\sin x}$

f) $\log\left(x+2+\sqrt{x^2+4x+1}\right)$ g) $\left(a+bx^2\right)^{1/2}$

Q.4: Find the differential coefficient of $\sin[\cos(\tan x)]$ with respect to x.

Q.5: If $y = \frac{x\sin^{-1}x}{\sqrt{1-x^2}} + \log\sqrt{1-x^2}$, prove that $\frac{dy}{dx} = \frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}$

Q.6: If $\tan y = \frac{2t}{1-t^2}$ and $\sin x = \frac{2t}{1-t^2}$, then prove that $\frac{dy}{dx} = 1$.

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UNIT 7: INDETERMINATE FORM

UNIT STRUCTURE

7.1 Learning Objectives

7.2 Introduction

7.3 Indeterminate Form

7.4 Let us Sum up.

7.5 Answers to Check Your Progress

7.6 Further Reading

7.7 Model Questions

7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- evaluate limits of indeterminate forms

  $\frac{0}{0}, \frac{\infty}{\infty}, \infty, -\infty, 1^{\infty}, 0.\infty, 0^{\infty}$ etc.

7.2 INTRODUCTION

We have already introduced the concept of limits in unit 2. In this unit, we also discuss the nature of the function whose numerator and denominator both approach to zero at a particular point (Indeterminate form). It is called indeterminate form. Some indeterminate forms are: $\frac{0}{0}, \frac{\infty}{\infty}, \infty, -\infty, 1^{\infty}, 0.\infty, 0^{\infty}$ etc.

7.3 INDETERMINATE FORMS

In unit 2, we simplified $\lim_{x\to a} \frac{f(x)}{g(x)}$ as $\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x)$ and $\lim_{x\to a} f(x)$ exist and $\lim_{x\to a} g(x) \neq 0$.

But if $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ then $\lim_{x\to a} \frac{f(x)}{g(x)}$ takes the indeterminate form $\frac{0}{0}$. The other indeterminate forms are $\frac{\infty}{\infty}$, $\infty - \infty$, $0.\infty$, $0^{o}, \infty^{o}$, 1^{∞}.

L' Hospital's Rule:

1) If f(a) = g(a) = 0, $f'(x)$ and $g'(x)$ exists near a (but not necessarily at a), then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

2) If f and g are two functions such that $f(x)\to\infty$ as $x\to\infty$, and $g(x)\to\infty$ as $x\to\infty$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Note: The indeterminate forms of types $0.\infty$ and $\infty-\infty$ can be evaluated by transforming them as a quotient of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ to which L' Hospital's rule is applicable.

Illustrative examples :

Example 1: Evaluate $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

Solution: $\lim_{x\to 0}\frac{x-\sin x}{x^3}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{1-\cos x}{3x^2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{\sin x}{6x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{\cos x}{6} = \frac{1}{6}$

Example 2: Evaluate $\lim_{x\to 0}\frac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}$

Solution: $\lim_{x\to 0}\frac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{e^x + e^{-x} - \frac{2}{1+x}}{\sin x + x\cos x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{e^x - e^{-x} + \frac{2}{(1+x)^2}}{\cos x + \cos x - x\sin x}$

$= \frac{1-1+2}{1+1-0} = 1$

Example 3: Evaluate $\lim_{x\to 0}\frac{\log\tan 7x}{\log\tan 2x}$

Solution: $\lim_{x\to 0}\frac{\log\tan 7x}{\log\tan 2x}$ $\left[\frac{\infty}{\infty}\text{form}\right]$

$$= \lim_{x\to 0}\frac{\frac{1}{\tan 7x}.\sec^2 7x.7}{\frac{1}{\tan 2x}.\sec^2 2x.2}$$

$$= \lim_{x\to 0}\frac{7\sec^2 7x}{2\sec^2 2x}.\lim_{x\to 0}\frac{\tan 2x}{\tan 7x}$$

$$= \frac{7}{2}\lim_{x\to 0}\frac{\tan 2x}{\tan 7x} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$= \frac{7}{2}.\lim_{x\to 0}\frac{2\sec^2 2x}{7\sec^2 7x}$$

$$= \frac{7}{2}.\frac{2}{7} = 1$$

Example 4: Find a and b if $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$

Solution: $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}$ $\left[\frac{0}{0}\text{form}\right]$

$$= \lim_{x\to 0}\frac{1+a\cos x+x(-a\sin x)-b\cos x}{3x^2}$$

$\because$ Denominator $\to 0$ as $x\to 0$ and the value of the limit is finite

$\therefore$ numerator $\to 0$ as $x\to 0$

$\Rightarrow$ 1 + a – b = 0 .....(1)

Thus,

$$\lim_{x\to 0}\frac{1+a\cos x-ax\sin x-b\cos x}{3x^2} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{x\to 0}\frac{-2a\sin x-ax\cos x+b\sin x}{6x} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{x\to 0}\frac{-2a\cos x-a\cos x+ax\sin x+b\cos x}{6}$$

$$= \frac{-2a-a+b}{6}$$

= 1 (given)

$\therefore$ –3a + b = 6

$\Rightarrow$ –3a+1+a = 6, using (1)

$\Rightarrow$ –2a = 5

$\Rightarrow a = -\frac{5}{2}$

$\therefore$ (1) $\Rightarrow$ $b = 1 + a = 1 - \frac{5}{2} = -\frac{3}{2}$

Example 5: Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{\cot x}{x}\right)$

Solution: $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{\cot x}{x}\right)$ $\left[\infty-\infty \text{ form}\right]$

$$= \lim_{x\to 0}\frac{\sin x - x\cos x}{x^2\sin x} \quad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{x\to 0}\frac{\sin x - x\cos x}{x^3}\cdot\lim_{x\to 0}\frac{x}{\sin x}$$

$$= \lim_{x\to 0}\frac{\sin x - x\cos x}{x^3} \quad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{x\to 0}\frac{\cos x - \cos x + x\sin x}{3x^2}$$

$$= \lim_{x\to 0}\frac{\sin x}{3x} \quad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{x\to 0}\frac{\cos x}{3} = \frac{1}{3}$$

Example 6: Evaluate $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$

Solution: $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$ [form $0.\infty$]

$$= \lim_{x\to 1}\frac{-(1-x)}{\cot\frac{\pi x}{2}} \quad \left[\text{form}\frac{0}{0}\right]$$

$$= \lim_{x\to 1}\frac{-1}{-\operatorname{cosec}^2\frac{\pi x}{2}\cdot\frac{\pi}{2}}$$

$$= \lim_{x\to 1}\frac{2}{\pi}\sin^2\frac{\pi x}{2} = \frac{2}{\pi}$$

Example 7: Evaluate $\lim_{x\to 0} x\log x$

Solution: $\lim_{x\to 0} x\log x$ $[\text{form } 0.\infty]$

$$= \lim_{x\to 0} \frac{\log x}{\frac{1}{x}} \qquad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0} (-x) = 0$$

Example 8: $\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$

Solution: $\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$ $[\text{form } \infty-\infty]$

$$= \lim_{x\to 0} \frac{x-e^x+1}{x\left(e^x-1\right)} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{1-e^x}{xe^x+e^x-1} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-e^x}{xe^x+e^x+e^x}$$

$$= \frac{-1}{0+1+1} = -\frac{1}{2}$$

Example 9: Evaluate $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]$

Solution: $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]$ $[\text{form } \infty-\infty]$

$$= \lim_{x\to 0} \frac{x-\log(1+x)}{x^2} \qquad \left[\text{form}\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{1-\frac{1}{1+x}}{2x}$$

$$= \lim_{x\to 0} \frac{1+x-1}{2x(1+x)}$$

$$= \lim_{x\to 0} \frac{1}{2(1+x)} = \frac{1}{2}$$

Example 10: Evaluate $\lim_{x\to\frac{\pi}{2}}(\sec x - \tan x)$

Solution: $\lim_{x\to\frac{\pi}{2}}(\sec - \tan x) \qquad [\text{form } \infty - \infty]$

$$= \lim_{x\to\frac{\pi}{2}}\left(\frac{1-\sin x}{\cos x}\right) \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to\frac{\pi}{2}}\left(\frac{-\cos x}{-\sin x}\right)$$

$$= \lim_{x\to\frac{\pi}{2}}\cot x$$

$$= 0$$

Indeterminate forms 0^o, 1^∞ and ∞^o: Limits involving exponential expressions of the above type may be evalutated by taking logarithms and expressing them in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Illustrative examples:

Example 1: Evaluate $\lim_{x\to 1} x^{\frac{1}{x-1}}$

Solution: The given limit is in the form 1^∞.

Let $f(x) = x^{\frac{1}{x-1}}$

$$\therefore \log f(x) = \frac{1}{x-1}\log x$$

$$\Rightarrow \lim_{x\to 1}\log f(x) = \lim_{x\to 1}\frac{\log x}{x-1} \quad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{x\to 1}\frac{\frac{1}{x}}{1} = 1$$

$$\therefore \lim_{x\to 1} f(x) = e^1$$

$$\Rightarrow \lim_{x\to 1} x^{\frac{1}{x-1}} = e$$

Example 2: Evaluate $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$

Solution: The given limit is in the form 1^∞.

Let $y = (\cos x)^{\frac{1}{x^2}}$

$$\therefore \log y = \frac{1}{x^2}\log(\cos x)$$

$$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log(\cos x)}{x^2} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$$

$$= \lim_{x\to 0} \frac{-\tan x}{2x} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\sec^2 x}{2}$$

$$= -\frac{1}{2}$$

$$\therefore \lim_{x\to 0} y = e^{-\frac{1}{2}} \Rightarrow \lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}}$$

Example 3: Evaluate $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$

Solution: Given limit is in the form 1^{∞}.

Let $y = \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$

$$\therefore \log y = \frac{1}{x}\log\left(\frac{\sin x}{x}\right)$$

$$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\frac{x}{\sin x}\,\frac{x\cos x - \sin x}{x^2}}{1}$$

$$= \left(\lim_{x\to 0} \frac{x}{\sin x}\right)\left(\lim_{x\to 0} \frac{x\cos x - \sin x}{x^2}\right)$$

$$= \lim_{x\to 0} \frac{x\cos x - \sin x}{x^2} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{2x}$$

$$= \lim_{x\to 0} \frac{-\sin x}{2} = 0$$

$$\therefore \lim_{x\to 0} y = e^{o} \Rightarrow \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = 1$$

Example 4: Evaluate $\lim_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x}$

Solution: Given limit is in the form 1^{∞}.

Let $y = (\tan x)^{\tan 2x}$

$\therefore \log y = \tan 2x . \log(\tan x)$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} \log y = \lim_{x\to\frac{\pi}{4}} \tan 2x . \log(\tan x) \qquad [\text{form} \infty . 0]$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\log(\tan x)}{\cot 2x} \qquad \left[\text{form} \frac{0}{0}\right]$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\frac{1}{\tan x}.\sec^2 x}{-\operatorname{cosec}^2 2x.2}$$

$$= \frac{1.(\sqrt{2})^2}{-2.1}$$

$$= -1$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} y = e^{-1}$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x} = \frac{1}{e}$$



CHECK YOUR PROGRESS

Q.1: Evaluate the following:

a) $\lim_{x\to 0} \frac{1-\cos x}{3x^2}$ b) $\lim_{x\to 0} \frac{x-\sin x}{x^3}$

c) $\lim_{x\to 0} \frac{1-\cos x}{x\log(1+x)}$ d) $\lim_{x\to 0} \frac{e^{ax}-e^{-ax}}{\log(1+bx)}$

e) $\lim_{x\to 0} \frac{x\cos x - \log(1+x)}{x^2}$

f) $\lim_{x\to 0} \frac{5\sin x - 7\sin 2x + 3\sin 3x}{\tan x - x}$

Q.2: If $\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^2}$ be finite, find a and the limit.

Q.3: Evaluate the followings.

a) $\lim\limits_{x\to\frac{\pi}{2}} (1-\sin x)\tan x$

b) $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

c) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}$

d) $\lim\limits_{x\to 0} x^x$

e) $\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

f) $\lim\limits_{x\to\frac{\pi}{2}}(\cos x)^{\frac{\pi}{2}-x}$

g) $\lim\limits_{x\to 0}\left(\frac{2x+1}{x+1}\right)^{\frac{1}{x}}$



7.4 LET US SUM UP

- If $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ is in $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exists, then

  $$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

  The process can be repeated any number of times provided the conditions hold.

- Limits of the form $\infty-\infty$ can be put in $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form on simplification.
- If limits takes the form 0^o, 1^∞, ∞^o, then by taking logarithm, it can be put in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

7.5 ANSWERS TO CHECK YOU PROGRESS

Ans. to Q. No. 1: a) $\lim\limits_{x\to 0}\frac{1-\cos x}{3x^2}\left[\frac{0}{0}\text{ form}\right]$

$= \lim\limits_{x\to 0}\frac{\sin x}{6x}\left[\frac{0}{0}\text{ form}\right]$

$= \lim\limits_{x\to 0}\frac{\cos x}{6}$

$= \frac{1}{6}$

b) $\lim_{x\to 0} \frac{x-\sin x}{x^3}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x\to 0} \frac{1-\cos x}{3x^2}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x\to 0} \frac{\sin x}{6x}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$

c) $\lim_{x\to 0} \frac{1-\cos x}{x\log(1+x)}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x\to 0} \frac{\sin x}{\log(1+x)+\frac{x}{1+x}}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x\to 0} \frac{\cos x}{\frac{1}{1+x}+\frac{1.(1+x)-x.1}{(1+x)^2}}$

$= \frac{1}{1+1} = \frac{1}{2}$

d) $\lim_{x\to 0} \frac{e^{ax}-e^{-ax}}{\log(1+bx)}$ $\left[\text{form } \frac{0}{0}\right]$

$= \lim_{x\to 0} \frac{ae^{ax}+ae^{-ax}}{\frac{1}{1+bx}.b}$

$= \frac{a+a}{b} = \frac{2a}{b}$

e) $\lim_{x\to 0} \frac{x\cos x-\log(1+x)}{x^2}$ $\left[\text{form } \frac{0}{0}\right]$

$= \lim_{x\to 0} \frac{\cos x-x\sin x-\frac{1}{1+x}}{2x}$ $\left[\text{form } \frac{0}{0}\right]$

$= \lim_{x\to 0} \frac{-\sin x-\sin x-x\cos x+\frac{1}{(1+x)^2}}{2}$

$= \frac{1}{2}$

f) $\lim\limits_{x\to 0} \dfrac{5\sin x - 7\sin 2x + 3\sin 3x}{\tan x - x} \left[\dfrac{0}{0} \text{ form}\right]$

$= \lim\limits_{x\to 0} \dfrac{5\cos x - 14\cos 2x + 9\cos 3x}{\sec^2 x - 1} \left[\dfrac{0}{0} \text{ form}\right]$

$= \lim\limits_{x\to 0} \dfrac{-5\sin x + 28\sin 2x - 27\sin 3x}{2\sec x.\sec x\tan x}$

$= \lim\limits_{x\to 0} \left[\dfrac{-5\sin x + 28\sin 2x - 27\sin 3x}{2\sec^2 x.x} . \dfrac{x}{\tan x}\right]$

$= \left(\lim\limits_{x\to 0} \dfrac{-5\sin x + 28\sin 2x - 27\sin 3x}{2x\sec^2 x}\right)\left(\lim\limits_{x\to 0} \dfrac{x}{\tan x}\right)$

$= \lim\limits_{x\to 0} \dfrac{-5\sin x + 28\sin 2x - 27\sin 3x}{2x\sec^2 x} \left[\text{form } \dfrac{0}{0}\right]$

$= \lim\limits_{x\to 0} \dfrac{-5\cos x + 56\cos 2x - 81\cos 3x}{2\sec^2 x + 2x.2\sec x.\sec x\tan x}$

$= \dfrac{-5 + 56 - 81}{2}$

$= -15$

Ans. to Q. No. 2: $\lim\limits_{x\to 0} \dfrac{\sin 2x + a\sin x}{x^2} \left[\dfrac{0}{0} \text{ form}\right]$

$= \lim\limits_{x\to 0} \dfrac{2\cos 2x + a\cos x}{2x}$

$\because$ Denominator $\to 0$ as $x \to 0$ and the limit is finite

$\therefore$ Numerator $\to 0$ as $x \to 0$

$\Rightarrow 2 + a = 0$

$\therefore a = -2$

$\therefore$ the given limit

$= \lim\limits_{x\to 0} \dfrac{2\cos 2x - 2\cos x}{2x} \left[\text{form } \dfrac{0}{0}\right]$

$= \lim\limits_{x\to 0} \dfrac{-4\sin 2x + 2\sin x}{2}$

$= 0$

Ans. to Q. No. 3: a) $\lim\limits_{x\to\frac{\pi}{2}} (1-\sin x)\tan x \left[\text{form } 0.\infty\right]$

$$= \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\cot x} \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}} \sin^2 x.\cos x = 0$$

b) $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$ [Form $\infty-\infty$]

$$= \lim_{x\to 0} \frac{\sin x - x}{x\sin x}$$

$$= \lim_{x\to 0}\left(\frac{\sin x - x}{x^2}\cdot\frac{x}{\sin x}\right)$$

$$= \left(\lim_{x\to 0}\frac{\sin x - x}{x^2}\right)\left(\lim_{x\to 0}\frac{x}{\sin x}\right)$$

$$= \lim_{x\to 0} \frac{\sin x - x}{x^2} \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\cos x - 1}{2x} \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\sin x}{2}$$

$$= 0$$

c) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}$. The given limit is in the form 1^{∞}

Let $y = (1+x)^{\frac{1}{x}}$

$$\therefore \log y = \frac{1}{x}\log(1+x)$$

$$\therefore \lim_{x\to 0}\log y = \lim_{x\to 0}\frac{\log(1+x)}{x} \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = 1$$

$$\therefore \lim_{x\to 0} y = e^1 \Rightarrow \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

d) $\lim_{x\to 0} x^x$. The given limit is in the form 0^o.

Let $y = x^x$

$\Rightarrow \log y = x\log x$

$\Rightarrow \lim_{x\to 0} \log y = \lim_{x\to 0} x \log x \ [0.\infty \text{ form}]$

$$= \lim_{x\to 0} \frac{\log x}{\frac{1}{x}} \left[\frac{\infty}{\infty} \text{form}\right]$$

$$\Rightarrow \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0} -x = 0$$

$\Rightarrow \lim_{x\to 0} y = e^o$

$\Rightarrow \lim_{x\to 0} x^x = 1$

e) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$. Given limit is in the form 1^∞

Let $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

$\therefore \log y = \frac{1}{x}\log\left(\frac{\tan x}{x}\right)$

$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x} \left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{\frac{x}{\tan x}\cdot\frac{x\sec^2 x - \tan x.1}{x^2}}{1}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^2} \quad \left(\because \lim_{x\to 0}\frac{x}{\tan x} = 1\right)$$

$$= \lim_{x\to 0} \frac{2x\sec^2 x\tan x + \sec^2 x - \sec^2 x}{2x}$$

$$= \lim_{x\to 0} \sec^2 x \tan x = 0$$

$\therefore \lim_{x\to 0} y = e^o \Rightarrow \lim_{x\to 0} (1+x)^{\frac{1}{x}} = 1$

f) $\lim_{x\to\frac{\pi}{2}} (\cos x)^{\frac{\pi}{2}-x}$. Given limit is in the form 0^o.

Let $y = (\cos x)^{\frac{\pi}{2}-x} \Rightarrow \log y = \left(\frac{\pi}{2} - x\right)\log(\cos x)$

$\therefore \lim_{x\to\frac{\pi}{2}} (\log y) = \lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)\log(\cos x)$ $[0.\infty$ form$]$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\log(\cos x)}{\frac{1}{\frac{\pi}{2}-x}} \quad \left[\frac{\infty}{\infty} \text{form}\right]$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\cos x}(-\sin x)}{\frac{-1}{\left(\frac{\pi}{2}-x\right)^2}(-1)}$$

$$= \lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)^2 (-\tan x) = 0$$

$\therefore \lim_{x\to\frac{\pi}{2}} y = e^o \Rightarrow \lim_{x\to\frac{\pi}{2}} (\cos x)^{\frac{\pi}{2}-x} = 1$

g) $\lim_{x\to 0} \left(\frac{2x+1}{x+1}\right)^{\frac{1}{x}}$. The limit is in the form 1^∞

Let $y = \left(\frac{2x+1}{x+1}\right)^{\frac{1}{x}}$

$\therefore \log y = \frac{1}{x}\log\left(\frac{2x+1}{x+1}\right)$

$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log(2x+1) - \log(x+1)}{x} \quad \left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x\to 0} \frac{\frac{2}{2x+1} - \frac{1}{x+1}}{1}$$

$$= \lim_{x\to 0} \left(\frac{2}{2x+1} - \frac{1}{x+1}\right) = 2 - 1 = 1$$

$\therefore \lim_{x\to 0} y = e^1 \Rightarrow \lim_{x\to 0} \left(\frac{2x+1}{x+1}\right)^{\frac{1}{x}} = e$



7.6 FURTHER READING

1) Gupta, Sudhir (2011) (1st edition): *Basics of Differentential Calculus*; New Delhi: Anmol publication Pvt. Ltd.
2) Habib, Amber (2011) (1st edition): *The Calculus of Finance*; Hyderabad: Universities Press India Pvt. Ltd.
3) Maron, I. A. (1993) (1st edition): *Problems in Calculus of One Variable*; New Delhi: CBS Publishers & Distributors Pvt. Ltd.
4) Sharma, A. K. (2004) (1st edition): *Text Book of Differential Calculus for B.A., B.Sc., B.Com., I.A.S., P.C.S.*; New Delhi: Discovery Publishing House.



7.7 MODEL QUESTIONS

Q.1: Evaluate the followings :

a) $\lim_{x\to 0}\frac{e^x - e^{\sin x}}{x - \sin x}$

b) $\lim_{x\to 0}\frac{\tan x - \sin x}{x^3}$

c) $\lim_{x\to 0}\frac{1+\sin x - \cos x + \log(1-x)}{x\tan^2 x}$

d) $\lim_{x\to 0}\frac{\sin x^2 - \sin^2 x}{x^4}$

e) $\lim_{x\to 0}\frac{a^x - b^x}{x}$

f) $\lim_{x\to 0}\frac{\log x}{\cos ecx}$

g) $\lim_{x\to 0}\cot x \log\frac{1+x}{1-x}$

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