UNIT 13: TESTING OF HYPOTHESIS

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13.1 LEARNING OBJECTIVES

After going through this unit you will be able to

- establish null and alternative hypotheses about a population parameter
- understand the two types of errors; interpret the confidence

level and the significance level

- develop hypothesis testing procedures for accepting or rejecting a null hypothesis
- learn how and when to use the normal and t distributions for hypothesis testing

13.2 INTRODUCTION

In today's dynamic business environment, decision makers need might answers to certain questions in order to take optimum decisions. For example a pharmaceutical concern might be interested in assessing whether a new drug is really effective in curing a disease; a personnel manager might be interested to study the job satisfaction level of employees, a marketing manager might be interested in knowing the customer satisfaction about a particular product, etc. In all the situations, the concerned manager has to take wise decisions since such decisions involve the risk of taking wrong decisions. It is here that modern theory of probability plays a vital role in decision making and the branch of Statistics that develops the criterion for such decisions is known as testing of hypothesis. Hypothesis testing begins with an assumption commonly termed as "hypothesis" that a researcher makes about an unknown population parameter. Hypothesis testing is a well-defined procedure which enables us to decide objectively whether to accept or reject the hypothesis based on the information obtained from the sample.

13.3 STATISTICAL HYPOTHESIS: NULL HYPOTHESIS AND ALTERNATIVE HYPOTHESIS

A statistical hypothesis is some assumption or statement, which may or may not be true, about an unknown population parameter. For example, suppose the Vice President (HR) of a company wants to examine the effectiveness of a training programme which the company has organized for all its employees located at different places of the country. The Vice President (HR) will then make an assumption that "training has not enhanced efficiency". To test such claim statistically, sample data are collected and

analyzed. The assumption is accepted or rejected on the basis of the sample findings.

There are two types of statistical hypothesis: (i) Null hypothesis and (ii) Alternative hypothesis. In hypothesis testing, to reach decisions we must start with a hypothesis, called null hypothesis which is symbolized H_0. The null hypothesis asserts that there exists no (significant) difference between the sample statistic and the population parameter and whatever observed difference is there is merely due to fluctuations of sampling from the same population. In the words of R.A. Fisher, "Null hypothesis is the hypothesis which is tested for possible rejection under the assumption that it is true." The null hypothesis that makes a claim regarding the specific value of the parameter is generally expressed by any one of the following mathematical statements

(i) $H_0 : \mu = \mu_0$ (ii) $H_0 : \mu \leq \mu_0$ (iii) $H_0 : \mu \geq \mu_0$

where μ is the population mean and denotes a hypothesized value of μ.

An alternative hypothesis, generally denoted by , is the opposite of the null hypothesis. Thus any hypothesis which is complementary to the null hypothesis is called an alternative hypothesis. It is obvious that both null and alternative hypotheses cannot be true simultaneously at the same time. In other words, when null hypothesis is found to be true, the alternative hypothesis is to be rejected or when null hypothesis is rejected, the alternative hypothesis must be true. For example, if we are interested in studying the hypothesis that the population has a specified mean, μ_0 (say), then we have the null hypothesis $H_0 : \mu = \mu_0$ and against this null hypothesis we may have any one of the following alternative hypotheses (depending upon the situation) given by,

(i) $(i.e., \mu > \mu_0 \ or \ \mu < \mu_0)$

(ii) $H_1 : \mu > \mu_0$ (iii) $H_1 : \mu < \mu_0$

The alternative hypothesis given by (i) is called two-tailed alternative and the alternatives given by (ii) and (iii) are called right-tailed and left-tailed alternatives respectively. Accordingly, the nature of the alternative hypothesis

enables us to decide whether we have to use an one-tailed (right or left) or two-tailed test.

Simple and Composite hypothesis: If the hypothesis completely specifies the population, then it is called simple hypothesis, otherwise it is called composite hypothesis. For example, from a normal population $N(\mu, \sigma^2)$, the hypothesis of the form

$H: \mu = \mu_0$ and $\sigma^2 = \sigma_0^2$ (both μ and are specified)

is a simple hypothesis because it completely specifies the distribution. On the other hand, the hypotheses of the forms

(i) $H: \mu = \mu_0$ (σ^2 is not specified), (ii) $H: \sigma^2 = \sigma_0^2$ (μ not being specified)

are composite since each of these does not completely specify the value of the parameter.

13.4 ERRORS IN HYPOTHESIS TESTING, LEVEL OF SIGNIFICANCE AND CRITICAL REGION

σ_0^2

In hypothesis testing, the null hypothesis should be accepted if it happens to be true and the null hypothesis H_0 should be rejected if it happens to be false. However, in practical situations, the correct decision is not always possible while testing a null hypothesis. Since the decision to reject or accept a null hypothesis depends only on sample data therefore a decision maker may commit two types of errors (i.e., take wrong decisions) which are diagrammatically shown in Fig.13.1.

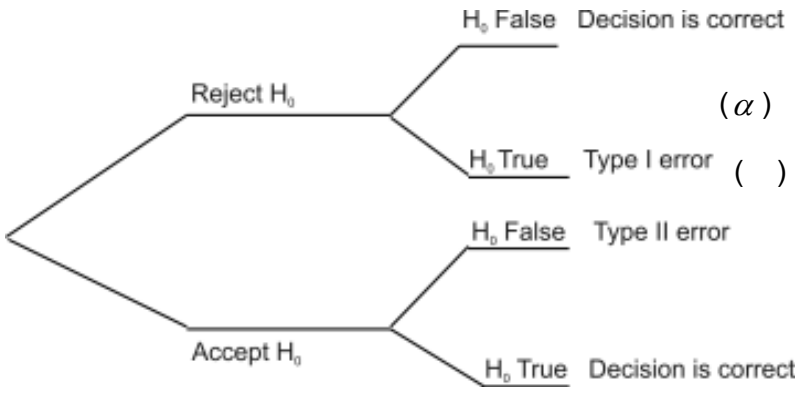


Figure 13.1: Type I and Type II errors

Type I error is committed if we reject a null hypothesis when it is true. The possibility of committing Type I error is denoted by the symbol . Thus,

(Type I error)= (Reject H_0 when H_0 is true)

On the other hand, Type II error is committed if we accept a null hypothesis when it is false. The possibility of committing Type II error is denoted by the symbol β. Thus we have

(Type II error)=(Accept H_0 when H_0 is false)

where and are also known as the sizes of Type I and Type II error respectively.

For example, a business firm, before introducing a new product, has to choose one of the two decisions- to introduce or not to introduce the product. Again if the product is introduced then the firm may face two situations- the failure of the product and the success of the product. The firm thus runs the risk of taking wrong decisions in two ways. The one risk of taking wrong decision known as Type II error, is that the firm does not introduce the product though it would have succeeded had it been introduced. The another risk of taking wrong decision known as Type I error is that the firm introduces the product but it failed.

In the terminology of Statistical Quality Control, Type I error is known as producer's risk while Type II error is known as consumer's risk.

In practice, it is not possible to make error free judgment in Statistics. Although both the errors should be very small yet it is not possible to minimize both these errors simultaneously at the same time on the same hypothesis test. Generally there exists an inverse relationship between and as a result of which an attempt to reduce type I error ( ) leads to an increase in Type II error ( ) and vice-versa. Since in most of decision making problems, it is more risky to accept a wrong hypothesis than to reject a correct one, i.e., consequences of Type II error are likely to be more serious than the consequences of Type I error therefore a compromise is made by minimizing the more serious error (i.e., β) after keeping fixed the less serious error (i.e.,).

In testing of hypothesis, the **Level of significance** is defined in terms of the probability of rejecting a null hypothesis when it is true. The level of significance is also known as the size of the critical region or the rejection region and generally denoted by α . It is to be noted that the level of significance should be determined before a test procedure so that obtained result is free from the influence of the decision maker. In practice, generally adopted level of significance by researchers is: 1 % (=0.01), 5% (=0.05) and 10% (=0.10). If a researcher use 5% level of significance, it would mean that in 5 samples out of 100 he is likely to reject a correct null hypothesis H_0. In other words the researcher is 95% confident that his decision to reject H_0 is correct. Thus higher the significance level adopted for testing a hypothesis, the higher the chance of rejecting a true null hypothesis.

In hypothesis testing, the area under the sampling distribution curve of the test statistic is divided into two mutually exclusive regions termed as acceptance region and rejection or critical region. The acceptance region is a range of values of the sample statistic that lead us to the acceptance of null hypothesis H_0 while critical region is the range of sample statistic values within which if values of the sample statistic falls lead to the rejection of null hypothesis H_0. In Fig. 8.2 if the calculated value of the sample statistic falls in the acceptance region, the null hypothesis is accepted otherwise it is rejected. For making a decision whether to accept or reject a null hypothesis, a researcher has to determine the critical value that separates the regions of acceptance and rejection. The determination of critical value depends on the size of the rejection region, which is directly associated with the risk involved in decision making.

13.5 ONE TAILED AND TWO TAILED TESTS

In any test, there are two types of tests of hypothesis. They are: (i) One-tailed tests and (ii) Two-tailed tests. Whether to use a one-tailed test or two tailed test depends entirely on the formulation of the alternative hypothesis. If the alternative hypothesis is one-tailed then the test to be applied is one-tailed and if the alternative hypothesis is two-tailed then the corresponding test is two-tailed. In case of two-tailed test, the critical or

significant values of the test statistic lie towards both the tails of the graph of the sampling distribution of the statistic. Figure 13.2 relates to standard normal test statistic The figure reveals that critical values of the standard normal test statistic $Z.$ lies on both sides of the mean i.e., in both the tails of the distribution. The shaded region under the normal curve is the rejection region (or critical region) at 5% level of significance, the region being 5% of the total area. In other words, the total area under the normal curve being unity, the size of the rejection region is 0.05.

Again sometimes we may encounter certain situations where alternative hypothesis states that either one process is better than the other or one process is worse than the other. But in two-tailed test the alternative hypothesis H_1 states that both may occur i.e., one process may be better or worse than the other. Figure 13.3 and Figure 13.4 below exhibits the critical values and critical regions of the standard normal test statistic for right-tailed test and left-tailed test respectively for 5% level of significance.

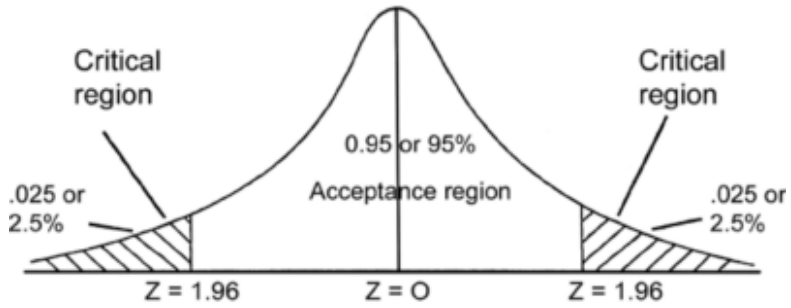


Fig. 13.2 Two-tailed test

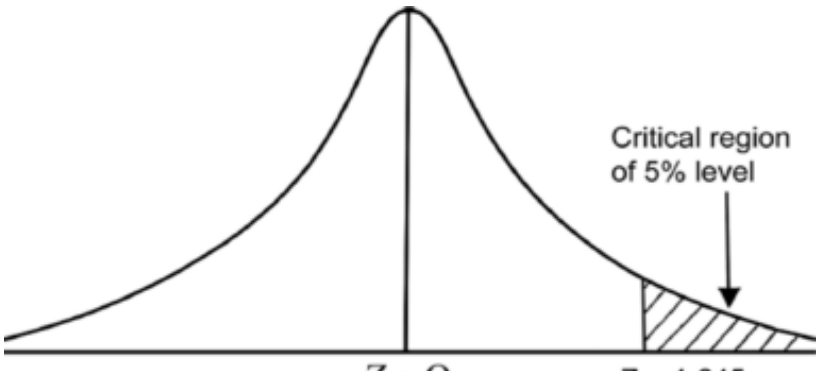
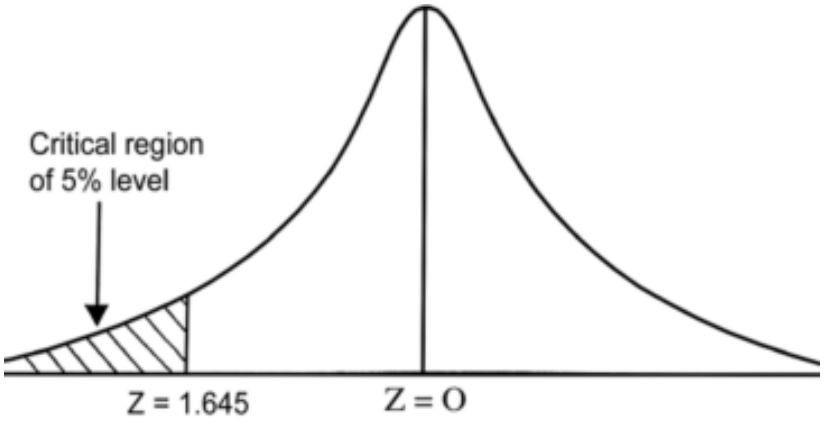


Fig. 13.3 Right-tailed test



13.4 Left-tailed test

Now, we present below a summary of certain critical values of test statistic Z (i.e., standard normal test statistic) for both one-tailed and two-tailed tests at some commonly used level of significance. These critical values of Z for different levels of significance have been obtained by using the table of area under normal curve.

Z

Table 8.1

Critical values (Z_α)	Level of significance (α)		
	5% or 0.05	1% or 0.01	10% or 0.010
Two-tailed test	$\lvert Z_\alpha \rvert = 1.96$	$\lvert Z_\alpha \rvert = 2.58$	$\lvert Z_\alpha \rvert = 1.645$
Right-tailed test	$Z_\alpha = 1.645$	$Z_\alpha = 2.33$	$Z_\alpha = 1.28$
Left-tailed test	$Z_\alpha = -1.645$	$Z_\alpha = -2.33$	$Z_\alpha = -1.28$

Note: (i) The notation Z_α stands for the critical value of Z at α level of significance.

(ii) Critical or significant values are also known as tabulated values.

(iii) If the size of the sample is small (usually less than 30) then the normality assumption of the sampling distribution of the test statistic is not valid and hence we cannot use the significant values given in the above table. In such situation we use the significant values based on the exact sampling distribution of , which turns out to be t, F or χ^2 (see units 10 and 11)

13.6 GENERAL PROCEDURE FOR HYPOTHESIS TESTING

Hypothesis testing is a well-defined systematic procedure that involves five different steps as given below:

1. The first step is to set up the null hypothesis H_0.

2. Set up the alternative hypothesis H_1. The nature of alternative hypothesis H_1 helps us to choose whether to be adopted one-tailed or two-tailed test.

3. To choose an appropriate level of significance. Commonly used levels of significance are: 0.01, 0.05 and 0.10.

4. After setting up the null and alternative hypotheses, the researcher needs to select an appropriate statistical test that will be used for analyzing statistical data. Type, number and the level of data may provide a platform in the selection of the statistical test. Apart from these, the statistics used in the study (mean, proportion and variance etc.) must also be considered when a researcher selects an appropriate test, which can be used for testing of hypothesis to obtain the best results. Some of the widely used testing procedures are: z, t, F and χ^2.

5. The final step in any hypothesis testing is to draw a statistical decision, which involves either to accept or to reject a null hypothesis. This depends on whether the calculated value of the test statistic falls in the acceptance region or in the rejection region at a given level of significance. In other words, if the computed value of Z (in modulus value) is less than the critical value Z_α then we say that it is not significant and null hypothesis may be accepted. On the other hand, if the computed value of Z (in modulus value) is greater than the critical value Z_α then we say that it is significant and null hypothesis is rejected at the level of significance α.

Therefore various steps to be followed in testing of a statistical hypothesis can be summarized as shown in Figure 13.5

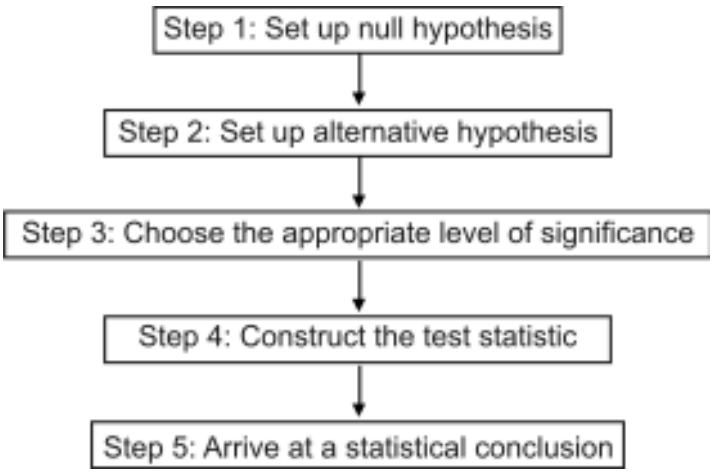


Figure 13.5: Steps in Hypothesis testing

Power of Test: Another way to evaluate the goodness of a statistical test is the complement of type II error, known as power of a test, which is defined as:

P

Power of a test = (rejecting H_0 when it is false)

= P (rejecting H_0 when H_1 is true)

= 1- P (accepting H_0 when H_1 is true)

= 1- P (type II error)

= 1- β

Where β is the probability of type II error. Among a class of tests, the best is the one which has the maximum power for the same size.



CHECK YOUR PROGRESS

Q 1: What is Hypothesis testing?

Q 2: What is Type I error?

Q 3: What is Type II error?

13.7 TESTING OF HYPOTHESIS IN CASE OF LARGE SAMPLES

Here we shall develop the tests of significance in case of large samples. In practice, a sample is to be considered as large if the sample size is greater than or equal to 30 i.e., . Now since the assumptions made in case of large samples are not valid for small samples therefore the tests of significance used to deal with problems relating to large sample are different from the ones used for small samples.

The following are the assumptions made in testing of hypothesis relating to large samples

(i) The population from which the sample is drawn is normally distributed. Consequently the sampling distribution of the sample statistic is also normal.

(ii) Since sample size is large, due to central limit theorem, the value given by any random sample can be used in place of population value for calculating the standard error of the estimate.

13.7.1 HYPOTHESIS TESTING FOR SINGLE POPULATION MEAN

Suppose a random sample of size $n(n \geq 30)$ with mean $\bar{x}$ is drawn from a population. We hold the claim that the population mean μ has the specified value .

Here we set up the null hypothesis $H_0 : \mu = \mu_0$. Against this hypothesis we may have any one of the following three alternative hypotheses

(i) $H_1 : \mu \neq \mu_0$ (Two-tailed test)

(ii) $H_1 : \mu > \mu_0$ (Right-tailed test)

(iii) $H_1 : \mu < \mu_0$ (Left-tailed test)

Under the null hypothesis the test statistic to be constructed is

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

where $\mu = \mu_0$, σ is the population standard deviation, is the sample size and is the sample mean.

In many real life situations, the population standard deviation remains unknown. To overcome such type of situations, in case of large sample ($n \geq 30$), the unknown population standard deviation σ can be replaced by a sample standard deviation .

For the comparison of the calculated value of Z with the critical value at the given level of significance, Table 8.1 is used and on the basis of the comparison decision to accept or reject the null hypothesis is taken as per the procedure discussed in 8.6.

We will now discuss the following examples.

Example 13.1 A business firm conducted a sample survey 15 years ago and claimed that the average household income of a particular geographic region is Rs. 10,000. Mr. Baruah, recently appointed as vice president in the firm has expressed doubts about the accuracy of the data. In order to verify the claim, a random sample of 200 households is tested. It is found that the mean household income is Rs. 11,000. Assume that the population standard deviation of the household income is Rs. 1200. Can we support the claim at 0.05 level of significance?

Solution: Here first we set up the null hypothesis H_0 and alternative hypothesis H_1 as follows:

$H_0 : \mu = 10{,}000$

$H_1 : \mu \neq 10{,}000$

Given, $n = 200$, $\bar{x} = 11{,}000$, $\sigma = 1200$ and

Since the sample size is large , therefore the Z test can be applied to test hypothesis.

Under the null hypothesis, the test statistic is given by

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$$= \frac{11{,}000 - 10{,}000}{\frac{1200}{\sqrt{200}}} = 11.79$$

Therefore the calculated value of Z is 11.79

Again the critical value of Z at 5% level of significance for two-tailed test =1.96 Since the calculated value is Z greater than the critical value of Z, therefore the statistical decision is to reject the null hypothesis and conclude that the claim is not valid. Mr. Baruah's doubt about average household income was right.

Example 13.2: A company conducted a survey in the past to its consumers which reveals that the average household income is Rs. 20,000 in a particular region. Due to the rapid growth of development the company thinks that the average income may have changed. To verify this, a random sample of 50 households is selected and found that sample mean is Rs. 30,000 with a standard deviation of Rs. 15,000. Use 5% level of significance to verify the finding.

Solution: In the light of the problem, the null hypothesis H_0 and alternative hypothesis H_1 are given as

$$H_0 : \mu = 20{,}000$$

$$H_1 : \mu \neq 20{,}000$$

Given, $n = 50$, $\bar{x} = 30{,}000$, $s = 15{,}000$ and $\alpha = 0.05$

Since the sample size is large , therefore the Z test can be applied to test hypothesis.

Under the null hypothesis, the test statistic is given by

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$$= \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}$$

(since the population S.D. σ is unknown, therefore it can be replaced by the sample S.D.)

Therefore the calculated value of Z is 4.714

$$= \frac{30,000 - 20000}{\frac{15,000}{\sqrt{50}}} = \frac{70.71}{15} = 4.714$$

Again the critical value of Z at 5% level of significance for two-tailed test =1.96 Since the calculated value Z is greater than the critical value of Z therefore the statistical decision is to reject the null hypothesis and conclude that average household income is significantly different from Rs. 20,000.

Example 13.3: The mean life of a sample of 80 fluorescent electric bulbs produced by a company is found to be 2000 hours with a standard deviation of 400 hours. Test the hypothesis that the mean life time of the bulbs produced in general is higher than the mean life of 1900 hours at $\alpha = 0.01$ level of significance.

Solution: For the problem, let us take the null hypothesis that mean life of bulbs is not more than 1900 hours. Then the null hypothesis H_0 and alternative hypothesis H_1 will be as follows

$H_1 : \mu > 1900$ (Right-tailed test)

Given, $n = 80$, $\bar{x} = 2000$, $s = 400$ and $\alpha = 0.01$

Since the sample size is large , therefore the test can be applied to test hypothesis.

Under the null hypothesis, the test statistic is given by

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$$= \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}$$

(since the population S.D. σ is unknown, therefore it can be replaced by the sample S.D.)

Therefore the calculated value of Z is 2.23

Again the critical value of Z at 1% level of significance for right-tailed test =2.33

Since the calculated value Z is less than the critical value of Z, therefore the statistical decision is to accept the null hypothesis. Hence we conclude that mean life of bulbs produced by the company is not more than 1900 hours.

13.7.2 HYPOTHESIS TESTING FOR SINGLE POPULATION PROPORTION

There are many areas where information that is highly dynamic in nature is generally expressed in terms of proportions or percentage

such as, we often find that the market share of a company is 40% or 25% of the customers have switched from one brand to another brand or 10% of the items are defective. Quality defects, consumer preferences, market share etc. are some of the common areas where researchers may be interested to check the hypothesis whether such proportions have changed. For this a random sample of size n is selected from a large population possessing a particular attribute of interest (also termed as success) then

$p =$ observed sample proportion of successes =

Here we are interested to test the hypothesis that the proportion P in the population has a specified value P_0.

We set up the null hypothesis $H_0 : P = P_0$

Depending upon the nature, the alternative hypothesis may be any one of the following:

$$Z = \frac{p - P}{\sqrt{\frac{PQ}{n}}} \sim N(0,1)$$

(i) $P \neq P_0$ (Two-tailed test)
(ii) $P > P_0$ (Right-tailed test)
(iii) $P < P_0$ (Left-tailed test)

Since n is large, therefore under the null hypothesis, the test statistic to be constructed is

where $P = P_0, Q = 1 - P = 1 - P_0$

To compare the calculated value of Z with the critical value at the given level of significance for the statistical decision, we use the Table 8.1.

Example 13.4: The production manager of a company that manufactures electric heaters claims that at least 10% of the heaters are defective. In order to test his believe, he takes a random sample of 100 heaters and finds that 12 heaters are defective. Use 5% level of significance to test the hypothesis.

Solution: Given $n = 100$, Number of defective heaters $= 12$

sample proportion of defective heaters

Null hypothesis $H_0 : P = 0.10$

Alternative hypothesis $H_1 : P \neq 0.10$

Since the sample size is large (≥ 30), therefore the Z test can be applied to test hypothesis.

Under the null hypothesis, the test statistic is given by

$$Z = \frac{p - P}{\sqrt{\frac{PQ}{n}}}$$

$$= \frac{0.12 - 0.10}{\sqrt{\frac{0.10 \times 0.90}{100}}} \quad (\because Q = 1 - P = 1 - 0.10 = 0.90)$$

$$= \frac{0.02}{0.03} = 0.67$$

$\therefore$ The calculated value of

The critical value of Z for two-tailed test at 5% level of significance=1.96 Since the calculated value is less than the critical value (0.67<1.96) therefore the statistical decision leads to acceptance of null hypothesis and conclude that the production manager's claim that at least 10% of the products are defective is valid. Consequently, the manufacturer can plan a strategy considering the fact that 10% of the products may be defective.

Example 13.5: A manufacturer of LCD TV claims that it is becoming quite popular in Guwahati, and that about 5% homes are having LCD TV. However, a dealer of conventional TVs claims that the percentage of homes with LCD TV is less than 5%. A sample of 500 household is surveyed, and it is found that only 18 households have LCD TV. Test at 1% level of significance whether the claim of the company is tenable.

Solution: Given, Given $n = 400$, Number of households having LCD TV

$p =$ sample proportion of LCD TV owners

Null hypothesis $H_0 : P = 0.05$

Alternative hypothesis $H_1 : P < 0.05$

Since the sample size is large (≥ 30), therefore the Z test can be applied to test hypothesis.

Under the null hypothesis, the test statistic is given by

$$Z = \frac{p - P}{\sqrt{\frac{PQ}{n}}}$$

$$= \frac{0.045 - 0.05}{\sqrt{\frac{0.05 \times 0.95}{500}}} \quad (\because Q = 1 - P = 1 - 0.05 = 0.95)$$

$= \frac{18}{500} = 0.045$

$$= \frac{-0.005}{0.0097}$$

$$= -0.515$$

$$\therefore \quad |Z| = 0.515$$

Therefore the calculated value of $|Z| = 0.515$

The critical value of Z for one-tailed test at 1% level of significance = 2.33 Since the calculated value is less than the critical value (0.515 < 2.33) therefore the statistical decision leads to the acceptance of null hypothesis and conclude that the manufacturer's claim that about 5% homes in Guwahati are having LCD TV is valid.

Example 13.6: The manager of a workshop claimed that at least 95% of the equipments which he supplied to a factory conformed to specifications. To verify his claim a sample of 200 pieces of equipments was selected and revealed that 18 were faulty. Test his claim at a significance level of 0.05.

Solution: Given n=200,

$P =$ Proportion of pieces conforming to specification in the lot = 95% =0.95

$Q =$ Proportion of pieces not conforming to specification in the lot = $1 - P = 0.05$

Sample proportion $\mathrm{p} = \frac{200 - 18}{200} = 0.91$

We take the null hypothesis that the proportion of pieces conforming to specification is 95% i.e., $H_0 : P = 0.95$

Alternative hypothesis $H_1 : P < 0.10$ (left-tailed test)

Since the sample size is large (≥ 30), therefore the test can be applied to test hypothesis.

Under the null hypothesis, the test statistic is given by

$$\mathrm{Z} = \frac{\mathrm{p} - \mathrm{P}}{\sqrt{\frac{\mathrm{PQ}}{\mathrm{n}}}}$$

$$= \frac{0.91 - 0.95}{\sqrt{\frac{0.95 \times 0.91}{200}}}$$

$$= \frac{-0.04}{0.015}$$

$$= -2.6$$

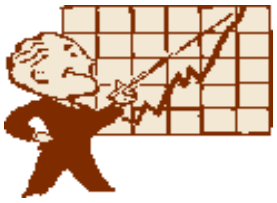
$$\therefore |Z| = 2.6$$

The critical value of $|Z|$ at $\alpha = 0.05$ for one-tailed test = 1.645.

Since the calculated value of is greater than the critical value (i.e., 2.6 >1.645) therefore we reject the null hypothesis at 5% level and conclude that the claim of the manager is not valid.

[Alternatively, without taking the modulus we may compare the calculated value of Z with the critical value of $Z = -1.645$ at $\alpha = 0.05$

for left-tailed test. Since , the null hypothesis is to be rejected and thus both types of comparison lead us to the same decision.]



CHECK YOUR PROGRESS

Q 4.State whether the following statements are true or false:

(i) A tentative assumption about a population parameter is called the null hypothesis.

(ii) The equality part of the expression (either or) always appears in the null hypothesis.

(iii) The estimated variance of sampling distribution of a statistic is called standard error.

(iv) The test statistic to test for large sample is z-test.

(v) The usual notation for the standard error of the sampling distribution is .

(vi) Suppose for a particular hypothesis and . Then the power of this hypothesis test is 0.90.

(vii) If 2.81 is the critical value of test-statistic , then the significance level of the test is 0.05.

(viii) The large sample test for testing for normal population is t-test.

(viii) For test of hypothesis against $H_1 : \mu > \mu_0$, the critical region at $\alpha = 0.05$ and , is $z > 1.96$.

(ix) When using the sample proportion $\bar{p}$ to test the hypothesis, the standard error of $\bar{p}$ is $\sqrt{pq/n}$.

(x) Sampling distribution will be approximately normal if sample size is small.

P – Value Approach for Hypothesis Testing: This approach is applied in hypothesis testing for large samples and sometimes referred to as the observed level of significance. This approach is very advantageous especially after the introduction of many statistical software packages. The P – value defines the smallest value of α for which

the null hypothesis can be rejected. The decision rule for accepting or rejecting a null hypothesis based on the value as discussed below:

Reject the null hypothesis H_0 when the value is < α, otherwise, accept the null hypothesis

For example, suppose a researcher conducting a hypothesis test with rejection region on the right tail of the distribution obtains an observed test statistic value of 2.12. From the normal table, the corresponding probability area for z value 2.12 is 0.4830. Therefore, the probability of getting a value greater than or equal to 2.12 is 5.0000-0.4830=0.017 (as shown in the following figure)

In this case, the value is 0.017. For =0.05 and =0.01, this value falls under the rejection region (=0.05, 0.017< 0.05 and =0.01, 0.017< 0.01), so the null hypothesis will be rejected at 0.05 0.01 levels of significance. At =0.01, the researcher cannot reject the null hypothesis for the value of equal to 0.017 because =0.01< 0.017.

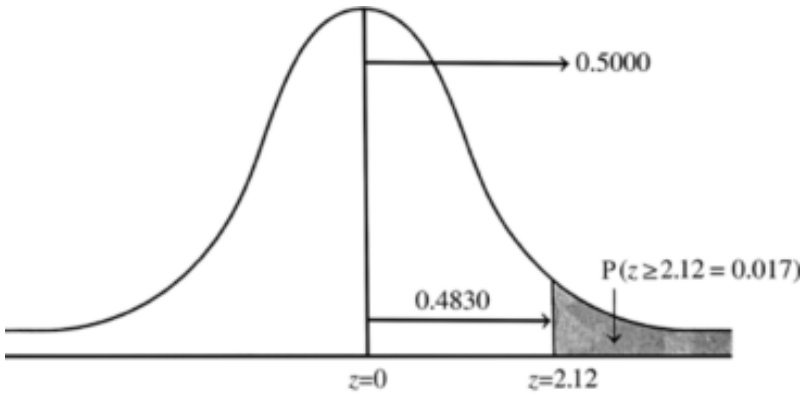


Fig. 13.5 Probability of p (z 2.12)

The procedure of calculating the value for a two-tailed test is slightly different. In a two-tailed test, the rejection region falls in both the tails of the normal distribution. For example, at =0.05, the rejection region is located in both the tail areas of the distribution in terms of 0.025% (0.025 +0.025=0.05) area in both the tails of the distribution. The

researcher compares this value to the computed value for accepting or rejecting the null hypothesis. For this comparison, instead of splitting the value, we double the value and then compare that value with the value. In the above mentioned example, the researcher conducted a hypothesis test with rejection region on both the tails of the distribution and obtained the observed test statistic as 2.12 and corresponding value as 0.017. So, instead of splitting value in two parts, the researcher should double the value, i.e., (0.017×2=0.034). So, for a two-tailed test, this value (0.034) is compared with the values 0.05 and 0.01. Hence, at α =0.05 and α =0.01 the null hypothesis will be rejected but at α =0.01 the null hypothesis will be accepted (0.01< 0.034).

Example 13.7: A marketing research firm conducted a survey 10 years ago and observed that the average household income of a particular geographic region is Rs. 10,000. Mr. Gupta, who has recently joined the firm as a vice president has expressed doubts about the accuracy of the data. For verifying the data, the firm has decided to take a random sample of 200 households that yield a sample mean (for household income) of Rs. 10,200. Assume that the population standard of the household income is Rs. 1200. Use the -value approach to verify Mr. Gupta's doubts. (Use as the level of significance.)

$H_0 : \mu = 10,000$

Solution: We set up the null and alternative hypothesis given by

Null hypothesis

Alternative hypothesis $H_1 : \mu \neq 10,000$

Since the sample size is large (≥ 30), therefore the Z test can be applied to test hypothesis.

Under the null hypothesis, the test statistic is given by

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$$= \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}$$

(since the population S.D. σ is unknown, therefore it can be replaced by the sample S.D.)

Therefore the observed test statistic is computed as 2.36.

From the normal table, the corresponding probability area for Z value = 2.36 is 0.4909. So, the probability of obtaining a Z value greater than or equal to 2.36 is 0.5000-0.4909=0.0091 (as obtained from the following figure).

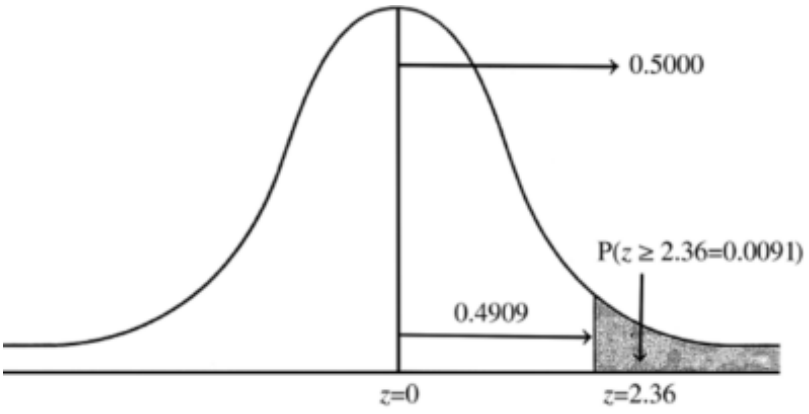


Fig. 13.6 Probability of p ($z \geq 2.36$)

For a two-tailed test, the obtained area value 0.0091 is to be multiplied by 2 and thus producing the value 0.0091×2=0.0182. Now since, , therefore we may accept the null hypothesis.

13.7.3 HYPOTHESIS TESTING FOR DIFFERENCE BETWEEN TWO POPULATIONS MEANS

In the real-world business environment, business researchers often encounter situations where they need to compare two means taken from two different samples drawn from two populations. On the basis of the samples researchers make an inference about the difference in the means of the two populations. For example, suppose a researcher is interested to conduct a survey to study the consumer satisfaction for a particular brand of product in two different cities, Guwahati and Dibrugarh. In order to accomplish this, the researcher selects two different random samples from the two cities taken in the study. By using the procedures of testing of hypothesis, finally, the researcher draws a statistical conclusion regarding the consumer satisfaction in the two cities. Now, let and $\overline{x_2}$ be the means of two large random samples of sizes n_1 and n_2 drawn from two populations with means μ_1 and μ_2 and standard deviations σ_1 and σ_2. We want to test the hypothesis whether the means of the two populations are equal or if there exists any significant difference between the population means.

$\overline{x_1}$

Thus the null hypothesis is $H_0 : \mu_1 = \mu_2$

The alternative hypothesis is $H_1 : \mu_1 \neq \mu_2$

Assuming the null hypothesis to be true, the test statistic

$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

The decision rule is same as discussed in 8.7.2 and 8.7.3.

Example 13.8: An investigation of the relative merits of two kinds of flashlight batteries showed that a random sample of 100 batteries of brand A lasted on the average for 35.5 hours with a standard deviation of 1.8 hours, while a random sample of 80 batteries of brand B lasted

on the average 36.8 hours with a standard deviation of 1.5 hours. Do the two brands differ significantly in quality? Use $\alpha = 0.05.$

Solution: Let and denote the means of two populations.

Let us set up the null hypothesis that the two brands do not differ significantly in quality, i.e.,

and an alternative hypothesis $H_1 : \mu_1 \neq \mu_2$

Since the sample size is large (≥ 30), therefore the Z test can be applied to test hypothesis.

Under the null hypothesis, the test statistic is given by

$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

$$= \frac{36.5 - 36.8}{\sqrt{\frac{3.24}{100} + \frac{2.25}{80}}} = \frac{-0.3}{\sqrt{0.0324 + 0.0281}} = -1.2195$$

$$\therefore |Z| = 1.2195$$

The critical value of $|Z|$ for two-tailed test at $\alpha = 0.05$ is 1.96.

Since the calculated value of is less than the critical value of $|Z|$ therefore we may accept the null hypothesis and conclude that there is no significant difference between the average life times of the two brands of batteries.

Example 13.9: The means of two large samples of size 1000 and 2000 are found to be 67.5 and 68.0 respectively. Test the equality of the two populations each with S.D. 2.5.

Solution: Let us set up null and alternative hypothesis as:

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$

Where μ_1 is the mean of the first population and μ_2 is the mean of the second population.

Since the sample size is large (≥ 30), therefore the test can be applied to test hypothesis.

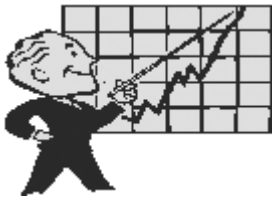
Under the null hypothesis, the test statistic is given by

$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

$$= \frac{67.5 - 68.0}{\sqrt{\frac{6.25}{1000} + \frac{6.25}{2000}}} = \frac{-0.5}{\sqrt{0.00625 + 0.00313}} = -5.16$$

$\alpha = 0.05$ $\therefore |Z| = 5.16$

Since the calculated value of $|Z|$ is greater than the critical value of $|Z|$ therefore we reject the null hypothesis and conclude that there is significant difference between population means.



CHECK YOUR PROGRESS

Q 5. A sample of 40 observations is selected from one population with S.D. (σ) 5. The sample mean is 102. A sample of 50 observations is selected from a second population with S.D. 6. The sample mean is 99. Conduct the following test of hypothesis (use)

(a) Is this a one tailed or a two-tailed test? (b) State the decision rule. (c) Compute the value of the test statistic. (d) What is your decision regarding ?

13.7.4 HYPOTHESIS TESTING FOR DIFFERENCE BETWEEN TWO POPULATION PROPORTIONS

At times, we come across certain situations where we are interested to study whether or not there exist significant differences between the proportions of two groups of, say, consumer in respect of a certain activity. Let p_1 and p_2 be the proportions of a certain attribute in two large random samples of sizes n_1 and n_2 drawn from two populations having proportions P_1 and P_2. On the basis of the samples, we want to test whether the two population proportions are equal or if there exist any significant difference between the population proportions. Thus we will have null and alternative hypotheses as:

$H_0 : P_1 = P_2$

$H_1 : P_1 \neq P_2$

Assuming the null hypothesis to be true, the test statistic

$$Z = \frac{P_1 - P_2}{\sqrt{\frac{P_1 Q_1}{n_1} + \frac{P_2 Q_2}{n_2}}} \sim N(0,1)$$

where $Q_1 = 1 - P_1, \quad Q_2 = 1 - P_2.$

The procedure of taking decision is same as discussed in 8.7.2 and 8.7.3.

Note: (i) If P_1 and P_2 remains unknown, then they can be estimated by the corresponding sample proportions p_1 and p_2.

(ii) Sometimes, population proportion can be estimated by pooling the two sample proportions p_1 and p_2 into a single proportion p as given

In this case, the test statistic Z can be restated as

$$Z = \frac{p_1 - p_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \quad q = 1 - p$$

Example 13.10: In two populated states there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations?

Solution: Given $n_1 = 1200, \quad n_2 = 900$

$$P_1 = 30\% = 0.30 \Rightarrow Q_1 = 0.70; \quad P_2 = 25\% = 0.25 \Rightarrow Q_2 = 0.75$$

Now we set up the null hypothesis H_0 that the difference in population proportions is likely to be hidden in samples i.e.,

$$H_0 : P_1 = P_2$$

And the alternative hypothesis is

$$H_1 : P_1 \neq P_2$$

Since the samples are large, under H_0 the test statistic Z is given by

$$Z = \frac{P_1 - P_2}{\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}} \sim N(0,1)$$

$$= \frac{0.30 - 0.25}{\sqrt{\frac{0.30 \times 0.70}{1200} + \frac{0.25 \times 0.75}{900}}}$$

$$= \frac{0.05}{\sqrt{0.000175 + 0.000208}} = \frac{0.05}{0.0196} = 2.55$$

Therefore the calculated value of Z =2.55

The critical value of Z at 5% level for two tailed test=1.96

Since the calculated value of Z > the critical value of Z, We reject the null hypothesis and hence, conclude that the difference in population

proportions is not likely to be hidden in these samples.

Example 13.11: A Company has the head office at Calcutta and a branch at Guwahati. The personnel director wanted to assess if the workers at the two places would like the introduction of a new plan of work to enhance the quality of services and a survey was conducted for the purpose. Out of a sample of 500 workers at Calcutta 62% favoured the new plan. At Guwahati out of a sample of 400 workers 41% were against the new plan. Is there a significant difference between the two groups in their in their attitude towards the new plan at 5% level? $n_2 = 400, \quad p_2 = 59\% = 0.59 \Rightarrow q_2 = 1 - 0.59 = 0.41$

Solution: Given $n_1 = 500, \quad p_1 = 62\% = 0.62 \Rightarrow q_1 = 1 - 0.62 = 0.38$

Now we set up the null hypothesis that the population proportion of workers favouring the new plan in Calcutta and Guwahati is same, i.e. $H_0 : P_1 = P_2$,

The alternative hypothesis is

Since the samples are large, under the test statistic is given by

Where $\mathrm{p} = \dfrac{\mathrm{n_1 p_1 + n_2 p_2}}{\mathrm{n_1 + n_2}}, \quad \mathrm{q} = 1 - \mathrm{p}$

$$Z = \frac{0.62 - 0.59}{\sqrt{0.007 \times 0.393\left(\frac{1}{500} + \frac{1}{400}\right)}},$$

$$= \frac{0.03}{0.0327} = 0.917$$

Therefore the calculated value of Z =0.917

The critical value of Z at 5% level for two tailed test=1.96

Since the calculated value of $Z <$ the critical value of , We may accept the null hypothesis and hence, conclude that there is no significant difference between the two groups in their attitude towards the new plan.

13.8 TESTING OF HYPOTHESIS IN CASE OF SMALL SAMPLES

We have already discussed that in case of large samples (i.e. $n \geq 30$,) the sampling distribution of the statistic is approximately normal. But in many decision making situations, due to time, money and other constraints, sometimes a researcher may have to take small sample of size less than 30 (i.e. $n < 30$,). In case of small samples also, the researcher can apply standard normal test for a specified population mean or equality of two population means provided the sample is drawn from a normal population where S.D. σ is known. But if remains unknown then standard normal test fails and in all such cases the sampling distribution turns out to be the student's 't' distribution. Sir William Gosset of Ireland in early 1900, under his pen name 'Student', developed a method for testing of hypothesis known as 't-test'. It is to be noted that the concept of degrees of freedom is associated with t-test. The graph of the 't' distribution approaches the normal curve as the degrees of freedom increase.

$\sigma_1^2, x_2, \ldots\ldots\ldots\ldots, x_n$

If be a random sample of size drawn from a normal population with mean μ and variance , then the Student's 't' statistic is defined by

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}},$$

where $\bar{x}$ = sample mean

and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$, sample mean square

is an unbiased estimator of σ^2 . If s^2 be the sample variance given by

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$$

Then $ns^2 = (n-1)S^2$

Thus we have, $\frac{S}{\sqrt{n}} = \frac{s}{\sqrt{n-1}}$

$\therefore \quad t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}}$

't' defined in (9.1) follows t-distribution with (n-1) d.f.

13.8.1 CHARACTERISTICS OF t-DISTRIBUTION

The characteristics of t distribution are given below:

(1) It is an exact distribution.

(2) The distribution is unimodal with

Mean=Median=Mode=0

(3) The t-distribution is less peaked than normal distribution at the centre and higher at both the ends.

(4) The shape of the -distribution depends on depends on the sample size . As increases, it approaches the normal distribution.

(5) The limits of is .

(6) The critical values of at level of significance vary with the degrees of freedom.

13.8.2 APPLICATIONS OF -DISTRIBUTION

The -distribution has a number of applications in Statistics some of which are given below

(i) It is used to test the significance of single mean, population standard deviation being unknown.

(ii) It is used to test the significant difference between two population means under the assumption that population standard deviations being equal but unknown.

(iii) It is used to test the significance of an observed sample correlation coefficient.

Assumptions for Student's -test: The following assumptions are made in Student's -test:

(i) The parent population from which the sample is drawn is normal.

(ii) The drawn sample is random.

(iii) The population standard deviation remains unknown.

13.8.3 HYPOTHESIS TESTING FOR SINGLE POPULATION MEAN

Suppose a random sample of size with mean $\bar{x}$ is drawn from a population μ. We want to test the hypothesis that the population has the specified value when the population S.D. is unknown.

Here we set up the null hypothesis .

Now depending upon the situation, against this hypothesis we may have any one of the following three alternative hypotheses

(i) $H_1 : \mu \neq \mu_0$ (Two-tailed test)

(ii) $H_1 : \mu > \mu_0$ (Right-tailed test)

(iii) $H_1 : \mu < \mu_0$ (Left-tailed test)

Since the sample is small, therefore under the null hypothesis the test statistic to be constructed is

$t = \dfrac{x - \mu}{\dfrac{s}{\sqrt{n-1}}}$ follows t distribution with d f.

where $s =$ sample S.D., , σ is the population standard deviation, is the sample size and is the sample mean.

Note: For large sample ($n \geq 30$), we have

S.E. ($\bar{x}$) = $\dfrac{s}{\sqrt{n}}$

On the other hand, for small sample ($n < 30$),

S.E. ($\bar{x}$) = $\dfrac{s}{\sqrt{n-1}}$

The critical values (i.e., tabulated values) of t distribution given in the table for 't' in the appendix are the critical values of for right-tailed test. The corresponding values for left-tailed test will just be the negative

of these values. If the values of for a two-tailed test at level are to be determined then instead of , we are to take .

If the alternative hypothesis is two-tailed as defined in (1), then the null hypothesis is to be rejected if the calculated value of . Again if the alternative hypothesis is one-tailed as defined in (2) or (3), the null hypothesis is to be rejected if the calculated value of $|t| > t_\alpha$.

Example 13.12: Royal Tyres has launched a new brand of tyres for tractors and claims that under normal circumstances the average life of the Tyres is 40,000 km. A retailer wants to test this claim and has taken a random sample of 8 Tyres that yields mean of 39,750 with a standard deviation of 2618.61. He tests the life of the Tyres under normal circumstance. (Use $\alpha = 0.05$)

Solution: Here the size of the sample is 8 (< 30) and the population standard deviation is unknown, so test can be applied to test the hypothesis.

Given and $s = 2618.61$

Now, we set up the null and alternative hypotheses as:

Null hypothesis $H_0 : \mu = 40{,}000$

Alternative hypothesis $H_1 : \mu \neq 40{,}000$

Under the null hypothesis the test statistic is

$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n-1}}}$$

$$= \frac{39750 - 40000}{\frac{2618.61}{\sqrt{8-1}}}$$

$$= -\frac{250}{1070.677132} = -0.233$$

$$\therefore \quad |t| = |-0.233| = 0.233$$

Degrees of freedom= $n-1=7$

The critical value of $|t_{0.05}(7)|$ for two-tailed test = 2.365

Since the critical value of $|t|$ is less than the critical value of $|t|$ at 5% level for 7 d.f., therefore we may accept the null hypothesis at 5% level of significance and conclude that the company's claim is valid under normal conditions.

Example 13.13: The mean price of shares of a company during the year 1992 was Rs. 85. In 1993, the mean price of shares of 20 randomly selected days is found to be Rs. 75 with a standard deviation of Rs. 10. Is there a significant difference in the prices of shares for the two years? (Given $|t_{0.05}(19)| = 2.093$)

Solution: Here the size of the sample $(i.e., n)$ is 20 (<30) and the population standard deviation is unknown, so test can be applied to test the hypothesis.

Given and $s = Rs.10$

Now, we set up the null and alternative hypotheses as:

Null hypothesis

Alternative hypothesis $H_0 : \mu \neq Rs.85$

Under the null hypothesis the test statistic is

$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n-1}}}$$

Which follows t distribution with d f.

Now $$t = \frac{75-85}{\frac{10}{\sqrt{20-1}}}$$

$$= \frac{-10}{\frac{10}{\sqrt{19}}} = -\sqrt{19} = -4.36$$

$\therefore \quad |t| = |-4.36| = 4.36$

Degrees of freedom = $20 - 1 = 19$

The critical value of for two-tailed test = 2.093

Since the critical value of $|t|$ is greater than the critical value of $|t|$ at 5% level for 19 d.f., therefore we reject the null hypothesis at 5% level of significance and conclude that there is significant difference between the mean prices of shares in the two years.

Example 13.14: A fertilizer mixing machine is set to give 12 kg. of nitrate for every 100 kg. of fertilizer. Ten bags of 100 kg. each are examined. The percentage of nitrate so obtained is: 11, 14, 13, 12, 13, 12, 13, 14,11 and 12. Is there reason to believe that the machine is defective? (Use $\alpha = 0.05$)

Solution: Given

Let us consider the null and alternative hypotheses as

$H_0 : \mu = 12$ and $H_1 : \mu \neq 12$

We now calculate mean and standard deviation of the given sample as shown below:

Table 8.1: Calculations of sample mean and standard deviation

Variable x	$x - \bar{x} = x - 12.5$	$(x - \bar{x})^2$
11	-1.5	2.25
14	1.5	2.25
13	0.5	.25
12	-0.5	.25
13	0.5	.25
12	-0.5	.25
13	0.5	.25
14	1.5	2.25
11	-1.5	2.25
12	-0.5	.25
125	0	10.5

$\bar{x} = \frac{\sum x}{n} = \frac{125}{10} = 12.5$ and

Under null hypothesis the test statistic is

$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n-1}}}$$

Which follows t distribution with d.f

Now $t = \frac{12.5 - 12}{\frac{1.025}{\sqrt{10-1}}}$

$$= \frac{0.5}{\frac{1.025}{\sqrt{9}}} = \frac{0.5}{0.3417} = 1.46$$

$\therefore \quad |t| = |1.46| = 1.46$

$s = \sqrt{\frac{1}{n}\sum (x - \bar{x})^2} = \sqrt{\frac{10.5}{10}}$

$= \sqrt{1.05} = 1.025$

Degrees of freedom= $10 - 1 = 9$

The critical value of for two-tailed test = 2.262

Since the critical value of $|t|$ is less than the critical value of $|t|$ at 5% level for 9 d.f., therefore we may accept the null hypothesis at 5% level of significance and conclude that the manufacturer's claim is valid, i.e., the machine is not defective.

13.8.4 HYPOTHESIS TESTING FOR DIFFERENCE BETWEEN TWO POPULATIONS MEANS

In case of two populations, our concern is to test if two independent samples have been drawn from two normal populations having the same means, the population standard deviations being equal. Let $\overline{x_1}$ and $\overline{x_2}$ be the means of two independent random samples of sizes n_1 and n_2 ($n_1 < 30, n_2 < 30$) drawn from two normal populations having means μ_1 and μ_2 respectively with equal but unknown standard

deviations σ_1 and .

Thus the null hypothesis is , i.e., samples have been drawn from the normal populations with same means.

The alternative hypothesis H_1 may be any one of the following depending upon the situation.

(1) $H_1 : \mu_1 \neq \mu_2$ (two-tailed)
(2) $H_1 : \mu_1 < \mu_2$ (left-tailed)
(3) $.H_1 : \mu_1 > \mu_2$ (right-tailed)

Assuming the null hypothesis to be true, the test statistic t is given by

$$t = \frac{\overline{x_1} - \overline{x_2}}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}$$

i.e., the test statistic t follows distribution with n_1+n_2-2 d.f

where

and $S^2 = \frac{1}{n_1 + n_2 - 2}\left[\sum\left(x_1 - \overline{x_1}\right)^2 + \sum\left(x_2 - \overline{x_2}\right)^2\right]$

$= \frac{1}{n_1 + n_2 - 2}\left[n_1 s_1^2 + n_2 s_2^2\right]$, where $s_1^2 = \frac{1}{n_1}\sum\left(x_1 - \overline{x_1}\right)^2$ and $s_2^2 = \frac{1}{n_2}\sum\left(x_2 - \overline{x_2}\right)^2$

is an unbiased estimate of the common population variance σ^2 based on both the samples.

The decision to reject or accept the null hypothesis is similar to that used in one-sample tests.

Example 13.15: The IQs (intelligence quotients) of 16 students from one area of a city showed a mean of 107 with a standard deviation of 10, while the IQs of 14 students from another area of the city showed a mean of 112 with a standard deviation of 8. Is there a significant difference between the IQs of the two groups at 5% level of significance?

Solution:

Given $n_1 = 16, \quad n_2 = 14, \quad \overline{x_1} = 107, \quad \overline{x_2} = 112, \quad s_1 = 10$ and $s_2 = 8$

We set up null and alternative hypothesis as

Null hypothesis is $H_0 : \mu_1 = \mu_2$

Alternative hypothesis is $H_0 : \mu_1 \neq \mu_2$

Since sizes of both the samples are small and population standard deviations are unknown therefore to test the hypothesis we apply t test given by

i.e., the test statistic t follows t distribution with n_1+n_2-2 d.f.

Now

$$S^2 = \frac{1}{n_1+n_2-2}\left[n_1 s_1^2 + n_2 s_2^2\right]$$

$$= \frac{1}{16+14-2}\left[16(10)^2 + 14(8)^2\right]$$

$$= 89.142857$$

Thus

$$\therefore \quad t = \frac{112-107}{9.44\sqrt{\frac{1}{16}+\frac{1}{14}}} = 1.45$$

The degrees of freedom = $n_1+n_2-2=28$

The critical value of $|t_{0.05}(28)|$ for two-tailed test = 2.02

Since the critical value of $|t|$ is less than the critical value of $|t|$ at 5% level for 28 d.f., therefore we may accept the null hypothesis at 5% level of significance and conclude that the difference between the two population means is not significant.

Example 13.16: In a test given to two groups of students, the marks obtained are as given below:

First group : 25 32 30 32 24 14 32

Second group: 24 34 22 30 42 31 40 30 32 35

Examine the significance of the difference between the arithmetic mean of the marks secured by the students of the above two groups.

Solution: Here we set up null and alternative hypotheses as

Null hypothesis is $H_0 : \mu_1 = \mu_2$ i.e., there is no significant difference between mean marks secured by students of the two groups.

Alternative hypothesis is $H_0 : \mu_1 \neq \mu_2$

Since sizes of both the samples are small and population standard deviations are unknown therefore to test the hypothesis we apply t test given by

i.e., the test statistic t follows distribution with n_1+n_2-2 d.f.

Now

Calculations of $\overline{x_1}$, $\overline{x_2}$ and pooled sample standard deviations are given in Table 8.2

Table 8.2: Calculation of $\overline{x_1}$, $\overline{x_2}$ and S

x_1	$x_1-\overline{x_1}$ $=x_1-27$	$(x_1-\overline{x_1})^2$	x_2	$x_2-\overline{x_2}$ $=x_2-32$	$(x_2-\overline{x_2})^2$
25	-2	4	24	-8	64
32	5	25	34	2	4
30	3	9	22	-10	100
32	5	25	30	2	4
24	-3	9	42	10	100
14	-13	169	31	-1	1
32	5	25	40	8	64
			30	-2	4
			32	0	0
			35	3	9
189		266	320		350

Here $\overline{x}_1=\dfrac{\sum x_1}{n_1}=\dfrac{189}{7}=27$

Sample variance,

$s_1^2=\dfrac{\sum(x_1-\overline{x}_1)^2}{n_1}=\dfrac{266}{7}=38$

Sample variance, $s_2^2=\dfrac{1}{n_2}\sum(x_2-\overline{x}_2)^2=\dfrac{350}{10}=35$

Now, $S=\sqrt{\dfrac{7\times38+10\times35}{7+10-2}}=6.41$

$\therefore\quad t=\dfrac{27-32}{6.41\sqrt{\dfrac{1}{7}+\dfrac{1}{10}}}$

$=\dfrac{-5}{6.41\sqrt{\dfrac{17}{70}}}=-\dfrac{5}{6.41\times0.49}=-1.592$

$\therefore\quad |t|=1.592$

The degrees of freedom = $n_1+n_2-2=15$

The critical value of $|t_{0.05}(15)|$ for two-tailed test = 2.131

Since the critical value of $|t|$ is less than the critical value of $|t|$ at 5% level for 28 d.f., therefore we may accept the null hypothesis at 5% level of significance and conclude that the mean marks obtained by the students of two groups do not differ significantly.

Example 13.17: Two salesmen, A and B, are employed by a company. To study if there is any significant difference between the average sales of the two salesmen, the HRD manager of the company conducted a survey yielding the following information:

	Salesman A	Salesman B
Number of sales	20	22
Average weekly sales (Rs. lakh)	30	25
Standard deviation (Rs. lakh)	10	7

Solution: First we set up the null hypothesis that there is no significant difference between the average sales of the two salesmen. That is, $H_0 : \mu_1 = \mu_2$

The alternative hypothesis is $H_1 : \mu_1 \neq \mu_2$

Since sizes of both the samples are small and population standard deviations are unknown therefore to test the hypothesis we apply t test given by

i.e., the test statistic t follows distribution with n_1+n_2-2 d.f.
Now

$$= \frac{1}{20+22-2}\left[20(10)^2 + 22(7)^2\right] = 76.95$$

Thus $S = 8.77$

The degrees of freedom = $n_1+n_2-2=40$

The critical value of $|t_{0.05}(40)|$ for two-tailed test = 2.021

Since the calculated value of $|t|$ is less than the critical value of $|t|$ at 5% level for 40 d.f., therefore we may accept the null hypothesis at 5% level of significance and conclude that the average sales by the two salesmen are not significantly different.



CHECK YOUR PROGRESS

Q 6.To the following information (a) State null and alternative hypotheses, (b) State the decision rule, (c) Compute the pooled estimate of the population variance, (d) Compute the test statistic. (e) What is your decision regarding H_0 ?

A random sample of 10 observations from one population revealed a sample mean of 23 and a S.D. of 4. A random sample of 8 observations from another population revealed a sample mean of 26 and a S.D. of 5. Use 5% level of significance to test if there is any significant difference between population means?

$\therefore$ $t = \dfrac{30-25}{8.77\sqrt{\dfrac{1}{20}+\dfrac{1}{22}}} = 1.84$

13.8.5 PAIRED t – TEST FOR DIFFERENCE BETWEEN TWO POPU LATIONS MEANS

In the test used for difference of means, the two samples were independent of each other whereas in case of paired test we have

(i) The sample sizes (and n_2) are equal.

(ii) The sample observations $(x_1, x_2, \ldots\ldots\ldots\ldots, x_n)$ and $(y_1, y_2, \ldots\ldots\ldots\ldots, y_n)$ are not completely independent but they are dependent in pairs i.e., the pairs $(x_1, y_1), (x_2, y_2), \ldots\ldots\ldots (x_n, y_n)$

correspond to the 1st, 2nd, ……...,nth unit respectively.

For example, the management of a company plagued by poor productivity realizes the need to provide technical training to employees. It hires a researcher to measure the productivity levels of a sample of 25 employees. The productivity levels are measured again after a one-month technical training programme. In this kind of pre- and post-training study, which are selected before and after the study cannot be treated as independent because each observation in sample 1 is related to the observation in sample 2. The productivity scores obtained before training is related to the scores obtained after training because the two measurements are apply to the same person.

Now we explain the paired $t-$ test for difference between the means of two related populations as follows:

Suppose a business concern is interested to know if advertisement is really effective in promoting sales of a particular product. Let ( ) be a sample of sales of the product in departmental stores for a certain period before advertisement campaign and ($y_1, y_2, ..., y_n$) be the sample of corresponding sales of the same product and for period in the same departmental stores respectively. Now $(x_i, y_i), i = 1,2,....., n$ is the pair of sales in the ith departmental store before and after advertisement. Suppose we want to test the significance of the difference between the sales, and.

Let $d_i = x_i - y_i, (i = 1,2,....., n)$

We set up the null hypothesis that there is no significant change in sales after advertisement campaign. i.e., $H_0 : \mu_1 = \mu_2$.

The alternative hypothesis is $H_1 : \mu_1 \neq \mu_2$.

Under the null hypothesis, the test statistic is

$$t = \frac{\bar{d}}{S/\sqrt{n}} \sim t_{n-1}$$

where $\mathrm{d} = \mathrm{x} - \mathrm{y}, \quad \bar{\mathrm{d}} = \frac{1}{\mathrm{n}} \sum \mathrm{d}$

and $S^2 = \frac{1}{n-1}\sum\left(d-\bar{d}\right)^2 = \frac{1}{n-1}\left[\sum d^2 - \frac{\left(\sum d\right)^2}{n}\right]$

Example 13.18: An electronic company arranged a special training programme for one segment of its employees. In order to measure the change in the attitude of its employees after the training, the company has prepared a well-designed questionnaire consisting of 10 questions on a 1 to 5 rating scale (1 is strongly disagree and 5 is strongly agree). The company selected a random sample of 10 employees. The scores are given below:

Employees	Scores before training	Scores after training
1	25	32
2	26	30
3	28	32
4	22	34
5	20	32
6	30	28
7	22	25
8	20	30
9	21	25
10	24	28

$H_0 : \mu_1 = \mu_2$

Test whether there is a significant change in the attitude of employees after the training programme. (Use $\alpha = 0.10$)

Solution: Here, scores before training ( ) and after training ( ) are not independent but are paired together. Hence, to test the difference between the means of two related populations we apply the paired test.

We set up the null and alternative hypotheses as:

The null hypothesis i.e., there is no significant effect of the training.

The alternative hypothesis $H_1 : \mu_1 \neq \mu_2$.

Under the null hypothesis, the test statistic is

$$t = \frac{\bar{d}}{S/\sqrt{n}} \sim t_{n-1}$$

where $d = x - y, \quad \bar{d} = \frac{1}{n}\sum d$

and $$S^2 = \frac{1}{n-1}\sum\left(d - \bar{d}\right)^2 = \frac{1}{n-1}\left[\sum d^2 - \frac{\left(\sum d\right)^2}{n}\right]$$

Calculations of $\bar{d}$ and S^2 are given below:

Employees	Scores before training(x)	Scores after training(y)	$d = x - y$	d^2
1	25	32	-7	49
2	26	30	-4	16
3	28	32	-4	16
4	22	34	-12	144
5	20	32	-12	144
6	30	28	2	4
7	22	25	-3	9
8	20	30	-10	100
9	21	25	-4	16
10	24	28	-4	16
Total			-58	514

$$\therefore d = \frac{\sum d}{n} = \frac{-58}{10} = -5.8$$

Thus $S = \sqrt{19.733333} = 4.44$

$$\therefore t = \frac{-5.8}{4.44/\sqrt{10}} = \frac{-5.8}{1.4047} = -4.13$$

$$|t| = 4.13$$

The degrees of freedom = 9

The critical value of $|t_{0.10}(9)|$ for two-tailed test = 1.833

Since the calculated value of $|t|$ is greater than the critical value of $|t|$ at 10% level for 9 d.f., therefore we reject the null hypothesis at 10% level of significance and conclude that there is a significant difference in the attitude of employees before and after the training.



CHECK YOUR PROGRESS

Q7. To the following (a) State null hypothesis, (b) State the decision rule, (c) Compute the test statistic. (e) What is your decision regarding H_0?

An IQ test was administered to 5 persons before and after they were trained. The results are given below:

Candidate:	I	II	III	IV	V
IQ before training:	110	120	123	132	125
IQ after training:	120	118	125	136	121

Test if there is any change in IQ level after the training programme.



13.9 LET US SUM UP

- Hypothesis testing is a well-defined procedure that helps us to decide objectively whether to accept or reject the hypothesis based on the information available from the sample. Testing of hypothesis involves five different steps.

- In case of testing of statistical hypothesis, there are two types of hypothesis tests namely two-tailed test and one-tailed test. Two-tailed tests contain the rejection region on both the tails of the sampling distribution of a test statistic. As different from a two-tailed test, a one-tailed test contains the rejection region on one tail of the sampling distribution of a test statistic.

- The testing of hypothesis involves two types of errors: Type I error

and Type II error. A Type I error is committed by rejecting a null hypothesis when it is true. The possibility of committing Type I error is termed as level of significance and is denoted by α . On the other hand, a type II error is committed by accepting a null hypothesis when it is false. The likelihood of committing Type II error is denoted by .

- The test can be applied for dependent samples and the testing procedure is termed as "paired test or test for related samples." Under this approach, observations in 1st sample are related to the observations in 2nd sample.

- Hypothesis can also be performed for single population proportion as well as for the difference in two population proportions. In case of two random samples, on the basis of the difference in sample proportions, a researcher can estimate the difference in population proportions.



13.10 FURTHER READINGS

1) Gupta S.C. (1994). Fundamentals of statistics. New Delhi, Himalayan Publishing House.

2) Sharma, J.K. (2007). Business Statistics. New Delhi. Pearson Education Ltd.

3) Hazarika, P.L. (2016). Essential Statistics For Economics And Business Studies. New Delhi. Akansha Publishing House.

4) Lind, D.A., Marshal, W.G., Wathen, S.A. (2009) Statistical Techniques in Business and Economics. New Delhi. Tata McGraw Hill Education Private Limited.



13.11 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q No 1: Hypothesis testing is the process of testing a statement or belief about a population parameter by the use of information collected from a sample (s).

Ans. to Q No 2: Type I error is the probability of rejecting the null hypothesis

when it is true. It is denoted by Greek letter .

Ans. to Q No 3: Type II error is the probability of accepting the null hypothesis when it is false. It is denoted by Greek letter .

Ans. to Q No 4: (i) T, (ii) T, (iii) F, (iv) T, (v) F, (vi) T, (vii) F, (viii) F, (ix) T, (x) F.

Ans. to Q No 5: (a) Two-tailed test, (b) Reject $|z| > 1.96$, (a)

$$z = \frac{102 - 99}{\sqrt{\frac{5^2}{40} + \frac{6^2}{50}}} = 2.59$$

(d) Reject H_0

Ans. to Q No 6: (a) Null hypothesis is $H_0 : \mu_1 = \mu_2$, Alternative hypothesis is $H_1 : \mu_1 \neq \mu_2$,

(b) Here $d.f. = 10 + 8 - 2 = 16$, Therefore tabulated value of $|t_{0.05}(16)|$ for two-tailed test = 2.120

The decision rule is to reject H_0 if $|t| > 2.120$.

H_0 (c) Pooled estimate of the population variance, $S^2 = \frac{(10)4^2 + (8)5^2}{10 + 8 - 2} = 22.5$

(d) $$t = \frac{23 - 26}{\sqrt{22.5\left(\frac{1}{10} + \frac{1}{8}\right)}} = -1.333$$

(e) H_0 may be accepted.

Ans. to Q No 7: (a) Null hypothesis is H_0: There is no change in IQ level after training programme.

(b) Reject H_0 if $t > 4.6$, the tabulated value of $|t_{0.05}(4)|$ for two-tailed test.

(c) $\bar{d} = \frac{10}{5} = 2, S = 5.477$ Therefore, $t = \frac{2}{5.477/\sqrt{5}} = 0.817$

(d) H_0 may be accepted.



13.12 MODEL QUESTIONS

1. Discuss the various steps involved in testing of hypothesis.

2. What do you understand by null hypothesis, alternative hypothesis and level of significance?

3. Define Type I error and Type II error.

4. What is meant by one-tailed test and two-tailed test?

5. The mean breaking strength of the cables supplied by a manufacturer is 1800 with a standard deviation of 100. By a new technique in the manufacturing process it is claimed that the breaking strength of the cables has increased. In order to test this claim a sample of 50 cables is tested. It is found that the mean breaking strength is 1850. Can we support the claim at 1% level of significance?

6. The MacBurger restaurant chain claims that the mean waiting time of customers is 3 minutes with a population standard deviation of 1 minute. The quality- assurance department found in a sample of 50 customers at the Warren Road MacBurger that the mean waiting time was 2.75 minutes. At the 5% significance level, can we conclude that the mean waiting time is less than 3 minutes?

7. A company manufacturing a certain type breakfast cereal claims that 60% of all housewives prefer that type to any other. A random sample of 300 housewives contains 165 who do prefer that type. At 5% level of significance, test the claim of the company.

8. A man buys 50 electric bulbs of ‘Philips’ and 50 electric bulbs of ‘HMT’. He finds that ‘Philips’ gave an average life of 1500 hours with a standard deviation of 60 hours and ‘HMT’ bulbs gave an average life of 1512 hours with a standard deviation of 80 hours. Is there a significant difference in the mean life of the two makes of bulbes?

9. 500 units from a factory are inspected and 12 are found to be defective, 800 units from another factory are inspected and 12 are found to be defective. Can it be concluded that at 5% level of significance production at the second factory is better than in first factory?

10. Ten oil tins are taken at random from an automatic filling machine. The mean weight of the tins is 15.8 kg. and the standard deviation is 0.50kg. Does the sample mean differ significantly from the intended weight of 16 kg?

11. Two salesmen A and B are working in a certain district. From a sample survey conducted by the Head Office, the following results were obtained. State whether there is any significant difference in the average sales between the two salesmen.

	Salesmen	
	A	B
Size of samples	20	18
Average sales (Rs. in thousands)	170	205
Standard deviation (Rs. in thousands)	20	25

12. The sales data of an item in six shops before and after a special promotional campaign are as under:

Shops	:	A	B	C	D	E	F
Before campaign	:	35	28	31	48	50	42
After campaign	:	58	29	30	55	56	45

Can the campaign be judged to be a success? Test at 5% level of significance?

(Significant values of $|t|$ at 1% level and 5% level for 5 d.f. respectively are 2.015 and 2.571)

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