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Project Management and Operations Research

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MASTER OF COMMERCE

BLOCK - 2

KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

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MASTER OF COMMERCE

PROJECT MANAGEMENT AND OPERATIONS RESEARCH

Block 2

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BLOCK INTRODUCTION:

This is the second block of the course "'Project Management and Operations Research'. The block consists of eight units.

Unit 8 provides the definition and a brief historical background of operations research.

Unit 9 is dedicated to formulation of linear programming problem.

The graphical method to solution of an LPP is explained with the help of a number of examples in unit 10. The cases of multiple, unbounded solution and infeasible are also illustrated graphically.

Unit 11 discuss the simplex method to solve LPPs involving more than two decision variables. Concept of Big M method and two phase method are also discussed in this unit.

The dual to an LPP is explained in unit 12.

In Unit 13, the formulation of Transportation problem is discussed. Three methods namely the north west corner method, Matrix minima method and Vogel's approximation method to determine an initial basic feasible solution are also discussed. In addition, the modified distribution (MODI) method for obtaining the optimal solution of a transportation problem are explained.

Unit 14 explains the formulation of an assignment problem with the help of examples. The Hungarian method for solving an assignment problem is also discussed in this unit.

Unit 15 deal with the game theory.

While going through a unit, you will notice some along-side boxes, which have been included to help you know some of the difficult, unseen terms. Again, we have included some relevant concepts in "LET US KNOW'' along with the text. And, at the end of each section, you will get "CHECK YOUR PROGRESS" questions. These have been designed to self-check your progress of study. It will be better if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with "ANSWERS TO CHECK YOUR PROGRESS" given at the end of each unit.

UNIT 8 : INTRODUCTION TO OPERATIONS RESEARCH

UNIT STRUCTURE

8.1 Learning Objectives

8.2 Introduction

8.3 Historical Background

8.4 Scope of Operations Research

8.5 Features of Operations Research

8.6 Phases of Operations Research

8.7 Types of Operations Research Model

8.8 Operations Research Techniques and Tools

8.9 General Methods for Solving Operations Research Model

8.10 Limitations of Operations Research

8.11 Let Us Sum Up

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8.13 Answers To Check Your Progress

8.14 Model Questions

8.1 LEARNING OBJECTIVES

After going through this unit, you will able to know

- historical background of operations research (OR)
- scope, features, phases of OR
- types of OR models
- OR techniques and tools
- general methods for solving OR model
- limitations of OR.

8.2 INTRODUCTION

The subject Operations Research (OR) is a branch of Mathematics/ Statistics. It is used to provide a scientific base for management to take timely and effective judgements to their problems. Initially it was used

successfully in military/army operations. Nowadays it is used in business, government and industry, and in many fields. [The Americans call the subject 'Operationg Research', whereas the British call it 'Operational Research'] A few opinions of some of the leading experts on the subject are given below.

1. Operational Research is a scientific method of providing executive departments with a quantitative basis for decision regarding the operations under their control.

   Morse and Kimball (1946)

   [Philip McCord Morse (1903-1985), American physicist, considered to be father of OR in the US; George Elbert Kimball (1906-1967), American professor of quantum chemistry, pioneer of OR algorithms]

2. Operational Research is a scientific method of providing executive with an analytical and objective basis for decisions.

   Blackett (1948)

   [Patrick Maynard Stuart Blackett (1897-1974), British experimental physicist, awarded Nobel Prize for physics in 1948, developed OR]

3. The terms 'Operations Research' have heretofore been used to connote various attempts to study operations of war by scientific methods. From a more general point of view OR can be considered to be an attempt to study those operations of modern society which imvolve organisations of men and machines.

   Morse (1948)

4. Operations Research is the art of giving bad answers to problems to which otherwise worse answers are given.

   Saaty (1958)

   [Thomas L Saaty (1926-2017), Distinguiahod University Professor of University of Pittsburgh]

5. Operations Research is a management activity pursued in two complementary ways - one half by the free and bold exercise of commonsense untrammeled by any routine and other half by the application of repertoire of well-established pre-created methods and techniques.

   Singh (1968)

   [Jagjit Singh (1912-2002), writer and science popularizer, awarded

the Kalinga prize from the UNESCO in 1963, famous for his work Great Ideas of Operations Research]

> Operations Research is the art of winning wars without actually fighting.
> – Sir Arthur Charles Clarke (1917-2008)

8.3 HISTORICAL BACKGROUND

It has been said that Operations Research is a 'war baby'. The terms (Operations Research) were coined during the Second World War. They connote various attempts to study certain operations of war by using scientific methods. After the war there was a scarcity of industrial material. Industrial productivity reached the nadir. Researchers observed that the models studied during war can be interpreted in industrial pursuit also. In early 1950's this activity acquired the status of a subject. Some universities and institutes started offering courses in OR. Nowadays it has become popular throughout the world. India is one of the few first countries in the world who started teaching OR in different universities. Delhi University started an MSc programme in OR in 1965. Now it has become an important branch of study in Mathematics, Statistics, Commerce, Management, Engineering etc.

8.4 SCOPE OF OPERATIONS RESEARCH

OR techniques may be used to solve several real world problems. We have listed below some of the sectors where OR techniques can be applied.

1. **Defence Operations :** It has been said that OR is a 'war baby'. Arthur C clarke has opined that 'Operations Research is the art of winning wars without actually fighting'. Many of defence operations involve scientific decision making. Techniques like Linear Programming, Game Theory, and Inventory Models etc can be applied in defence establishments.
2. **Industry :** R. Panneerselvan in his outstanding book Operations

Research (2006) has listed the different techniques that can be used in four functional areas of industrial management, viz, production, marketing, personnel and finance.

Some of the techniques that can be used in the production function are given below :

(i) Linear programming for aggregate planning,

(ii) Integer programming for shop floor production scheduling.

(iii) Network based techniques for live balancing and project management, etc.

Some of the techniques that can used in the marketing function are given below :

(i) Linear programming for product mix problem,

(ii) Game theory for order bidding decision.

(iii) Distance network related techniques for shipping finished goods.

Some of the techniques that can be used in the personnel function are given below :

(i) Linear programming for manpower planning.

(ii) Queueing theory for the size of maintenance crews.

(iii) Scheduling techniques for manpower scheduling.

Some of the functions that can be used in the finance function are given below :

(i) Integer programming for capital budgeting.

(ii) Linear programming for break-even analysis.

(iii) Integer programming or dynamic programming for portfolio selection.

3. **Planning :** OR techniques can be used for planning purpose. Government has to prepare plans for balanced growth. Various departments of the government prepare different plans. For this purpose OR techniques are immensely useful.

4. **Agriculture :** Population is increasing rapidly. Demand for food production is increasing day by day. Availability of land is limited. So, agricultural planning must be done in such a way that various crops can be cultivated in the same land throughout the year, and farmers

do not face any problem and get the desired price, and the consumers get the desired items at the desired cost at the earliest. OR techniques can be used for this purpose.

5. **Public Utilities :** OR techniques can be used for the betterment of public. For example, we find long-queues of out-door patients in hospitals. Using Queueing Theory, this problem can be removed similarly, there are dearth of ambulances, shortage of medicines, beds etc. These problems can also be effectively solved by the application of OR techniques. OR techniques can also be applied in traffic control. It can be used in government offices, banks, post offices, railways, airways etc.

8.5 FEATURES OF OPERATIONS RESEARCH

Some of the salient features of Operations Researce (OR) are given below :

1. OR helps a manager in making quick and better decisions. A manager, devoid of knowledge of OR, has to depend on thumb rules or guess work while taking decisions. A manager, with sound knowledge of OR, has better ability to take decisions.
2. OR is an interdisciplinary team approach. An OR team requires mathematician/statistician, economist, engineer, management personnel, computer programmer etc.
3. OR uses techniques of scientific research to reach the optimum solution.
4. OR tries to optimise total return by maximising profit and by minimising cost.
5. OR is a system approach. An organisation has different subsystems. OR tries to analyse the whole system and prepares models in such a way that there is a coordination between the subsystems.



CHECK YOUR PROGRESS

Q 1: Write the scope of Operations Research.

.....

.....

Q 2: Write down the salient features of Operations Research.

.....

.....

8.6 PHASES OF OPERATIONS RESEARCH

An OR study passes through various phases.

First Phase : Observation

At the beginning, environment of the problem should be observed. This can be done by visiting the sites, by holding conferences etc. It will help the researchers to formulate the problem.

Second Phase : Analysis and Definition of the Problem

Secondly, the problem is to be analysed and defined. According to Hamdy A Taha, three principal elements of the decision problem are to be identified, viz, (1) description of the decision alternatives, (2) determination of the objective of the study, and (3) specification of the limitations under which the modelled system operates.

Third Phase : Development of a Model

Thirdly, a model is to be developed. Models are basically mathematical models. Generally, these are in the form of equations under different constraints. A model should be modified if necessary.

Fourth Phase : Selection of Appropriate Data Input

A model works smoothly provided there is appropriate data input. Therefore, selection of appropriate data input is important.

Fifth Phase : Solution of the Problem

Fifthly, a solution of the model is obtained.The solution is tested before it is implemented. After studing the pros and cons of the solution, the model is updated if necessary.

Sixth Phase : Implementation

Sixthly and lastly, after considering all aspects of the solution, it is implemented. If it is properly implemented, it gets support from all concerned.

Hamdy A Taha has opined that 'the success of the phases leading to the solution of the mathematical model depends largely on the creativity and experience of the operations research team.' (Operations Research: An Introduction, 2007)

8.7 TYPES OF OPERATIONS RESEARCH MODEL

Model is said to be 'an abstraction of reality'. one of the simplest examples of a model is a linear equation to forecast the demand of a product. An OR model is a defined as an idealized (simplified) representation of a real-life system'. (R. Panneerselvam)

There are different types of models in every area of business or industry.

A typical budget is an account model. Here 'business accounts are referred to with the intention of providing measurements such as rate of expenses, quantity sold etc.' (Kanti Swarup, P. K. Gupta, Man Mohan)

A mathematical equation may be considered to be a mathematical model. Here a relation between constants and variables is represented.

A model which has the possibility of measuring observations is called a quantitative model.

A physical model is a product or a device used for experimentation.

8.8 OPERATIONS RESEARCH TECHNIQUES AND TOOLS

There is not a single general technique in Operations Research (OR) to solve all mathematical models. It uses different tools/techniques for solving various problems. Some of the frequently used tools/techniques are listed below :

1. **Linear Programming (LP) :** LP is the most prominent OR technique. It optimise the objective function (profit or cost, return on investment

etc.) with certain constraints. It is used for solving problems involving cost diet, transportation, assignment of jobs to machines etc.

2. **Dynamic Programming :** It is used in which the original model can be decomposed into more manageable subproblems.
3. **Integer Programming :** It is used in which the variables assume integer values.
4. **Network Programming :** It is used in which the problem can be modelled as a network.
5. **Nonlinear Programming :** It is used in which the functions of the models are nonlinear.
6. **Queuing and Simulation Models :** Queuing and simulation are used for studying waiting lines. They determine measures of performance of the waiting lines, such as average waiting time in queue, average waiting time for service, and utilization of service facilities' (Hamdy A. Taha)
7. **Game Theory :** Game Theory is used for making decisions in which two opponents/players with conflicting objectives are trying to outdo each other.
8. **Decision Thory :** Decision Theory is used for making decisions under condition of complete certainty about the future outcomes.
9. **Inventory Models :** Inventory Models make decisions that minimise total inventory cost. They reduce the total cost of purchasing, carrying, and out of stock inventory.
10. **Markov Analysis :** Markov Analysis (after the name of Russian mathematician A Markov) helps to analyse the behaviour of some variable in order to predict the future behaviour of that variable. It was first used to describe and predict the behaviour of gas particles in a closed container.

8.9 GENERAL METHODS FOR SOLVING OPERATIONS RESEARCH MODEL

There are three types of methods that are generally used for solving Operations Research models.

1. **Analytic method :** An OR model may be solved by using the tools of classical mathematics, for example, differential calculus and finite differences. It is known as analytic method.
2. **Iterative method :** Iterative method is used if classical methods fail because of the number of variables or of the complexity of constraints. Iterative method starts with a trial solution. The trial solution is replaced by an improved solution. The process is repeated until no further improvement is possible.
3. **Monte carlo method :** Monte carlo method is based on random sampling of variable's values from a distribution of that variable.

The main steps of Monte carlo method are as follows.

1. To draw a flow chart of the system.
2. To take correct sample observations to select some suitable model for the system, and to compute the probability distributions for the variables of our interest.
3. To convert the probability distributions to a cumulative distributive function.
4. To select a sequence of random numbers with the help of random number tables.
5. To determine the sequence of values of variables of interest with the sequence of random numbers obtained in 4.
6. To construct some standard mathematical function to the values obtained in 5.

8.10 LIMITATIONS OF OPERATIONS RESEARCH

Despite immense applications of OR, it has certain limitations also. Some of the limitations are listed below :

1. There is a communication gap between a manager of an industry and a researcher in OR. Generally a researcher in OR is a mathematician/ statistician he may be unaware of the problems of industry. On the other hand, a manager may not easily grasp the complex problems of OR. As a result of it, there may be resistance from the management itself.

2. Basic data in OR models are subjected to frequent changes. These changes are to be incorporated in the models. Therefore, OR models become expensive and time consuming.
3. OR provides solution only when all the factors related to a problem can be quantifiable. Non quantifiable factors, i.e., qualitative factors or emotional factors do not find any place in OR models.
4. OR models cannot be applied to all categories of problems. They can be applied only to specific categories of problems.
5. The aim of an OR model is to obtain optimal solution considering all the factors. There are numerous factors. These factors are to be expressed in quality. It involves huge calculations.
6. The implementation of an OR model is a delicate task. It must take into account all the related complexities.



LET US KNOW

1. OR approach is multidisciplinary.
2. A model in OR is
   (i) an approximation,
   (ii) an idealization,
   (iii) an assence of reality.
3. OR techniques are not applicable if resources available are unlimited.
4. A model should be simple and coherent.
5. A model should not take much time in its construction for any problem.



8.11 LET US SUM UP

- Operations Research (OR) is used to provide a scientific base for management to take timely and effective judgements to their problems.
- It has been said that OR is a 'war baby'. The terms OR connote various attempts to study certain operations of second world war by using

scientific methods. After the war there was a scarcity of industrial material. Researchers observed that the models studied during war can be interpreted in industrical pursuit also. In early 1950's this activity acquired the status of a subject.

- OR techniques may be used to solve several real world problems. Some of the sectors where OR techniques can be applied are as follows,– Defence operations, Industry, Planning, Agriculture, Public utilities such as hospitals, traffic control, government offices, banks, post offices etc.
- OR helps a manager in making quick and better decisions.
- OR is an interdisciplinary team approach.
- OR uses techniques of scientific research to reach the optimum solution.
- An OR study passes through various phases, viz, Observation, Analysis and Definition of the problem, Development of a Model, Selection of Appropriato Data Input, Solution of the Problem, Implementation.
- There are different types of models in every area of business or industry.
- OR uses different tools/techniques for solving various problems some of the frequently used tools/techniques are Linear Programming, Dynamic Programming, Integer Programming, Network Programming, Nonlinear Programming Queuing and Simulation Models etc.
- Despite immense applications of OR, it has certain limitations also.



8.12 FURTHER READING

1) Kanti Swarup, P. K. Gupta, Man Mohan : Operations Research (2017), Sultan Chand and Sons, New Delhi.

2) Prem Kumar Gupta, D. S. Hira : Operations Research (2007), S Chand and Company Limited, New Delhi.

3) R. Panneerselvam : Operations Research (2006), Prentice Hall of India Private Limited, New Delhi.

4) S. D. Sharma : Operations Research (2012), Kedar Natu Ram Nath, Meerut.



8.13 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: See Section 8.4

Ans to Q No 2: See Section 8.5



8.14 MODEL QUESTIONS

Q 1: What is Operations Research?

Q 2: State the features of Operation Research.

Q 3: Write a note on the origin and development of Operations Research.

Q 4: Describe briefly the different phases of Operations Research.

Q 5: Describe briefly the tools and techniques of Operations Research.

Q 6: What are the limitations of Operations Research?

Q 7: Discuss the role of OP in Management

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UNIT 9 : LINEAR PROGRAMMING PROBLEM (LPP)

UNIT STRUCTURE

9.1 Learning Objectives

9.2 Introduction

9.3 Linear programming problem (LPP)

9.3.1 General form of Linear programming problem (LPP)

9.3.1 Some important definitions

9.4 Basic assumption of LPP

9.5 Application of LPP

9.6 Limitations of LPP

9.7 Mathematical formulation of a LPP

9.7.1 Illustrative examples of maximization cases

9.7.2 Illustrative examples of minimization cases

9.8 Let Us Sum Up

9.9 Further Reading

9.10 Answers To Check Your Progress

9.11 Model Questions

9.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- understand the meaning of linear programming problem (LPP)
- understand the standard form of LPP
- define the important definition associated with LPP
- know the basic assumptions and limitations
- formulate LPP from practical problems
- know the applications of LPP in the various field.

9.2 INTRODUCTION

Linear programming problem is one of the popular O.R techniques to optimize performance (e.g. profit or loss) under set of resource constraints (e.g. money, materials, man power, time, machine capacity etc) as specified

by an organization. L.P was developed by George B. Dantzig and his associates during world war II (1943-1945) when the war operations had to be planned to minimize expenditure and maximize damage to the enemy. Linear programming was first used by U.S. Air Force as an aid in decision making.Linear programming is one of the most powerful and useful techniques for making various managerial decisions.

Clearly, linear programming problem is most widely used to solve problem in management and commerce

In this unit, we shall introduce you to linear programming problem where you will first learn the various definitions related to LPP. In this unit,you will also learn how to formulate LPP and know about the standard forms of LPP.

9.3 LINEAR PROGRAMMING PROBLEM

The term 'programming' means planning and refers to the process of determining a particular course of action and term 'linear' means that all the variables involved must have unit power only.e.g. the expression $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n$ is linear.Higher powers variables or their products donot appear in the expressions for the objective function and the constraints.e.g.the expression like $x_1^4, x_2^{5/3}, x_1x_2$ etc. Linear Programming is a mathematical technique which is used to optimize (maximize or minimize) the objective function with limited resources. The resources may be in the form of men, money, raw materials, machine, time, labour etc.

9.3.1 General form of Linear programming problem (LPP):

The general form of a linear programming problem with n decision variables and m constraints can be written in the following form :

Maximize or Minimize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subject to the constraints

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq \text{ or} = \text{or} \geq b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq \text{ or} = \text{or} \geq b_2$

..... (1)

.....

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq \text{ or} = \text{or} \geq b_m$$

and the non-negative restrictions

$$x_1, x_2, x_3, \cdots x_n \geq 0 \quad .... \quad (2)$$

where

(i) $x_1, x_2, x_3, \cdots, x_n$ the variables whom we want to determine are called the decision variables.

(ii) the function Z which is to be optimized (maximized or minimized) is called the objective function of the LPP.

(iii) the inqualities (1) are called the constraints of the LPP.

(iv) the inequation (2) is known as the non-negative restrictions.

Another form of LPP :

Maximize or Minimize $Z = \sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \geq \text{or} = \text{or} \leq b_i$, $i = 1,2,3,\cdots,m$
$j = 1,2,3,\cdots,n$

and $x_j \geq 0$.

In martix form, the above LPP can be written as

Maximize (or Minimize) $Z = \begin{pmatrix} c_1 & c_2 & \cdot & \cdot & \cdot & c_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$

Subject to the constraints

$$\begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \leq \text{or} = \text{or} \geq \begin{pmatrix} \text{b}_1 \\ \text{b}_2 \\ \cdot \\ \cdot \\ \cdot \\ \text{b}_\text{m} \end{pmatrix}$$

9.3.2 Some important definitions :

Objective function: A function $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ which is to be optimized (maximized or minimized) is called objective function.

Decision variables: The variables $x_1, x_2, x_3, \cdots, x_n$ involved in the objective function are called decision variables.

Constraints: There are certain limitations on the use of limited resources. These limitations are called as constraints. Signs $\geq, =, \leq$ are used in constraints.

Non-negative restrictions: All decision variables are restricted to non-negative values i.e $x_1, x_2, x_3, \cdots x_n \geq 0$.

9.5 BASIC ASSUMPTIONS OF LPP

The Linear programming problem are formulated on the basis of the following assumptions :

1. **Proportionality :** The contribution of each variable in the objective function or its usage of the resources is directly proportional to the value of the variable.(e.g. if one unit product requires 2 hours of machine time, 6 units of it will require 12 hours).
2. **Certainity or Deterministic:** This assumption means that all co-efficients in the objective function and constraints are completely known and do not change during the period being stated.
3. **Divisibility :** Values of decision variables and resources can take any non-negative values i.e. fractional values of the decision variables are permitted.
4. **Additivity :** Sum of the resource used by different activities must be equal to the total amount of resource used by each activity.

9.4 APPLICATIONS OF LPP

LPP is a widely used tool of OR in almost every area of a business and management. LPP is used in the following areas :

(i) **Manufacturing Problem :** To find the number of items of each type that should be manufactured so as to maximize the profit subject to production restrictions imposed by limitation on the use of machinery, law material and labour.

(ii) **Transportation Problem :** To find the least costly way of transportation from the warehouses to customers.

(iii) Investment Problem : To find the amount to be invested in variety of shares and debentures so as to maximize return on investment.

(iv) Advertising Media Problem : To find the optimum allocation of advertisements in various media subject to total advertising budget to maximize effective number of customers.

(v) Diet Problem : To find the minimum requirement of nutrients subject to availability of foods and their prices.

(vi) Production Problem : To decide the production schedule that could satisfy future demands (seasonal or otherwise) and minimize cost in face of fluctuating rates and storage (inventory) cost.

(vii) Job Assigning Problem : To assign job to workers for maximum effectiveness and optimum results subject to restrictions of wages and other cost.

9.6 LIMITATIONS OF LPP

Inspite of having wide applicability of LPP, it has certain limitations also. Some of the limitations are as follows :

(i) L.P.P treats all relationships as linear but it is not true in many real life situation.

(ii) The decision variables in L.P.P would be meaningful only if they have integer values. But sometimes we get fractional values to the optimal solution where only integer values are meaningful.

(iii) The problems are complex if the number of variables and constraints are quite large.

(iv) It deals with only single objective function whereas in real life situations, there may be more than one objective.

9.7 MATHEMATICAL FORMATION OF A LPP

The procedure for mathematical formulation of LPP consists of the following steps :

Step I : Identify the unkown decision variables to be determined and assign symbols to them.

Step II : Identify all the restrictions or constraints in the problem and express them as linear inequalities in decision variables.

Step III : Formulate the objective function and represent it also as a linear function of decision variables.

Step IV : The non-negativity constraint is added as the negative values of the decision variables do not have any physical meaning.

Step V : Express the complete formulation of LPP as a standard mathematical model.

Note :The linear programming involves only one objective namely profit maximization or cost minimiztion.

9.7.1 Illustrative Examples of Maximization cases

Example 1: A firm manufactures two types of products A and B and sells them at a profit of Rs. 3 on type A and Rs. 4 on type B. Each product is processed in two machines M_1 and M_2. Type A requires 4 minutes of processing time on M_1 and 5 minutes on M_2. Type B requires 1 minute on M_1 and 3 minute on M_2. Machine M_1 is available for not more than 5 hours 20 minutes while machine M_2 is available for 8 hours during working hours. Formulate the LPP so as to maximize the profit.

Solution : (i) Variables : Let $x =$ number of A type products and $y =$ number of B type products.

(ii) Objective function :

Profit on x products $= 3x$

Profit on y products $= 4y$

As the objective function is to maximize the profit, the objective function is maximize $Z = 3x + 4y$.

(iii) Constraints : To produce these units type A and type B products, it requires

$4x + y$ processing minutes on M_1

$5x + 3y$ processing minutes on M_2

Since machine M_1 is available for not more than 5 hours and 20 minutes and machine M_2 is available for 8 hours during working hours.

the constraints are

$$4x + y \le 320$$

$$5x + 3y \le 480$$

(iv) Non-negative restrictions : Since the number of products type A and type B cannot be negative, we have $x, y \ge 0$.

$\therefore$ The complete formulation of the LPP is

Maximize $Z = 3x + 4y$

Subject to the constraints

$$4x + y \le 320$$

$$5x + 3y \le 480$$

and $x, y \ge 0$.



LET US KNOW

A linear programming problem consists of the objective function, the set of constraints and the non-negative constraint.

Example 2 : A company manufactures two types of printed circuits A and B.

The requirement of transistors,resistors and capacitors for each type of printed circuits along with other data are given below :

	Circuit		
	A	B	Stock avaiable
Transistor	5	10	160
Resistor	10	20	300
Capacitor	15	10	180
Profit	Rs 3	Rs. 5	

Formulate the LPP.

Solution : (i) Variable **:** Let $x =$ number of type A printed circuit and $y =$ number of type B printed circuit.

(ii) Objective function :

Profit on x type printed circuit $= 3x$

Profit on y type printed circuit $= 5y$

As the objective function is to maximize the profit, the objective function is maximize $Z = 3x + 5y$.

(iii) Constraints : Stock available transistor = 160

Stock available resistor = 300

Stock available capacitor =180

Required transistor for circuit A =5

Required transistor for circuit B =10

$\therefore$ The constraint is $5x + 10y \leq 160$

Required resistor for circuit A =10

Required resistor for circuit B =20

$\therefore$ The constraint is $10x + 20y \leq 300$

Required capacitor for circuit A =15

Required capacitor for circuit B =10

$\therefore$ The constraint is $15x + 10y \leq 180$

(iv) Non-negative restrictions : Since the number of printed circuit of type A and type B cannot be negative, we have $x, y \geq 0$.

$\therefore$ The complete formulation of the LPP is

Maximize $Z = 3x + 5y$

Subject to the constraints

$$5x + 10y \leq 160$$
$$10x + 20y \leq 300$$
$$15x + 10y \leq 180$$

and $x, y \geq 0$.

Example 3 : A firm engaged in producing two models A and B performs three operations painting, assembly and testing. The relevant data are as follows :

Model	Unit sale Price	Hours required for each unit		
		Assembly	Painting	Testing
A	Rs. 50	1.0	0.2	0.1
B	Rs.80	1.5	0.2	0.1

Total number of hours available are :

Assembly 600, painting 100, testing 30. Determine weekly production schedule to maximize the profit.

Solution: (i) Variable : Let $x =$ number of A type model and $y =$ number of B type model.

(ii) Objective function :

Profit on x type model $= 50x$

Profit on y type model $= 80y$

As the objective function is to maximize the profit, the objective function is maximize $Z = 50x + 80y$.

(iii) Constraints : Hours available for assembly = 600

Hours available for painting = 100

Hours available for testing =30

Hours required for assembly for model A=1.0

Hours required for assembly for model B=1.5

$\therefore$ The constraint is $x + 1.5y \leq 600$

Hours required for painting for model A=0.2

Hours required for painting for model B=0.2

$\therefore$ The constraint is $0.2x + 0.2y \leq 100$

Hours required for testing for model A=0.1

Hours required for testing for model B=0.1

$\therefore$ The constraint is $0.1x + 0.1y \leq 30$

(iv) Non-negative restrictions : Since the number of model type A and type B cannot be negative, we have $x, y \geq 0$

$\therefore$ The complete formulation of the LPP is

Maximize $Z = 50x + 80y$

Subject to the constraints

$$x + 1.5y \leq 600$$
$$0.2x + 0.2y \leq 100$$
$$0.1x + 0.1y \leq 30$$

and $x, y \geq 0$.

Example 4 **:** A company produces refrigerators in unit I and heaters in Unit II. The two products are produced and sold on a weekly basis. The weekly production cannot exceed 25 in unit I and 36 in unit II, due to constraints 60 workers are employed. A refrigerators requires 2 man week of labour,while a heater requires 1 man week of labour.

The profit available is Rs.800 per refrigerator and Rs.600 per heater. Formulate the LPP problem.

Solution : (i) Variable : Let $x =$ number of refrigerators in unit I and $y =$ number of refrigerators in unit I.

(ii) Objective function : Profit on x refrigerators $= 800x$

Profit on y heaters $= 600y$

As the objective function is to maximize the profit, the objective function is maximize $Z = 800x + 600y$.

(iii) Constraints : The weekly production cannot exceed 25 in unit I.

$\therefore \quad x \leq 25$

The weekly production cannot exceed 36 in unit II.

$\therefore \quad y \leq 36$

A refrigerator requires 2 man week of labour, while a heater requires 1 man week of labour and 60 workers are employed weekly.

$\therefore \quad 2x + y \leq 60$

(iv) Non-negative restrictions : Since the number of refrigerators and heaters cannot be negative ,we have $x, y \geq 0$

$\therefore$ The complete formulation of the LPP is

Maximize $Z = 800x + 600y$

Subject to the constraints

$$x \leq 25$$
$$y \leq 36$$
$$2x + y \leq 60$$

and $x, y \geq 0$.

Example 5 **:** A company manufacture two types of products A and B and sell them at a profit of Rs.2 on type A and Rs.3 on type B. Each product is processed on two machines M_1 and M_2. Type A requires one minute of processing time on M_1 and two minute on M_2 while type B requires one minute on M_1 and one minute on M_2. The machine M_1 is available for not more than 6 hours 40 minute and machine M_2 is available for 10 hours during any working day. Formulate the LPP.

Solution: (i) Variable **:** Let $x =$ number of products of type A and

$y =$ number of products of type B.

(ii) Objective function :

Profit on x product of type A $= 2x$

Profit on y product of type B $= 3y$

As the objective function is to maximize the profit, the objective function is maximize $Z = 2x + 3y$.

(iii) Constraints : Type A requires one minute processing time on M_1 and type B requires one minute on M_1. But hours available for machine M_1 during any working day = 6 hours 40 minutes=400 minutes.

$\therefore \quad x + y \leq 400$

Similarly, Type A requires two minute processing time on M_2 and type B requires one minute on M_2. But hours available for machine M_2 during any working day = 10 hours =600 minutes.

$\therefore \quad 2x + y \leq 600$

(iv) Non-negative restrictions : Since the number of products of type A and type B cannot be negative,we have $x, y \geq 0$.

$\therefore$ The complete formulation of the LPP is

Maximize $Z = 2x + 3y$

Subject to the constraints

$x + y \leq 400$

$2x + y \leq 600$

and $x, y \geq 0$.

9.7.2 Illustrative Examples of Minimization cases :

Example 6 **:** A pharmaceutical company manufactures headache pills in two sizes A and B. Size A contains 2 mgs.of aspirin, 5 mgs. of bicarbonate and 1 mg.of codeine. Size B contains 1 mgs.of aspirin, 8 mgs.of bicarbonate and 6 mg.of codeine. It is found that it requires atleast 12 mgs. of aspirin,74 mgs.of bicarbonate and 24 mgs.of codeine for providing immediate relief. It is required to determine the least number of pills a patient should take to get immediate relief.Formulate the problem as a standard LPP.

Solution : (i) Variable : Let $x =$ number of pills in size A and $y =$ number of pills in size B.

(ii) Objective function : Minimize $Z = x + y$

(iii) Constraints : Size A contains 2 mgs. of aspirin and size B contains 1 mgs. of aspirin. But Company requires atleast 12 mgs. of aspirin.

$\therefore \quad 2x + y \geq 12$

Similarly, size A contains 5 mgs.of bicarbonate and size B contains 8 mgs. of bicarbonate. But Company requires atleast 74 mgs.of bicarbonate.

$\therefore \quad 5x + 8y \geq 74$

Also, size A contains 1 mg. of codeine and size B contains 6 mg. of codeine. But Company requires atleast 24 mgs. of codeine.

$\therefore \; x + 6y \geq 24$

(iv) Non-negative restrictions :Since the number of pills cannot negative, we have $x, y \geq 0$

$\therefore$ The complete formulation of the LPP is

Minimize $Z = x + y$

Subject to the constraints

$2x + y \geq 12$

$5x + 8y \geq 74$

$x + 6y \geq 24$

and $x, y \geq 0$.

Example 7 : A person requires atleast 10,12 and 12 units of chemicals A,B,C repectively for his garden. A liquid product contains 5,2 and 1units of A, B and C repectively per jar. A dry product contains 1,2 and 4 units of A, B and C per carton.The liquid product sells for Rs.5 per jar and dry product sells for Rs.4 per carton. Formulate the LPP.

Solution : (i) Variable : Let x_1 = the number of units liquid product A,B and C per jar

and x_2 =the number of units dry product A,B and C per carton.

(ii) Objective function : The selling cost of liquid product = Rs 5 per jar and the selling cost of dry product =Rs 4 per carton.

The aim is to minimize the cost.

$\therefore$ The objective function will be

Min. $Z = 5x_1 + 4x_2$

(iii) Constraints : Requirement of chemical A =10 unis

Number of units of chemical A in liquid product =5 units

Number of units of chemical A in dry product =1 units

$\therefore \quad 5x_1 + x_2 \geq 10$

Requirement of chemical B =12 unis

Number of units. of chemical B in liquid product =2 units

Number of units of chemical B in dry product =2 units

$\therefore \quad 2x_1 + 2x_2 \geq 12$

Requirement of chemical C=12 unis

Number of units.of chemical C in liquid product = 1 units

Number of units of chemical C in dry product = 4 units

$\therefore \quad x_1 + 4x_2 \geq 12$

(iv) Non-negative restrictions : Since the number of liquid and dry product cannot negative, we have $x_1, x_2 \geq 0$

$\therefore$ The complete formulation of the LPP is

Min. $Z = 5x_1 + 4x_2$

Subject to the constraints

$$5x_1 + x_2 \geq 10$$
$$2x_1 + 2x_2 \geq 12$$
$$x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

Example 8 : A soft drink firm has two bottling plants,one located at P and the other at Q. Each plant produces three different soft drinks A, B and C. The capacities of the two plants in number of bottles per day are as follows

Products	Plants	
	P	Q
A	3000	1000
B	1000	1000
C	2000	6000

A market survey indicates that during the month of April, there will be a demand for 24000 bottles of A,16000 bottles of B and 48000 bottles of C. The operating costs per day of running plants P and Q respectively are Rs.6000 and Rs.4000.Formulate the LPP problem.

Solution : (i) Variable: Let $x =$ number of days the firm runs plant A in April and $y =$ number of days the firm runs plant B in April.

(ii) Objective function : Cost of running the plant A = Rs.6000 per day and Cost of running the plant B = Rs.4000 per day.

The aim is to minimize the cost.

$\therefore$ The objective function will be

Minimize $Z = 6000x + 4000y$

(iii) Constraints : Plant P produces 3000 bottles of A and plant Q produces 1000 bottles of A.

During the month of April, there will be a demand for 24000 bottles of A.

$$\therefore \ 3000x + 1000y \geq 24000$$

$$\Rightarrow 3x + y \geq 24$$

Plant P produces 1000 bottles of B and plant Q produces 1000 bottles of B.

During the month of April, there will be a demand for 16000 bottles of B.

$$\therefore \ 1000x + 1000y \geq 16000$$

$$\Rightarrow \ x + y \geq 16$$

Plant P produces 2000 bottles of C and plant Q produces 6000 bottles of C.

During the month of April, there will be a demand for 16000 bottles of C.

$\therefore \quad 2000x + 6000y \geq 48000$

$\Rightarrow x + 3y \geq 24$

(iv) Non-negative restrictions : Since the number of days for running the plant cannot negative ,we have $x, y \geq 0$

$\therefore$ The complete formulation of the LPP is

Min $Z = 6000x + 4000y$

Subject to the constraints

$3x + y \geq 24$

$x + y \geq 16$

$x + 3y \geq 24$

$x, y \geq 0$

Example 9 : A diet is to contain at least 400 units of carbohydrat, 500 units of fat and 300 units of protein. Two foods are available : F_1 which costs Rs. 2 per unit, F_2 which costs Rs. 4 per unit. A unit of food F_1 contains 10 units of carbohydrate, 20 units of fat and 15 units of protein. A unit of food F_2 contains 25 units of carbohydrate, 10 units of fat and 20 units of protein.

Formulate the problem as a LPP.

Solution : (i) Variable :

Let x = Number of units of food F_1 to be purchased

y = Number of units of food F_2 to be purchased.

The information supplied in the given problem may be summarised in the tabular form as follows :

Type of food	Number of units of food	Carbohydrate	Nutrients Fat	Protein	Cost per unit (Rs.)
F_1	x	10	20	15	2
F_2	y	25	10	20	4
Minimum requirement		400	500	300	

(ii) Objective function : Here we observe that our objective is to minimize cost–

The objective function is Z = 2x + 4y.

(iii) Constraints : The total amount of carbohydrates from foods F_1 and F_2 is 10x + 25y and this is required to be at least the minimum amount of carbohydrates required, namely 400 units. Therefore, we must have $10x + 25y \geq 400$

Similarly, for the fat constraint, we get $20x + 10y \geq 500$

For the protein constraint, we get $15x + 20y \geq 300$

(iv) Non-negative restrictions : Units of food manufactured can not be negative,we have $x, y \geq 0$

$\therefore$ The complete formulation of the LPP is

Minimize Z = 2x + 4y

Subject to the constraints

$10x + 25y \geq 400$

$20x + 10y \geq 500$

$15x + 20y \geq 300$

$x , y \geq 0$

Example 10 : Vitamins V_1 and V_2 are found in two different foods F_1 and F_2. One unit of food F_1 contains 2 units of vitamins V_1 and 5 units of vitamin V_2 and one unit of food F_2 contains 4 units of vitamin V_1 and 2 units of vitamin V_2. One unit of food F_1 and F_2 cost Rs.30 and 25 respectively.The minimum daily requirement of vitamin V_1 and V_2 is 40 and 50 units repectively. Formulate the LPP.

Solution : (i) Variable :

Let x = Number of units of food F_1

y = Number of units of food F_2 .

(ii) Objective function : Cost of x units of food F_1 = Rs.30x

and Cost of y units of food F_2 = Rs.40y

The aim is to minimize the cost.

$\therefore$ The objective function will be

Minimize $Z = 30x + 40y$

(iii) Constraints : One unit of food F_1 contains 2 units of vitamins V_1 and one unit of food F_2 contains 4 units of vitamin V_1. The minimum daily requirement of vitamin V_1 is 40 units.

$\therefore \quad 2x + 4y \geq 40$

Similarly, one unit of food F_2 contains 5 units of vitamins V_2 and one unit of food F_2 contains 2 units of vitamin V_2. The minimum daily requirement of vitamin V_1 is 50 units.

$\therefore \quad 5x + 2y \geq 50$

(iv) Non-negative restrictions : Units of foods F_1 and F_2 can not be negative, we have $x, y \geq 0$

$\therefore$ The complete formulation of the LPP is

Minimize $Z = 30x + 40y$

Subject to the constraints

$$2x + 4y \geq 40$$

$$5x + 2y \geq 50$$

and $x, y \geq 0$



CHECK YOUR PROGRESS

Q 1: What do you mean by linear programming problem?

Q 2: Say whether the following statements are true (T) or false (F)

(i) Linear programming technique can be applied only in those situations where the constraints and the objective function are linear.

(ii) LPP deals with problems involving only a single variable.

(iii) Decision variables in a standard LPP should have either zero or positive values.

Q 3: Fill in the blanks.

(i) The variables involved in a LPP are called ..... variable.

(ii) In a Linear Programming , the linear function to be optimized is called ..... function.

(iii) In a Linear Programming, all relationship among decision variables must be .....

(iv) Most of the constraints in an LPP are expressed as .....

(v) ..... of the decision variables is one of the assumptions of a LPP.

Q 4: Choose the correct alternatives

(i) In LPP the conditions to be satisfied

(a) Constraints have to be linear (b) Objective function have to linear

(c) both (a) and (b) (d) none of the above.

(ii) The linear function of unknown variables which is to be maximized or minimized is called

(a) constraints (b) objective function

(c) feasible solution (d) none of the above

(iii) Mathematical formulation of LPP is important because

(a) It helps in converting the verbal description and numerical data into mathematical expression

(b) decision makers prefer to work with formal models

(c) it captures the relevant relationship among decision factors

(d) it enables the use of algebraic techniques.

(iv) A constraint in an LP model restricts

(a) value of the objective function (b) use of the available resources

(c) value of the decision variable (d) all of the above

(v) The first step in formulating a linear programming problem is

(a) Identify any upper or lower bound on the decision variables

(b) State the constraints as linear combinations of the decision variables

(c) Understand the problem

(d) Identify the decision variables

Q 5: A company produces two products A and B which possess raw materials 400 quintals and 450 labour hours. It is known that 1 unit of product A requires 5 quintals of raw materials and 10 man hours and yields a profit of Rs.45. Product B requires 20 quintals of raw materials,15 man hours and yields a profit of Rs. 80. Formulate the LPP.

Q 6: A manufacturer produces two different types of products A and B. Each product requires three different operations grinding, assembling and testing. Each unit of product A requires 15, 20 and 10 minute to grinding, assembling and testing respectively, where as each unit of product B requires 7.5, 40 and 45 minutes for these three operations. respectively. The production run calls for at least 7.5 hours of grinding time, at least 20 hours of assembling time and at least 15 hours of testing time. Each unit of product A costs Rs. 60 and each unit of product B costs Rs. 90 to manufacture, determine the number of units of each product the firm should produce to minimize the cost.

Q 7: A company produces two types of pens, Hightech and Smoothflow. Pen Hightech is of superior quality and smoothflow is of lower quality. Profits from Hightech and Smoothflow are Rs. 3 per pen respectively Raw materials required for each pen of Hightech type is twice as that of pen of Smoothflow type. The supply of raw materials is sufficient only for 1000 pens of Smoothflow type. Pen hightech requires a special clip and 400 such clips are available per day. Smooth flow pen requires a particular type of clip and 700 such clips are available per day.

Formulate the above information as an LPP.

Q 8: A small firms build two types of garden shed A and B. Type A requires 2 hours of machine time and 5 hours of craftman time and type B requires 3 hours of machine time and 5 hours of craftman time. Each day there are 30 hours of machine time available and 60 hours of craftman time. The profit on

each type A shed is Rs.60 and on each type B shed is Rs.84.Formulate the LPP.



9.8 LET US SUM UP

- Linear programming problem deals with the optimization (Maximization or Minimization) of a function of decision variables.
- Formulation of a LPP involves defining the decision variables, constructing the objective function and writing down the constraints. The procedure for the mathematical formulation of LPP are discussed.
- Various terms associated with a LPP are defined.
- The basic assumptions and limitations of LPP are discussed.
- Linear programming techniques are applied in product mix, blending, diet, transportation and assignment problems.



9.9 FURTHER READING

1) Gupta, Prem kumar, Hira,Dr. D. S. *Problems in operation Research (principles and Solutions),* S.Chand & Company Ltd.New Delhi - 110055.

2) Swarup, Kanti,Gupta, P.K,Mohan,Man, *Operations Research,* Sultan chand & Sons,New Delhi.

3) Taha, Hamdy A. *Operation Research : An Introduction,* Published by Pearson Education Inc.

4) Ravindran, A. Phillips, Don T. Solberg, James J. *Operation Research* wiley India pvt. Ltd. New Delhi - 110002.

5) Vohra, N.D.Quantitiative Techniques in management, Tata McGraw Hill, New Delhi



9.10 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: Refer section 9.3

Ans to Q No 2: (i) True (ii) True (iii) True

Ans to Q No 3: (i) decision (ii) objective (iii) linear (iv) inequality (v) certainity.

Ans to Q No 4 : (i) (c) (ii) (c) (iii) (b) (iv) (a) (v) (c) (vi) (d)

Ans to Q No 5 : (i) Variable :

Let x be the number of units of product A and y be the number of units of product B.

(ii) Objective function :

Profit on x unit of product A $= 45x$

Profit on y unit of product B $= 80y$

As the objective function is to maximize the profit,the objective function is maximize $Z = 45x + 80y$.

(iii) Constraints :

Labour hours availability of product A and B = 450 hours

Product A requires 5 quintals of raw materials and Product B requires 20 quintals of raw materials

But, raw material availability of product A and B = 400 quintals

$\therefore$ The constraint is $5x + 20y \leq 400$

Product A requires 10 man hours and product B requires 15 man hours.

$\therefore$ The constraint is $10x + 15y \leq 450$

(iv) Non-negative restrictions : Since the number units of product A and product B cannot be negative,we have $x, y \geq 0$

$\therefore$ The complete formulation of the LPP is

Maximize $Z = 45x + 80y$

Subject to the constraints

$5x + 20y \leq 400$

$10x + 15y \leq 450$

and $x, y \geq 0$.

Ans to Q No 6 : Minimize Z = 60x + 90y

Subject to 15x + 7.5y ≥ 450

20x + 40y $\geq$ 1200

10x + 45y $\geq$ 900

x, y ≥ 0

Ans to Q No 7:

Let x = No of hightech pen produced perdas

y = No of smoothflow pen produced perdas

Maximize Z = 5x + 3y

Subject to 2x + y ≤ 1000

x ≤ 400

y ≤ 700

x, y ≥ 0

Ans to Q No 8 : (i) Variable : Let x =number of type A sheds produced each day and y = number of type B sheds produced each day.

(ii) Objective function : Profit on x type A sheds $= 60x$

Profit on y type B sheds $= 84y$

As the objective function is to maximize the profit,the objective function is maximize $Z = 60x + 84y$.

(iii) Constraints : Machine time : $2x + 3y \leq 30$

Craftsman time : $5x + 5y \leq 60$

(iv) Non-negative restrictions :Since the number of type A shed and type B shed cannot be negative,we have $x, y \geq 0$

$\therefore$ The complete formulation of the LPP is

Maximize $Z = 60x + 84y$

Subject to the constraints

$$2x + 3y \leq 30$$

$$5x + 5y \leq 60$$

and $x, y \geq 0$.



9.11 MODEL QUESTIONS

Q 1: Explain the linear programming problem giving two examples

Q 2: State four applications of an linear programming.

Q 3: A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs Rs.100 for preparation,require 7 man - day of work and yields a profit Rs 30.An acre of wheat costs Rs.120 to prepare,require 10 man - days of work and yields a profit Rs.40. An acre of soyabeans costs Rs. 70 to prepare,require 8 man - days of work and yields Rs. 20. The farmer has Rs 1,00,000 for preparation and 8000 man - days of work. Formulate this as a LPP.

Q 4: What do you understand by Linear Programming ? What are its major assumptions ?

Q 5: Write a note on the limitations of linear programming ?

Q 6: Discuss the scope of linear programming in solving Business Problem.

Q 7: When we say that an LPP have (i) unique solution (ii) an infinite number of solutions (iii) no solution.

Q 8: What do you mean by the term ‘ constraints’ ? Explain with examples.

Q 9: What does the non-negativity restriction mean ?

Q 10: Choose the correct alternatives

(a) A constraint in an LP model restricts.

(i) value of objective function.

(ii) value of decision variables.

(iii) use of the available resources.

(iv) all of the above.

(b) The distingushing feature of a LPP is

(i) the relationship among all variables is linear.

(ii) value of decision variables is non-negative.

(iii) it has single objective function and constraints.

(iv) all of the above.

(c) Which of the following is an assumption of a LPP.

(i) divisibility (ii) certainity (iii) additivity (iv) all the above.

(d) Which of the following is not the characteristic of LPP.

(i) resources must be linited.

(ii) only one objective function.

(iii) all the relationship involved must be linear.

(iv) the problem must be of minimization type.

(e) Non-negativity condition in an LPP implics

(i) a positive co-efficient of variables in the objective function.

(ii) non-negative value of resources.

(iii) a positive co-efficient of variables in any constraint.

(iv) none of the above.

Q 11: A company produces two types of products say type A and B. Profits on the types of product are Rs 40 and Rs 50 per kg repectively. The data on resource required and availability of resource are given bleow.

Requirment	Capacity available		
	product A	product B	per month
Raw materials (Kg)	60	120	12000
Machinig hours/piece	8	5	600
Assembling (man hours)	3	4	500

Formulate the LPP as a linear programming problem to maximize the profit.

Q 12: A firm manufacturer produces two products A and B on which the profits earned per unit are Rs. 5 and Rs.6 repectively. Each product is proceesed on two machines M_1 and M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2, while product B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hours 30 minutes while M_2 is available for 10 hrs during any working day.Formulate the LPP to maximize the profit.

Q 13: A company produces two products: juicers and washing machine. Production happens in two different departments I and II. Juicers are made in Department I and washing machine in Department II. These

two items are sold weekly. The weekly production should not cross 25 juicers and 35 washing machines. The company always hires a total of 60 employees in two department. A juicers requires two man week's labour while a washing machine needs one man week's labour. A juicers makes a profit of Rs. 60 and a washing machine contributes a profit of Rs.40. Formulate this as an LPP.

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UNIT 10 : GRAPHICAL METHOD OF SOLUTION FOR LINEAR PROGRAMMING PROBLEM

UNIT STRUCTURE

10.1 Learning Objectives

10.2 Introduction

10.3 Some basic definitions

10.4 Graphical methods to solve LPP

10.4.1 Illustrative Examples

10.5 Some special cases

10.6 Limitations of graphical method

10.7 Let Us Sum Up

10.8 Further Reading

10.9 Answers To Check Your Progress nd

10.10 Model Questions

10.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- understand some important definitions
- know the graphical method
- find solutions to LPP using graphical methods
- learn how to find the feasible region of an LPP of two variable from the graph
- obtain the optimal solution of an LPP of two variable from the graph.

10.2 INTRODUCTION

In unit 9,We discussed some problems that led to a linear programming problem in which a linear function of two variables is to be maximized(or minimized) subject to a number of linear inqualities.In this unit,we shall discuss how a LP problem is solved.There are two methods of solving an LPP.One is graphical method and other is simplex

method.Graphical method is used for solving a LPP which involves only two variables only. In this unit, we shall study graphical method only.

10.3 SOME IMPORTANT DEFINITIONS

Following are some important definitions associated to solution of Linear Programming Problem.

Solution : A set of values of the decision variables which satisfy the constraints of a general LPP is called a solution to the general LPP.

Feasible solution : A solution which satisfy the non-negative conditions of the problem is called a **feasible solution.**

Non feasible solution : A solution which does not satisfy the non-negative conditions of the problem is called non-feasible solution.

Optimal solution : Any feasible solution which optimizes (maximizes or minimizes) the objective function of a general LPP is called an **optimum solution.**

Feasible region : The common region determined by all the constraints and non-negativity restriction of an LPP is called a feasible region.

10.4 GRAPHICAL METHOD TO SOLVE LPP

Linear programming problem with two variables can be solved by graphical method.

Procedure for solving LPP by graphical method

Steps involved in the graphical method are as follows :

Step I : We form the problem mathematically.

Step II: We write the constraints in the form of equations.

First we consider each inequality constraint as an equation.

Step III: To draw the line,we have to find two points. By putting $x = 0$ we get one point. By putting $y = 0$ we get another point.We plot the points and draw the lines accordingly.

Step IV: Now, we identify the feasible region.If the inequality constraint corresponding to that line is '$\leq$', then the region below the line lying in the first quadrant is shaded.Similarly,if the inequality constraint

corresponding to the line is '$\geq$', then the region above in the first qudrant is shaded.The common region 'thus 'obtained is called the feasible region.

Step V: We find the corner points of the feasible region in step IV. The intersecting points on the boundary of the feasible region are called as corner points.

Step VI: We calculate the value of the objective function by substituting corner points in the objective function..

Step VII: We select the corner point which gives the optimum solution. The optimum solution occurs in a maximization case at the corner point yielding the largest value of the objective function and in a minimization case at the corner point yielding the smallest value of the objective function.

Note : 1. Due to the non negativity of variables,the feasible region is shaded in the first quadrant of the graph.

2. The graphical method is also known as corner point method.

10.4.1 Illustative Examples

Example 1 : A machine produces two products A and B.Machine can produce A by using 2 units of chemicals and 1 unit of a compound and can produce B by using 1 unit of chemicals and 2 units of compound. Only 800 units of chemicals and 1000 units of the compound are available.The profits available per unit of A and B are respectively Rs.30 and Rs.20. Find the maximum profit.

Solution : The above information is summarized in the following table :

Product	Chemicals	Compound	profit (in Rs.) per unit
A	2	1	30
B	1	1	20
Available units	800	1000	--

Let the machine produce x units of product A and y unit of product B.

Profit per unit of products A and B are Rs.30 and Rs.20 repectively.The total profit of producing x units of product A and y unit of product B is $30x + 20y$ .The formulation of the LPP is

$Z = 30x + 20y$

Subject to the constraints

$2x + y \le 800$

$x + 2y \le 1000$

$x, y \ge 0$

The solution to the LPP can be obtained by using graphical method.

First we consider the equation $2x + y = 800$.

When $x = 0$, $y = 800$ and $y = 0$, $x = 40$.

$\therefore$ $2x + y = 800$ passes through the points (0,800) and (400,0).

We consider the equation $x + 2y = 1000$.

When $x = 0$, $y = 500$ and $y = 0$, $x = 100$.

$\therefore$ $x + 2y = 1000$ passes through the points (0,500) and (1000,0).

Now we draw the graph of these two equations.

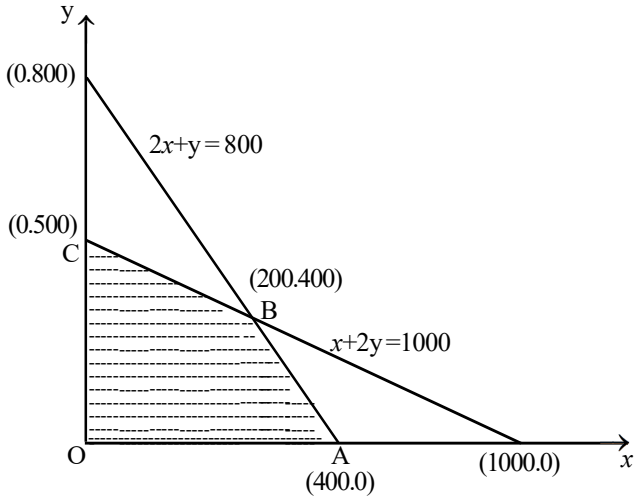


Fig 10.1

The feasible region is the shaded region OABC.The coordinate of the vertex

B are obtained by solving the equations.

$2x + y = 800$ ... (1)

$x + 2y = 1000$ ..... (2)

Multiplying (1) by 2

$4x + 2y = 1600$ ..... (3)

Multiplying (2) by 1

$x + 2y = 1000$ ..... (4)

Subtracting (4) from (3),we get $3x = 600$

$\Rightarrow x = 200$

Subtituting the value of $x = 200$ in (1) we obtain $y = 400$

$\therefore$ B is (200,400).

The corner points of the feasible region OABC are O(0,0), A (400,0), B(200,400) and C(0,500).

Corner points	Value of $Z = 20x + 10y$
O(0,0)	0
A (400,0)	12000
B (200, 400)	14000 (Maximum value)
C (0,500)	10000

Therefore the optimal solution is $x = 200, y = 400$ and the maximum profit = Rs 14000.

Example 2 : Solve the following LPP by the graphical method.

Minimize $Z = 20x + 10y$

Subject to $x + 2y \leq 40$

$3x + y \geq 30$

$4x + 3y \geq 60$

$x, y \geq 0$

Solution : We consider each inequality of the constraints as an equation.

$x + 2y = 40$ ..... (1)

$3x + y = 30$ ..... (2)

$4x + 3y = 60$ ..... (3)

For the equation, $x + 2y = 40$

If $x = 0 \Rightarrow 2y = 40$

$\Rightarrow y = 20$

If $y = 0 \Rightarrow x = 20$

$\therefore$ $x + 2y = 40$ passes through the points (0,20) and (40,0)

Similarly, $3x + y = 30$ passes through the points (0,30) and (10,0),

$4x + 3y = 60$ passes through the points (0,20) and (15,0).

Now, we plot each equation on the graph.

From the figure 10.2, we get the feasible region ABCD. C and D are the points of intersection of lines : $x + 2y = 40$, $3x + y = 30$ and $4x + 3y = 60$.

On solving the equations, we get (4,18) and (6,12).

The corner points of the feasible region ABCD are A (15,0), B(40,0), C(4,18) and D(6,12).

Corner points	Value of $Z = 20x + 10y$
A(15,0)	300
B(40,0)	800
C(4,18)	260
D(6,12)	240 (Minimum value)

$\therefore$ The minimum value of Z occurs at D(6,12). Hence, the optimal solution is $x = 6$, $y = 12$.

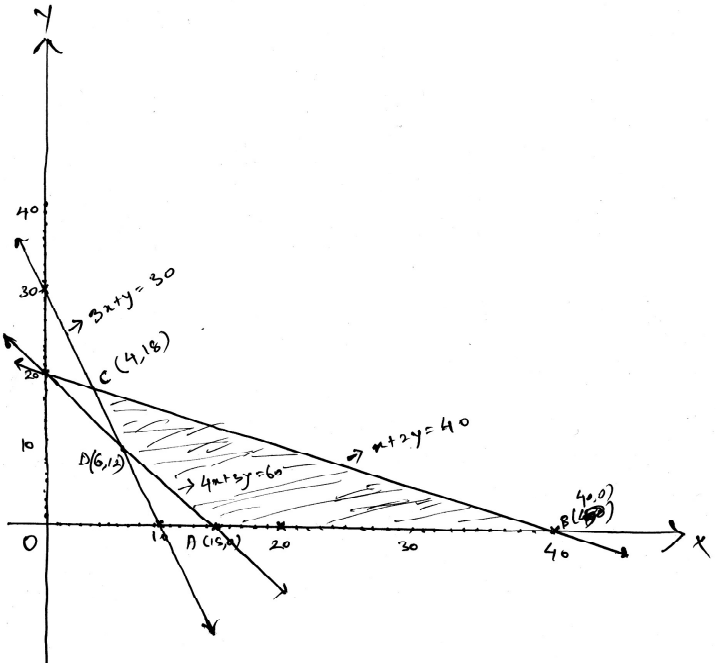


Fig 10.2

Example 3 : Solve the following LPP using graphical method

Maximize $Z = 5x + 3y$

Subject to $3x + 5y \le 15$

$x + 2y \le 10$

$x, y \ge 0$

Solution : We consider each inequality of the constraints as an equation.

$3x + 5y = 15$ ..... (1)

$5x + 2y = 10$ ..... (2)

From (1) We put $x = 0 \Rightarrow y = 3$ and $y = 0 \Rightarrow x = 5$

$\therefore$ (1) passess through the points (0,3) and (5,0).

From (2) we put $x = 0 \Rightarrow y = 5$ and $y = 0 \Rightarrow x = 2$

$\therefore$ (2) passess throgh the points (0, 5) and (2, 0).

Now, we plot each equation on the graph.

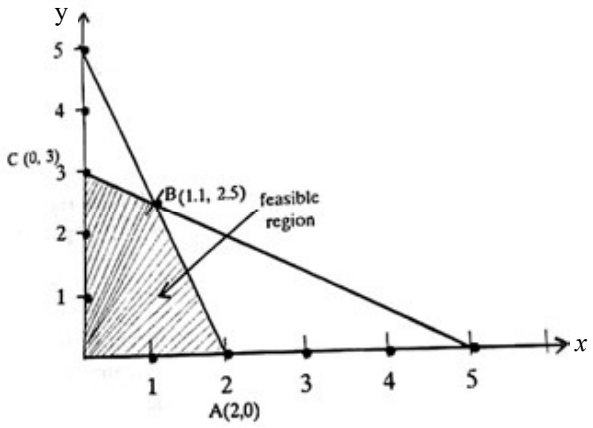


Fig 10.3

From the figure 10.3, we get the feasible region OABC. B is the point of intersection of lines (1) and (2).

On solving (1) and (2), we get the point B(1.1,2.5).

The corner points of the feasible region OABC are O(0,0), A(2,0), B(1.1,2.5) and C(0, 3).

Corner ponts	Value of $Z = 5x + 3y$
O(0,0)	0
A(2,0)	9
B(1.1,2.5)	10
C(0,3)	13

$\therefore$ The minimum value of Z occurs at C(0,3).Hence,the optimal solution is $x = 0$, $y = 3$.

Example 4 : Solve graphically

Maximize $Z = 6x + 3y$

Subject to $4x + 3y \leq 6$

$x + 4y \geq 4$

$x, y \geq 0$

Solution : We consider each inequality of the constraints as an equation.

$4x + 3y = 6$ ..... (1)

$x + 4y = 4$ ..... (2)

From (1), we put $x = 0 \Rightarrow y = 2$ and $y = 0 \Rightarrow x = \frac{3}{2}$

$\therefore$ (1) passess through the points (0,2) and ($\frac{3}{2}$,0).

From (2) we put $x = 0 \Rightarrow y = 1$ and $y = 0 \Rightarrow x = 4$

$\therefore$ (2) passess throgh the points (0,5) and (2,0).

Now, we plot each equation on the graph.

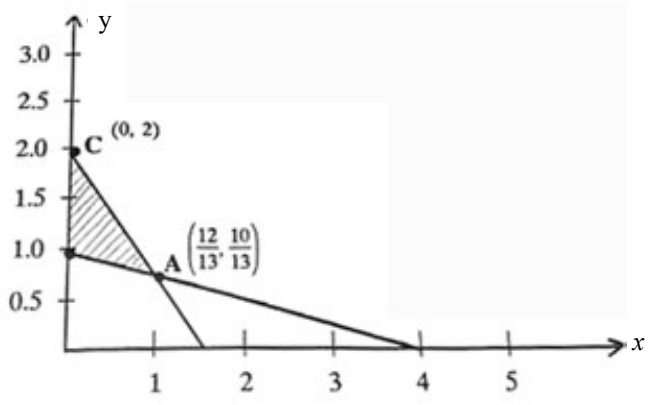


Fig 10.4

To find A using (1) and (2). This is the point of intersection of the two lines.

$4x + 3y = 6$ ..... (1)

$x + 4y = 4$ ..... (2)

Multiplying the equation (2) by 4, we get $4x + 16y = 16$ ..... (3)

Subtracting (1) from (3), $13y = 10$

$$\Rightarrow y = \frac{10}{13}$$

Putting $y = \frac{10}{13}$ in (2)

$$x + \frac{40}{13} = 4$$

$$\Rightarrow x = 4 - \frac{40}{13}$$

$$\Rightarrow x = \frac{52 - 40}{13} = \frac{12}{13}$$

The point A is $\left(\frac{12}{13}, \frac{10}{13}\right)$

From the figure 14.4, we get the feasible region ABC. The corner points are

A$\left(\frac{12}{13}, \frac{10}{13}\right)$, B(0,1) and C(0, $\frac{3}{2}$).

Value of the objective function $Z = 6x + 3y$ at the point A$\left(\frac{12}{13}, \frac{10}{13}\right)$

$$= 6\left(\frac{12}{13}\right) + 3\left(\frac{10}{13}\right)$$

$$= \frac{72}{13} + \frac{30}{13}$$

$$= \frac{102}{13}$$

Value of the objective function $Z = 6x + 3y$ at the point B(0,1)

$$= 6(0) + 3(1)$$

$$= 3$$

Value of the objective function $Z = 6x + 3y$ at the point C(0, $\frac{3}{2}$)

$$= 6(0) + 3\left(\frac{3}{2}\right)$$

$$= 0 + \frac{9}{2}$$

$$= \frac{9}{2}$$

$\therefore$ The optimum solution is $Z = \frac{102}{13}$ at $x = \frac{12}{13}$ and $y = \frac{10}{13}$.

10.5 SOME SPECIAL CASES

The constraints generally give region of feasible solution which may be bounded or unbounded.In general, a linear programming problem may have :

(i) a unique optimal solution

(ii) an infinite number of optimal solutions

(iii) an unbounded solution

(iv) no solution.

We now discuss some examples to illustrate the above cases.

Example 5 : Solve the following LPP graphically.

Maximize $Z = 100x + 40y$

Subject to $5x + 2y \leq 1000$

$3x + 2y \leq 900$

$x + 2y \leq 500$

and $x, y \geq 0$

Solution : We consider each inequality of the constraints as an equation.

$5x + 2y = 1000$ ..... (1)

$3x + 2y = 900$ ..... (2)

$x + 2y = 500$ ..... (3)

From (1),we put $x = 0 \Rightarrow y = 500$ and $y = 0 \Rightarrow x = 200$

$\therefore$ (1) passess through the points (0,500) and (200,0).

From (2) we put $x = 0 \Rightarrow y = 450$ and $y = 0 \Rightarrow x = 300$

$\therefore$ (2) passess throgh the points (0,450) and (300,0).

From (3), we put $x = 0 \Rightarrow y = 250$ and $y = 0 \Rightarrow x = 500$

$\therefore$ (2) passess throgh the points (0,250) and (500,0).

Now, we plot each equation on the graph.

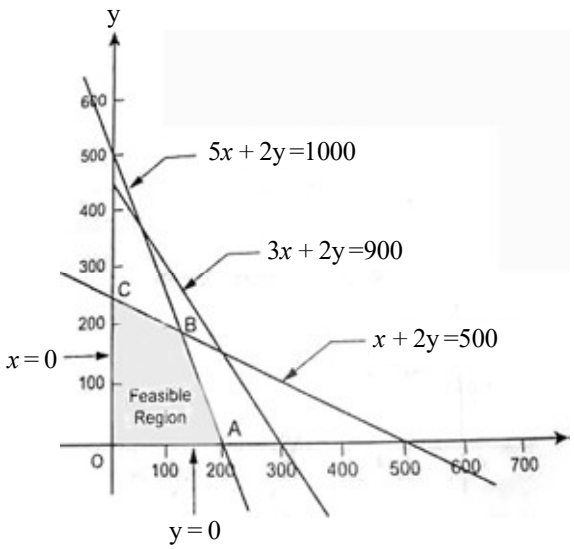


Fig 10.5

From the figure 10.5, we get the feasible region OABC. B is the point of intersection of lines (1) and (3). On solving (1) and (2), we get the point B(125,187.5).

The corner points of the feasible region OABC are O(0,0), A(200,0), B(125,187.5) and C(0,250).

Corner ponts	Value of $Z = 5x + 3y$
O(0,0)	0
A(200,0)	20,000
B(125,187.5)	20,000
C(0,250)	10,000

Here the maximum value of Z occurs at two vertices A and B. Any point on the line joining A and B will also give the same value of Z.

Since, there are infinite number of points between any points, there are infinite number of points on the line joining A and B gives the same maximum value of Z. Thus,there are infinite number of optimal solutions for this LPP.

Example 6 : Using graphical method, solve the following LPP

Maximize $Z = 2x + 3y$

Subject to $x - y \leq 2$

$x + y \geq 4$

and $x, y \geq 0$

Solution : We consider each inequality of the constraints as an equation.

$x - y = 2$ ..... (1)

$x + y = 4$ ..... (2)

From (1), we put $x = 0 \Rightarrow y = -2$ and $y = 0 \Rightarrow x = 2$

$\therefore$ (1) passess through the points (0,-2) and (2,0).

From (2) we put $x = 0 \Rightarrow y = 4$ and $y = 0 \Rightarrow x = 4$

$\therefore$ (2) passess throgh the points (0,450) and (300,0).

Now, we plot each equation on the graph.

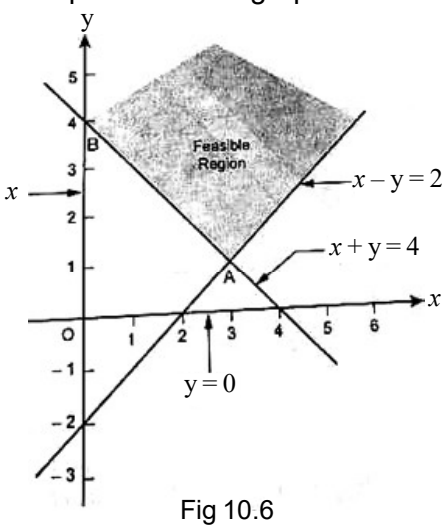


Fig 10.6

From the figure 10.6, we observe that the feasible region is unbounded. The vertices of the feasible region are A(3,1) and B(0,4).

Value of the objective function $Z = 2x + 3y$ at these vertices are 9 and 12.

But there are points in this feasible region for which Z will have much higher values. In fact, the maximum value of Z occurs at infinity. Hence, this problem has an unbounded solution.

Example 7 : Solve graphically the following LPP :

$$Z = 4x + 3y$$

Subject to $x - y \leq -1$

$-x + y \leq 0$

and $x, y \geq 0$

Solution : Since both decision variables x, y are non negative,the solution lies in the first quadrant of the plane.

We consider each inequality of the constraints as an equation.

$$x - y = -1 \text{ ..... (1)}$$

$$-x + y = 0 \text{ ..... (2)}$$

From (1), we put $x = 0 \Rightarrow y = 1$ and $y = 0 \Rightarrow x = -1$

$\therefore$ (1) passess through the points (0,1) and (-1,0).

From (2) we put $y = 0 \Rightarrow x = 0$

$\therefore$ (2) passess throgh the points (0,0).

Now, we plot each equation on the graph.

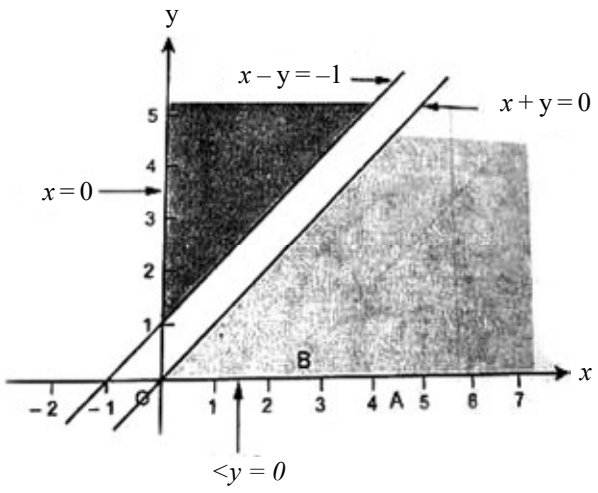


Fig 10.7

The two solution spaces are shown shaded in the following figure.

There is no common point to both the shaded regions. So, we cannot find a feasible region for this problem. So, the problem cannot be solved. Hence, the problem have no feasible solution.

10.6 LIMITATIONS OF GRAPHICAL METHOD

Since graphical method is used to solve a LPP having only two variables,but most of the practical situations do involve more than two variables.So, Graphical method is not a powerful technique for solving a LPP.

Graphical method can deal with any number of constraints but since each constraint is shown as a line on a graph,a large number of lines make the graph difficult to read.So, Graphical method is not a suitablel for solving a LPP having more than three constraints..



CHECK YOUR PROGRESS

Q 1: Define feasible solution.

Q 2. Define optimal solution.

Q 3: Define feasible region.

Q 4: Say whether the following statements are true (T) or false (F)

(i) Graphical method of linear programming is useful when the problem involves two variables.

(ii) Optimal solution in the graphical method of linear programming always lie at one of the vertices in the feasible region.

(iii) An LPP may have more than one optimal solution.

(iv) Constraints appears as straight line when plotted on a graph.

Q 5: Solve the following LPP graphically

Maximize $Z = x + y$

Subject to $2x + y \leq 6$

$x + 2y \leq 6$

and $x, y \geq 0$

Q 6: Solve the following LPP graphically

Maximize $Z = 60x + 15y$

Subject to the constraints

$x + y \leq 50$

$3x + y \leq 90$

and $x, y \geq 0$

Q 7: Solve the LPP graphically

Maximize $Z = 2x + 3y$

Subject to the constraints

$x + y \leq 30$

$y \leq 12$

$x \leq 20$

and $x, y \geq 0$

Q 8: Solve the following LPP

Maximize $Z = x + y$

Subject to the constraints

$x + y \leq 1$

$-3x + y \geq 3$

and $x, y \geq 0$

Q 9: Solve the following LPP using graphical method

Maximize $Z = 100x + 80y$

Subject to the constraints $5x + 10y \leq 50$

$8x + 2y \leq 16$

$3x - 2y \leq 6$

and $x, y \geq 0$

Q 10: Solve the following LPP graphically

Maximize $Z = 10x + 3y$

Subject to the constraints

$2x + 3y \leq 18$

$6x + 5y \leq 60$

and $x, y \geq 0$

Q 11: Solve the LPP by graphical method.

Minimize $Z = 20x + 10y$

Subject to the constraints

$x + 2y \leq 40$

$3x + y \geq 30$

$4x + 3y \geq 60$

and $x, y \geq 0$



10.7 LET US SUM UP

- A set of values of the decision variables which satisfies the constraints of a general LPP is called a solution to the general LPP.
- A solution which satisfies the non-negative conditions of the problem is called a feasible solution.
- Any feasible solution which optimizes(maximizes or minimizes) the objective function of a general LPP is called an optimum solution.
- The common region determined by all the constraints and non-negativity restriction of an LPP is called a feasible region.
- LPP involving two variables can be solved by graphical method.
- Steps involved in the graphical methods for solving linear programming problems in two variables are as follows :
  - (a) we identify the feasible region
  - (b) we compute the coordinates of the corner points
  - (c) we substitute the coordinates of the corner points into the objective function to find the optimal value.
  - (d) We select the corner point which gives the optimum solution.The optimum solution occurs in a mazimization case at the corner point yielding the largest value of the objective function and in a minimization case at the corner point yielding the smallest value of the objective function.



10.8 FURTHER READING

1) Gupta,Prem kumar, Hira,Dr. D. S. *Problems in operation Research(principles and Solutions),* S.Chand & Company Ltd.New Delhi - 110055.

2) Swarup,Kanti,Gupta,P.K,Mohan,Man, *Operations Research* ,Sultan chand & Sons,New Delhi.

3) Taha,Hamdy A. *Operation Research : An Introduction* , Published by Pearson Education Inc.

4) Ravindran,A,Phillips,Don T,Solberg, James J. *Operation Research* wiley India pvt. Ltd. New Delhi - 110002.

10.9 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: Refer to section 10.3

Ans to Q No 2: Refer to section 10.3

Ans to Q No 3: Refer to section 10.3

Ans to Q No 4: (i) True (ii) True (iii) True (iv) True.

Ans to Q No 5: Max Z =4, $x = 2$, $y = 2$

Solution to Q No 6: Let us consider the constraint $x + y \le 50$ as an equation $x + y = 50$ represents a line which pases through (0,50) and (50,0) in XY - plane.

The origin (0,0) gives $0 + 0 = 0 < 50$.

$\therefore$ All the points below and on this line satisfy the inequation $x + y \le 50$.

Let us consider the inquality $3x + y \le 90$ as an equation $3x + y = 90$,represnts a line passes through (0,90) and (30,0) in xy-plane.

$\therefore$ All the points below and on this line satisfy the inequation $3x + y \le 90$.

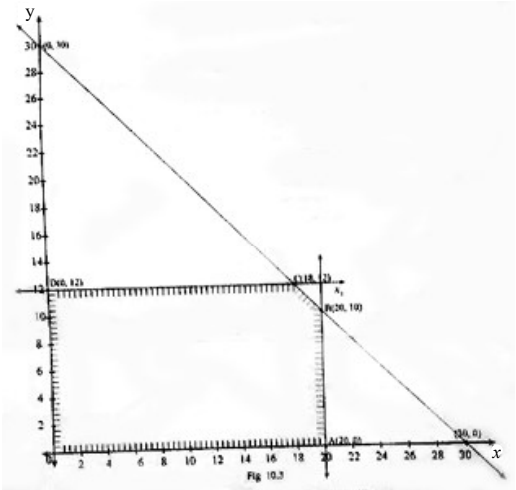


Fig 10.8

From the figure 14.2, we get the feasible region OABC. B is the point of intersection of lines (1) and (2). On solving (1) and (2), we get the point B(20,30).

The corner points of the feasible region OABC are O(0,0), A(30,0), B(20,30) and C(0,50).

Corner ponts	Value of $Z = 5x + 3y$
O(0,0)	0
A(30,0)	1800
B(20,30)	1650
C(0,50)	750

The maximum value of Z is 1800 at A(30,0).

$\therefore$ The optimal solution to the problem is $x = 30, y = 0$ and MaxZ=1800.

Ans to Q No 7: Try yourself. Max Z=72, $x=18, y=12$.

Solution to Q No 8 : Let us consider the constraint $x + y \leq 1$ as an equation $x + y = 1$ represents a line which pases through (0,1) and (1,0) in XY - plane.

The origin (0,0) gives $0 + 0 = 0 < 1$.

$\therefore$ All the points below and on this line satisfy the inequation $x + y \leq 1$.

Let us consider the inquality $-3x + y \leq 3$ as an equation $-3x + y = 3$, represnts a line passes through (0,3) and (-1,0) in xy-plane.

The origin (0,0) gives $-3.0+0=0\geq 1$.

$\therefore$ All the points above and on this line satisfy the inequation $-3x + y \leq 3$

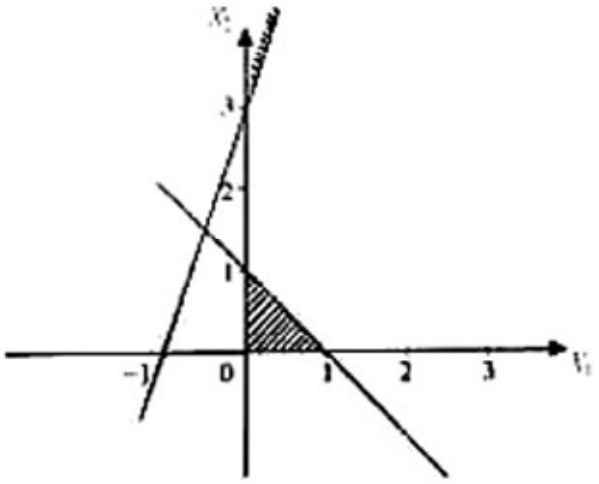


Fig 10.9

There being no point (x, y) common to both the shaded regions, we cannot find a feasible region for this problem. So the problem cannot be solved. Hence the peoblem has no solution.

Solution to Q No 9 : We consider each inequality of the constraints as an equation.

$5x+10y=50$ ..... (1)

$8x+2y=16$ ..... (2)

$3x-2y=6$ ..... (3)

From (1). We get $y=5$ when $x=0$ and $x=10$ when $y=0$.

$\therefore$ (1) passess through the points (0,5) and (10,0).

From (2). We get $y=8$ when $x=0$ and $x=2$ when $y=0$.

$\therefore$ (2) passess through the points (0,8) and (2,0).

From (3). We get $y=-3$ when $x=0$ and $x=2$ when $y=0$.

$\therefore$ (3) passess through the points (0,-3) and (2,0).

Now, we plot each equation on the graph.

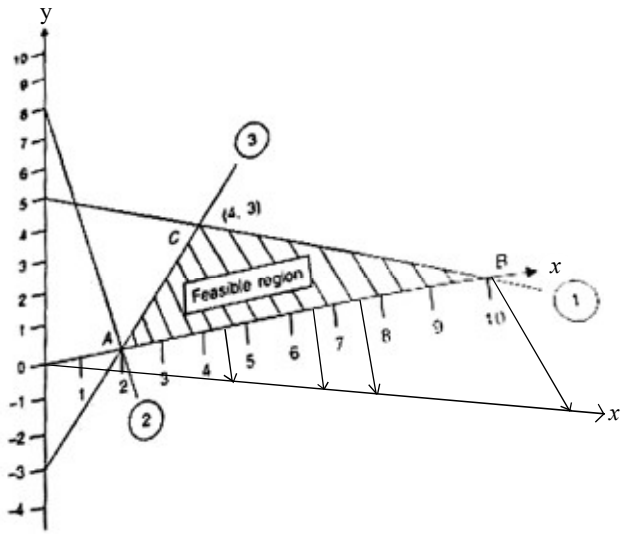


Fig 10.10

From thr figure 14.4, we get the feasible region ABC. C is the point of intersection of lines (1) and (3). On solving (1) and (3), we get the point C(4,3).

The corner points of the feasible region OABC are A(2,0), B(10,0) and C(4,3).

Corner ponts	Value of $Z = 5x + 3y$
A(2,0)	200
B(10,0)	1000
C(4,3)	640

The maximum value of Z is 1000 at B(10,0).

$\therefore$ The optimal solution to the problem is $x = 10, y = 0$ and Max Z=1000.

Solution to Q No 10 : We consider each inequality of the constraints as an equation.

$$2x + 3y = 18 \quad \text{..... (1)}$$

$$6x + 5y = 60 \quad \text{..... (2)}$$

From (1). We get $y = 6$ when $x = 0$ and $x = 9$ when $y = 0$.

$\therefore$ (1) passess through the points (0,6) and (9,0).

From (2). We get $y = 12$ when $x = 0$ and $x = 10$ when $y = 0$.

$\therefore$ (2) passess through the points (0,12) and (10,0).

Now, we plot each equation on the graph.

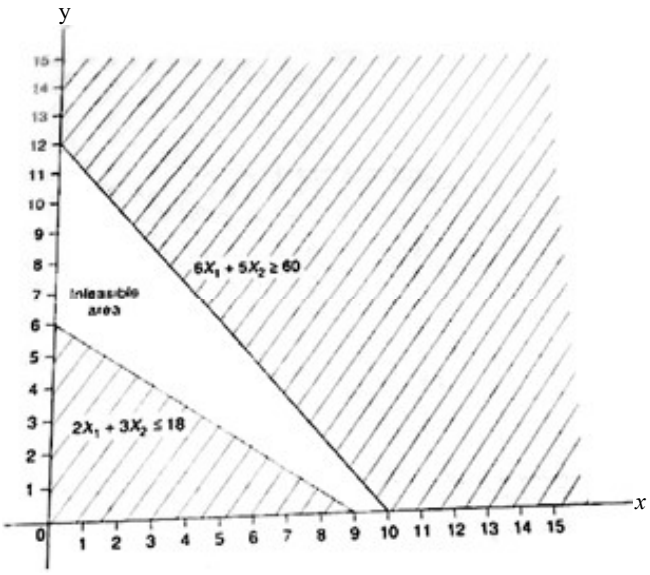


Fig 10.11

The two solution spaces are shown shaded in the following figure.

There is no common point to both the shaded regions. So, we cannot find a feasible region for this problem. So, the problem cannot be solved.

Hence, the problem have no feasible solution.

Answer to QNo 11 : Min Z=240, $x = 6$, $y = 12$.



10.10 MODEL QUESTIONS

Q 1: Explain the graphical method to solve LPP.

Q 2: State the limitations of the graphical method of solving a LPP.

Q 3: Solve the following LPP graphically.

(i) Maximize $Z = 40x + 80y$

Subject to $2x + 3y \leq 48$

$x \leq 15$

$y \leq 10$

$x, y \geq 0$

(ii) Maximize $Z = 5x + 6y$

Subject to $2x + 3y \leq 18$

$2x + y \leq 12$

$3x + 3y \leq 24$

$x, y \geq 0$

(iii) Maximize $Z = 20x + 10y$

Subject to $x + 2y \leq 40$

$3x + y \geq 30$

$3x + 3y \geq 60$

$x, y \geq 0$

(iv) Maximize $Z = 10x_1 + 15x_2$

Subject to $15x_1 + 25x_2 \leq 180$

$x_1 + x_2 \leq 10$

$x_1 \geq 2,\ x_2 \geq 3$

$x_1, x_2 \geq 0$

(v) Maximize $Z = x + 5y$

Subject to $x + 4y \leq 12$

$x + y \geq 2$

$x \leq 8$

$x, y \geq 0$

(vi) Maximize $Z = 8x_1 + 5x_2$

Subject to $x_1 \leq 150$

$x_2 \leq 250$

$2x_1\ x_2 \leq 500$

$x_1, x_2 \geq 0$

(vii) Maximize $Z = x + y$

Subject to $2x + y \leq 8$

$2x + 3y \leq 12$

and $x, y \geq 0$

(viii) Maximize $Z = 4x + y$

Subject to $x + y \leq 50$

$3x + y \leq 90$

and $x, y \geq 0$

(ix) Maximize $Z = 7x + 3y$

Subject to the constraints

$2x + 6y \leq 24$

$6x + 2y \leq 24$

and $x, y \geq 0$

(x) Minimize $Z = 10x + 6y$

Subject to the constraints

$4x + y \geq 80$

$2x + y \geq 60$

and $x, y \geq 0$

Q 4. Apply graphical method to find non-negative values of x and y which minimize $Z = 10x + 25y$ subject to $x + y \geq 50$, $x \geq 20$ and $y \leq 40$.

Q 5. A dealer wishes to purchase a number of fans and sewing machines.He has only Rs.5760 to invest and has the space at most for 20 items.A fan costs him Rs.360 and a sewing machine at a profit of Rs.18.Assuming that he can sell all the items that he can buy,how should he invest his money in order to maximize his profit ? Formulate this problem as a linear programming problem and then use graphical method to solve it.

*** ***** ***

UNIT 11 : SIMPLEX METHOD

UNIT STRUCTURE

11.1 Learning Objectives

11.2 Introduction

11.3 Standard Form of Linear Programming Problem (LPP)

- 11.3.1 General LPP
- 11.3.2 Canonical and Standard Forms of LPP

11.4 Solution of LPP

- 11.4.1 Fundamental Theorem of LPP
- 11.4.2 The Simplex Algorithm
- 11.4.3 Big-M Method
- 11.4.4 Two-Phase Method

11.5 Let Us Sum Up

11.6 Further Reading

11.7 Answers To Check Your Progress

11.8 Model Questions

11.1 LEARNING OBJECTIVES

After going through this unit, you will able to know

- standard form of LPP
- fundamental theorem of LPP
- simplex algorithm fo solving LPP
- Big-M method and two-phase method.

11.2 INTRODUCTION

Linear programming is one of the most important optimisation (maximisation or minimisation) techniques. It is a mathematical programming technique to optimise performance (for example profit, cost etc.) under a set of resource constraints (for example man-hours, machine hours, capital, raw material etc.) This technique (or method) can be used if the following conditions are satisfied.

(i) The objective function (profit, cost) which is to be optimised must be well-defined. It must be a linear function of the decision variables. (The vaiables associated with a linear programming problem are called decision variables).

(ii) The constraints on the amount or extent of the attainment of the objective must be expressed as linear equalities or inequalities in terms of variables.

(iii) There must be alternative courses of action. A product may be processed in two different machines. The problem is how much of the product can be allocated to each machine.

(iv) Decision variables must be non-negative so that real life situations can be dealt with.

(v) The resources must be in limited supply.

11.3 STANDARD FORM OF LINEAR PROGRAMMING PROBLEM (LPP)

11.3.1 General LPP

The general LPP is stated as follows:

To find the values of the variables $x_1, x_2, \ldots . x_n$ which optimise (maximise or minimise) an objective function

$z = c_1x_1 + c_2 x_2 + \ldots + c_nx_n$

subject to the constraints

$a_{11}x_1 + a_{12}x_2 + \ldots .+ a_{1n}x_n \ (\leq \text{or} = \text{or} \geq) \ b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots .+ a_{2n}x_n \ (\leq \text{or} = \text{or} \geq) \ b_2$

. . . . . .

. . . . . .

$a_{m1}x_1 + a_{m2} + \ldots + a_{mn}x_n (\leq \text{or} = \text{or} \geq) \ b_m$, and $x_1, x_2, \ldots, x_n \geq 0$

In sigma notation it can be expressed as follows.

maximise (or minimise) $z = \sum_{j=1}^{n} c_j x_j$

subject to the constaints $\sum_{j=1}^{n} a_{ij} x_j (\leq or = or \geq) \ b_i, \ i = 1, 2, \ldots, m$

and $x_j \geq 0$, j = 1, 2, . . ., n.

It is to be noted that for each constraint, one of the signs '$\leq$', '=', '$\geq$' holds goods. The sign may vary from constraint to constraint.

Variables x_j((j = 1, 2, . . ., n) are called **decision variables.**

Constants c_j are called **objective function coefficients.**

Constants b_i (i = 1, 2, . . ., m) are called **stipulations.**

Constants a_{ij} are called **structural coefficients.**

Matrix Form of LPP

Let $A = \begin{pmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{m1} & a_{m2} & .... & a_{mn} \end{pmatrix}$

be an mxn matrix.

$x = (x_1\ x_2 \ldots x_n)^T$ be n x 1 column matrix (vector)

$b = (b_1\ b_2 \ldots b_m)^T$ be m x 1 column matrix (vector).

$c = (c_1\ c_2 \ldots c_n)$ be 1 x n row matrix (vector).

Then the LPP becomes

Maximise (or minimise) $z = cx$

Subject to $Ax\ (\leq \text{or} = \text{or} \geq)\ b$, and $x \geq 0$

A is called the coefficient matrix,

x is the decision vector,

b is the requirement vector,

c is the cost (or price or profit) vector,

o is an n x n null column matrix (vector).

11.3.2 Canonical and Standard Forms of LPP

(A) Canonical Form

Maximise $z = \sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, i = 1, 2, . . ., m

$x_j \geq 0$, j = 1, 2, . . . , n.

The characteristics of cannonical form are as follows.

(a) Objective function is in maximisation form.

(b) All the constraints are of the 'less than or equal to' form.

(c) all the decision variables are non-negative.

Note : 1. If the objective function is in minimisation form, it is converted to maximisation form by multiplying by -1. For example the LPP 'Minimise $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$' is converted to 'Maximise $-z = -c_1 x_1 - c_2 x_2 - \ldots - c_n x_n$'.

2. An inequality constraint of 'greater than or equal to' type is converted to 'less than or equal to' type. For example the constraint $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$ is converted to $-a_{11}x_1 - a_{12}x_2 - \ldots - a_{1n}x_n \leq -b_1$.

3. An equality is replaced by two inequalities of opposite signs. For example

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$ is replaced by

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

and $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$

i.e., $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

and $-a_{11}x_1 - a_{12}x_2 - \ldots - a_{1n}x_n \leq -b_1$

4. Decision variables $x_1, x_2, \ldots, x_n$ are assumed to be non-negative. If a variable is unrestricted in sign, then it may be positive, or zero, or negative. For example, let x_1 be unrestricted in sign. Then x_1 can be expressed as $x_1 = x_1' - x_1''$, where x_1', x_1'' are non-negative variables. If $x_1' > x_1''$, $x_1 > 0$; if $x_1' = x_1''$, $x_1 = 0$; if $x_1' < x_1''$, $x_1 < 0$.

B. Standard Form

Maximise (or minimise)

$$z = \sum_{j=1}^{n} c_j x_j$$

Subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$, i = 1, 2, . . ., m

$x_j \geq 0$, j = 1, 2, . . ., n

and $b_i \geq 0$

In matrix form it becomes Maximise (or minimise) z = cx

Subject to Ax = b,

$x \geq 0, \quad b \geq 0.$

The characteristics of standard form are as follows.

(a) Objective function may be in maximisation form or in minimisation form.

(b) All constraints are expressed as equations.

(c) All the variables are non-negative.

(d) The right hand side of each constraint is non-negative.

Note : Any LPP can be expressed in standard form by application of some elementary transformations.

(i) Any unrestricted variable can be expressed as the difference of two non-negative variables.

(ii) If the right hand side of a constraint is negative, it is made positive by multiplying both sides by –1.

(iii) If some of the constraints are inequations, they are to be converted to equations by adding or subtracting non-negative variables to the left hand side of each constraint. These variables are called **slack variables** when added to the constraints ($\leq$); and **surplus** variables when subtracted from the constraints ($\geq$). Decision variables, slack variables, surplus variables are called admissible variables.

Example 1: Express the following LPP in standard form.

Maximise $z = 3x_1 + 5x_2 + 4x_3$.

Subject to the constraints

$2x_1 - 5x_2 \leq 6$

$2x_1 + 3x_2 + x_3 \geq 5$

$3x_1 + 4x_3 \leq 3$; $x_1, x_2, x_3 \geq 0$

Solution :We introduce slack variables s_1, s_3 ;and surplus variable s_2.

The LPP in standard form is as follows

Maximise $z = 3x_1 + 5x_2 + 4x_3$.

Subject to the constraints

$2x_1 - 5x_2 + s_1 = 6$

$$2x_1 + 3x_2 + x_3 - s_2 = 5$$

$$3x_1 + 4x_3 + s_3 = 3$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Example 2: Express the following LPP in standard form.

Minimise

$$z = 2x_1 + x_2 + 3x_3$$

Subject to the constraints

$$-5x_1 + 2x_2 \leq 5$$

$$3x_1 + 2x_2 + 4x_3 \geq 7$$

$$2x_1 + 5x_3 \leq 3$$

$x_1, x_2 \geq 0$, x_3 is unrestricted

Solution : We write x_3 as the difference of two non-negative variables y_1, y_2, i.e, $x_3 = y_1 - y_2$. The the LPP becomes.

Minimise $z = 2x_1 + x_2 + 3(y_1 - y_2)$

Subject to the constraints

$$-5x_1 + 2x_2 \leq 5$$

$$3x_1 + 2x_2 + 4(y_1 - y_2) \geq 7$$

$$2x_1 + 5(y_1 - y_2) \leq 3$$

$$x_1, x_2, y_1, y_2 \geq 0.$$

We introduce slack variables s_1, s_3; and surplus variable s_2.

The LPP in standard form is as follows.

Minimise $z = 2x_1 + x_2 + 3y_1 - 3y_2$

Subject to the constraints

$$-5x_1 + 2x_2 + s_1 = 5$$

$$3x_1 + 2x_2 + 4y_1 - 4y_2 - s_2 = 7$$

$$2x_1 + 5y_1 - 5y_2 + s_3 = 3,$$

$x_1, x_2, y_1, y_2 s_1, s_2, s_3 \geq 0$.



CHECK YOUR PROGRESS 1

Q 1: Express the following LPPs in standard form.

(i) Maximise $z = 5x_1 + 7x_2$

Subject to $3x_1 + 2x_2 \leq 20$

$x_1 + 3x_2 \geq 10$

$x_1, x_2 \geq 0$

(ii) Maximise $z = 4x_1 - x_2 - 3x_3$

Subject to $x_1 + 5x_2 - 3x_3 \leq 20$

$2x_1 + 7x_2 + 2x_3 \leq 10$

$x_1 - 5x_2 - 3x_3 \geq 3$

$x_1, x_3 \geq 0$, x_2 is unrestricted.

(iii) Maximise $z = 2x_1 + 5x_2$

Subject to $x_1 + 2x_2 - x_3 = -4$

$3x_1 + 4x_2 - x_4 = -7$

$x_1 + 2x_2 + x_5 = 9$

$5x_1 - 2x_2 + x_6 = 17$

x_1, x_2 are unrestricted, $x_3, x_4, x_5, x_6 \geq 0$

Q 2: Express the following LPP in standard form

Maximise $z = 3x_1 + 5x_2 - 2x_3$

Subject to $x_1 + 2x_2 - x_3 \geq -4$

$-5x_1 + 6x_2 + 7x_3 \geq 5$

$2x_1 + x_2 + 3x_5 = 10$

$x_1, x_2 \geq 0$, x_3 is unrestricted in sign.

11.4 SOLUTION OF LPP

If an LPP contains more than 2 variables, then it cannot be solved with the help of Graphical Method. Of course, an LPP containing 3 variables can be solved by Graphical Method, but the procedure is tedious as it involves intersection of planes in three dimensions. Simplex method can be used for solving any linear model for which the solution exists. It was developed by American mathematician George Bernard Dantzig (1914-2005) in 1947.

Before discussing the simplex method, we state some definitions.

Let us consider the general LPP involving n variables and m constraints ($m \leq n$)

Maximise $z = c_1x_1 + c_2x_2 + \ldots + c_n x_n$

Subject to $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$$

- -

- -

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$$

$$x_1, x_2, \ldots, x_n \geq 0$$

We introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$. The LPP in standard form is as follows.

Maximise $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ (1)

Subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + x_{n+1} = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + x_{n+2} = b_2 \\ - \\ - \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + x_{n+m} = b_m \end{array}\right\} \quad (2)$$

$x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+m} \geq 0$ (3)

Solution : A set of variables.

$[x_1, x_2 \ldots x_{n+m}]$ (column vector) is called a **solution** to LPP if it satisfies the constraints (2).

Feasible solution : A set of variables $[x_1 \ x_2 \ldots x_{n+m}]$ is called a **feasible solution** to the above LPP if it satisfies the constraints (2) and the non-negativity restrictions (3).

Basic solution : A solution obtained by equating any n variables (among the m+n variables) equal to zero, and solving the remaining m variables (provided the determinant of the coefficients of these m variables is non-zero) is called a **basic solution**. These m variables (some of which may be zero) are called **basic** variables, and the remaining n variables are called **non-basic** variables.

The number of basic solutions is at most ${}^{m+n}C_m$, i.e., $\frac{(m+n)!}{m!\ n!}$

Basic Feasible Solution : A basic solution that also satisfies the non-negativity restrictions (3) is called a basic feasible solution (BFS).

Non-degenerate basic feasible solution : A basic feasible solution in which all the m basic variables are positive and the remaining n variables are zero each is called a non-degenerate basic feasible solution.

Degenerate basic feasible solution : A basic feasible solution in which one or more of the m basic variables are equal to zero is called degenerate basic feasible solution.

Optimal basic feasible solution : A basic feasible solution that also optimises the objective function (1) is called optimal basic feasible solution.

Unbounded solution : A solution is called unbounded solution if the value of the objective function can be increased or decreased indefinitely.

Example 3 : Find the basic solutions of the system of equations

$x_1 + x_2 + 2x_3 = 9$

$3x_1 + 2x_2 + 5x_3 = 22$

Solution : Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $b = \begin{pmatrix} 9 \\ 22 \end{pmatrix}$

$\therefore$ the system can be written as $Ax = b$.

Here number of variables = n = 3

number of equations = m = 2

$\therefore$ the system has at most ${}^3C_2$, i.e, 3 basic equations. $\left({}^3C_2 = \frac{3!}{2!\,1!} = 3 \right)$

Let $\alpha_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

The square submatrices by taking two at a time are

$B_1 = (\alpha_1 \quad \alpha_2)$, $B_2 = (\alpha_1 \quad \alpha_3)$, $B_3 = (\alpha_2 \quad \alpha_3)$

$|B_1| = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = 2 - 3 = -1 \neq 0$ $\quad \therefore B_1^{-1}$ exists

$|B_2| = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0$ $\quad \therefore B_2^{-1}$ exists

$|B_3| = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1 \neq 0$ $\quad \therefore B_3^{-1}$ exists

Now $XB_1 = B_1^{-1} b = \frac{1}{-1} \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 22 \end{pmatrix}$

$= \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ 22 \end{pmatrix}$

$$= \begin{pmatrix} -18 & +22 \\ 27 & -22 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$XB_2 = B_2^{-1}b = \frac{1}{-1}\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 9 \\ 22 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 9 \\ 22 \end{pmatrix}$$

$$= \begin{pmatrix} -45 & +44 \\ 27 & -22 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$$XB_3 = B_3^{-1}b = \frac{1}{1}\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 9 \\ 22 \end{pmatrix}$$

$$= \begin{pmatrix} 45 & -44 \\ -18 & +22 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Thus, the basic solutions are

$x_1 = (4, 5, 0)$, $x_2 = (-1, 0, 5)$, $x_3 = (0, 1, 4)$.

Example 4 : Find the basic feasible solutions of the system of equations

$2x_1 + 6x_2 + 2x_3 + x_4 = 3$

$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$

Solution : Let $A = \begin{pmatrix} 2 & 6 & 2 & 1 \\ 6 & 4 & 4 & 6 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\therefore$ the system can be written as $Ax = b$.

Here number of variables = n = 4

number of equations = m = 2.

$\therefore$ the system has at most ${}^4C_2$, i.e., 6 basic solutions.

Let $\alpha_1 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

The square submatrices by taking 2 at a time are.

$B_1 = (\alpha_1 \ \ \alpha_2)$, $B_2 = (\alpha_1 \ \ \alpha_3)$, $B_3 = (\alpha_1 \ \ \alpha_4)$, $B_4 = (\alpha_2 \ \ \alpha_3)$,

$B_5 = (\alpha_2 \ \ \alpha_4)$, $B_6 = (\alpha_3 \ \ \alpha_4)$

$|B_1| = \begin{vmatrix} 2 & 6 \\ 6 & 4 \end{vmatrix} = 8 - 36 = -28 \neq 0$ $\quad \therefore B_1^{-1}$ exists

$$|B_2| = \begin{vmatrix} 2 & 2 \\ 6 & 4 \end{vmatrix} = 8 - 12 = -4 \neq 0 \qquad \therefore\ B_2^{-1} \text{ exists}$$

$$|B_3| = \begin{vmatrix} 2 & 1 \\ 6 & 6 \end{vmatrix} = 12 - 6 = 6 \neq 0 \qquad \therefore\ B_3^{-1} \text{ exists}$$

$$|B_4| = \begin{vmatrix} 6 & 2 \\ 4 & 4 \end{vmatrix} = 24 - 8 = 16 \neq 0 \qquad \therefore\ B_4^{-1} \text{ exists}$$

$$|B_5| = \begin{vmatrix} 6 & 1 \\ 4 & 6 \end{vmatrix} = 36 - 4 = 32 \neq 0 \qquad \therefore\ B_5^{-1} \text{ exists}$$

$$|B_6| = \begin{vmatrix} 2 & 1 \\ 4 & 6 \end{vmatrix} = 12 - 4 = 8 \neq 0 \qquad \therefore\ B_6^{-1} \text{ exists}$$

Now $XB_1 = B_1^{-1} b = \frac{1}{-28}\begin{pmatrix} 4 & -2 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$= \frac{1}{-28}\begin{pmatrix} 12 & -12 \\ -18 & +4 \end{pmatrix} = \frac{1}{-28}\begin{pmatrix} 0 \\ -14 \end{pmatrix} = \begin{pmatrix} -2 \\ \frac{1}{2} \end{pmatrix}$$

$$XB_2 = B_2^{-1} b = \frac{1}{-4}\begin{pmatrix} 4 & -2 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \frac{1}{-4}\begin{pmatrix} 12 & -4 \\ -18 & +4 \end{pmatrix} = \frac{1}{-4}\begin{pmatrix} 8 \\ -14 \end{pmatrix} = \begin{pmatrix} -2 \\ \frac{7}{2} \end{pmatrix}$$

$$XB_3 = B_3^{-1} b = \frac{1}{6}\begin{pmatrix} 6 & -1 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 18 & -2 \\ -18 & +4 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 16 \\ -14 \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ -\frac{7}{3} \end{pmatrix}$$

$$XB_4 = B_4^{-1} b = \frac{1}{16}\begin{pmatrix} 4 & -2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} 12 & -4 \\ -12 & +12 \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 8 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$$XB_5 = B_5^{-1} b = \frac{1}{32}\begin{pmatrix} 6 & -1 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \frac{1}{32}\begin{pmatrix} 18 & -2 \\ -12 & +12 \end{pmatrix} = \frac{1}{32}\begin{pmatrix} 16 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$$XB_6 = B_6^{-1}b = \frac{1}{8}\begin{pmatrix} 6 & -1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} 18 & -2 \\ -12 & +4 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 16 \\ -8 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$\therefore$ the basic solutions are

$$x_1 = \left(0, \frac{1}{2}, 0, 0\right), x_2 = \left(-2, 0, \frac{7}{2}, 0\right), x_3 = \left(\frac{8}{3}, 0, 0, \frac{-7}{3}\right)$$

$$x_4 = \left(0, \frac{1}{2}, 0, 0\right), x_5 = \left(0, \frac{1}{2}, 0, 0\right), x_6 = (0, 0, 2, -1)$$

of these 6 basic solutions, the basic deasible solutions are

$$x_1 = \left(0, \frac{1}{2}, 0, 0\right), x_4 = \left(0, \frac{1}{2}, 0, 0\right), x_5 = \left(0, \frac{1}{2}, 0, 0\right)$$

The other 3 basic solutions contain negative numbers.



CHECK YOUR PROGRESS 2

Q 1: Find the basic feasible solutions of the system of equations

$2x_1 + 3x_2 + x_3 = 8$

$x_1 + 2x_2 + 2x_3 = 5$

Q 2: How many basic solutions are there in the given set of linearly independent equations?

$2x_1 - 5x_2 + 6x_3 = 9$

$6x_1 + x_2 + 18x_3 = 11$

Find all of them.

Q 3: Find the basic feasible solutions of the system of equations

$2x_1 - 3x_2 + x_3 = 8$

$x_1 + 2x_2 + 3x_3 = 15$

11.4.1 Fundamental Theorem of LPP

We state, without proof Fundamental Theorem of LPP.

Let us consider the LPP

Maximise z = cx

subject to Ax = b

$x \geq 0$

If the LPP admits of an optimal solution, then the optimal solution will coincide with at least one basic feasible solution of the problem.

11.4.2 The Simplex Algorithm(or the simplex Method)

The simplex Algorithm is an algebraic procedure. Here each iteration involves a system of equations to obtain a new trial solution for the optimality test. Accordingly, we take the following three steps.

I First Step : We start with a basic feasible solution of a given LPP.

II Second Step (Iterative Step) : We move to a better basic feasible solution.

III Third Step (Optimality Test Step) : The present basic feasible solution is optimal.

IV Fourth Step : If 'Yes', we stop. If 'No', we go to second step.



LET US KNOW

An algorithm is 'a mathematical procedure or method to obtain a result'. (An Introduction to The Theory of Numbers (1991) : I Niven and H. S. Zuckerman)

We give a detailed description of the above algorithm with the help of the following example.

Example 5 :

Maximise $z = 60x_1 + 70x_2$

Subject to $2x_1 + x_2 \leq 300$

$3x_1 + 4x_2 \leq 509$

$4x_1 + 7x_2 \leq 812$

$x_1, x_2 \geq 0$

by simplex method.

Solution : We introduce slack variables x_3, x_4, x_5.

$\therefore$ the problem becomes

Maximise

$$z = 60x_1 + 70x_2 + 0x_3 + 0x_4 + 0x_5$$

Subject to $2x_1 + x_2 + x_3 = 300$

$3x_1 + 4x_2 + x_4 = 509$

$4x_1 + 7x_2 + x_5 = 812$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

We take $x_1 = 0$, $x_2 = 0$

$\therefore$ $x_3 = 300$, $x_4 = 509$, $x_5 = 812$

$\therefore$ (0, 0, 300, 509, 812) is a basic solution. Moreover, it is a basic feasible solution, since all the variables are non-negative.

x_3, x_4, x_5 are basic variables.

x_1, x_2 are non-basic variables.

Corresponding to this basic feasible solution,

$$z = 60 \times 0 + 70 \times 0 + 0 = 0$$

We represent by CB the coefficients of the basic variables in the objective function.

We represent by XB the numerial values of the basic variables.

The numerical values of the basic variables are $XB_1 = 300$, $XB_2 = 509$, $XB_3 = 812$.

The objective function

$$Z = 60x_1 + 70x_2 + 0x_3 + 0x_4 + 0x_5$$

can also be expressed as

$$Z - 60x_1 - 70x_2 - ox_3 - ox_4 - ox_5 = 0$$

The simplex computation starts with the first simplex table given by

		c_j	60	70	0	0	0
CB	Basic Variables	XB	x_1	x_2	x_3	x_4	x_5
0	x_3	300	2	1	1	0	0
0	x_4	509	3	4	0	1	0

0	x_5	812	4	7	0	0	1
	z		–60	–70	0	0	0

Table 1

The topmost row of Table 1 gives the coefficients (c_j) of the variables x_1, x_2, x_3, x_4, x_5 in the objective function respectively. The column under x_1 gives the coefficients of x_1 in the three constraint equations respectively, and so on.

We have $z = 60x_1 + 70x_2$. Here x_1, x_2 are non-basic variables. If either x_1 or x_2 is taken as basic variable, value of z will increase. Now, coefficient of x_2 is greater than the coefficient of x_1 in the objective function. So, we choose x_2 as a basic variable in the next iteration. We can choose this new basic variable by looking at the last row of Table 1 as well. The entry –70 corresponding to x_2 is smaller than the other negative value –60. Therefore, x_2 is considered as a basic variable in the next iteration.

Thus x_2 is the new basic variable.

There are three constraints. So, there will be three basic variables. Now, one of the existing basic variables will be non-basic. Let us identify this non-basic variable.

Let us consider the first constraint equation.

$2x_1 + x_2 + x_3 = 300$

$\Rightarrow 2x_1 + x_3 = 300 - x_2$

But $x_1 = 0$ $\quad \therefore x_3 = 300 - x_2$

$x_3 \geq 0 \Leftrightarrow 300 - x_2 \geq 0$

$\Leftrightarrow 300 \geq x_2$

$\Leftrightarrow x_2 \leq 300$

Again, from the second constraint equation.

$3x_1 + 4x_2 + x_4 = 509$

$\Rightarrow 3x_1 + x_4 = 509 - 4x_2$

But $x_1 = 0$ $\quad \therefore x_4 = 509 - 4x_2$

$x_4 \geq 0 \Leftrightarrow 509 - 4x_2 \geq 0$

$\Leftrightarrow 509 \geq 4x_2$

$\Leftrightarrow 4x_2 \leq 509$

$$\Leftrightarrow x_2 \leq \frac{509}{4} = 127.25$$

Again, from the third constraint equation.

$$4x_1 + 7x_2 + x_5 = 812$$

But $x_1 = 0$ $\quad \therefore 7x_2 + x_5 = 812$

$$\Leftrightarrow x_5 = 812 - 7x_2$$

$$x_5 \geq 0 \Leftrightarrow 812 - 7x_2 \geq 0$$

$$\Leftrightarrow 7x_2 \leq 812$$

$$\Leftrightarrow x_2 \leq \frac{812}{7} = 116$$

$\therefore x_2 = \min (x_2 \leq 300, x_2 \leq 127.25, x_2 \leq 116)$

$= 116$

When $x_2 = 116$, from the 3rd constraint equation

$$7x_2 + x_5 = 812$$

$\Rightarrow 7 \times 116 + x_5 = 812 \qquad \Rightarrow x_5 = 0$

$\therefore$ in the next iteration, x_5 becomes non-basic.

The revised values of the other two basic variables are as follows.

$x_3 = 300 - x_2 = 300 - 116 = 184$

$x_4 = 509 - 4 \times 116 = 509 - 464 = 45$

Thus, the revised basic variables are x_3, x_4 x_2. Accordingly $CB_1 = 0$, $CB_2 = 0$, $CB_3 = 70$ respectively.

x_2 is the incoming basic variable. We make the coefficient of x_2 one in Table 2 by dividing each element of the 3rd row of Table 1 by 7.

Thus the 3rd row in Table 2 becomes

$$\frac{4}{7} \quad 1 \quad 0 \quad 0 \quad \frac{1}{7}$$

The incoming basic variable should appear only in the 3rd row.

Multiplying every element of the 3rd row of Table 2 by 1, and subtracting from the 1st row of Table 1, we get the 1st row of Table 2 as follows.

$2-1\times\frac{4}{7}=\frac{10}{7}$ 0 0 0 $0-\frac{1}{7}=-\frac{1}{7}$

Similarly multiplying every element of the 3rd row of Table 2 by 4, and subtracting from the 2nd row of Table 1, we get the 2nd row of Table 2 as follows.

$3-\frac{4\times4}{7}=\frac{5}{7}$ 0 0 1 $0-\frac{4}{7}=-\frac{4}{7}$

		c_j	60	70	0	0	0
CB	Basic Variables	XB	x_1	x_2	x_3	x_4	x_5
0	x_3	184	$\frac{10}{7}$	0	0	0	$-\frac{1}{7}$
0	x_4	45	$\frac{5}{7}$	4	0	1	$-\frac{4}{7}$
70	x_2	116	$\frac{4}{7}$	1	0	0	$\frac{1}{7}$
	z_j-c_j		–20	0	0	0	10

Table 2

Let CB_1, CB_2, CB_3 be the coefficients of the basic variables in the objective function. For Table 2, $CB_1 = 0$, $CB_2 = 0$, $CB_3 = 70$.

Let the corresponding elements of the jth row column be denoted by $Y1_j$, $Y2_j$, $Y3_j$ respectively.

Corresponding to variables j, let z_j be defined by

$z_j = CB_1 . Y1_j + CB_2 . Y2_j + CB_3 . Y3_j$

Then the z-row can be represented by $z_j - c_j$

Now $z_1 - c_1 = CB_1 .Y1_j + CB_2 . Y2_1 + CB_3 . Y3_1 - C_1$

$= 0\times\frac{10}{7}+0\times\frac{5}{7}+70\times\frac{4}{7}-60$

$= 40 - 60 = -20$

$z_2 - c_2 = CB_1 . Y1_2 + CB_2 . Y2_2 + CB_3 . Y3_2 - C_2$

$= 0 \times 0 + 0 \times 0 + 70 \times 1 - 70 = 0$

$z_3 - c_3 = CB_1 . Y1_3 + CB_2 . Y2_3 + CB_3 . Y3_3 - C_3$

$= 0 \times 0 + 0 \times 0 + 70 \times 0 - 0 = 0$

$z_4 - c_4 = CB_1 . Y1_4 + CB_2 . Y2_4 + CB_3 . Y3_4 - C_4$

$= 0 \times 0 + 0 \times 1 + 70 \times 0 - 0 = 0$

$z_5 - c_5 = CB_1 \cdot Y1_5 + CB_2 \cdot Y2_5 + CB_3 \cdot Y3_5 - C_5$

$$= 0 \times \left(-\frac{1}{7}\right) + 0 \times \left(-\frac{4}{7}\right) + 70 \times \frac{1}{7} - 0$$

$= 10$

In Table 2, the only negative $z_j - c_j$ is $z_1 - c_1 = -20$. Therefore z_1 will be a basic variable in the next iteration.

We calculate the minimum of the ratios

$$\frac{184}{\frac{10}{7}}, \frac{45}{\frac{5}{7}}, \frac{116}{\frac{4}{7}}$$

$$\text{Thus min} \left(\frac{184 \times 7}{10}, \frac{45 \times 7}{5}, \frac{116 \times 7}{4}\right)$$

$$= \min \left(\frac{644 \times 7}{5}, 63, 203\right)$$

$= 63$

We get this minimum corresponding to x_4. Therefore x_4 becomes a non-basic variable in the next iteration.

Now, proceeding as above we construct the following table (Table 3)

		c_j	60	70	0	0	0
CB	Basic Variables	XB	x_1	x_2	x_3	x_4	x_5
0	x_3	94	0	0	1	–2	1
60	x_1	63	1	0	0	$\frac{7}{5}$	$-\frac{4}{5}$
70	x_2	80	0	1	0	$-\frac{4}{5}$	$\frac{3}{5}$
	$z_j - c_j$		0	0	0	28	– 6

Table 3

We note that $z_5 - c_5 = -6 < 0$. sherefore, x_5 will be a basic variable in the next iteration.

We calculate minimum of the ratios $\frac{94}{1}, \frac{80}{\frac{3}{5}}$

$\min\left(94, \frac{80\times 5}{3}\right) = 94$

The ratio $\frac{63}{\frac{-4}{5}}$ is not consider, since $-\frac{4}{5}$ is negative

$\therefore$ The variable x_3 will be non-basic in the next iteration.

Now, proceeding as above, we construct the following table (Table 4)

		c_j	60	70	0	0	0
CB	Basic Variables	XB	x_1	x_2	x_3	x_4	x_5
0	x_5	94	0	0	1	–2	1
60	x_1	$\frac{691}{5}$	1	0	$\frac{4}{5}$	$-\frac{1}{5}$	0
70	x_2	$\frac{118}{5}$	0	1	$-\frac{3}{5}$	$\frac{2}{5}$	0
	$z_j - c_j$		0	0	6	16	0

Table 4

We note that $z_j - c_j \geq 0$ for all j. Thus, the current basic feasible solution is optimal.

Thus, the maximum value of z is obtained when

$x_1 = \frac{691}{5}, \quad x_2 = \frac{118}{5}$

$\therefore$ Z(maximum)

$= 60\times\frac{691}{5} + 70\times\frac{118}{5}$

= 12 x 691 + 14 x 118

= 8292 + 1652

= 9944

Note :

1. If all the quantities corresponding to $z_j - c_j$ are positive, the inclusion of any non-basic variable will not increase the value of

the objective function. The current solution will give the optimal value of the objective function. If there are more than one negative values, we choose the variable as a basic variable corresponding to which z-value is least.

2. Let x_j be the incoming basic variable, and the corresponding elements of the j th row column be denoted by $Y1_j$, $Y2_j$, $Y3_j$ respectively. If the present values of the basic variables are XB_1, XB_2, XB_3 respectively, we can calculate.

$$\min \left(\frac{XB_1}{Y1_j}, \frac{XB_2}{Y2_j}, \frac{XB_2}{Y3_j} \right)$$

$Y1_j, Y2_j, Y3_j > 0$

We note that if any one of $Y1_j$, $Y2_j$, $Y3_j$ is negative, this is not considered in the comparison. If the minimum occurs corresponding to $\frac{XB_r}{Yr_j}$, then the r th basic variable will become non-basic in the next iteration.



CHECK YOUR PROGRESS 3

Q 1: Solve the following LPP by simplex method.

Maximise $z = 2x_1 + x_2$

Subject to $4x_1 + 3x_2 \leq 12$

$4x_1 + x_2 \leq 8$

$4x_1 - x_2 \leq 8$

$x_1, x_2 \geq 0$



LET US KNOW

1. Simplex method is an iterative process. It involves the substitution of variables for obtaining successively better solutions.
2. For solving an LPP by simplex method, all the variables involved must be non-negative.
3. For solving an LPP by simplex method, all the constraints must be converted into equations.

4. For maximisation LPP, the simplex method is terminated when all the $z_j - c_j$'s are non-negative.
5. For minimisation LPP, the simplex method is terminated when all the $z_j - c_j$' s are non-positive.

11.4.3 Big-M Method (or Method of Penalties)

There are many LPPs where slack variables cannot provide a solution. In such type of problems, at least one of the constraints is of '$\geq$' or '=' type. In such problems, a kind of variables is introduced. These are known as artificial variables. They take the role of slack variables in the first iteration, only to be replaced at a later iteration, With the help of it we get the starting basic feasible solution so that the simplex algorithm can be applied as usual to get optimal solution. There are two techniques to solve such type of problems, viz, (1) Big-M Method, (2) Two-Phase Method.

The Big-M Method is due to American mathematician Abraham Charnes (1917-1992).

The iterative procedure of the Big-M Method is as follows.

Step I : We express the LPP in standard form by introducing slack/ surplus variables.

Step II : We add non-negative variables to the left hand sides of all the constriants of initially '$\geq$' or '=' type, called artificial variables. We introduce these artificial variables in order to obtain an initial basic feasible solution. These variables will not be allowed to appear in the final solution. Therefore, these variables are assigned a very large per unit penalty in the objective function. This penalty is denoted by – M for maximisation problems and + M for minimisation problems, where M > 0. Value of M is much higher than the cost coefficients of other variables and for hand calculations, it is not necessary to assign any specific value to it

Step III : We solve the modified LPP by the simplex method.

11.4.4 Two-Phase Method

Two phase simplex method is used for problems involving '$\geq$' or '=' type constraints. The method is due to Dantzig, orden, and wolfe.

In Two phase somplex method the whole procedure of solving an LPP involving artificial variables is divided into two phases.

In summary, the two-phase simplex method is as follows.

1. We bring the constraints into equality form. For each constraint in which the slack variable and the right-hand side have opposite signs, or in which there is no slack variables, we add a new artificial variable that has the same sign as the right hand side.
2. Phase I : We minimise the sum of the artificial variables, starting from the basic feasible solution where the absolute value of the artificial variable for each constraint, or of the slack variable in case there is no artificial variable, is equal to that of the right hand side.
3. If some artificial variable has a positive value in the optimal solution, the original problem is infeasible, stop.
4. Phase II : We solve the original problem, starting from the basic feasible solution found in phase I.



11.5 LET US SUM UP

- Linear programming is one of the most important optimisation (maximisation or minimisation) techniques. It is a mathematical programming technique to optimise performance (for example profit, cost etc) under a set o resource constraints (for example man-hours, machine hours, capital, raw material etc.).
- The General Linear Programming Problem (LPP) is as follows.

  Maximise (or minimise)

$$z = \sum_{j=1}^{n} c_j x_j$$

subject to the constraints

$$z = \sum_{j=1}^{n} a_{ij} x_j \ (\leq \text{ or } = \text{ or } \geq) \ b_i, \ i = 1, 2, \ldots, m$$

and $x_j \geq o$, $j = 1, 2, \ldots, n$

The Canonical Form of LPP is as follows :

Maximise $z = \sum_{j=1}^{n} c_j x_j$

Subject to : $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, $i = 1, 2, \ldots, m$

$x_j \geq 0, j = 1, 2, \ldots, n$

The characteristics of canonical form are as follows.

(a) Objective function is in maximisation form.

(b) All the constraints are of the 'less than or equal to' form.

(c) all the decision variables are non-negative.

- The standard form of LPP is as follows Maximise (or minimise)

$$z = \sum_{j=1}^{n} c_j x_j$$

Subject to $z = \sum_{j=1}^{n} a_{ij} x_j = b_i$, $i = 1, 2, \ldots, m$

$x_j \geq 0$, $j = 1, 2, \ldots, n$

and $b_i \geq 0$

The characteristics of standard form are as follows.

(a) Objective function may be in maximisation form or in minimisation form.

(b) All the constraints are expressed as equations.

(c) All the variables are non-negative.

(d) The right hand side of each constraint is non-negative.

- If an LPP contains more than 2 variables, then it cannot be solved with the help of Graphical Method. Of course, an LPP containing 3 variables

can be solved by Graphical Method; but the procedure is tedious as it involves intersection of planes in three dimensions. Simplex method can be used for solving any linear model for which the solution exists. It was developed by American mathematician George Bernard Dantzig (1914-2005) in 1947.

- *The Simplex Algorithm is as follows.*

  First Step : We start with a basic feasible solution of a given LPP

  Second Step : We move to a better basic feasible solution.

  Third Step : The present basic feasible solution is optimal.

  Fourth Step : If 'Yes', we stop

  If 'No', we go to

  Second Step.

- The Big-M method extends the simplex algorithm to problems that contain 'greater-than' constraints. It is a modified version of the simplex method in which we assign a very large value M to each of the artificial variables.
- Two-phase simplex method is used for problems involving '$\geq$' or '=' type constraints. In Two-phase simplex method the whole procedure of solving an LPP involving artificial variables is divided into two phases.



11.6 FURTHER READING

1) Prem Kumar Gupta, D. S. Hira : *Operations Research* (2007), S Chand and Company Limited, New Delhi-110055.

2) Kanti Swarup, P. K. Gupta, Man Mohan : *Operations Research* (2017), Sultan Chand and Sons, New Delhi.

3) Sandip Banajee : *Topics in Mathematies* (2007), Books and Allied (P) Ltd, Kolkata 9.

4) R. Panneerselvam : *Operations Research* (2006), Prentice Hall of India Private Limited, New Delhi-110001.



11.7 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

Ans to Q No 1: (i) Do yourself.

(ii) The problem is as follows.

Maximise $z = 4x_1 - x_2 - 3x_3$

Subject to $x_1 + 5x_2 - 3x_3 \leq 20$

$2x_1 + 7x_2 + 2x_3 \leq 10$

$x_1 - 5x_2 - 3x_3 \geq 3$

$x_1, x_3 \geq 0$, x_2 is unrestricted.

We write x_2 as the difference of two non-negative variables y_1, y_2, i.e.,

$x_2 = y_1 - y_2$. Then the LPP becomes.

Maximise

$z = 4x_1 - (y_1 - y_2) - 3x_3$

Subject to

$x_1 + 5(y_1 - y_2) - 3x_3 \leq 20$

$2x_1 + 7(y_1 - y_2) - 2x_3 \leq 10$

$x_1 - 5(y_1 - y_2) - 3x_3 \geq 10$

$x_1, x_3, y_1, y_2 \geq 0$

We introduce slack variables s_1, s_2 , and surplus variables s_3,

The LPP in standard form is as follows :

Maximise $z = 4x_1 - y_1 + y_2 - 3x_3$

Subject to $x_1 + 5y_1 - 5y_2 - 3x_3 + s_1 = 20$

$2x_1 + 7x_1 - 7y_2 + 2x_3 + s_2 = 0$

$x_1 - 5y_1 + 5y_2 - 3x_3 - s_3 = 0$

$x_1, x_3, y_1, y_2, s_1, s_2 s_3 \geq 0$

(iii) Do yourself.

Ans to Q No 2: The problem is as follows :

Maximise $z = 3x_1 + 5x_2 - 2x_3$

Subject to $x_1 + 2x_2 - x_3 \geq -4$

$-5x_1 + 6x_2 + 7x_3 \geq 5$

$2x_1 + x_2 + 3x_5 = 10$

$x_1, x_2 \geq 0$, x_3 is unrestricted in sign.

We write x_2 as the difference of two non-negative variables y_1, y_2, i.e., $x_3 = y_1 - y_2$. Then the LPP becomes.

Maximise $z = 3x_1 + 5x_2 - 2(y_1 - y_2)$

Subject to $x_1 + 2x_2 - (y_1 - y_2) \geq -4$

$-5x_1 + 6x_2 + 7(y_1 - y_2) \geq 5$

$2x_1 + x_2 + 3(y_1 - y_2) = 10$

$x_1, x_2, y_1, y_2 \geq 0$,

We multiply both sides of the first constraint by –1 in order to make the right hand side positive. Thus the first constraint becomes.

$-x_1 - 2x_2 + (y_1 - y_2) \leq 4$

We introduce slack variable s_1 and samplus variables s_2.

Thus, the LPP in standard form is as follows.

Maximise $z = 3x_1 + 5x_2 - 2x_1 + 2y_2$

Subject to $-x_1 - 2x_2 + y_1 - y_2 + s_1 = 4$

$-5x_1 + 6x_2 + 7y_1 - 7y_2 - s_2 = 5$

$2x_1 + x_2 + 3y_1 - 3y_2 = 10$

$x_1, x_2, y_1, y_2, s_1, s_2 \geq 0$

Check Your Progress 2

Ans to Q No 1: The system of equations is

$2x_1 + 3x_2 + x_3 = 8$

$x_1 + 2x_2 + 2x_3 = 5$

Let $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $b = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$

$\therefore$ the system can be written as $Ax = b$.

Here number of variables = 3

number of equations = 2

$\therefore$ the system has at most ${}^3C_2$, i.e 3 basic equations.

Let $\alpha_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The square submatrices by taking two at a time are

$B_1 = (\alpha_1 \quad \alpha_2),\ B_2 = (\alpha_1 \quad \alpha_3),\ B_3 = (\alpha_2 \quad \alpha_3)$

$|B_1| = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1 \neq 0 \qquad \therefore\ B_1^{-1}$ exists

$|B_2| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0 \qquad \therefore\ B_2^{-1}$ exists

$|B_3| = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = 6 - 2 = 4 \neq 0 \qquad \therefore\ B_3^{-1}$ exists

Now $XB_1 = B_3^{-1}b = \frac{1}{1}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 16 & -15 \\ -8 & +10 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$XB_2 = B_2^{-1}b = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ 5 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 16 & -5 \\ -8 & +10 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 11 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{11}{3} \\ \frac{2}{3} \end{pmatrix}$

$XB_3 = B_3^{-1}b = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 8 \\ 5 \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} 16 & -5 \\ -16 & +15 \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} 11 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{11}{4} \\ -\frac{1}{4} \end{pmatrix}$

Thus, the basic solutions are

$x_1 = (1, 2, 0),\ x_2 = \left(\frac{11}{3},\ 0,\ \frac{2}{3}\right),\ x_3 = \left(0,\ \frac{11}{4},\ \frac{1}{4}\right)$

Of these 3 basic solutions, the basic feasible solutions are

$(1, 2, 0),\ \left(\frac{11}{3},\ 0,\ \frac{2}{3}\right)$; the other basic solution contains negative numbers.

Ans to Q No 2: Do yourself.

Ans to Q No 3: Do yourself.

Check Your Progress 3

Ans to Q No 1:

Maximise $z = 2x_1 + x_2$

Subject to $4x_1 + 3x_2 \leq 12$

$4x_1 + x_2 \leq 8$

$4x_1 - x_2 \leq 8$

$x_1, x_2 \geq 0$

We introduce slack variables x_3, x_4, x_5.

$\therefore$ the problem becomes

Maximise $z = 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5$

Subject to $4x_1 + 3x_2 + x_3 = 12$

$4x_1 + x_2 + x_4 = 8$

$4x_1 - x_2 + x_5 = 8$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

We take $x_1 = 0, x_2 = 2$

$\therefore x_3 = 12, x_4 = 8, x_5 = 8$

$\therefore$ (0, 0, 12, 8, 8) is a basic feasible solution.

Corresponding to this basic feasible solution,

$z = 0$

We represent by CB the coefficients of the basic variables in the objective function. We represent by XB the numberical values of the basic variables. Here $XB_1 = 12$, $XB_2 = 8$, $XB_3 = 8$.

We construct Table 1 as follows. The topmost row of Table 1 gives the coefficients (c_j) of the variables x_1, x_2, x_3, x_4, x_5 in the objective function respectively.

		c_j	2	1	0	0	0
CB	Basic Variables	XB	x_1	x_2	x_3	x_4	x_5
0	x_3	12	4	3	1	0	0
0	x_4	8	4	1	0	1	0
0	x_5	8	4	–1	0	0	1
	z		–2	–1	0	0	0

Table 1

Min (–2, –1) = –2

$\therefore$ x_1 is considered as a basic variable in the next iteration.

$$\text{Min}\left(\frac{12}{4}, \frac{8}{4}, \frac{8}{4}\right) = 2$$

$\therefore$ x_4 or x_5 is the new non-basic variable. We consider x_4 as the new non-basic variable.

The next table (Table 2) is as follows.

		c_j	2	1	0	0	0
CB	Basic Variables	XB	x_1	x_2	x_3	x_4	x_5
0	x_3	4	0	2	1	–1	0
2	x_1	2	1	$\frac{1}{4}$	0	$\frac{1}{4}$	0
0	x_5	0	0	–2	0	–1	1
	$z_j - c_j$		0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0

Table 2

Clearly, x_2 is considered as basic variable in the next iteration.

$$\min\left(\frac{4}{2}, \frac{2}{\frac{1}{4}}\right) = 2$$

$\therefore$ x_3 is the new non-basic variable.

The next table (Table 3) is as follows.

		c_j	2	1	0	0	0
CB	Basic Variables	XB	x_1	x_2	x_3	x_4	x_5
1	x_2	2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0
2	x_1	$\frac{3}{2}$	1	0	$-\frac{1}{8}$	$\frac{3}{8}$	0
0	x_5	4	0	0	1	–1	1
	$z_j - c_j$		0	0	$\frac{1}{4}$	$\frac{1}{4}$	0

Here $z_j - c_j \geq 0$, for all j

Thus, the current basic feasible solution is optimal.

Thus, the maximum value of z is obtained when

$$x_1 = \frac{3}{2}, \quad x_2 = 2$$

$$\text{maximum} = 2 \times \frac{3}{2} + 2 = 3 + 2 = 5$$



11.8 MODEL QUESTIONS

Q 1: Define the following terms

(a) basic solution

(b) basic variable

(c) basic feasible solution

(d) degenerate solution

(e) artificial variable.

Q 2: How many basic feasible solutions are there to a given system of 3 simultneous linear equations in 4 unknowns?

Q 3: Solve the following LPPs by simplex method.

(a) Maximise $z = 4x_1 + 7x_2$

Subject to $2x_1 + x_2 \leq 10$

$x_1 + x_2 \leq 6$

$x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$

(b) Maximise $z = 5x_1 + 2x_2 + 2x_3$

Subject to $x_1 + 2x_2 - 2x_3 \leq 30$

$x_1 + 3x_2 - x_3 \leq 36$

$x_1, x_2, x_3 \geq 0$

(c) Maximise $z = x_1 - 3x_2 + 2x_3$

Subject to $3x_1 + x_2 + 2x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

(d) Maximise $z = 5x_1 + 10x_2 + 8x_3$

Subject to $3x_1 + 5x_2 + 2x_3 \leq 60$

$4x_1 + 4x_2 + 4x_3 \leq 72$

$2x_1 + 4x_2 + 5x_3 \leq 100$

$x_1, x_2, x_3 \geq 0$

(e) Maximise $z = 20x_1 + 6x_2 + 8x_3$

Subject to $8x_1 + 2x_2 + 3x_3 \leq 250$

$4x_1 + 3x_2 \leq 150$

$2x_1 + x_3 \leq 50$

$x_1, x_2, x_3 \geq 0$

Q 4: When simplex method is used? State the simplex algorithm.

Q 5: Write notes on:

(a) Big-M method

(b) Two phase Method.

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UNIT 12 : DUALITY IN LINEAR PROGRAMMING PROBLEM

UNIT STRUCTURE

12.1 Learning Objectives

12.2 Introduction

12.3 Importance of Duality Concepts

12.4 Formulation of Dual Problem

12.5 Economic Interpretation of Duality

12.6 Sensitivity Analysis

12.7 Let Us Sum Up

12.8 Further Reading

12.9 Answers To Check Your Progress

12.10 Model Questions

12.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- know the concept of duality in linear programming problem
- know the importance of duality concepts
- formulate the dual problem
- know the economic interpretation of duality
- know the sensitivity analysis.

12.2 INTRODUCTION

Every Linear Programming Problem (LPP) is associated with another LPP. The original LPP is called primal, and the corresponding LPP is called dual. Either of the problems is primal with the other one as dual.

Let us take an example. Let the minimum daily requirements of vitamins A and B for a patient be 10 units, and 15 units respectively. Let us assume that food F_1 contains 6 units of vitamin A and 5 units of vitamin B. On the other hand food F_2 contains 3 units of A, and 4 units of B. It is required to find the minimum cost if the cost per unit of F_1 is Rs. 5, and the cost per

unit of F_2 is Rs. 8.

The problem can be put as follows :

Vitamins	Food F_1	Food F_2	Daily requirements
A	6	3	10
B	5	4	15
Cost per unit	5	8	

Now we formulate the problem mathematically. Let x_1 units of food F_1, and x_2 units of food F_2 be required to get the minimum amount of vitamins. Then the LPP is as follows.

Minimise $z = 5x_1 + 8x_2$

Subject to $6x_1 + 3x_1 \geq 10$

$5x_1 + 4x_2 \geq 15$

$x_1 \geq 0, \quad x_2 \geq 0$

Now we look at the problem from a wholesaler's point of view. Let the wholesaler sell vitamins A and B separately. The retailers buy the vitamins from him and form two foods F_1 and F_2 as given above. The wholesaler fixes the maximum per unit selling prices for the two vitamins A and B in such a manner that the resulting prices of food F_1 and food F_2 do not exceed their existing market prices.

Let w_1, and w_2 be the prices per unit of A, B respectively. Then the LPP is as follows.

Maximise $w = 10w_1 + 15w_2$

Subject $6w_1 + 5w_2 \leq 5,$

$3w_1 + 4w_2 \leq 8;$

$w_1 \, w_2 \geq 0$

Thus, we see that corresponding to every LPP, there exists another LPP More over, the primal-dual relatiouship is connected not only mathematically, but also in economic sense.

12.3 IMPORTANCE OF DUALITY CONCEPTS

Duality has immense importance. Some of the advantages of duality are given below.

1. If a primal problem contains a large number of rows (constraints) and a smaller number of columns (variables), the computation can be reduced by converting it into dual, and their solving it.
2. If a problem doesnot give any solution in primal, it can be verified with dual.
3. In economics, it is helpful in formulating input and output systems.
4. In game theory, dual is used by the column player in order to minimise his maximum loss. On the other hand, the row player applies primal to maximise his minimum gains.
5. In physics, dual is used in parallel circuit and series circuit theory.
6. Economic interpretations of dual can be made. It will help the management in making decisions.

12.4 FORMULATION OF DUAL PROBLEM

Let the primal problem be

Maximise $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

Subject to $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$

.....

.....

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$

$x_1, x_2, \ldots x_n \geq 0$

The corresponding dual is defined as

Minimise $w = b_1w_1 + b_2w_2 + \ldots + b_mw_m$

Subject to $a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m \geq c_1$

$a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m \geq c_2$

.....

.....

$a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_n \geq c_n$

$w_1w_2, \ldots, w_m \geq 0$

Matrix Form

Primal Let $c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$$A = \begin{pmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2} \ldots \ldots a_{mn} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Then the primal problem becomes

Maximise $z = c^T x$

Subject to $Ax \leq b,$

$x \geq 0$

Dual

Let $\upsilon = \begin{pmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_m \end{pmatrix}$

Then the dual problem becomes

Minimise $w = b^T \upsilon$

Subject to $A^T \upsilon \leq c$

$\upsilon \geq 0$

Note :

1. The maximisation problem in the primal becomes the minimisation problem in the dual, and vice versa.
2. The maximisation problem has '$\leq$' constraints, while the minimisation problem has '$\geq$' constraints.
3. The number of variables in the dual is the same as the number of constraints in the primal and vice versa.
4. The transpose of the coefficient matrix of the primal becomes the coefficient matrix in the dual.
5. The variables in both the problems are non negative.
6. The constants in the objective function of the primal become the constants in the constraints of the dual, and vice versa.

Reduction to Standard Primal Form

1. We change the objective function to Maximization form.
2. If the constraints have an inequality sign '$\geq$', we multiply both sides by -1 and convert the inequality sign to '$\leq$'.
3. If the constraint has an '=' sign, we replace it by two constraints involving inequalities of opposite signs. For example the constraint $2x_1 + 3x_2 - 5x_3 = 6$ will be replaced by $2x_1 + 3x_2 - 5x_3 \leq 6$, $2x_1 + 3x_2 - 5x_3 \geq 6$, i.e., $-2x_1 - 3x_2 + 5x_3 \leq -6$.
4. We replace every unrestricted variable by the difference of two non-negative variables.

Now :

(i) If the objective function is of maximisation form, all constraints will have '$\leq$' sign.

(ii) If the objective function is of minimisation form, all constraints will have '$\geq$' sign.

(iii) By 'a variable unrestricted in sign', we mean the difference of two non-negative variables. Let x_1, x_2 be two non-negative variables. Then the variable x_3 ($= x_1 - x_2$) is unrestricted in sign. If $x_1 > x_2$, $x_3 > 0$; if $x_1 = x_2$, $x_3 = 0$; if $x_1 < x_2$, $x_3 < 0$.

Example 1 : Find the dual of the following LPP.

$$\begin{aligned} \text{Maximise} \quad & z = 2x_1 + 3x_2 - x_3 \\ \text{Subject to} \quad & x_1 - x_2 + x_3 \geq 5 \\ & 2x_1 - x_2 + 3x_3 \leq 6 \\ & 3x_1 - 2x_2 + x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution : First we express all the constraints in the 'less than' form.

The LPP becomes

$$\begin{aligned} \text{Maximise} \quad & z = 2x_1 + 3x_2 - x_3 \\ \text{Subject to} \quad & -x_1 + x_2 - x_3 \leq -5 \\ & 2x_1 - x_2 + 3x_3 \leq 6 \\ & 3x_1 - 2x_2 + x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Let w_1, w_2, w_3 be the dual variables. The dual problem is as follows.

Minimise $w = 5w_1 + 6w_2 + 8w_3$

Subject to $-w_1 + 2w_2 + 3w_3 \geq 2$

$w_1 - w_2 - 2w_3 \geq 3$

$-w_1 + 3w_2 + w_3 \geq -1$

$w_1, w_2, w_3 \geq 0$

Example 2 : For the following LPP, verify that the dual of the dual is the primal.

Maximise $z = 3x_1 + 2x_2 + x_3$

Subject to $x_1 - 3x_2 + 2x_3 \leq 5$

$x_1 + 5x_2 + x_3 \geq 8$

$x_1, x_2, x_3 \geq 0$

Solution : First we express all the constraints in the 'less than' form. The LPP becomes

Maximise $z = 3x_1 + 2x_2 + x_3$

Subject to $x_1 - 3x_2 + 2x_3 \leq 5$

$-x_1 - 5x_2 - x_3 \leq -8$

$x_1, x_2, x_3 \geq 0$

Let w_1, w_2 be the dual variables. The dual problem is as follows.

Minimise $w = 5w_1 - 8w_2$

Subject to $w_1 - w_2 \geq 3$

$-3w_1 - 5w_2 \geq 2$

$2w_1 - w_2 \geq 1$

$w_1, w_3 \geq 0$

Let y_1, y_2, y_3 be the variables of the dual of the dual. Then the dual of the dual is as follows.

Maximise $u = 3y_1 + 2y_2 + y_3$

Subject to $y_1 - 3y_2 + 2y_3 \leq 5$

$-y_1 - 5y_2 - y_3 \leq -8$

$y_1, y_2 y_3 \geq 0$

which is the same as the original LPP.

Example 3: Find the dual of the following LPP.

Maximise $z = 2x_1 + 3x_2 + x_3$

Subject to $4x_1 + 3x_2 + x_3 = 6$

$x_1 + 2x_2 + 5x_3 = 4$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Also express the dual in terms of unrestraicted variables.

Solution : First we express all the constaints in the 'less than' form.

Maximise $z = 2x_1 + 3x_2 + x_3$

Subject to $4x_1 + 3x_2 + x_3 \leq 6$

$4x_1 + 3x_2 - x_3 \geq 6$

$x_1 + 2x_2 + 5x_3 \leq 4$

$x_1 + 2x_2 + 5x_3 \geq 4$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

i.e. subject to $4x_1 + 3x_2 + x_3 \leq 6$

$-4x_1 - 3x_2 - x_3 \leq -6$

$x_1 + 2x_2 + 5x_3 \leq -4$

$-x_1 - 2x_2 - 5x_3 \leq -4,$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Let w_1, w_2, w_3, w_4 be the dual variables.

The dual problem is as follows.

Minimise $w = 6w_1 - 6w_2 + 4w_3 - 4w_4$

Subject to $4w_1 - 4w_2 + w_3 - w_4 \geq 2$

$3w_1 - 3w_2 + 2w_3 - 2w_4 \geq 3$

$w_1 - w_2 + 5w_3 - 5w_4 \geq 1$

$w_1, w_2, w_3, w_4 \geq 0$

Let $w' = w_1 - w_2$ and $w'' = w_3 - w_4$. Now w' and and w'' are unrestricted variables. The above dual problem can be stated as follows involving w' and w'' .

Minimise $w = 6w' + 4w''$

Subject to $4w' + w'' \geq 2$

$3w' + 2w'' \geq 3$

$w' + 5w'' \geq 2$

w' and w'' are unrestricted in sign.

Theerems on Duality

We state, without proof, some theorems on duality.

1. The dual of the dual is the primal.
2. The value of the objective function z for any feasible solution of the primal is less than (or equal to) the value of the objective function w for any feasible solution of the dual.
3. If either the primal or the dual problem has an unbounded solution, the solution of the other problem is infeasible.
4. If both the primal and the dual problems have feasible solutions, then both have optimal solutions, and max z = min w. (Fundamental theorem on Duality).
5. (i) If a primal variable is positive, then the corresponding dual constraint is an equation at the optimum.
   (ii) If the primal constraint is a strict inequality, then the corresponding dual variable is zero at the optimum.
   (iii) If a dual variable is positive, then the corresponding primal constraint is an equation at the optimum.
   (iv) If a dual constraint is a strict inequality, then the corresponding primal variable is zero at the optimum.



CHECK YOUR PROGRESS

Q 1: Find the duals of the following LPPs.

(i) Maximise $z = x_1 - x_2 + 3x_3$

Subject to $x_1 + x_2 + x_3 \leq 10$

$2x_1 - x_3 \leq 2$

$2x_1 - 2x_2 + 3x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

(ii) Minimise $z = 3x_1 - 2x_2$

Subject to $2x_1 + x_2 \leq 1$

$-x_1 + 3x_2 \geq 4$

$x_1, x_2 \geq 0$

Q 2: Find the dual of the following LPP

Maximise $z = 6x_1 - 3x_2$

Subject to $3x_1 + 2x_2 \leq 25$

$x_1 \geq 4$

$x_1, x_2 \geq 0$

Q 3: Find the dual of the following LPP

Maximise $z = 2x_1 + 5x_2 + 3x_3$

Subject to $2x_1 + 4x_2 - x_3 \leq 8$

$-2x_1 - 2x_2 + 3x_3 \geq -7$

$x_1 + 3x_2 - 5x_3 \geq -2$

$4x_1 + x_2 + 3x_3 \leq 4,$

$x_1, x_2, x_3 \geq 0$

Q 4: Find the dual of the following LPP

Maximise $z = 4x_1 + 5x_2 + 12x_3$

Subject to $2x_1 + x_2 + x_3 \leq 4$

$3x_1 - 2x_2 + x_3 = 3$

$x_1, x_2, x_3 \geq 0$

Q 5: Find the dual of the following LPP

Maximise $z = 6x_1 + 6x_2 + 10x_3$

Subject to $4x_1 + 5x_2 + 7x_3 \leq 6$

$3x_1 + 7x_2 \leq 10$

$2x_1 + x_2 + 8x_3 = 20$

$2x_2 + 9x_3 \geq 5$

$x_1, x_3 \geq 0$; x_2 is unrestricted in sign.

Q 6: Find the dual of the following LPP

Maximise $z = x_1 + 4x_2 + 3x_3$

Subject to $2x_1 + 3x_2 - 5x_3 \leq 2$

$3x_1 - x_2 + 6x_3 \geq 1$

$x_1 + x_2 + x_3 = 4$

$x_1, x_2 \geq 0$; x_3 is unrestricted in sign.

Q 7: Find the dual of the following LPP.

Minimise $z = 2x_2 + 5x_3$

Subject to $x_1 + x_2 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 4$

$x_1, x_2, x_3 \geq 0$

12.5 ECONOMIC INTERPRETATION OF DUALITY

Let us consider the following primal problem.

Maximise $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$

.....

.....

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$

$x_1, x_2, \ldots x_n \geq 0$

Economic interpretation of this problem is as follows :

n stands for economic activities, m stands for resources.

c_j is revenue per unit of activity j.

b_i is maximum availability of resource i.

a_{ij} is consumption of resource i per unit of activity j.

The corresponding dual of the above primal is given by

Minimise $w = b_1w_1 + b_2w_2 + \ldots + b_mw_m$

Subject to $a_{11}w_1 + a_{12}w_2 + \ldots + a_{m1}w_m \geq c_1$

$a_{21}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m \geq c_2$

.....

.....

$a_1 + a_{2n}w_2 + \ldots + a_{mn}w_m \geq c_n$

$w_1, w_2, \ldots w_m \geq 0$

Let $(x_1, \ldots x_n)$ be optimal for the primal, and $(w_1, w_2, \ldots, w_m)$ be optimal for the dual. Then, we have

$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = b_1w_1 + b_2w_2 + \ldots + b_mw_m$

Left hand side = Maximal resource

Right hand side = Σ resources i (availability of resource i) x (revenue per unit of resource i)

Thus, optimal value of w_i is dual price of resource i.

Again, if we have $c_1x_1 + c_2x_2 + \ldots + c_nx_n < b_1w_1 + b_2w_2 + \ldots + b_mw_m$

Left hand side = current (suboptimal) revenue

Right hand side = Σ resources i (worth of resource i)

Since resources are not being fully utilised, solution is not optimal.

Again, if $(x_1, x_2, \ldots, x_n)$ is feasible (not necessarily optimal) for the primal, and $(w_1, w_2, \ldots, w_m)$ is the corresponding collection of dual values, then we have current objective coefficient of x_j

= left hand side of dual constraint j – right hand side

$= (a_{1j}\, w_i + \ldots + a_{mj} w_j) - c_j$

c_j is revenue per unit of activity j

$\therefore$ $a_{ij} w_1 + \ldots + a_{mj} w_j$ is an imputed (implicit) cost per unit of activity j.

Also, w_i is imputed cost per unit of resource i in a unit of activity j.

If objective coefficient of x_j (= cost – revenue = reduced cost) is strictly negative, Then revenue > cost.

12.6 SENSITIVITY ANALYSIS

The parameters and characteristics of an LPP may undergo change in course of time. Moreover, the researcher may be interested to know the effect of changing the parameters and characteristics of the problem on optimality. This kind of sensitivity analysis (or post optimality analysis) can be done by (i) making changes in the right hand side constants of the constraints, (ii) making changes in the objective function coefficients, (iii) adding a new constraint, (iv) adding a new variable.

After the introduction of such changes in an LPP, there may be any of the following cases.

(i) The optimum solution remains unchanged. Basic variable and their values remain unchanged.

(ii) Basic variables remain unchanged, but their values undergo changes.

(iii) Basic solution gets changed completely.



CHECK YOUR PROGRESS

Q 8: Write a note on Sensitivity Analysis.

.....

.....



12.7 LET US SUM UP

- Corresponding to every LPP, there exists another LPP, – one is the primal, the other is the dual
- If a primal problem contains a large number of rows (constraints) and a smaller number of columns (variables), the computation can be reduced by converting it into dual, and then solving it.
- If a problem does not give any solution in primal, it can be verified in dual.
- Dual is used in physics, economics, game theory etc.
- The maximisation problem in the primal with '$\leq$' constraints becomes the minimisation problem with '$\geq$' constraints in the dual, and vice versa.
- The number of variables in the dual is the same as the number of constraints in the primal, and vice versa.
- The variables in both the problems are non-negative.
- The transpose of the coefficient matrix of the primal becomes the coefficient matrix in the dual.



12.8 FURTHER READING

1) Linear programming (2009) by J G Chakravorty, and P. R. Ghosh, Moulik Library, Kolkata 200073.

2) Operations Research (2006) by R. Panneerselvam, Prentico - Hall of India Private Limited, New Delhi 110001.



12.9 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (i) Primal

(i) Maximise $z = x_1 - x_2 + 3x_3$

Subject to $x_1 + x_2 + x_3 \leq 10$

$2x_1 - x_3 \leq 2$

$2x_1 - 2x_2 + 3x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Dual

Let w_1, w_2, w_3 be the dual variable. The dual problem is as follows.

Minimise $w = 10w_1 + 2w_2 + 6w_3$

Subject to $w_1 + 2w_2 + 3w_3 \geq 1$

$w_1 - 2w_3 \geq -1$

$w_1 - w_2 + 3w_3 \geq 3$

$w_1, w_2, w_3, w_4 \geq 0$

(ii) Do yourself

Ans to Q No 2: Do yourself

Ans to Q No 3: First we express all the constaints in the 'less than' form.

The LPP becomes

Maximise $z = 2x_1 + 5x_2 + 3x_3$

Subject to $2x_1 + 4x_2 - x_3 \leq 8$

$2x_1 + 2x_2 - 3x_3 \leq 7$

$-x_1 - 3x_2 + 5x_3 \leq 2$

$4x_1 + x_2 + 3x_3 \leq 4,$

$x_1, x_2, x_3 \geq 0$

Let w_1, w_2, w_3, w_4 be the dual variables. The dual problem is as follows.

Minimise $w = 8w_1 + 7w_2 + 2w_3 + 4w_4$

Subject to $2w_1 + 2w_2 - w_3 + 4w_4 \geq 2$

$4w_1 + 2w_2 - 3w_3 + w_4 \geq 5$

$-w_1 - 3w_2 + 5w_3 + 3w_4 \geq 4$

$w_1, w_2, w_3 \geq 0$

Ans to Q No 4: First we express all the constaints in the 'less than' form.

Maximise $z = 4x_1 + 5x_2 + 12x_3$

Subject to $2x_1 + x_2 + x_3 \leq 4$

$3x_1 - 2x_2 + x_3 \leq 3$

$3x_1 - 2x_2 + x_3 \geq 3$

$x_1, x_2, x_3 \geq 0$

i.e. Subject to $2x_1 + x_2 + x_3 \leq 4$

$3x_1 - 2x_2 + x_3 \leq 3$

$-3x_1 + 2x_2 - x_3 \leq -3$

$x_1, x_2, x_3 \geq 0$

Let w_1, w_2, w_3 be the dual variables, The dual problem is as follows.

Minimise $w = 4w_1 + 3w_2 - 3w_3$

Subject to $2w_1 + 3w_2 - 2w_3 \geq 4$

$w_1 - 2w_2 + 2w_3 \geq 5$

$w_1 + w_2 - w_3 \geq 12$

$w_1, w_2, w_3 \geq 0$

Ans to Q No 5: Here x_2 is unrestricted in sign.

Let $x_2 = y_2 - y_4$, where $y_2, y_4 \geq 0$

The LPP becomes

Maximise $x = 6x_1 + 6(y_2 - y_4) + 10x_3$

Subject to $4x_1 + 5(y_2 - y_4) + 7x_3 \leq 6$

$3x_1 + 7(y_2 - y_4) \leq 10$

$2x_1 + (y_2 - y_4) + 8x_3 = 20$

$2(y_2 - y_4) + 9x_3 \geq 5$

$x_1, y_2, y_4, x_3 \geq 0$

i.e. subject to $4x_1 + 5y_2 - 5y_4 + 7x_3 \leq 6$

$3x_1 + 7y_2 - 7y_4 \leq 10$

$2x_1 + y_2 - y_4 + 8x_3 \leq 20$

$2x_1 + y_2 - y_4 + 8x_3 \geq 20$

$2y_2 - 2y_4 + 9x_3 \geq 5$

i.e subject to $4x_1 + 5y_2 - 5y_4 + 7x_3 \leq 6$

$3x_1 + 7y_2 - 7y_4 \leq 10$

$2x_1 + y_2 - y_4 + 8x_3 \leq 20$

$-2x_2 - y_2 + y_4 - 8x_3 \leq -20$

$-2y_2 + 2y_4 - 9x_3 \leq -5$

$x_1, y_2, y_4, x_3 \geq 0$

Let w_1, w_2, w_3, w_4, w_5 be dual variables. The dual problem is as follows.

Minimise $w = 6w_1 + 10w_2 + 20w_3 - 20w_4 - 5w_5$

Subject to $4w_1 + 3w_2 + 2w_3 - 2w_4 \geq 6$

$5w_1 + 7w_2 + w_3 - w_4 - 2w_5 \geq 6$

$-5w_1 - 7w_2 - w_3 + w_4 + 2w_5 \geq -5$

$7w_1 + 8w_2 - 8w_4 - 9w_5 \geq 10$

$w_1, w_2, w_3, w_4\ w_5 \geq 0$

Ans to Q No 6: Do yourself

Ans to Q No 7: The given LPP is expressed in standard form as shown below :

Minimise $z = 0x_1 + 2x_2 + 5x_3$

Subject to $x_1 + x_2 + 0x_3 \geq 2$

$-2x_1 - x_2 - 6x_2 \geq -6$

$x_1 - x_2 + 3x_3 \geq 4$

$-x_1 + x_2 - 3x_2 \geq -4$

$x_1, x_2, x_3 \geq 0$

The coefficient matrix A is :

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & -6 \\ 1 & -1 & 3 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 0 & -6 & 3 & -3 \end{bmatrix}$$

Since there are four constraints in the primal hence there will be four dual variables. Denoting these dual variables by w_1, w_2, w_3 and w_4, we get the dual of the given LPP (primal) as :

Maximise $w = 2x_1 - 6w_2 + 4w_3 - 4w_4$

Subject to $w_1 - 2w_2 + w_3 - w_4 \leq 0$

$w_1 - w_2 - w_3 + w_4 \leq 2$

$0w_1 - 6w_2 + 3w_3 - 3w_4 = 5$

$w_1, w_2, w_3, w_4 \geq 0$

Writing $w' = w_3 - w_4$, the above dual can also be expressed as follows:

Maximise $W = 2w_1 + 6w_2 + 4w'$

Subject to $w_1 - 2w_2 + w' \leq 0$

$w_1 - w_2 - w' \leq 2$

$6w_2 + 3w' \leq 5$

where $w_1, w_2 \geq 0$, w' is an unrestricted variable.

Ans to Q No 8: Book Article.



12.10 MODEL QUESTIONS

Q 1: Define dual. Write a note on the importance of duality concepts.

Q 2: Give the economic interpretation of dual variables.

Q 3: Find the duals of the following LPPs.

(i) Maximise $z = 4x_1 + 2x_2$

Subject to $x_1 + x_2 \geq 3$

$x_1 - x_2 \geq 2$; $x_1, x_2 \geq 0$

(ii) Maximise $z = 3x_1 + 5x_2 + 7x_3$

Subject to $x_1 + x_2 + 3x_3 \leq 10$

$4x_1 - x_2 + 2x_3 \geq 15$

$x_1, x_2 \geq 0$; x_3 is unrestricted.

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UNIT 13 : TRANSPORTATION PROBLEM

UNIT STRUCTURE

13.1 Learning Objectives

13.2 Introduction

13.3 Formulation of Transportation Problem (TP)

13.4 Basic Theorems on TP

- 13.4.1 Existence of Feasible Solution
- 13.4.2 Basic Feasible Solution

13.5 Initial Basic Feasible Solution

- 13.5.1 North-West Corner Method
- 13.5.2 Matrix-Minima Method
- 13.5.3 Vogel Approximation Method

13.6 Optimality Test

- 13.6.1 Algorithm of MODI Method

13.7 Let Us Sum Up

13.8 Further Reading

13.9 Answers To Check Your Progress

13.10 Model Questions

13.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- know a Transportation Problem (TP)
- formulate a TP
- know the basic theorems on TP
- obtain initial basic feasible solutions using various methods
- know the MODI method for obtaining optimal solution.

13.2 INTRODUCTION

Let there be a tyre manufacturing company. Let us suppose that the company has m factories located in m different cities of the country. Furthermore, let us suppose that the manufactured product is supplied to n

different retail dealers in n different cities of the country. Now the problem is to determine the transportation schedule in such a way that the cost of transporting the finished product from different factories to different retail dealers is minimum.

A Transportation Problem (TP) is a particular type of Linear Programming Problem (LPP). Here, various amounts of single homogeneous commodity that one initially stored at various sources are to be transported to different destination in such a way that the total transportation cost is minimum.

13.3 FORMULATION OF TRANSPORTATION PROBLEM (TP)

A TP can be expressed as an LPP as follows.

Minimise $z = \sum_{i-1}^{m} \sum_{i-1}^{m} c_{ij}\ x_{ij}$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad \text{(i = 1, 2, . . ., m)}$$

$$\sum_{i=1}^{n} x_{ij} = b_j, \quad \text{(j = 1, 2, . . ., n)}$$

$$x_{ij} \geq 0$$

Where

a_i = availability at i th source,

b_j = demand of j th destination,

c_{ij} = per unit transportation cost from i th source to j th destination.

The above TP can be expressed in a tabular form as follows.

		Destination						
		1	2	.....			n	Supply
		x_{11}	x_{12}	-	-	-	x_{1n}	a_1
	1	c_{11}	c_{12}	-	-	-	c_{1n}	
		x_{21}	x_{22}	-	-	-	x_{2n}	a_2
	2	c_{21}	c_{22}	-	-	-	c_{2n}	
Source		- -	- -	-	-	-	-	
		- -	- -	-	-	-	-	
		- -	- -	-	-	-	-	
		x_{m1}	x_{m2} -	-	-	-	x_{mn}	a_m
	m	c_{m1}	c_{m2}	-	-	-	- c_{mn}	
Demand		b_1	b_2	-	-	-	b_n	

Note : 1. The transportation table represents a matrix within a matrix.

2. The matrix of transportation variables x_{ij} is superimposed on the cost matrix.

3. If the total available quantity is equal to total demand, i.e., if $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, the TP is called a balanced TP.

Assumptions

1. The cost of transportation on a given route is directly proportional to the number of units transported. Thus, the cost of transportation is linear. For example, the cost of transportation of two objects will be twice that of a single object, even though it is not true in practical situations.
2. The unit transportation cost of the product from all sources to all destinations is precisely known.
3. Total quantity of products at different sources is equal to the total requirements at different destinations.
4. Products can be transported from all sources to all destinations.
5. The objective is to minimise the total transportation cost as a whole and not for individual supply and distribution centres.

13.4 BASIC THEOREMS ON TP

13.4.1 Existence of Feasible Solution

A necessary and sufficient condition for the existence of a feasible solution to the transportation problem given in 13.3 is

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \text{ i.e., total supply = total demand}$$

Proof : The condition is necessary

Let there exist a feasible solution to the transportation problem. Then we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} b_j$$

$$\therefore \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

The condition is sufficient

We assume that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = k$ (say)

We assert that there exists a feasible solution given by

$x_{ij} = \frac{a_i b_j}{k}$ for all i and j.

$\because\ a_i > 0$ and $b_j > 0,\ \therefore\ x_{ij} \geq 0$ for all i and j.

Again $\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \left(\frac{a_i b_j}{k} \right)$

$$= \frac{a_i}{k} \sum_{j=1}^{n} b_j$$

$$= \frac{a_i}{k} k = a_i, \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \left(\frac{a_i b_j}{k} \right)$$

$$= \frac{b_j}{k} \sum_{i=1}^{m} a_i$$

$$= \frac{b_j}{k} k = b_j, \qquad j = 1, 2, \ldots, n.$$

$\therefore$ x_{ij} satisfies all the constraints of the transportation problem.

Thus, a feasible solution exists

Corollary : There always exists an optimum solution to a transportation problem.

Proof : Let $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

$\therefore$ a feasible solution x_{ij} exists

We know that $0 \le x_{ij} \le \min (a_i, b_j)$

i.e., each x_{ij} is bounded.

$\therefore$ the feasible region of the problem is closed, bounded, and non-empty.

$\therefore$ there exists an optimum solution.

13.4.2 Theorem (Basic Feasible Solution)

The number of basic (decision)variables in an m x n balanced transportation problem is m+n–1.

Proof : We consider the first (m+n–1) constraints of the transportation problem.

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots, n-1 \text{ ((n–1) equations)} \qquad (1)$$

$$\sum_{j=1}^{n} x_{ij} = a_j, \quad i = 1, 2, \ldots, m \text{ (m equations)} \qquad (2)$$

Now, adding (n–1) destination – constraints (1), we get

$$\sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j \qquad (3)$$

Again, adding m source – constraints (2), we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \qquad (4)$$

Subtracting (3) from (4), we get

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n-1} b_j$$

$$\text{or } \sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} x_{ij}\right) = \sum_{j=1}^{n} b_j - \sum_{j=1}^{n-1} b_j \qquad (5)$$

Now

$$\sum_{j=1}^{n} x_{ij} = x_{i1} + x_{i2} + x_{i3} + \ldots + x_{i(n-1)} + x_{in}$$

$$\sum_{j=1}^{n-1} x_{ij} = x_{i1} + x_{i2} + x_{i3} + \ldots + x_{i(n-1)}$$

$$\sum_{j=1}^{n} b_j = b_1 + b_2 + b_3 + \ldots + b_{n-1} + b_n$$

$$\sum_{j=1}^{n-1} b_j = b_2 + b_2 + b_3 + \ldots + b_{n-1}$$

$\therefore$ from (5)

$\sum_{i=1}^{m} x_{in} = b_n$, which is the last

i.e., n th destinations – constraint.

Thus, if the first (m+n–1) constrants are satisfied, then $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ ensures that the (m+n)th constraint will be automatically satisfied.

Thus, out of m+n equations, one is redundant, the remaining (m+n–1) equations form a linearly independent set.

Note :

1. A feasible solution to an m–source, n–destination problem is basic if the number of positive allocations is m+n–1.
2. When the number of positive allocations in any basic feasible solution is less than m+n–1, the solution is said to degenerate.
3. By the Fundamental Theorem of LPP, one of the basic feasible solutions will be optimal solution.

Remarks :

1. The number of constraints in a balances TP with m sources and n destinations = m+n.
2. The number of variables in a balanced TP with m sources and n destinations = mn.
3. A constraint of a balanced TP is an equation.
4. The number of independent constraints in a balanced TP = m+n–1.
5. The number of basic variables in a balanced TP = m+n–1.

13.5 INITIAL BASIC FEASIBLE SOLUTION

There are many interesting methods for determining basic feasible solution of a balanced TP.

We will discuss three of them, viz.

(i) North-West Corner Method,

(ii) Matrix–Minima Method,

(iii) Vogel Approximation Method (VAM)

13.5.1 North-West Corner Method

This method is the simplest amongst all the methods. This method does not take into account the cost of transportation c_{ij}. First the method is explained by numerical examples. Then we will state the method.

Example 1 : Obtain an initial basic feasible solution to the following TP by North-West Corner Method.

		Destination				
		D_1	D_2	D_3	D_4	Supply
Source	S_1	11	13	17	14	250
	S_2	16	18	14	10	300
	S_3	21	24	13	10	400
Demand		200	225	275	250	

Solution : Here $a_1 + a_2 + a_3 = 250 + 300 + 400 = 950$

$b_1 + b_2 + b_3 + b_4 = 200 + 225 + 275 + 250 = 950$

$\therefore$ the problem is balanced.

$\therefore$ there exists a basic feasible solution.

Now we write the problem in tabular form.

	Destination				
	D_1	D_2	D_3	D_4	Supply
S_1	200 11	50 13	x 17	x 14	250
S_2	x 16	175 18	125 14	x 10	300
S_3	x 21	x 24	150 13	250 10	400
Demand	200	225	275	250	

We start with the extreme North-West corner cell (1, 1). It connects source S_1 to destination D_1. In this route, supply at S_1 is 250 units, demand at D_1 is 200 units. min (250, 200) = 200. 200 units are transported to D_1. We write it in the North-West corner in a box. Demand for D_1 is met. There is no need to transport from S_2 to D_1, S_3 to D_1. So, we put cross in the cells (2, 1), (3, 1).

Now 250–200 = 50. Now 50 units are available at S_1. Demand at D_2 is 225 units. min (225, 50) = 50. 50 units are transported from S_1 to D_2. We write it in the corner of the cell (1, 2) in a box. Supply at source S_1 is exhausted. So, we put cross in the cells (1, 3), (1, 4).

Now 225–50 = 175. So demand at D_2 is 175 units. Supply at S_2 in 300 units min (300, 175) = 175. 175 units are transported from S_2 to D_2. We write 175 in the corner of the cell (2, 2) in a box. 175+50 = 225. Demand for D_2 is met. There is no need to transport from S_3 to D_2. So we put a cross in the cell (3, 2).

Now 300–175 = 125. Now 125 units are available at S_2. Demand at D_3 is 275. min (275, 125) = 125. 125 units are transported from S_2 to D_3. We write 125 in the corner of the cell (2, 3) in a box. Supply at S_2 is exhausted we put a cross in the cell (2, 4).

Now 275 – 125 = 150. So demand at D_3 is 150 units. Supply at S_3 is 400 units min (400, 150) = 150. 150 units are transported from S_3 to D_3. We write it in the corner of the cell (3, 3). Now 150+125 = 275. Demand for D_3 is met.

Now 400–150 = 250. 250 units are available at S_3. Demand at D_4 is 250. 250 units are transported from S_3 to D_4. We write it in the corner or the cell (4, 3) in a box.

Thus, we have obtained a basic feasible solution of the TP with x_{11} = 200, x_{12} = 50, x_{22} = 175, x_{23} = 125, x_{33} = 150, x_{34} = 250. [These are basic variables. All other variables are non basic and zero where we have put a cross.]

The transportation cost by the above route

= 200 x 11 + 50 x 13 + 175 x 18 + 125 x 14 + 150 x 13 + 250 x 10

= 2200 + 650 + 3150 + 1750 + 1950 + 2500

= 12200

Note : Number of basic variables

= 6 = (4+3) – 1

Example 2 : Obtain an initial basic feasible solution to the following TP by North-West corner Method.

		Destination					
		D_1	D_2	D_3	D_4	D_5	Supply
	S_1	2	11	10	3	7	4
Source	S_2	1	4	7	2	1	8
	S_3	3	9	4	8	12	9
Demand		3	3	4	5	6	

Solution : $\Sigma a_i = a_1 + a_2 + a_3 = 4 + 8 + 9 = 21$

$\Sigma b_i = b_1 + b_2 + b_3 + b_4 + b_5 = 3 + 3 + 4 + 5 + 6 = 21$

= 3 + 3 + 4 + 5 + 6 = 21

$\therefore$ the problem is balanced.

$\therefore$ there exists a feasible solution.

Now we write the problem in tabular form.

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	3 2	1 11	x 10	x 3	x 7	4
S_2	x 1	2 4	4 7	2 2	x 1	8
S_3	x 3	x 9	x 4	3 8	6 12	9
Demand	3	3	4	5	6	

We start with North-West corner cell (1, 1).

min (a_1, b_1) = min (4, 3) = 3

We set $x_{11} = 3$

Demand for D_1 is met. So, we set $x_{21} = x_{31} = 0$, and put a cross.

Availability at $S_1 = 4 – 3 = 1$.

Demand for $D_2 = 3$

min (3, 1) = 1

We set $x_{12} = 1$

Supply at S_1 is exhausted. So, we set $x_{13} = x_{14} = x_{15} = 0$, and put a cross.

Again, demand for $D_2 = 3 – 1 = 2$.

Supply at $S_2 = 8$

min (8, 2) = 2

We set $x_{22} = 2$

Demand for D_2 is met. We set

$x_{32} = 0$, and put a cross.

Availability at $S_2 = 8 –2 = 6$

Demand for $D_3 = 4$

min (6, 4) = 4

We set $x_{23} = 4$

Demand for D_3 is met. We set $x_{33} = 0$, and put a cross.

Demand for $D_4 = 5$

Availability at $S_2 = 8 – (2+4) = 2$

min (5, 2) = 2

We set $x_{24} = 2$

Supply at S_2 is exhausted. We set $x_{25} = 0$, and put a cross.

Again, demand for $D_4 = 5 – 2 = 3$

Availability at $S_3 = 9$

min (3, 9) = 3

We set $x_{34} = 3$

Demand for D_4 is met.

Demand for $D_5 = 6$

Availability at $S_3 = 9 – 3 = 6$

We set $x_{33} = 6$

Thus, we have obtained a basic feasible solution of the TP given by $x_{11} = 3$, $x_{12} = 1$, $x_{22} = 2$, $x_{23} = 4$, $x_{24} = 2$, $x_{34} = 3$, $x_{35} = 6$. [These are basic variables. All other variables are non basic and zero where we have put a cross.]

The transportation cost by the above

route = 3 x 2 + 1 x 11 + 2 x 4 + 4 x 7 + 2 x 2 + 3 x 8 + 6 x 12

= 6 + 11 + 8 + 28 + 4 + 24 + 72 = 153

Note : Number of basic variables = 7 = (5 + 3) – 1

Now we state the North-West Corner Method,–

Step 1: We check if the TP is balanced, i.e., if $\Sigma a_i = \Sigma b_j$

Step 2: We start with the extreme North-West corner cell (1,1)

We compute min (a_1 b_1), If $a_1 < b_1$ min (a_1, b_1) = a_1, and if $b_1 < a_1$ min (a_1, b_1)= b_1,

We select x_{11} = min (a_1, b_1) and assign the value of x_{11} in the cell (1, 1).

Step 3: If $a_1 < b_1$, we move down vertically to the second row. We select x_{21} = min (a_2, $b_1 – x_{11}$) in the cell (2, 1).

If $b_1 < a_1$, we move right horizontally to the second column. We select x_{12} = min ($a_1 - x_{11}$, b_2) in the cell (1, 2).

If $a_1 = b_1$, there is a tie for the second allocation. We break the tie arbitrarily. We can either $x_{12} = 0$ or $x_{21} = 0$.

Step 4: We repeat steps 2 and 3 moving down towards the lower right corner of the transportation table until all availabilities are exhausted and all the demands are met.

13.5.2 Matrix-Minima Method

No consideration is given to the cost of production c_{ij} in the North-West Corner Method. In Matrix – Minima Method, we start transporting with the cheapest route and use the routes in ascending order of costs of production. First we explain the method by numerical examples. Then we will state the method.

Example 3: Obtain an initial basic feasible solution to the following TP by Matrix – Minima Method.

		D_1	D_2	D_3	D_4	D_5	Supply
	S_1	2	11	10	3	7	4
Source	S_2	1	4	7	2	1	8
	S_3	3	9	4	8	12	9
Demand		3	3	4	5	6	

Solution : We have

$$\Sigma a_i = a_1 + a_2 + a_3 = 4 + 8 + 9 = 21$$

$$\Sigma b_j = b_1 + b_2 + b_3 + b_4 + b_5 = 3 + 3 + 4 + 5 + 6 = 21$$

$\therefore$ the problem if balanced.

$\therefore$ there exists a feasible solution

Now we write the problem in tabular form.

Destination

		D_1	D_2	D_3	D_4	D_5	Supply
		x	x	x	4	x	
	S_1	2	11	10	3	7	4
Source		3	x	x	x	5	
	S_2	1	4	7	2	1	8
		x	3	4	1	1	
	S_3	3	9	4	8	12	9
Demand		3	3	4	5	6	

We see that there are 2 cheapest routes, – one from S_2 to D_1, the other from S_2 to D_5. We select any one,from S_2 to D_1.

Availability at $S_2 = 8$

Demand at $D_1 = 3$

min (8, 3) = 3

$\therefore$ S_2 can supply all the 3 units required by D_1.

We set $x_{21} = 3$ and write it in the cell (2, 1) in a box.

Demand for D_1 is met. We set $x_{11} = 0$, $x_{31} = 0$, and put a cross.

Now, availability at $S_2 = 8 - 3 = 5$

The next route is from S_2 to D_5.

Demand at $D_5 = 6$

min of (5, 6) = 5

We set $x_{25} = 5$ and write it in the cell (2, 5) in a box.

Availability of S_2 is exhausted.

We set $x_{22} = x_{23} = x_{24} = 0$, and put a cross in the cells (2, 2), (2, 3), (2, 4).

The next route is from S_1 to D_4

Availability at $S_1 = 4$

Demand at $D_4 = 5$

min (4, 5) = 4.

We set $x_{14} = 4$, and write it in the cell (1, 4) in a box.

Now, availability of S_1 is exhausted. We set $x_{11} = x_{12} = x_{13} = x_{15} = 0$, and put a cross in the cells (1, 1), (1, 2), (1, 3), (1, 5).

The next route is from S_3 to D_3.

Availability at $S_3 = 9$

Demand at $D_3 = 4$

min (9, 4) = 4

$\therefore$ S_3 can supply all the 4 units required by D_3. We set $x_{33} = 4$, and write it in the cell (3, 3) in a box.

Demand for D_3 is met. We have already set $x_{13} = x_{23} = 0$, and put a cross.

The next route is from S_3 to D_4.

Now, demand for $D_4 = 5 - 4 = 1$

Availability at $S_3 = 9 - 4 = 5$

min (1, 5) = 1

We set $x_{34} = 1$, and write it in the cell (3, 4) in a box.

Demand for D_4 is met.

We have already put a cross in the cell (2, 4)

The next route is from S_3 to D_2

Availability at $S_3 = 9 – (4+1) = 4$

Demand at $D_2 = 3$

min (4, 3) = 3

We set $x_{32} = 3$, and write it in the cell (3, 2) in a box.

The next route is from S_3 to D_5

Availability at $S_3 = 9 – (3+4+1) = 1$

Demand for $D_5 = 6 – 5 = 1$

We set $x_{35} = 1$, and write it in the cell (3, 5) in a box.

Thus, we have obtained a basic feasible solution of the TP with $x_{14} = 4$, $x_{21} = 3$, $x_{25} = 5$, $x_{32} = 3$, $x_{33} = 4$, $x_{34} = 1$, $x_{35} = 1$.

[These are basic variables. All other variables are non basic and zero where we have put a cross.]

The transportation cost by the above route $= 4 \times 3 + 3 \times 1 + 5 \times 1 + 3 \times 9 + 4 \times 4 + 1 \times 8 + 1 \times 12$

$= 12 + 3 + 5 + 27 + 16 + 8 + 12$

$= 8$

Note : Number of basic variables = 7 = (5+3) – 1

Now, we state the Matrix – Minima Method.

Step 1: We check if the TP is balanced, i. e., if $\Sigma a_i = \Sigma b_j$

Step 2: We find out the cell with the cheapest cost in the cost matrix. [If the cheapest cost is not unique, we make an arbitrary choice among the cheapest costs.] We allocate in that cell the maximum allowable amount.

Step 3: We cross out the satisfied row or column, and adjust the amounts of supply and demand accordingly.

Step 4: We repeat the process.

Step 5: At the end we are left with exactly one uncrossed row or column.

Note : Matrix – Minima Method is also called Least Cost Entry Method.

13.5.3 Vogel Approximation Method

We state the method below.

Step 1: We check if the TP is balanced i.e., if $\Sigma a_i = \Sigma b_j$.

Step 2: We identify the cheapest cost and the next cost for each row in the transportation table. We determine the difference between them for each row. It is known as 'penalty'. We write each penalty alongside the table near 'supply' column with parentheses against the respective row. [If there are two identical cheapest costs, the penalty will be zero.]

Step 3: We compute penalty for each column.

Step 4: We identify the row or column with the maximum penalty among all the rows and columns. If there is a tie, we break the tie arbitrarily.

Let the maximum penalty correspond to the i th row, and let c_{ij} be the cheapest cost in the i th row. We allocate the maximum possible amount $x_{ij} = \min(a_i, b_j)$ in the cell (i, j) and cross out the i th row or the j th column.

Step 5: We compute the column and row penalties for the reduced transportation table, and go to step 4. We repeat the procedure until all availabilities are exhausted and all the demands are met.

Note: Vogel Approximation Method is also to called unit cost Penalty Method.

Example 4: Obtain an initial basic feasible solution to the following TP by Vogel Approximation Method.

		Destination				
		D_1	D_2	D_3	D_4	Supply
Source	S_1	1	2	1	4	30
	S_2	3	3	2	1	50
	S_3	4	2	5	7	20
Demand		20	40	30	10	

Solution: We have

$$\Sigma a_i = a_1 + a_2 + a_3 = 30 + 50 + 20 = 100$$

$$\Sigma b_j = b_1 + b_2 + b_3 + b_4 + b_5 = 20 + 40 + 30 + 10 = 100$$

$$\therefore \ \Sigma a_i = \Sigma b_j$$

$\therefore$ the problem is a balanced TP.

$\therefore$ there exists a basic feasible solution.

	D_1	D_2	D_3	D_4	
S_1	1	2	1	4	30(0)
				10	
S_2	3	3	2	1	50(1)
S_3	4	2	5	7	20(2)
	20	40	30	10	
	(2)	(0)	(1)	(3)	

Table 1

penalty for 1st row = 1 – 1 = 0

penalty for 2nd row = 2 – 1 = 1

penalty for 3rd row = 4 – 2 = 2

penalty for 1st column = 3 – 1 = 2

penalty for 2nd column = 2 – 2 = 0

penalty for 3rd column = 2 – 1 = 1

penalty for 4th column = 4 – 1 = 3

We note the penalties in Table 1.

The maximum penalty occurs in the 4th column.

The cheapest cost in the 4th column is 1 and it is in the (2, 4) cell.

We set x_{24} = min (50, 10) = 10

Demand for D_4 is met. So we delete the 4th column.

We get the reduced transportation table (table 2).

	D_1	D_2	D_3	
	20			
S_1	1	2	1	30(0)
S_2	3	3	2	40(1)
S_3	4	2	5	20(2)
	20	40	30	
	(2)	(1)	(1)	

Table 2

The maximum penalty occurs in 1st column as well as in 3rd row.

The cheapest cost in the 1st column is 1, and it is in the (1, 1) cell.

We set x_{11} = min (30, 20) = 20

Demand for D_1 is met. So we delete the 1st column.

We get the reduced transportation (table 3).

We note the penalties in Table 3.

	D_2	D_3	
S_1	2	1	10(1)
S_2	3	2	40(1)
S_3	[20] 2	5	20(3)
	40	30	
	(0)	(1)	

Table 3

The maximum penalty occurs in the 3rd row.

The cheapest cost in the 3rd row occurs in the (3, 2) cell.

We set x_{32} = min (20, 40) = 20

Supply at S_3 is exhausted. So we delete the 3rd row.

We get the reduced transportation table(table 4)

We see that all the rows and columns have the same penalty.

We select the 2nd column.

	D_2	D_3	
S_1	[10] 2	1	10(1)
S_2	3	2	40(1)
	20	30	
	(1)	(1)	

Table 4

The cheapest cost in the 2nd column occurs in the (1, 2) cell.

We set x_{12} = min (10, 20) = 10

Supply at S_1 is exhausted. So, we delete the 1st row.

We get the reduced transportation table (table 5).

The maximum penalty is at D_2.

	D_3		
S_2	10 3	30 2	40 (1)
	10	30	
	(3)	(2)	

Table 5

Clearly x_{22} = min (40, 10) = 10

x_{23} = min (30, 30)= 30

Thus, we have obtained Table 6

	D_1	D_2	D_3	D_4	
S_1	20 1	10 2	1	4	30
S_2	3	10 2	30 2	10 1	50
S_3	4	20 2	5	7	20
	20	40	30	10	

Table 6

Thus, an initial basic feasible solution of the TP is

x_{11} = 20, x_{12} = 10, x_{22} = 10, x_{23} = 30, x_{24} = 10, x_{32} = 20

The transportation cost by the above route is given by

z = 20 x 1 + 10 x 2 + 10 x 3 + 30 x 2 + 10 x 1 + 20 x 2

= 20 + 20 + 30 + 60 + 10 + 40 = 180

Note : Number of basic variables = 6 = (3+4) – 1



CHECK YOUR PROGRESS

Q 1: Obtain an initial basic feasible solution for each of the following TPs by North-West Corner Method, Matrix-Minima Method, Vogel Approximation Method.

(i)

		Destination			
		D_1	D_2	D_3	Supply
Source	S_1	5	15	10	50
	S_2	10	8	20	55
Demand		40	30	35	

(ii)

		Destination				
		D_1	D_2	D_3	D_4	Supply
Source	S_1	4	6	9	5	16
	S_2	2	6	4	1	12
	S_3	5	7	2	9	15
Demand		12	14	9	8	

(iii)

		Destination			
		D_1	D_2	D_3	Supply
	S_1	10	9	8	8
Source	S_2	10	7	10	7
	S_3	11	9	7	9
	S_4	12	14	10	4
Demand		10	10	8	

13.6 OPTIMALITY TEST

We have discussed methods to obtain an initial basic feasible solution of a TP. After determination of an initial basic feasible solution by any method, we will try to obtain an optimal solution from this initial basic feasible solution. An important method used for a balanced TP to obtain optimal solution is MODI Method (MOdified - DIstribution Method), which is also known as U-V Method.

13.6.1 Algorithm of MODI Method

Let us consider a transportation problem with m sources, and n destinations given in Art 11.3, where a_i denotes availability at i th source, b_j denotes demand of jth destination, c_{ij} is per unit transportation cost from i th source to j th destination, x_{ij} is number of units transported from i th source to j th destination.

We verify if the TP is balanced.

Step 1: We find a basic feasible solution of the given TP by anyone of the methods we have discussed. There will be (m+n–1) independent positive allocations for non-degenerate problems. These

allocations are shown in the upper left corner of each basic cell.

Step 2: We assign row 1, row 2, . . ., row m of the cost matrix with variables u_1, u_2, . . ., u_m respectively, and column 1, column 2, . . ., column n with variables $v_1, v_2, \ldots, v_n$ respectively.

Step 3: We compute the values for u_1, u_2, . . ., u_m, and $v_1, v_2, \ldots, v_n$ by using the formula $u_i + v_j = c_{ij}$ to all the basic cells only.

To find u_i and v_j, we put anyone of them equal to zero.

Step 4: We calculate the cell evaluations A_{ij} for each non-basic (i, j) cells by the formula $A_{ij} = u_i + v_j - c_{ij}$ and put them in the upper right corners of the corresponding non-basic cells.

Step 5: Now three cases arise

(i) All A_{ij} < 0. In this case, the solution is optimal and unique. We stop.

(ii) A_{ij} < 0 with at least one A_{ij} = 0. In this case, the solution is optimal, but not unique. We stop.

(iii) At least one A_{ij} > 0. The solution is not optimal. We go to step 6.

Step 6: In order to find a new basic deasible solution, we select that (r, s) cell for which the net evaluation.

$A_{rs} = u_r + v_r - c_{rs}$ is positive maximum, as entering cell.

We draw a closed loop consisting of honzontal and vertical line segments connecting this entering (r, s) cell and those basic cells which will be required for this. [Change of direction of the loop will be 90^0 at some basic cell.] Such unique loop exists for any non-degenerate basic solution.

We start from this (r, s) cell. We allocate an amount θ with alternative positive and negative signs to all the end points of the closed loop so that supply and demand requirements are always satisfied. We choose θ as the minimum quantity which will give non-negative values for all the basic variables in the new solution. The value of θ can be seen from the allocations made in the cells of the

loop. The cell where evaluation becomes zero is considered as leaving basic cell.

Step 7: We repeat the steps 3-6 until all the evaluations to the empty sells are negative.

Example 5: Find the optimal basic feasible solution of the following TP by MODI method.

	D_1	D_2	D_3	Supply
S_1	10	9	8	8
S_2	10	7	10	7
S_3	11	9	7	9
S_4	12	14	10	4
Demand	10	10	8	

Solution : We have obtained an initial basic feasible solution of the TP given by $x_{11} = 6$, $x_{12} = 2$, $x_{22} = 7$, $x_{32} = 1$, $x_{33} = 8$, $x_{41} = 4$. (Check your progress 1, 1(iii)

	D_1	D_2	D_3	
S_1	6 10	2 9	8	8
S_2	10	7 7	10	7
S_3	11	1 9	8 7	9
S_4	4 12	14	10	4
	10	10	8	

Now,we calculate u_i , v_j , A_{ij}

The 1st row as well as the 3rd row contains maximum number of basic cells.So,we take $u_1 = 0$ (we may also take $u_3 = 0$)

Basic Variables	$u_i + v_j = c_{ij}$	Solutions
x_{11}	$u_1 + v_1 = 10$	$u_1 = 0,\ v_1 = 10$
x_{12}	$u_1 + v_2 = 9$	$u_1 = 0,\ v_2 = 9$
x_{22}	$u_2 + v_2 = 7$	$u_2 = -2,\ v_2 = 9$

x_{32} $u_3 + v_2 = 9$ $u_3 = 0,\ v_2 = 9$

x_{33} $u_3 + v_3 = 7$ $u_3 = 0,\ v_3 = 7$

x_{41} $u_4 + v_1 = 12$ $u_4 = 2,\ v_1 = 10$

Cell evaluations for non-basic cells

Non-basic variables $A_{ij} = u_i + v_j - c_{ij}$

x_{13} $A_{13} = v_1 + v_3 - c_{13} = 0 + 7 - 8 = -1 < 0$

x_{21} $A_{21} = u_2 + v_1 - c_{21} = -2 + 10 - 10 = -2 < 0$

x_{23} $A_{23} = u_2 + v_3 - c_{23} = -2 + 7 - 10 = -5 < 0$

x_{31} $A_{31} = u_3 + v_1 - c_{31} = 0 + 10 - 11 = -1 < 0$

x_{42} $A_{42} = u_4 + v_2 - c_{42} = 2 + 9 - 14 = -3 < 0$

x_{43} $A_{43} = u_4 + v_3 - c_{43} = 2 + 7 - 10 = -1 < 0$

Thus, we see that for all non-basic variables, $A_{ij} < 0$.

$\therefore$ the solution given by

$x_{11} = 6, x_{12} = 2, x_{22} = 7, x_{32} = 1, x_{33} = 8, x_{41} = 4$ is optimal and unique.

Minimum transportation cost

= 6 x 10 + 2 x 9 + 7 x 7 + 1 x 9 + 8 x 7 + 4 x 12

= 60 + 18 + 49 + 9 + 56 + 48

= 240



CHECK YOUR PROGRESS

Q 2: Solve the following transportation problem

	D_1	D_2	D_3	D_4	Supply
S_1	2	2	2	1	3
S_2	10	8	5	4	7
S_3	7	6	6	8	5
Demand	4	3	4	4	

Q 3: Find the optimal solution of the following transportation problem.

	D_1	D_2	D_3	D_4	Supply
S_1	5	4	6	14	15
S_2	2	9	9	6	4
S_3	6	11	7	13	8
Demand	9	7	5	6	

Q 4: Obtain an initial basic feasible solution to the following TP by Vogel Approximation Method. Also obtain the optimal solution.

	D_1	D_2	D_3	D_4	Supply
S_1	3	7	6	4	5
S_2	2	4	3	2	2
S_3	4	3	8	5	3
Demand	3	3	2	2	



13.7 LET US SUM UP

- A Transportation Problem (TP) is a particular type of linear programming problem (LPP). Here, various amounts of single homogeneous commodity that are initially stored at various sources are to be transported to different destinations in such a way that the total transportation cost is minimum.
- If the total available quantity is equal to total demand, the TP is called a balanced TP.
- The cost of transportation on a given route is directly proportional to the number of units transported.
- A necessary and sufficient condition for the existence of a feasible solution to a TP is 'total supply = total demand'.
- The number of basic (decision) variables in an m x n balanced TP is m+n-1.
- There are many methods for determining basic feasible solution of a balanced TP.
- An important method used for a balanced TP to obtain optimal solution is MODI Method (MOdified-DIstribution Method).



13.8 FURTHER READING

1) Operations Research (2018) by Kanti Swarup, P. K. Gupta, Man Mohan (Sultan Chand & SOns, New Delhi)

2) Linear Programming and Game Theory (2008) by DC Sanyal, K. Das (U N Dhur & Sons Private Ltd. Kolkata 700073)



13.9 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (i) We find an initial basic feasible solution of the TP, first by North-West Corner Method, and then by Matrix-Minima Method.

The TP is

		Destination			
		D_1	D_2	D_3	Supply
Source	S_1	5	15	10	50
	S_2	10	8	20	55
Demand		40	30	35	

We have

$\Sigma a_i = a_1 + a_2 = 50 + 55 = 105$

$\Sigma b_j = b_1 + b_2 + b_3 = 40 + 30 + 35 = 105$

$\therefore \ \Sigma a_i = \Sigma b_j$

$\therefore$ the problem is balanced.

$\therefore$ that exists a basic feasible solution.

North-West Corner Method

We write the problem in tabular form

	D_1	D_2	D_3	
S_1	40 5	10 15	x 10	50
S_2	x 10	20 8	35 20	55
	40	30	35	

We start with the extreme North-West Corner Cell (1, 1)

Supply at $S_1 = 50$

Demand at $D_1 = 40$

min (50, 40) = 40

We set $x_{11} = 40$

Demand for D_1 is met. So, we set $x_{21} = 0$ and put a cross.

Availability at $S_1 = 50 – 40 = 10$

Demand for $D_2 = 30$

Min (10, 30) = 10

We set $x_{12} = 10$

Supply at S_1 is exhausted. So we set $x_{13} = 0$, and put a cross.

Again demand for $D_2 = 30 – 10 = 20$

Availability at $S_2 = 55$

Min (20, 55) = 20

We set $x_{22} = 20$

Demand for D_2 is met

Availability at $S_2 = 55 – 20 = 35$

Demand for $D_3 = 35$

We set $x_{23} = 35$

Thus, we have obtained a basic feasible solution of the TP given by x_{11} = 40, $x_{12} = 10$, $x_{22} = 20$, $x_{23} = 35$

The transportation cost by the above

route = 40 x 5 + 10 x 15 + 20 x 8 + 35 x 20

= 200 + 150 + 160 + 700 = 1210

Matrix - Minima Method

We write the problem in tabular form

	D_1	D_2	D_3	
S_1	40 5	x 15	10 10	50
S_2	x 10	30 8	25 20	55
	40	30	35	

The cheapest route is from S_1 to D_1

Availability at S_1 = 50

Demand at D_1 = 40

Min (50, 40) = 40

$\therefore$ S_1 can supply all the 40 units required by D_1. We set x_{11} = 40

Demand for D_1 is met. We set x_{21} = 0 and put a cross.

Now, availability at S_1 = 50 – 40 = 10

The next route is from S_2 to D_2

Availability at S_2 = 55

Demand for D_2 = 30

Min (55, 30) = 30

We set x_{22} = 30

Demand for D_2 is met. We set x_{12} = 0 and put a cross.

The next route is from S_1 to D_3

Availability at S_1 = 10

Demand for D_3 = 35

Min (10, 35) = 10

We set x_{13} = 10

Supply at S_1 is exhausted.

Now, availability at S_2 = 55 – 30 = 25

Demand for D_3 = 35 – 10 = 25

We set x_{23} = 25

Thus, we have obtained a basic feasible solution given by x_{11} = 40, x_{13} =10, x_{22} = 30, x_{23} = 25

The transportation cost by the above route

= 40 x 5 + 10 x 10 + 30 x 8 + 25 x 20

= 200 + 100 + 240 + 500 = 1040

Note : Transport cost by North-West Corner Method = 1210

Transport cost by Maxima–Minima Method = 1040

Thus, Matrix-Minima Method gives a better solution than the North-West Corner method. [Learners are advised to work out by VAM]

(ii) Do yourself.

(iii) We find an initial basic feasible solution of this problem by Vogel Approximation Method. [Learners are advised to find an initial basic feasible solution by North-West Corner Method, Matrix-Minima Method Separately.]

The TP is

	D_1	D_2	D_3	Supply
S_1	10	9	8	8
S_2	10	7	10	7
S_3	11	9	7	9
S_4	12	14	10	4
Demand	10	10	8	

We have

$\Sigma a_i = a_1 + a_2 + a_3 + a_4 = 8 + 7 + 9 + 4 = 28$

$\Sigma b_j = b_1 + b_2 + b_3 = 10 + 10 + 8 = 28$

$\therefore$ the problem is balance TP.

$\therefore$ that exists a basic feasible solution.

We note the penalties in Table 1. The maximum penalty occurs in the 2nd row.The cheapest cost in the 2nd row is 7. It is in the (2, 2) cell.We set x_{22} = min (7, 10) = 7

	D_1	D_2	D_3	
S_1	10	9	8	8 (1)
		7		
S_2	10	7	10	7 (3)
S_3	11	9	7	9 (2)
S_4	12	14	10	4 (2)
	10	10	8	
	(0)	(2)	(1)	

Supply at S_2 is exhausted. So, we delete the 2nd row.

We get the reduced transportation table (table 2).

	D_1	D_2	D_3	
S_1	10	9		8 (1)
			8	
S_3	11	9	7	9 (2)
S_4	12	14	10	4 (2)
	10	3	8	
	(1)	(0)	(1)	

Table 2

The maximum penalty occurs in the 3rd row as well as in the 4th row.

We select the 3rd row.

The cheapest cost in the 3rd row is 7, and it is in the (3, 3) cell.

We set x_{33} = min (9, 8) = 8

Demand for D_3 is met. So we delete the 3rd column.

We get the reduced transportation table (table 3).

	D_1	D_2	
S_1	10	9	8 (1)
		1	
S_3	11	9	1 (2)
S_4	12	14	4 (2)
	10	3	
	(1)	(0)	

Table 3

The maximum penalty occurs in the 3rd row as well as in the 4th row.

We select the 3rd row.The cheapest cost in the 3rd row is 9, and it is in the (3, 2) cell.

We set x_{32} = min (1, 3) = 1

Supply at S_2 is exhausted. So, we delete the 2nd row.

We get the reduced transportation table (table 4).

	D_1	D_2	
S_1	10	2 9	8 (1)
S_4	12	14	4 (2)
	10	2	
	(2)	(5)	

Table 4

The maximum penalty occurs in the 2nd column. The cheapest cost in the 2nd column is 9, and it is in the (1, 2) cell.

We set x_{12} = min (8, 2) = 2

Demand for D_2 is met. We delete the 2nd column.

The reduced transportation table is as follows (Table 5).

	D_1	
S_1	2 10	6
S_4	4 12	4
	10	
	(2)	

Table 5

Clearly x_{11} = 6 ,x_{14} = 4

Thus we have obtained Table 6

	D_1	D_2	D_3	Supply
S_1	6 10	2 9	8	8
S_2	10	7 7	10	7
S_3	11	1 9	8 7	9
S_4	4 12	14	10	4
Demand	10	10	8	

Thus, an initial basic feasible solution of the TP is

$X_{11}= 6, x_{12}= 2, x_{22}= 7, x_{32}= 1, x_{33}= 8, x_{41}= 4$

The transportation cost by the above route is given by

$Z = 6 \times 10 + 2 \times 9 + 7 \times 7 + 1 \times 9 + 8 \times 7 + 4 \times 12$

$= 60 + 18 + 49 + 9 + 56 + 48$

$= 240$

Ans to Q No 2: Try yourself.

The optimal solution is $x_{11}= 3, x_{23}= 3, x_{24}= 4, x_{31}= 1, x_{32}= 3, x_{33}=1$

Minimum transportation cost = 68

Ans to Q No 3: Try yourself.

The optimal solution is $x_{11}= 8, x_{12}= 7, x_{24}= 4, x_{31}= 1, x_{33}= 5, x_{34}= 2$

Minimum transportation cost = 159

Ans to Q No 4: Try yourself.



13.10 MODEL QUESTIONS

Q 1: What is a transportation problem? Give the formulation of a Transportation Problem.

Q 2: Prove that a necessary and sufficient condition for the existence of a feasible solution to an mxn transportation problem is

$$\sum_{i=1}^{m} a_i = \sum_{j}^{n} b_j .$$

Where a_i and b_j denote the availability and demand at the i th source and j th destination respectively.

Q 3: Explain

(a) North-West Corner Method,

(b) Matrix-Minima Method,

(c) Vogel Approximation Method.

for finding an initial basic feasible solution for a transportation problem.

Q 4: Obtain initial solution for the following TP by North-West corner Method, and VAM, and compare the results.

		Destination				
		D_1	D_2	D_3	D_4	Supply
	S_1	1	2	1	4	20
Source	S_2	4	3	2	1	50
	S_3	4	2	5	9	30
Demand		20	40	30	10	

Q 5: Point out the differences between a transportation problem and an assignment problem.

Q 6: Give a brief outline of the procedure of solving a transportation problem.

Q 7: Describe any two methods to obtain an initial possible solution to a transportation problem

Q 8: A company has four plants P_1, P_2, P_3, P_4 from which it supplies to three markets M_1, M_2, M_3. Determine the optimal transportation plan from the following data giving the plan to market shifting costs, quantities available at each plant and quantities required at each market.

Market	P_1	P_2	P_3	P_4	Required at Market
M_1	19	14	23	11	11
M_2	15	16	12	21	13
M_3	30	25	16	39	19
Available at Plant	6	10	12	15	43

Q 9: Solve the following transportation problem :

		To				
		S_1	S_2	S_3	S_4	Supply
	O_1	1	2	1	4	30
From	O_2	3	3	2	7	50
	O_3	4	2	5	9	20
Demand		20	40	30	11	100

Q 10: Solve the following transpertation problem :

		To				
		1	2	3	4	Supply
	1	21	16	25	13	11
From	2	17	18	14	23	13
	3	32	27	18	41	19
Demand		6	10	12	15	43

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UNIT 14: ASSIGNMENT PROBLEM

UNIT STRUCTURE

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14.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- know the notion of an Assignment Problem
- formulate an Assignment Problem
- know the Hungarian Method of Solution of an Assignment Problem
- know the Travelling Salesman Problem.

14.2 INTRODUCTION

An assignment problem is a special case of transportation problem. Let there be equal number of jobs (n) and machines (n) in a factory. The problem is to assign n jobs to n machines such that the total cost incurred is minimised. The determination of this optimal assignment is known as cost minimising assignment problem or simply an assignment problem. In this unit we will formulate the assignment problem and discuss the solution of it.

14.3 FORMULATION OF AN ASSIGNMENT PROBLEM

Let us suppose that there is one machine (M_1) in a printing press, and there is one operator (job, J_1) to operate it. Clearly one job will be assigned to one machine.

Now suppose that there are two machines (M_1, M_2), and there are two jobs (J_1, J_2), These two different jobs are to be assigned to two different machines such that the total cost incurred is minimised. Clearly there are 2! = 2 ways to assign 2 different jobs to 2 different machines. Costs incurred for doing these jobs are given below.

	M_1	M_2
J_1	5	6
J_2	2	4

Now let us enumerate all the possible assignments together with the total cost.

Number	Assignment	Total Cost
1	$J_1 – M_1$	5 + 4 = 9
	$J_2 – M_2$	
2	$J_1 – M_2$	2 + 6 = 8
	$J_2 – M_1$	

Out of these 2 possible assignments, the minimum total cost is 8 which corresponds to $J_1 – M_2$, $J_2 – M_1$ i.e if job J_1 is assigned to machine M_2, and job J_2 is assigned to machine M_1, the total cost is minimised. This the optimal assignment.

Let us take another example. Let there be three machines (M_1, M_2, M_3), and let there be three jobs (J_1, J_2, J_3). These 3 different jobs are to be assigned to 3 different machines such that the total cost incurred is minimised. Clearly there are 3! = 3 x 2 x 1 = 6 ways to assign 3 different jobs to 3 different machines. Costs incurred for doing these jobs are given below.

	M_1	M_2	M_3
J_1	2	4	5
J_2	3	6	7
J_3	8	3	3

Now let us enumerate all the possible assignments together with the total cost.

Number	Assignment	Total Cost
1	$\begin{cases} J_1 - M_1 \\ J_2 - M_2 \\ J_3 - M_3 \end{cases}$	2 + 6 + 3 = 11
2	$\begin{cases} J_1 - M_1 \\ J_2 - M_3 \\ J_3 - M_2 \end{cases}$	2 + 7 + 3 = 12
3	$\begin{cases} J_1 - M_2 \\ J_2 - M_3 \\ J_3 - M_1 \end{cases}$	4 + 7 + 8 = 19
4	$\begin{cases} J_1 - M_2 \\ J_2 - M_1 \\ J_3 - M_3 \end{cases}$	4 + 3 + 3 = 10
5	$\begin{cases} J_1 - M_3 \\ J_2 - M_2 \\ J_3 - M_1 \end{cases}$	5 + 6 + 8 = 19
6	$\begin{cases} J_1 - M_3 \\ J_2 - M_1 \\ J_3 - M_2 \end{cases}$	5+ 3 + 3 = 11

Out of these 6 possible assignments, the minimum total cost is 10 which corresponds to $J_1 - M_2$, $J_2 - M_1$,$J_3 - M_3$, i.e, if job J_1 is assigned to machine M_2, job J_2 is assigned to machine M_1, and job J_3 is assigned to machine M_3, the total cost is minimised. This is the optimal assignment.

Thus, in assignment problem, a job is assigned in such a way that the maximum profit is gained on minimum investrment.

Now we generalise the problem.

Let there be n jobs J_1, J_2, . . ., J_n; and let there be n machines M_1, M_2, . . . M_n. Let the cost of doing the J_i th job by the M_j machine be c_{ij}. Then the cost matrix is given by

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{ij} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2j} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{i1} & c_{i2} & \dots & c_{ij} & \dots & c_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nj} & \dots & c_{nn} \end{bmatrix}$$

Now the problem is which job is to be assigned to which machine such that the total cost will be minimum.

Mathematically the problem is stated as follows.

To Minimise

$$z(\text{cost}) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\ x_{ij}$$

i = 1, 2, . . ., n ; j = 1, 2, . . ., n ;

Where

$$x_{ij} = \begin{cases} 1, & \text{if } i\text{th job is assigned to } j\text{th machine} \\ 0, & \text{if } i\text{th job is assigned to } j\text{th machine} \end{cases}$$

with the restrictions

$$\sum_{i=1}^{n} x_{ij} = 1,\ j = 1, 2, \dots, n$$

$$\sum_{j=1}^{n} x_{ij} = 1,\ i = 1, 2, \dots, n$$

In each of these summations, one term on the left hand side is equal to one, and the rest one zeroes.

Note :

1. Number of jobs is equal to the number of machines.
2. Not more than one job is assigned to the same machine.
3. Not more than one machine processes the same job.

14.3.1 Assignment Problem as a Special Case of Transportation Problem

We take an mxn transportation problem.

Minimise $z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\ x_{ij}$

Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, ..., m$$

$$\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, ..., n$$

$x_{ij} \geq 0$, for all i and j,

Where a_i = availability at the i th source,

b_j = demand of the j th destination,

c_{ij} = per unit transportation cost from the i th source to the j th destination.

x_{ij} = number of units transported from the i th source to the j th destination.

Assignment problem is a special case of transportation problem. Here 'source' is represented by 'job' and 'destination is represented by 'machine'. Number of sources is equal to the number of destinations. Supply at each source is unity (a_i = 1, for all i), and demand at each destination is also unity (b_j = 1, for all j). Cost of 'transporting' (assigning) job i to machine j is c_{ij}, and the number of units allocated to a cell can be either 1 or 0.

Thus, the cost matrix of an assignment problem is a square matrix. The optimum solution is such that there is only one assignment in a given row or column of the cost matrix.

14.4 SOLUTION OF AN ASSIGNMENT PROBLEM

We know that an assignment problem is a special case of transportation problem. Therefore, an assignment problem is a linear programming problem. So it can be solved by simplex method. But an expedient method has been evolved for its solution. It is known as Hungarian method. We will discuss it in 12.5. We prove two important theorems below.

Theorem : If a constant is added to (or subtracted from) any row or column of the cost matrix of an assignment problem, the optimal solution

remains the same.

Proof : Let $[c_{ij}]$ be the cost matrix of an assignment problem. Let the constants u_i be added to (or subtraited from) all the elements of the i th row of $[c_{ij}]$, and the constants b_j be added to (subtracted from) all the elements of the jth column of $[c_{ij}]$.

Let $[\bar{c}_{ij}]$ be the now cost matrix where $\bar{c}_{ij} = c_{ij} \pm u_i \pm v_j$.

Let $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\ x_{ij}$ be the objective function of the original assignment problem.

Let $\bar{z}$ be the objective function of the resulting assignment problem.

Therefore, $\bar{z} = \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{c}_{ij}\ x_{ij}$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(c_{ij} \pm u_i \pm v_j\right) x_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\ x_{ij} \pm \sum_{i=1}^{n} \left(u_i \sum_{j=1}^{n} x_{ij} \right) \pm \sum_{j=1}^{n} \left(v_j \sum_{j=1}^{n} x_{ij} \right)$$

$$= z \pm \sum_{i=1}^{n} u_i \pm \sum_{j=1}^{n} v_j \qquad \because \sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1$$

$= z \pm K$ where $\pm K = \pm \sum_{i=1}^{n} u_i \pm \sum_{j=1}^{n} v_j$

Thus, $\bar{z}$ is minimum when z is minimum.

Theorem : If in any assignment problem all $c_{ij} \geq 0$, then a set of solution $x_{ij} = \bar{x}_{ij}$ which satisfies $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\ x_{ij}\ \bar{x}_{ij} = 0$ will be an optimal solution.

Proof : We know that $c_{ij} \geq 0$, and x_{ij}=1 or 0.

$\therefore z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\ x_{ij}$ cannot be negative.

$\because$ for $x_{ij} = \bar{x}_{ij}$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\ \bar{x}_{ij} = 0$$

$$\therefore\ \bar{z} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\ x_{ij}\ \bar{x}_{ij} = 0$$

This is the minimum value of z.

Thus, the solution $x_{ij} = \bar{x}_{ij}$ satisfying the condition $\Sigma c_{ij}\, x_{ij} = 0$ will be an optimal solution of the problem.

Note : From the above two theorems, we can conclude that by adding or subtracting selected constants u_i, v_j to or from the elements of the cost matrix $[c_{ij}]$, we can always find another cost matrix $[\bar{c}_{ij}]$ with elements $\bar{c}_{ij}$ such that $\bar{c}_{ij} \geq 0,$ for all i, and j. It will not affect the optimal solution.

Next we choose u_i and v_j in such a manner that there will be at least one $\bar{c}_{ij} = 0$ in each row, and in each column. Then we shall try to find a feasible assignment for which the corresponding c_{ij} will be equal to zero. It will be optimal if there is exactly one assigned 0 in each row and in each column.

14.5 THE HUNGARIAN METHOD OF SOLUTION OF AN ASSIGNMENT PROBLEM

The Hungarian mathematician Denes Konig (1844 – 1944) published a theorem on graphs in 1931. It was generalised by another Hungarian mathematician Jeno Egervary (or Eugene Egervary, 1891 – 1958) in the same year. This theorem was applied by American mathematician Harold William Kuhn (1925 – 2014) to solve the assignment problem. This method is popularly known as Hungarian method. The Hungarian method is an algorithm which gives an optimal assignment for a given cost matrix. The algorithm is stated below.

Step I :

(i) **Row reduction :** We subtract the minimum element of each row from all elements of that row.

(ii) **Column reduction :** We subtract the minimum element of each column from all elements of that column.

[Clearly this reduced matrix will contain **at least** one zero in each row and in each column]

Step II : Zero assignment

(i) We start with the first row of the reduced matrix. We examine the rows successively until a row with exactly one zero is found. We assign this zero by encircling it and cross all other zeroes in its column.

(ii) We examined the columns successively until a column with exactly one zero is found. We encircle this zero and cross all other zeroes in its row.

We continue these operations successively on rows and columns until all zeroes have either been assigned or crossed out.

Now two possibilities arise.

Either

All the zeroes are assigned or crossed out. In this case we get an optimal solution.

or

At least two rows are left out from assignment or from crossing in a row or column. In this case we try to exclude some of the zeroes by trial and error method.

Thus, we get two situations (a) Number of assigned zeroes = n, the number rows (or columns) of the cost matrix. In this case, optimal assignment can be made. (b) Number of assigned zeroes < n. In this case, we should go to step III.

Step III : Drawing of minimum lines to cover zeroes

(i) We mark (✓) the rows that do not have assignments.

(ii) We mark (✓) the columns (not already marked) that have zeroes in marked rows.

(iii) We mark (✓) the rows (not already marked) that have assignments in marked columns.

(iv) We repeat the procedure (ii) and (iii) till no more rows or columns can be marked.

(v) We draw lines through unmarked rows and marked columns.

This gives the minimum number of lines crossing all zeroes. If this number is equal to the number of rows (or columns) of the cost matrix, them it is an optimal solution; otherwise we go to step IV.

Step IV :

(i) We select the smallest element from the uncovered elements.

(ii) We subtract the smallest element from all uncovered elements.

(iii) We add this smallest element to all those elements which are at the intersection of two lines.

Step V :

We have increased the number of zeroes. We modify the matrix with the help of step II and find the required assignment.

Example 1 : Solve the cost-minimising assignment problem whose cost matrix is given below.

	M_1	M_2	M_3
J_1	25	40	35
J_2	40	60	35
J_3	20	40	25

Solution : I. (i) Row Reduction : We subtract the smallet element of each row from all elements of that row.

0	15	10
5	25	0
0	20	5

(ii) Column Reduction : We subtract the smallest element of each column from all elements of that column.

0	0	10
5	10	0
0	5	5

II. Zero assignment

~~0~~	(0)	10
5	10	(0)
(0)	5	5

Thus, all the zeroes are either assigned or crossed out.

Again, number of assigned zeroes = 3, the number of rows (or columns) of the cost matrix.

$\therefore$ the assignment is optimal.

Optimal assignment : J_1M_2, J_2M_3, J_3M_4

Minimum assignment cost

= 40 + 35 + 20 = 95

Example 2 : Solve the cost minimising assignment problem whose cost matrix is given below.

	M_1	M_2	M_3	M_4
J_1	1	19	10	4
J_2	6	21	7	19
J_3	31	12	11	8
J_4	12	19	17	3

Solution :

I. (i) Row Reduction

0	18	9	3
0	15	1	13
23	4	3	0
9	16	14	0

(ii) Column Reduction

0	14	8	3
0	11	0	13
23	0	2	0
9	12	13	0

II. Zero assignment

(0)	14	8	3
~~0~~	11	(0)	13
23	(0)	2	~~0~~
9	12	13	(0)

Thus, all the zeroes are either assigned or crossed out.

Again, number of assigned zeroes = 4, the number of rows (or columns) of the cost matrix.

$\therefore$ the assignment is optimal. Optimal assignment : J_1M_1, J_1M_3, J_3M_2, J_4M_4

Minimum assignment cost = 1 + 7 + 12 + 3 = 23



CHECK YOUR PROGRESS

Q 1: Solve the cost-minimising assignment problems whose cost matrices are given below.

(a)

	M_1	M_2	M_3
J_1	12	10	8
J_2	9	8	11
J_3	11	14	12

(b)

	M_1	M_2	M_3	M_4
J_1	10	12	19	11
J_2	5	10	7	8
J_3	12	14	13	11
J_4	8	15	11	9

(c)

	M_1	M_2	M_3	M_4
J_1	8	10	17	9
J_2	3	8	5	6
J_3	10	12	11	9
J_4	6	13	9	7

(d)

	M_1	M_2	M_3	M_4
J_1	10	12	9	11
J_2	5	10	7	8
J_3	12	14	13	11
J_4	8	15	11	9

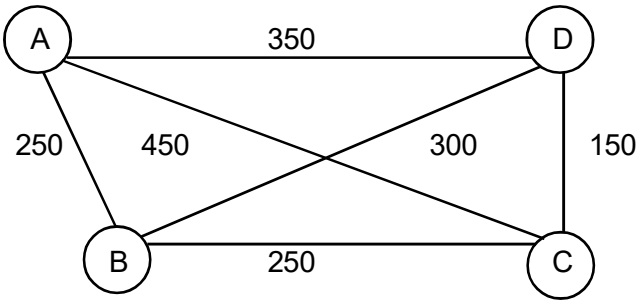
(e)

	M_1	M_2	M_3	M_4	M_5
J_1	6	12	3	11	15
J_2	4	2	7	1	10
J_3	8	11	10	7	11
J_4	16	19	12	23	21
J_5	9	5	7	6	10

14.6 THE TRAVELLING SALESMAN PROBLEM (SHORTEST CYCLIC ROUTE MODEL)

The travelling salesman problem (or shortest cyclic route model) consists of a salesman and a set of cities. The salesman has to visit each one of the cities once and only once starting from a certain one and returning to the same city. The problem is that the travelling salesman wants to minimise the total length (or total cost or total time) of the trip.

Let there be 4 cities A, B, C, D. Each one of the cities is connected with the other by route as shown below.



There one (4 – 1) ! = 3! = 6 possible routes, viz ABCDA, ABDCA, ADBCA, ADCBA, ACBDA, ACDBA. The distances between the cities (in km) are shown in the figure.

The length of the trip ABCDA is (200 + 250 + 150 + 350)km i.e., 950 km, whereas the length of the trip ABDCA is (200 + 300 + 150 + 450) km, i.e. 1100 km. The problem is to select the shortest route. This type of problem can be solved by considering it as an assignment problem.

There are two types of such problems, – symmetrical, and asymmetrical. A problem is said to be symmetrical if the distance (or cost or time) between any two cities does not depend on the direction of the journey. On the other hand, a problem is said to be asymmetrical if for one or more pair of cities, the distance (or cost or time) depends on the direction of the journey.

Now we formulate the Travelling Salesman Problem (TSP) as an assignment problem.

Formulation of TSP as an assignment problem

Let c_{ij} denote the distance (or cost or time) from the i th city to the j th city. Let

x_{ij} = 1, if the salesman travels from the i th city to the j th city

= 0, otherwise

Another restriction is also imposed. x_{ij} must be chosen in such a way that no city is visited twice till the completion of travelling of all the cities. It is not allowed to travel from the j th city to the i th city directly. To avoid this possibility c_{ij} is taken to be ∞. Furthermore, only one x_{ij} = 1, for each value of j.

Thus, the problem is as follows. Minimise $z = \sum_i \sum_j c_{ij}\ x_{ij}$

subject to the above constraints.

The distance (or cost or time) matrix of the problem is as follows.

From

	A_1	A_2	A_3	.....	A_n
A_1	∞	c_{12}	c_{13}	.....	c_{1n}
A_2	c_{21}	∞	c_{23}	.....	c_{2n}
A_3	c_{31}	c_{32}	∞	.....	c_{3n}
!	.....	.....	.....	.....	.....
A_n	c_{n1}	c_{n2}	c_{n3}	.....	∞

Example 3 : Choose the correct answer. A salesman has to visit n cities such that he will have to visit each of the cities once and only once starting from a certain city and returning to the same city. The number of ways of his tour plan is (a) n, (b) n!, (c) (n + 1) !, (d) (n – 1) !

Solution : (d) (n – 1) !



CHECK YOUR PROGRESS

Q 2: Write a short note on the Travelling Salesman Problem.

Q 3: Formulate the Travelling Salesman Problem.



14.7 LET US SUM UP

- Let there be equal number of jobs (n) and machines (n) in a factory. The problem is to assign n jobs to n machines such that the total cost incurred is minimised. The determination of this optimal assignment is known as cost minimising assignment problem or simply an assignment problem.
- Assignment problem is a special case of Transportation Problem.
- If a constant is added to (or subtracted from) any row or column of the cost matrix of an assignment problem, the optimal solution remains the same.
- The Hungarian method is an algorithm which gives an optimal assignment for a given cost matrix.
- The travelling salesman problem (TSP) consists of a salesman and a set of cities. The salesman has to visit each one of the cities once and only once starting from a certain one and returning to the same city. The problem is that the travelling salesman wants to minimise the total length (or total cost or total time) of the trip.



14.8 FURTHER READING

1) Operations Research : Prem Kuman Gupta, D. S. Hira (S. Chand & Company Ltd.

2) Operations Research : Kanti Swarup, P. K. Gupta, Man Mohan (Sultan Chand and Sons)

14.9 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (a) The Assignment Problem is

	M_1	M_2	M_3
J_1	12	10	8
J_2	9	8	11
J_3	11	14	12

We subtract the smallest element of each row from all elements of that row.

4	2	0
1	0	3
0	3	1

We subtract the smallest element of each column from all elements of that column.

4	2	0
1	0	3
0	3	1

We assign the zeroes as follows.

4	2	⓪
1	⓪	3
⓪	3	1

Number of assigned zeroes = 3 the number of rows (or columns) of the cost matrix is also 3.

$\therefore$ the assignment is optimal.

Optimal assignment : J_1M_3, J_2M_2, J_3M_1

Minimum assignment cost = 8 + 8 + 11 = 27

(b) The Assignment Problem is

	M_1	M_2	M_3	M_4
J_1	10	12	19	11
J_2	5	10	7	8
J_3	12	14	13	11
J_4	8	15	11	9

We subtract the smallest element of each row from all elements of that row.

0	2	9	1
0	5	2	3
1	3	2	0
0	7	3	1

We subtract the smallest element of each column from all elements of that column.

0	0	7	1
0	3	0	3
1	1	0	0
0	5	1	1

We assign the zeroes as follows.

~~0~~	(0)	7	1
~~0~~	3	(0)	3
1	1	~~0~~	(0)
(0)	5	1	1

Thus, all the zeroes are either assigned or crossed out.

Again, number of assigned zeroes = 4 ,Also the number of rows (or cloumns) of the cost matrix = 4.

$\therefore$ the assignment is optimal.

Optimal assignment : J_1M_2, J_2M_3, J_3M_4, J_4M_1

Minimum assignment cost = 12 + 7 + 11 + 8 = 38

(c) Try yourself.

(d) The Assignment Problem is

	M_1	M_2	M_3	M_4
J_1	10	12	9	11
J_2	5	10	7	8
J_3	12	14	13	11
J_4	8	15	11	9

Row Reduction

1	3	0	2
0	5	2	3
1	3	2	0
0	7	3	1

Column Reduction

1	0	0	2
0	2	2	3
1	0	2	0
0	4	3	1

Zero Assignment

1	0	0	2
(0)	2	2	3
1	0	2	0
~~0~~	4	3	1

We see that all the zeroes are not assigned or crossed out. We cover all the zeroes by minimum number of horizontal and vertical lines.

~~1~~	~~0~~	~~0~~	~~2~~
(0)	2	2	3
~~1~~	~~0~~	~~2~~	~~0~~
~~0~~	4	3	1

The number of lines is 3 < 4, the number of rows (or columns) of the cost matrix.

Now the smallest element among the uncovered elements is 1.

We subtract 1 from all the uncovered elements.

We add 1 to all the elements which are at the junction of two lines.

The complete table is as follows.

2	0	0	2
0	1	1	2
2	0	2	0
0	3	2	0

Zero Assignment

2	~~0~~	(0)	2
(0)	1	1	2
2	(0)	2	~~0~~
~~0~~	3	2	(0)

Thus, all the zeroes are either assigned or crossed out.

Again, number of assigned zeroes = 4, the number of rows (or columns) of the cost matrix.

$\therefore$ the assignment is optimal.

Optimal assignment : J_1M_3, J_2M_1, J_3M_2, J_4M_4

Minimum assignment cost = 9 + 5 + 14 + 9 = 37

(e) The Assignment Problem is

	M_1	M_2	M_3	M_4	M_5
J_1	6	12	3	11	15
J_2	4	2	7	1	10
J_3	8	11	10	7	11
J_4	16	19	12	23	21
J_5	9	5	7	6	10

Row Reduction

3	9	0	8	12
3	1	6	0	9
1	4	3	0	4
4	7	0	11	9
4	0	2	1	5

Column Reduction

2	9	0	8	8
2	1	6	0	5
0	4	3	0	0
3	7	0	11	5
3	0	2	1	1

Zero Assignment

2	9	(0)	8	8
2	1	6	(0)	5
(0)	4	3	~~0~~	~~0~~
3	7	~~0~~	11	5
3	(0)	2	1	1

All the zeroes are either assigned or crossed out.

But the number of assigned zeroes is 4 < 5, the number of rows (or columns) of the cost matrix. We cover all the zeroes by minimum number of horizontal and vertical lines.

2	9	(0)	8	8
~~2~~	~~1~~	~~6~~	~~(0)~~	~~5~~
~~(0)~~	~~4~~	~~3~~	~~0~~	~~0~~
3	7	~~0~~	11	5
~~3~~	~~(0)~~	~~2~~	~~1~~	~~1~~

The number of lines is 4 < 5, the number of rows (or columns) of the cost matrix.

Now the smallest element among the uncovered elements is 2.

We subtract 2 from all the uncovered elements.

We add 2 to all the elements which are at the junction of two lines.

The complete table is as follows.

0	7	0	6	6
2	1	8	0	5
0	4	5	0	0
1	5	0	9	3
3	0	4	1	1

Zero Assignment

(0)	7	~~0~~	6	6
2	1	8	(0)	5
~~0~~	4	5	~~0~~	(0)
1	5	(0)	9	3
3	(0)	4	1	1

Thus, all the zeroes are either assigned or crossed out.

Again, number of assigned zeroes = 5, the number of rows (or columns) of the cost matrix.

$\therefore$ the assignment is optimal.

Optimal assignment : J_1M_1, J_2M_4, J_3M_5, $J_4M_{3|}$, J_5M_2

Minimum assignment cost = 6 + 1 + 11 + 12 + 5 = 35

Ans to Q No 2: 1, 2 Try yourself. (Already explained)



14.10 MODEL QUESTIONS

Q 1: What is an assignment problem. Give the mathematical formulation of an assignment problem.

Q 2: What is the difference between a transportation problem and an assignment problem?

Q 3: Explain the Hungarian algorithm for solving an assignment problem.

Q 4: Solve the cost-minimising assignment problem whose cost matrix is given below.

	M_1	M_2	M_3	M_4
J_1	18	26	17	11
J_2	13	28	14	26
J_3	38	19	18	15
J_4	19	26	24	10

[minimum assignment cost = 59]

Q 5: A company has 5 jobs to be done on 5 machines. Any job can be done by any machine. The cost of doing the jobs in different machines are given below. Assign the jobs for different machines so as to minimise the total cost.

Jobs	Machines				
	A	B	C	D	E
1	13	8	16	18	19
2	9	15	24	9	12
3	12	9	4	4	4
4	6	12	10	8	13
5	15	17	18	12	20

[Optimal assignment : 1 $\rightarrow$ B, 2 $\rightarrow$ E, 3 $\rightarrow$ C, 4 $\rightarrow$ A, 5 $\rightarrow$ D
Minimum total cost = 42 units (one unit may imply one rupee, one dollar, one thousand rupee, one hundred dollar, etc. Note that in the given problem, the unit of cost is not mentioned.]

Q 6: Solve the following assignment problem in order to minimise the total cost. The cost matrix given below gives the assignment cost when different operators are assigned to various mechines.

operators

Machines

	I	II	III	IV	V
A	30	29	38	23	26
B	23	29	38	23	26
C	30	27	22	22	22
D	25	31	29	27	32
E	27	29	30	24	32

[The optimum assignment is :

Operators :	I	II	III	IV	V
Machines :	D	A	C	E	B

Optimum cost = 122 units]

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UNIT 15 : GAME THEORY

UNIT STRUCTURE

15.1 Learning Objectives

15.2 Introduction

15.3 Competitive Situations

15.4 Characteristics of Competitive Games

15.5 Two-Person Game, Two-Person Zero-Sum Game

15.6 Pay-off Matrix

15.7 Strategy

15.8 Maximin and Minimax Principle

15.9 Saddle Point, Value of a Game, and Symmetric Game

15.10 Dominance Property

15.11 Let Us Sum Up

15.12 Further Reading

15.13 Answers To Check Your Progress

15.14 Model Questions

15.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- competitive situation, games, and characteristics of games
- strategy, two-person game, pay-off matrix
- maximin-minimax principle
- saddle point and value of a game
- dominance property.

15.2 INTRODUCTION

Game Theory was developed in the first half of the 20th century. John von Neumann (1903-1957) is regarded as the father of Game Theory.

In our day-to-day life we face situations, where there are two (or more) opposite parties with conflicting interests, and the action of one depends upon the action which the opponent takes. It is required to take

decisions in such competitive situations. Competitive problems can be dealt with the help of Game Theory. The theory was put forward by John Von Neumann in 1928, and was expanded in collaboration with economist Oskar Morgenstern (1902-1977) in their book Theory of Games and Economic Behaviour (1944). Game Theory is used in problems of economics and business, biology, sports, military strategy, and other fields.

In this unit we will study characteristics of game and other related terms, maximin-minimax principle, dominance.

15.3 COMPETITIVE SITUATIONS

In practical life, we face situations, where there are two (or more) opposite parties with conflicting interests, and the action of one depends upon the action which the opponent takes. The outcome is controlled by the decision of all parties concerned. Such a situation is called as competitive situation.

15.4 CHARACTERISTICS OF COMPETITIVE GAMES

A competitive situation is called a Game if it has the following characteristics.

1. The number of competitors or participants, called players, is finite.
2. Each player has a finite number of choices or courses of action, called strategies. The number of strategies need not be the same for each player.
3. Complete relevant information, i.e, the different strategies of each player and the amount of gain on an individual's strategy is known to each player in advance.
4. A game takes place when each player executes his strategies.
5. Each player acts rationally to maximise his gain or minimise his loss.
6. The outcome of the play determines a set of payments which may be positive (gain), negative (loss), or zero (drow).
7. Each player makes individual decision without direct communication.

Example : Let there be two players A and B. They draw a card from a pack

of cards simultaneously with the condition that if both of them draw red or black cards, A will win and B will have to pay Rs. 10 to A. Otherwise B will win and A will have to pay Rs. 10 to B.

In the above game, There are two strategies for each player,– drawing red card or black card. It can be shown with the following table.

		Player B	
		red	black
Player A	red	10	–10
	black	–10	10

We note that the number of players is finite (2). Each player has finite number of choices, i.e., strategies (2) at his disposal.

A is called row-player or maximising player. B is called column player or minimising player. Positive entries indicate gain for A (loss for B). Negative entries indicate loss for A (gain for B).

15.5 TWO-PERSON GAME, TWO-PERSON ZERO-SUM GAME

A game involving two players is called a Two-Person Game. Of course, each player may have a number of possible choices for each play of the game.

A Two-Person Game is called a Two-person Zero-Sum Game if the gains of one player are the losses of the other player and vice versa.

Note : In a Two-Person Zero-Sum Game, the algebraic sum of the gains to both the players after the game is zero.

15.6 PAY-OFF MATRIX

The matrix which shows the outcome of a game as the players select their strategies is called the Pay-off Matrix.

Example : Let there be two players A and B. Let there be three strategies A_1, A_2, A_3 at the disposal of A, and two strategies B_1, B_2 at the disposal of B. Payments are made according to the following table

		Player B	
		B_1	B_2
Player A	A_1	15	10
	A_2	–20	10
	A_3	0	–12

In the above table, rows indicate strategies for A, whereas columns indicate strategies for B. Now, if the player A selects strategy A_1 (first row), and the player B selects strategy B_2 (second column), A gains an amount of 10, whereas B loses an amount of 10 (entry in the intersection of the first row and the second column). Now, if the player A selects strategy A_2 and B selects strategy B_1, A gains an amount of –20, i.e., A loses 20, whereas B gains an amount of 20. Again, if A selects strategy A_3, and B selects strategy B_1, there is no loss or gain for either player.

The pay-off matrix of the above game is given by the following matrix.

$$\begin{matrix} & B \\ A & \begin{bmatrix} 15 & 10 \\ -20 & 10 \\ 0 & -20 \end{bmatrix} \end{matrix}$$

Note : 1. Rows indicate strategies for A.

2. Columns indicate strategies for B.
3. Positive elements indicate gain for A, loss for B.
4. Negative elements indicate loss for A, gain for B.
5. A is called the row player or maximising player. He wants to maximise his gain.
6. B is called the column player or minimising player. He wants to minimise his loss.

15.7 STRATEGY

Strategy is the predetermined rule by which a player decides his choice from his list of choices during the game.

There are two types of strategy, pure strategy, and mixed strategy.

Pure strategy : The decision rule of selecting a particular course of action is called a pure strategy. The player knows in advance that he will choose that particular course of action.

Mixed strategy : If a player decides to select a particular course of action for sometime, and then decides to change that course of action, and selects another course of action, and so on, then it is called a Mixed Strategy.

Note : In this unit we will consider only pure strategies.

15.8 MAXIMIN AND MINIMAX PRINCIPLE

Let us consider the following pay-off matrix of a game

$$\begin{array}{cc} & \text{Play B} \\ & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player A} \quad \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 15 & 10 \\ -20 & 12 \\ 6 & -12 \end{bmatrix} \end{array}$$

In the above game A has 3 strategies A_1, A_2, A_3 at his disposal and B has 2 strategies B_1, B_2 at his disposal.

Let us assume that the player A adopts strategy A_1. In that case B will definitely adopt strategy B_2 so that A gets the minimum. Thus, A will get at least 10 (minimum of 15, 10) if he adopts strategy A_1. similary, A will get at least –20 (minimum of –20, 12) if he adopts strategy A_2. Similarly A will get at least –12 (minimum of 6, –12) if he adopts strategy A_3. Surely A will try to improve his gain. So he will select the best of the worst outcomes so that he gets maximum out of his minimum gains. This is known as the maximum principle. In the above case maximum of the minimum gains for A is 10 (maximum of 10, –20, –12). 10 is the maximin value of the game. Strategy A_1 is the optimal strategy for A. A is the maximising player.

Now, let us consider the case for B. Let us assume that the player B selects the strategy B_1. In that case A will definitely adopt strategy A_1 so that A gets the maximum gain 15 (maximum loss for B). Similarly, if B adopts strategy B_2, A will select strategy A_2 so that A gets maximum gain 12 (maximum loss for B). Thus, maximum losses for B are 15, 12. So B will adopt that

strategy so that his loss becomes minimum. This is known as the minimax principle. In the above case minimum of the maximum losses for B is 12 (minimum of 15, 12). 12 is the minimax value of the game strategy B_2 is the optimal strategy for B. B is the minimising player.

Thus,according to Maximin, and Minimax Principle, a player lists his worst possible outcomes, and then he selects that particular strategy which corresponds to the best of these worst outcomes.

Example 1 : Find the maximin, and minimax values of the following pay-off matrix

Player B

		B_1	B_2	B_3
Player A	A_1	1	7	2
	A_2	6	2	7
	A_3	5	1	6

Solution :

Player B

		B_1	B_2	B_3	Row minimum
Player A	A_1	1	7	2	1
	A_2	6	2	7	②
	A_3	5	1	6	1
Column maximum		⑥	7	7	

Maximin value = max (1, 2, 1) = 2

Minimax value = min (6, 7, 7) = 6

Note : Optimal strategy for A is A_2, whereas optimal strategy for B is B_1.

Example 2 : Find the maximin, and minimax values of the following pay-off matrix

Player B

		B_1	B_2	B_3	B_4
Player A	A_1	-5	3	1	6
	A_2	5	6	3	8
	A_3	2	0	1	-4

Solution :

		Player B				
		B_1	B_2	B_3	B_4	Row minimum
Player A	A_1	-5	3	1	6	-5
	A_2	5	6	3	8	(3)
	A_3	2	0	1	-4	-4
Column maximum		5	6	(3)	8	

Here A has 3 strategies A_1, A_2, A_3 at his disposal, whereas B has 4 strategies B_1, B_2, B_3, B_4 at his disposal.

Maximin value = max (–5, 3, –4)

= 3

Minimax value = min (5, 6, 3, 8)

= 3

Note : 1. Optimal strategy for A is A_2, whereas optimal strategy for B is B_3.

2. Maximin value = 3 = Minimax values.

This common value is attained at the intersection of the second row and the third column, i.e., (2, 3) th element of the pay-off matrix is the common value.

15.9 SADDLE POINT AND VALUE OF A GAME

Let us consider a game with two players A, and B, where A is the row player (maximising player), and B is the column player (minimising player). The game is called a game with a saddle point if the maximin value coincides with the minimax value. Thus, a saddle point (or equilibrium point) is the point of intersection of the optimal strategies of the two players. The value (pay-off) at the saddle point is called the value of the game Generally, it is denoted by v.

Note : 1. A pay-off matrix may not have a saddle point.

If maximin value $\neq$ minimax value, the maximin value is called the lower value of the game, whereas the minimax value is called the upper value of the game. Generally, the lower value is denoted by $\underline{v}$ and the upper value is denoted by $\overline{v}$.

2. If maximin value $\neq$ minimax value, i.e, if $\underline{v} \neq \overline{v}$, the game is said to be non-strictly determinable.
3. If $\underline{v} = v = \overline{v}$, the game is said to be strictly determinable.
4. If $\underline{v} = v = \overline{v} = 0$ i.e., if the value of the game is zero, the game is said to be a fair game.
5. If $\underline{v} = \overline{v} = v > 0$, the game is said to be in favour of the maximising player (or biased to the maximising player).
6. If $\underline{v} = \overline{v} = v < 0$ the game is said to be in favour of the minimising player (or biased to the minimising player).
7. A pay-off matrix may have more than one saddle point. In that case the game will have more than one optimum solution.

Working Rule for determining a saddle point

1. We select the minimum element of each row of the pay-off matrix, and write the minimum values on the right of the pay-off matrix.
2. We select the maximum element of each column of the pay-off matrix, and write the maximum values below the pay-off matrix.
3. We calculate maximin value and minimax value, and encircle them. If they are equal, then that particular value is the value of the game, and the position of that particular value i.e., the point of intersection of the optimal strategies is the saddle point. If maximin value and minimax value are unequal, saddle point does not exist.

Example 3: The pay-off matrix of a game is given below. Examine if the game is a fair game.

Player B

Player A	I	II	III	IV
I	2	0	4	3
II	1	-3	5	2
III	4	-3	-1	1
IV	3	-4	2	-1

Solution :

Player B

Player A	I	II	III	IV	Row minimum
I	2	0	4	3	⓪
II	1	–3	5	2	–3
III	4	–3	–1	1	–3
IV	3	–4	2	–1	–4
Column maximum	4	⓪	5	3	

Maximin value = max (0, –3, –3, –4)

= 0

Minimax value = min (4, 0, 5, 3)

= 0

We have maximin value = minimax value = 0

$\therefore$ the given game is a fair game.

Example 4 : The pay-off matrix of a game is given below. Find the saddle points, and the value of the game.

Player B

Player A	I	II	III
I	1	2	1
II	0	–4	–1
III	1	3	–2

Solution :

Player B

Player A	I	II	III	Row minimum
I	1	2	1	①
II	0	–4	–1	–4
III	1	3	–2	–2
Column maximum	①	3	①	

Maximin value = max (1, –4, –2)

= 1

Minimax value = min (1, 3, 1)

= 1

We have maximin value = minimax value = 1

$\therefore$ the value of the game is 1.

We see that the value of the game is attained at two points, viz, the point of intersection of the first row and the first column; and the point of intersection of the first row and the third column. Thus, (1, 1) th element, and (1, 3) th element of the pay-off matrix are the saddle points. Thus, the pay-off matrix has more than one saddle point.

Example 5 : The pay-off matrix of a game is given below. Find the value of λ so that the game is strictly determinable.

Player B

Player A	I	II	III
I	λ	7	5
II	-2	λ	-8
III	-3	4	λ

Solution : $\because$ The game is strictly determinable.

$\therefore$ the maximin value must be equal to the minimax value.

Ignoring the value of λ, we find the maximin, and minimax values of the given pay-off matrix.

Player B

Player A	I	II	III	Row minimum
I	λ	7	5	5
II	-2	λ	-8	-8
III	-3	4	λ	-3
Column maximum	–2	7	5	

maximin value = max (5, – 8, – 3) = 5

(value corresponding to the first row)

minimax value = min (– 2, 7, 5) = –2

(value corresponding to the first column)

$\therefore$ the saddle point must be the intersection of the first row and the first column, i.e., the (1, 1) th element, i.e., λ must be the value of the game.

For a strictly determinable game, $-2 \le \lambda \le 5$.



CHECK YOUR PROGRESS

Q 1: The pay-off matrices of certain games are given below. Find the values of the games.

(a) Player A / Player B

	I	II
I	3	2
II	−2	−3
III	−4	−5

(b) Player A / Player B

	I	II	III
I	1	13	11
II	−9	5	−11
III	−0	−3	12

Q 2: The pay-off matrix of a game is given below. Solve the game. Is the game fair?

Player A / Player B

	I	II	III
I	−3	−2	6
II	2	0	2
III	5	−2	−4

Q 3: The pay-off matrix of a game is given below.

Player A / Player B

	I	II
I	−2	λ
II	2	6

Show that the game is strictly determinable, whatever may be the value of λ. Find the value of the game.

Q 4: The pay-off matrix of a game is given below. Find the value of the game so that the game is strictly determinable.

Player A / Player B

	I	II	III
I	λ	6	2
II	−1	λ	−7
III	−2	4	λ

Q 5: The pay-off matrix of a game is given below. Find the ranges of the values of λ, and μ which will give the (2, 2) th element as the saddle point for the game.

Player B

$$\text{Player A} \quad \begin{matrix} & \text{I} & \text{II} & \text{III} \\ \text{I} \\ \text{II} \\ \text{III} \end{matrix} \begin{bmatrix} 2 & 4 & 5 \\ 10 & 7 & \mu \\ 4 & \lambda & 6 \end{bmatrix}$$

Symmetric Game

A game is said to be symmetric if its pay-off matrix is a skew-symmetric matrix.

Note : 1. The pay-off matrix of a symmetric game will remain unchanged if the positions of the two players are interchanged.

For the pay-off matrix of a symmetric game, $a_{ij} = -a_{ji}$, $i \neq j$ and $a_{ii} = 0$

2. A symmetric game is a fair game and the two optimal strategies are idential.

15.10 DOMINANCE PROPERTY

Let us consider the following pay-off matrix of a game

Player B

$$\text{Player A} \quad \begin{matrix} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \text{I} \\ \text{II} \\ \text{III} \end{matrix} \begin{bmatrix} -5 & 3 & 1 & 15 \\ 5 & 5 & 4 & 6 \\ -4 & -2 & 0 & -5 \end{bmatrix}$$

We see that every element of the 2nd row for strategy II of player A is greater than the corresponding element of the 3rd row. Clearly A will not like to play strategy III, because he will gain more by playing strategy II. Thus, strategy III of A is inferior to strategy II. In other words, strategy III of A is dominated by strategy II. Therefore, the 3rd row can be eliminated. Thus, the pay-off matrix is reduced to

B

$$\text{A} \quad \begin{matrix} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \text{I} \\ \text{II} \end{matrix} \begin{bmatrix} -6 & 3 & 1 & 15 \\ 5 & 5 & 4 & 6 \end{bmatrix}$$

Now, we see that every element of the 2nd column is greater than

the corresponding element of the 3rd column. Also, every element of the 4th column is greater than the corresponding element of the 3rd column. Clearly B will not like to play column II or column IV, because he will have to pay more by playing strategy II or strategy IV. Therefore, the 2nd column and the 4th column can be eliminated. Thus, the pay-off matrix is reduced to

$$\begin{array}{cc} & \begin{matrix} B \\ I \quad II \end{matrix} \\ \text{Player A} \begin{matrix} I \\ II \end{matrix} & \begin{bmatrix} -5 & 1 \\ 5 & 4 \end{bmatrix} \end{array}$$

Now, we see that every element of strategy II for A is greater than strategy I, i.e., strategy I for A is dominated by strategy II. Therefore, strategy I for A can be eliminated. Thus, the pay-off matrix is reduced to

$$\begin{array}{cc} & \begin{matrix} B \\ I \quad II \end{matrix} \\ A \quad II & \begin{bmatrix} 5 & 4 \end{bmatrix} \end{array}$$

Now the player B will not select strategy I, because he will have to pay more by playing strategy I. Thus strategy I for B is dominated by strategy III. Thus, strategy I for B can be eliminated. Thus, the pay-off matrix is reduced to

$$\begin{array}{cc} & \begin{matrix} B \\ III \end{matrix} \\ A \quad II & \begin{bmatrix} 4 \end{bmatrix} \end{array}$$

Thus, we are left with the 1 x 1 matrix consisting of the (2, 3)th element of the original pay-off matrix, which is the saddle point.

Thus, the solution of the game is given by

(a) A will select strategy II,

(b) B will select strategy III,

(c) the value of the game is 4.

The principle explained above is known as the Dominance Property. It can be summarised as follows.

1. If every element of a row, say i-th row of a pay-off matrix is less than or equal to the corresponding element of any other row, say j-th row, then the row-player, i.e., the maximising player will never select the i-

th row, because he wants to gain more. Thus, i-th row is dominated by the j-th row. i-th row is eliminated.

2. If every element of a column, say r-th column, of a pay-off matrix is greater than or equal to the corresponding element of any other column, say s-th column, then the column-player, i-e. The minimising player will never select the r-th column, because he does not want to pay more. Thus, r-th column is dominated by the s-th column. r-th column is eliminated.

Thus, by using Dominance Property, the size of the pay-off matrix is reduced by deleting from it those strategies which are dominated by some other strategies.

Example 6 : The pay-off matrix of a game is given below. Solve the game by using Dominance Property.

Player B

$$\text{Player A} \quad \begin{matrix} & B_1 & B_2 & B_3 & B_4 \\ A_1 & 4 & 2 & 3 & 2 \\ A_2 & -2 & -1 & 4 & -3 \\ A_3 & 5 & 2 & 3 & 3 \\ A_4 & 4 & 0 & 0 & 1 \end{matrix}$$

Solution : We see that every element of the 4th row, i.e, strategy A_4 for player A is less than the corresponding element of the 3rd row. Thus, the 4th row is dominated by the 3rd row. So, the 4th row is deleted. Similarly every element of the 1st row is less than or equal to every element of the 3rd row. So, the player will not choose strategy A_1. Thus, the first row is eliminated. So the pay-off matrix is reduced to

B

$$\text{A} \quad \begin{matrix} & B_1 & B_2 & B_3 & B_4 \\ A_2 & -2 & -1 & 4 & -3 \\ A_3 & 5 & 2 & 3 & 3 \end{matrix}$$

Again, every element of the 3rd-column is greater than every element of the 2nd column. So, B will not choose strategy B_3. Therefore, the 3rd column is eliminated. So, the pay-off matrix is reduced to

$$\begin{array}{cc} & \begin{matrix} B_1 & B_2 & B_4 \end{matrix} \\ \begin{matrix} A \\ \\ \end{matrix} & \begin{matrix} A_2 \\ A_3 \end{matrix} \begin{bmatrix} -2 & -1 & -3 \\ 5 & 2 & 3 \end{bmatrix} \end{array}$$

Again every element of the strategy A_3 of the player A is greater than the corresponding element of the strategy A_2. So, A will not select A_2. Therefore, A_2 is deleted. The pay-off matrix is reduced to

$$\begin{array}{cc} & \begin{matrix} B_1 & B_2 & B_3 \end{matrix} \\ \begin{matrix} A & A_3 \end{matrix} & \begin{bmatrix} 5 & 2 & 3 \end{bmatrix} \end{array}$$

Now, the element of B_3 is greater than the element of B_2. Also, the element of B_1 is greater than the element of B_2. Therefore, B will not select strategy B_1 and strategy B_3. So, we are left with the 1 x 1 matrix.

$$\begin{array}{cc} & \begin{matrix} B_2 \end{matrix} \\ \begin{matrix} A & A_3 \end{matrix} & \begin{bmatrix} 2 \end{bmatrix} \end{array}$$

Clearly (3, 2) th element of the original pay-off matrix is the saddle point.

The solution of the game is given by

(a) A will select strategy A_3,

(b) B will select strategy B_2,

(c) the value of the game is 2.



CHECK YOUR PROGRESS

Q 5: Using Dominance Property, solve the games whose pay-off matrices are given by

(a)
$$\begin{array}{cc} & \begin{matrix} \text{I} & \text{II} \end{matrix} \\ \begin{matrix} A \\ \\ \end{matrix} & \begin{matrix} \text{I} \\ \text{II} \\ \text{III} \end{matrix} \begin{bmatrix} 3 & 2 \\ -2 & -3 \\ -4 & -5 \end{bmatrix} \end{array}$$

(b)
$$\begin{array}{cc} & \begin{matrix} \text{I} & \text{II} & \text{III} \end{matrix} \\ \begin{matrix} A \\ \\ \end{matrix} & \begin{matrix} \text{I} \\ \text{II} \\ \text{III} \end{matrix} \begin{bmatrix} 1 & 13 & 11 \\ -9 & 5 & -11 \\ -0 & -3 & 12 \end{bmatrix} \end{array}$$

(c)

		B				
		I	II	III	IV	V
A	I	3	5	4	9	6
	II	5	6	3	7	8
	III	8	7	9	8	7
	IV	4	2	8	5	3



15.11 LET US SUM UP

- A competitive situation is called a **Game** if it has the following characteristics
  - (i) The number of competitors or participants, called players, is finite.
  - (ii) Each player has a finite number of choices or courses of action, called strategies. The number of strategies need not be the same for each player.
  - (iii) Complete relevant information, i.e, the different strategies of each player and the amount of gain on each player and the amount of gain on an individual's strategy is known to each player in advance.
  - (iv) A game takes place when each player executes his strategies.
  - (v) Each player acts rationally to maximise his gain or minimise his loss.
  - (vi) The outcome of the play determines a set of payments which may be positive (gain), negative (loss), or zero (draw).
  - (vii) Each player makes individual decision without direct communication.
- A game involving two players is called a Two-Person Game.
- A Two-Person Game is called a Two-person Zero-Sum Game if the gains of one player are the losses of the other player and vice versa.
- The matrix which shows the outcome of a game as the players select their strategies is called the Pay-off Matrix.

- The decision rule of selecting a particular course of action is called a pure strategy. The player knows in advance that he will choose that particular course of action.
- If a player decides to select a particular course of action for sometime, and then decides to change that course of action, and selects another course of action, and so on, then it is called a Mixed Strategy.
- According to Maximin and Minimax Principle, a player lists his worst possible outcomes, and then he selects that particular strategy which corresponds to the best of these worst outcomes.
- A game is called a game with a saddle point if the maximin value coincides with the minimax value.
- A saddle point (or equilibrium point) is the point of intersection of the optimal strategies of two players.
- The value at the saddle point is called the value of the game.
- Using Dominance Property, the size of the pay-off matrix is reduced by deleting from it those strategies which are dominated by some other strategies.



15.12 FURTHER READING

1) Operations Research : Prem Kuman Gupta, D. S. Hira (S. Chand & Company Ltd., New Delhi 110055).

2) Operations Research : Kanti Swarup, P. K. Gupta, Man Mohan (Sultan Chand and Sons)



15.13 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (a) Player B

		I	II	Row minimum
Player A	I	3	2	②
	II	−2	−3	−3
	III	−4	−5	−5

Column maximum 3 ②

maximin value = max (2, – 3, – 5) = 2

minimax value = min (3, 2) = 2

maximin value = minimax value = 2

∴ the value of the game is 2

(b) Do yourself.

Ans to Q No 2: Player B

$$\text{Player A}\quad \begin{matrix} & \text{I} & \text{II} & \text{III} & \\ \text{I} & 3 & -2 & 6 & -3 \\ \text{II} & 2 & 0 & 2 & \textcircled{0} \\ \text{III} & 5 & -2 & -5 & -4 \end{matrix}$$

(Row minimum)

Column maximum 5 ⓪ 6

maximin value = max (–3, 0, –4) = 0

minimax value = min (5, 0, 6) = 0

maximin value = minimax value = 0

The solution of the game is given by

(a) A will select strategy II.

(b) B will select strategy II.

(c) the value of the game is 0.

∴ the value of the game is 0,

∴ the game is a fair game.

Ans to Q No 3: We ignore the value of λ.

Player B

$$\text{Player A}\quad \begin{matrix} & \text{I} & \text{II} & \\ \text{I} & -2 & \lambda & -2 \\ \text{II} & 2 & 6 & 2 \end{matrix}$$

(Row minimum)

Column maximum 2 6

minimax value = maximin value = 2

∴ the game is strictly determinable, whatever may be the value of λ.

The value of the game is 2.

Ans to Q No 4: Player B

$$\begin{array}{c} \\ \text{Player A} \end{array} \begin{array}{c} \begin{array}{cccc} & \text{I} & \text{II} & \text{III} \end{array} \\ \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} \end{array} \begin{array}{c} \text{Row minimum} \\ 2 \\ -7 \\ -2 \end{array}$$

Column maximum $-1 \quad 6 \quad 2$

$\because$ the game is strictly determinable,

$\therefore$ the maximin value must be equal to the minimax value.

Ignoring the value of λ, we find the maximin, and minimax values of the given pay-off matrix.

Player B

$$\begin{array}{c} \\ \text{Player A} \end{array} \begin{array}{c} \begin{array}{cccc} & \text{I} & \text{II} & \text{III} \end{array} \\ \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} \end{array} \begin{array}{c} \text{Row minimum} \\ 2 \\ -7 \\ -2 \end{array}$$

Column maximum $-1 \quad 6 \quad 2$

maximin value = max (2, – 7, – 2) = 2

(value corresponding to the first row)

minimax value = min (– 1, 6, 2) = –1

(Value corresponding to the first volumn),

$\therefore$ the saddle point must be the intersection of the first row and the first column, i.e, the (1, 1)th element, i.e, λ must be the value of the game.

For a strictly determinable game, $-1 \le \lambda \le 2$.

Ans to Q No 5: Ignoring the values of λ, and μ, we find the maximin, and the minimax values of the given pay-off matrix.

Player B

$$\begin{array}{c} \\ \text{Player A} \end{array} \begin{array}{c} \begin{array}{cccc} & \text{I} & \text{II} & \text{III} \end{array} \\ \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{bmatrix} 2 & 4 & 5 \\ 10 & 7 & \mu \\ 4 & \lambda & 6 \end{bmatrix} \end{array} \begin{array}{c} \text{Row minimum} \\ 2 \\ 7 \\ 4 \end{array}$$

Column maximum $10 \quad 7 \quad 6$

$\because$ The (2, 2) th element is the saddle point for the game,

$\therefore$ maximin value = minimax value = 7

$\therefore$ the conditions to be imposed upon λ, and μ are

(i) $\lambda \leq 7$ (ii) $\mu \geq 7$.

Ans to Q No 5: (a) Do yourself

(b) Do yourself

(c)

Player B

Player A	I	II	III	IV	V
I	3	5	4	9	6
II	5	6	3	7	8
III	8	7	9	8	7
IV	4	2	8	5	3

We see that every element of the 4th row, i.e, strategy IV for the player A is less than the corresponding element of the 3rd row. So, A will not select strategy IV. Eliminating the 4th row, the pay-off matrix is reduced to

Player B

Player A	I	II	III	IV	V
I	3	5	4	9	6
II	5	6	3	7	8
II	8	7	9	8	7

Now, every element of the 2nd column is less than the corresponding element of the 4th column. Also, every element of the 2nd column is less than or equal to the corresponding element of the 5th column. So, B will not select strategy IV and strategy V. Eliminating the 4th and the 5th columns, the pay-off matrix is reduced to

B

A	I	II	III
I	3	5	4
II	5	6	3
III	8	7	9

Now, every element of the first row, is less than the corresponding element of the 3rd row. Also, every element of the second row is less than the corresponding element of the 3rd row. So, A will not select

strategy I, and strategy II. Eliminating the first and the 3rd rows, the pay-off matrix is reduced to

$$\begin{array}{cc} & \begin{matrix} \text{I} & \text{II} & \text{III} \end{matrix} \\ \text{A} \quad \text{III} & \begin{bmatrix} 8 & 7 & 9 \end{bmatrix} \end{array}$$

(B)

Now the element of I is greater than the element of II. Also the element of III is greater than the element of II. Therefore, B will not choose strategy I and strategy III.

So, we are left with the 1 x 1 matrix.

$$\begin{array}{cc} & \begin{matrix} \text{II} \end{matrix} \\ \text{A} \quad \text{I} & \begin{bmatrix} 7 \end{bmatrix} \end{array}$$

(B)

Clearly (3, 2) th element of the original pay-off matrix is the saddle point.

The solution of the game is given by

(a) A will select strategy III,

(b) B will select strategy II,

(c) the value of the game is 7.



15.14 MODEL QUESTIONS

Q 1: What is a game? Write down the characteristics of a game.

Q 2: What is a saddle point in connection with a game? The pay-off matrix of a game is given below. Show that it has two saddle points.

(B)

$$\begin{array}{cc} & \begin{matrix} \text{I} & \text{II} & \text{III} & \text{IV} \end{matrix} \\ \text{I} & \begin{bmatrix} 4 & 2 & 3 & 5 \end{bmatrix} \\ \text{II} & \begin{bmatrix} -2 & -1 & 4 & -3 \end{bmatrix} \\ \text{A} \quad \text{III} & \begin{bmatrix} 5 & 2 & 3 & 3 \end{bmatrix} \\ \text{IV} & \begin{bmatrix} 4 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Q 3: The pay-off matrix of a game is given below. What is the value of the game? Is the game fair?

$$\begin{array}{c} \\ A \end{array} \begin{array}{c} \\ I \\ II \end{array} \begin{array}{c} \begin{array}{ccc} I & II & III \end{array} \\ \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix} \end{array}$$

Q 4: Explain the Dominance Property of a game. Using Dominance Property, solve the game whose pay-off matrix is given by

$$\begin{array}{c} \\ A \end{array} \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{cccc} I & II & III & IV \end{array} \\ \begin{bmatrix} -5 & 2 & 1 & 4 \\ 5 & 6 & 3 & 7 \\ 4 & 0 & 1 & -3 \end{bmatrix} \end{array}$$

Q 5: Fill up the blanks

(a) A game involving two players is called a ..... Game.

(b) In a Two-person Zero-Sum Game, the algebraic sum of the gains to both the players after the game is .....

(c) The maximising player adopts the ..... criterion.

(d) The minimising player adopts the ..... criterion.

(e) A game is said to be a fair game if the value of the game is .....

Q 6: Solve the game whose pay-off matrix is given below :

$$\begin{array}{c} \\ \text{Player A} \end{array} \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix} \end{array}$$

Player B

Q 7: Solve the game whose pay of matrix is given by

$$\begin{array}{c} \\ A \end{array} \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} -2 & 15 & -2 \\ -5 & -6 & -4 \\ -5 & 20 & -8 \end{bmatrix} \end{array}$$

B

Q 8: Using dominance rule reduce the following pay off matrix to a 2 x 2 matrix and hence find the solution to the game.

$$
\begin{array}{cc} & B \\ A & \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{bmatrix} B_1 & B_2 & B_3 \\ -1 & -2 & 8 \\ 7 & 5 & -1 \\ 6 & 0 & 12 \end{bmatrix} \end{array}
$$

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