UNIT 1 : INTEGRATION AND METHODS OF INTEGRATION

UNIT STRUCTURE

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1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- understand about integration of simple functions from the view point of inverse of differentiation
- know the properties of indefinite integrals
- understand the various methods of integrations viz
  - – method of substitution
  - – integration by parts
  - – integration by partial fractions.

1.2 INTRODUCTION

The development of integral calculus can be viewed from two different directions which are related to solving of problems of the following two types.

(i) to find the function when its derivative is known

(ii) to find the area bounded by the curve (or graph) of a function under certain given conditions.

The process related to the first category leads to the development of "indefinite integration" and the other leads to "definite integration".

The word "integration" literally means "summation". It is, in fact, the process of finding the "limit of a sum" of a certain number of elements, as the number of elements tends to infinity and each of the elements becomes infinitesimally small.

In this unit, we shall confine ourselves to the study of indefinite integral including some process of integration.

1.3 INTEGRATION OF SIMPLE FUNCTIONS

We have already learnt how to find the derivative *of* a functional of defined in an interval. If $y = f(x)$ is a function then its derivative is denoted by $f'(x)$ or $\frac{dy}{dx}$. Now, here in this unit, we will learn how to find f whenever f' is given. Integration is the inverse process of differentiation, i.e., to find a function when the differential coefficient is given.

Let us consider the following examples.

We know that (i) $\frac{d}{dx}\left(\frac{x^4}{4}\right) = x^3$

and (ii) $\frac{d}{dx}(\sin x) = \cos x$

Here we say that, in case of (i) $\frac{x^4}{4}$ is an antiderivative (or an integral or primitive) of x^3. Similarly, in (ii) $\sin x$ is an antiderivative of $\cos x$.

Let us take two other examples :

$$\text{(iii)}\ \frac{d}{dx}\left(\frac{x^4}{4} + C\right) = x^3$$

$$\text{and (iv)}\ \frac{d}{dx}(\sin x + C) = \cos x$$

Here C is any real number

It is evident from (iii), that $\frac{x^4}{4}+C$ is also an antiderivative of x^3 and in (iv) $\sin x + C$ is also an antiderivative of $\cos x$. Therefore, we have understood that the antiderivative of a function is not unique.

Definition : The set of functions that could possibly have the given function as a derivative are called "antiderivatives" (or integrals or primitives) of the function.

More generally, if there is a function $\phi(x)$ defined in an interval I such that $\frac{d}{dx}[\phi(x)]=f(x),\ x\in I$, then for any real number C $\frac{d}{dx}[\phi(x)+C]=f(x),\ x\in I$. Thus $\phi(x)+C$ denotes the family of anti-derivatives of $f(x)$. We introduce the symbol $\int f(x)dx$ which will represent the whole class of anti-derivatives, i.e. $\int f(x)dx=\phi(x)+C$.

The symbol $\int f(x)dx$ is read as the indefinite indegral of $f(x)$ with respect to *x*. The process of finding the anti-derivatives is known as "integration". The sign '$\int$' is called the integral sign. The symbol '$\int$' is an elongated S which is taken from the first letter of 'sum'.

Some Basic Results

Based upon the above definition and also on some standard derivatives, we obtain the following integration formulae.

	Derivatives	**Integrals (or anti derivatives)**
1.	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$	$\int x^n dx=\frac{x^{n+1}}{n+1}+C,\ (n\neq -1)$
2.	$\frac{d}{dx}(\sin x)=\cos x$	$\int \cos x dx=\sin x+C$
3.	$\frac{d}{dx}(\cos x)=-\sin x$	$\int \sin x dx=-\cos x+C$
4.	$\frac{d}{dx}(\tan x)=\sec^2 x$	$\int \sec^2 x dx=\tan x+C$
5.	$\frac{d}{dx}(\sec x)=\sec x\tan x$	$\int \sec x\tan x dx=\sec x+C$
6.	$\frac{d}{dx}(\cot x)=-\text{cosec}^2 x$	$\int \text{cosec}^2 x dx=-\cot x+C$

7. $\frac{d}{dx}(\text{cosec}x) = -\text{cosec}x \cot x$ $\quad \int \text{cosec}x \cot x dx = -\text{cosec}x + C$

8. $\frac{d}{dx}(e^x) = e^x$ $\quad \int e^x dx = e^x + C$

9. $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x$ $\quad \int a^x dx = \frac{a^x}{\log_e a} + C$

10. $\frac{d}{dx}(\log x) = \frac{1}{x}$ $\quad \int \frac{1}{x} dx = \log|x| + C$

11. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\quad \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

$= -\cos^{-1} x + C$

12. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

13. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\quad \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

$= -\cot^{-1} x + C$

14. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ $\quad \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

$= -\text{cosec}^{-1} x + C$

16. $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

Example : Evaluate (i) $\int x^4 dx$ (ii) $\int \frac{1}{\sqrt{x}} dx$ (iii) $\int 3^x dx$

Solution : (i) $\int x^4 dx = \frac{x^{4+1}}{4+1} + C = \frac{x^5}{5} + C$ (using formula 1)

(ii) $\int \frac{1}{\sqrt{x}} dx = \int x^{-½} dx = \frac{x^{-½+1}}{-½+1} + C = 2\sqrt{x} + C$

(iii) $\int 3^x dx = \frac{3^x}{\log_e 3} + C$ (using formula 9)

1.4 PROPERTIES OF INDEFINITE INTEGRAL

P–1 Differentiation and integration are inverse processes of each other in the sense of the results given below–

$\frac{d}{dx}\int f(x)dx = f(x)$ and $\int f'(x)dx = f(x) + C$, C is a constant.

Proof : Let $\phi(x)$ be any antiderivative of $f(x)$ i.e. $\frac{d}{dx}[\phi(x)] = f(x)$

Then $\int f(x)dx = \phi(x) + C$

$\therefore \frac{d}{dx}\int f(x)dx = \frac{d}{dx}[\phi(x) + C] = \frac{d}{dx}[\phi(x)] + \frac{d}{dx}(C) = f(x)$

(first part is proved)

Similarly, we know that $f'(x) = \frac{d}{dx} f(x)$

Hence $\int f'(x)dx = f(x) + C$, C is any arbitrary constant.

P-2 Two identical integrals with the same derivative lead to the same family of curves and hence both are equivalent.

Proof : Let $\frac{d}{dx}\int f(x)dx = \frac{d}{dx}\int g(x)dx$

Then $\frac{d}{dx}\int f(x)dx - \frac{d}{dx}\int g(x)dx = 0$

or, $\frac{d}{dx}[\int f(x)dx - \int g(x)dx] = 0$

or, $\int f(x)dx - \int g(x)dx = C$

or, $\int f(x)dx = \int g(x)dx + C$

It is customarily expressed as $\int f(x)dx = \int g(x)dx$ without mentioning the constant of integration.

P-3 $\int[f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

Proof : By property P-1, We have

$\frac{d}{dx}\int [f(x) + g(x)]dx = f(x) + g(x)$ ... (a)

Also we know that

$\frac{d}{dx}[\int f(x)dx + \int g(x)dx] = \frac{d}{dx}\int f(x)dx + \frac{d}{dx}\int g(x)dx$

$= f(x) + g(x)$ ... ...(b)

In view of property P-2, from (a) and (b) we can write

$$\int[f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

Corollary

(A) $\int[f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$

(B) We can generalize property P-3 as below -

$$\int [f(x)+g(x)+h(x)+....+\phi(x)]dx$$
$$= \int f(x)dx + \int g(x)dx + \int h(x)dx + ... + \int \phi(x)dx$$

P-4 For any real number K,

$$\int K f(x)dx = K\int f(x)dx$$

Proof : We know that $\frac{d}{dx}\int K f(x)dx = K f(x)$, by property P-1

Also $\frac{d}{dx}\left[K\int f(x)dx\right] = K\frac{d}{dx}\int f(x)dx = K f(x)$

So by property P-2

$$\int K f(x)dx = K\int f(x)dx$$

Illustrative Examples

Example 1 : Evaluate (i) $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$

(ii) $\int\left(x^{-3/2} - 4e^x + \cos x\right)dx$ (iii) $\int\left(2x^2 + 4\sec^2 x - \sqrt[3]{x}\right)dx$

Solutions : (i) $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$

$$= \int\left(x + 2 + \frac{1}{x}\right)dx$$

$$= \int x dx + \int 2dx + \int \frac{1}{x}dx \qquad \text{(by property P-3)}$$

$$= \frac{x^{1+1}}{1+1} + C_1 + 2\int dx + \log x + C_2$$

$$= \frac{x^2}{2} + C_1 + 2x + C_3 + \log x + C_2$$

$$= \frac{x^2}{2} + 2x + \log x + C \qquad (\text{Here } C_1 + C_2 + C_3 = C)$$

(ii) $\int\left(x^{-3/2} - 4e^x + \cos x\right)dx$

$$= \int x^{-3/2} - dx - \int 4e^x dx + \int \cos x\, dx$$

$$= \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + C_1 - 4\int e^x + \sin x + C_2$$

$$= \frac{x^{-1/2}}{\frac{-1}{2}} + C_1 - 4e^x + C_3 + \sin x + C_2$$

$$= -2x^{-1/2} - 4e^x + \sin x + C$$

(iii) $\int \left(2x^2 + 4\sec^2 x - \sqrt[3]{x}\right)dx$

$$= \int 2x^2 dx + \int 4\sec^2 x\, dx - \int \sqrt[3]{x}\, dx$$

$$= 2\int x^2 dx + 4\int \sec^2 x\, dx - \int x^{1/3}\, dx$$

$$= \frac{2}{3}x^3 + C_1 + 4\tan x + C_2 - \frac{x^{4/3}}{4/3} + C_3$$

$$= \frac{2}{3}x^3 + 4\tan x - \frac{3}{4}x^{4/3} + C$$

Example 2 : Evaluate (i) $\int \frac{x^3 - x^2 + x - 1}{x - 1}\, dx$

(ii) $\int \frac{3x^4 + 7x + 5}{2\sqrt{x}}\, dx$ (iii) $\int \frac{1 + \sin x}{\cos^2 x}\, dx$

Solution : (i) $\int \frac{x^3 - x^2 + x - 1}{x - 1}\, dx$

$$= \int \frac{(x-1)(x^2+1)}{(x-1)}dx$$

$$= \int \left(x^2 + 1\right)dx$$

$$= \frac{x^3}{3} + x + C$$

(ii) $\int \frac{3x^4 + 7x + 5}{2\sqrt{x}}\, dx$

$$= \int \left(\frac{3}{2}x^{7/2}\, dx + \frac{7}{2}x^{1/2} + \frac{5}{2}x^{-1/2}\right)dx$$

$$= \frac{3}{2}.\frac{2}{9}x^{9/2} + \frac{7}{2}.\frac{2}{3}x^{3/2} + \frac{5}{2}.\frac{2}{1}x^{1/2} + C$$

$$= \frac{1}{3}x^{9/2} + \frac{7}{3}x^{3/2} + 5\sqrt{x} + C$$

(iii) $\int \frac{1 + \sin x}{\cos^2 x}\, dx$

$$= \int \frac{1}{\cos^2 x}\, dx + \int \frac{\sin x}{\cos^2 x}\, dx$$

$$= \int \sec^2 dx + \int \tan x \sec x\, dx$$

$$= \tan x + \sec x + C$$

Example 3 : Write an antiderivative of $\cos 2x$.

Solution : We know that $\frac{d}{dx}(\sin 2x) = 2\cos 2x$

or $\cos 2x = \frac{1}{2}\frac{d}{dx}(\sin 2x) = \frac{d}{dx}\left(\frac{1}{2}\sin 2x\right)$

$\therefore$ antiderivative of $\cos 2x$ is $\frac{1}{2}\sin 2x + C$.



CHECK YOUR PROGRESS - 1

Q.1 : Find an antiderivative of each of the followings–

(i) $\frac{1}{x}, (x \neq 0)$ (ii) $\sin 3x$ (iii) $(ax - b)^2$

Q.2 : Find the following integrals

(i) $\int x^3 \left(x + \frac{1}{x}\right)^2 dx$ (ii) $\int (\cos x + \sin x)^2 dx$

(iii) $\int \left(\frac{2\cos x}{5\sin^2 x} + \frac{1}{3\cos^2 x}\right) dx$ (iv) $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$

(v) $\int \left(e^{x \log a} + e^{a \log x} + e^{a \log a}\right) dx$ (vi) $\int \sqrt{1 + \cos 2x}\, dx$

(vii) $\int \frac{(1+x)^2}{\sqrt{x}}\, dx$ (viii) $\int \left(\frac{x}{m} + \frac{m}{x} + x^m + m^x\right) dx$

Q.3 : Evaluate (i) $\int \left(x^4 + x^2 + 1\right) d\left(x^2\right)$ (ii) $\int \frac{x^4 + x^2 + 1}{x^2 - x + 1}\, dx$

(iii) $\int \frac{x^{-1/3} + \sqrt{x} + 2}{\sqrt[3]{x}}\, dx$

Q.4 : If the derivative of $f(x)$ w.r.t. x is $\frac{1}{x} + \frac{1}{1+x^2}$ and $f(1) = \pi/4$, then find $f(x)$.

Q.5 : The gradient at a point (x,y) of a curve is $3x^2 - 12$ and the curve passes through the point (2,–7). Find the equation of the curve.

1.5 METHODS OF INTEGRATION

In our earlier discussion, we have considered integrals of those functions which are readily obtainable from derivatives of some standard functions. This method is based on inspection, i.e. it is dependent on the search of a function F whose derivative is *f* which lead us to the integrals of *f.* But integrals of certain functions cannot be obtained directly if they are not in one of the standard forms. Hence we need to develop some other techniques or methods for finding the integrals by reducing the given functions to standard forms. Some of the methods are– (i) Integration by substitution, (ii) Integration by parts ,(iii) Integration by partial fractions.

1.5.1 Integration by Substitution

The method of evaluating an integral by reducing it to standard form by a proper substitution is called "integration by substitution".

Suppose $\phi(x)$ is a differentiable function. Then to evaluate integrals of the form $\int f(\phi(x))\phi'(x)dx$, we substitute $\phi(x) = t$ which then gives $\phi'(x)dx = dt$. This substitution reduces the above integral to the form $\int f(t)dt$.

If now $\int f(t)dt$ is the standard form then we can find the integrals. Finally we replace t by x with the help of the above substitution.

It is important to guess what will be the useful substitution. Usually we make a substitution for a function whose derivative also occurs in the integrand.

A few examples are given below to illustrate the method.

Illustrative Examples

Example 1 : Evaluate $\int (ax + b)^n dx$

Solution : Let $ax + b = t$, then a $\frac{dx}{dt} = 1$

or $dx = \frac{1}{a}dt$

$$\therefore \int (ax+b)^n dx = \int t^n \frac{1}{a} dt = \frac{1}{a}\int t^n dt = \frac{1}{a}\frac{1}{n+1}t^{n+1} + C$$

$$= \frac{1}{a(n+1)}(ax+b)^{n+1} + C$$

Example 2 : Evaluate $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$

Solution : Let $\sqrt{x} = t$, then $\frac{1}{2\sqrt{x}}dx = dt$, or $\frac{1}{\sqrt{x}}dx = 2dt$

Now $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx = \int 2\sin t\, dt = 2(-\cos t) + C = -2\cos\sqrt{x} + C$

Example 3 : Evaluate $\int \tan^4\theta \sec^2\theta d\theta$

Solution : Let $\tan\theta = t$, then $\sec^2\theta d\theta = dt$

Now $\int \tan^4\theta \sec^2\theta d\theta = \int t^4 dt = \frac{1}{5}t^5 + C = \frac{1}{5}\tan^5\theta + C$

Example 4 : Evaluate $\int \frac{10x^9 + 10^x \log_e^{10}}{10^x + x^{10}}\, dx$

Solution : Let $10^x + x^{10} = t$, then $(10^x \log_e 10 + 10x^9)dx = dt$

Now $\int \frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}} dx = \int \frac{1}{t} dt = \log t + C = \log(10^x + x^{10}) + C$

A Few Special Results

1. $\int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + C$

Proof : Let $f(x) = z$, then $f'(x)dx = dz$

Therefore, $\int \frac{f'(x)}{f(x)} dx = \int \frac{dz}{z} = \log|z| + C = \log|f(x)| + C$

This result can be applied in case of the followings

(a) $\int \tan x\, x = \log|\sec x| + C$ (b) $\int \cot x\, dx = \log|\sin x| + C$

(c) $\int \sec x\, dx = \log|\sec x + \tan x| + C$

(d) $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$

Proof (a) : $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{-\sin x}{\cos x}\, dx$

$$= -\log|\cos x| + C = \log|\sec x| + C$$

Proof (c) : $\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$= \log|\sec x + \tan x| + C$$

N.B. : (b) and (d) are left for the learners



CHECK YOUR PROGRESS - 2

Q. Find the integrals

1. $\int \frac{x}{9-4x^2}\, dx$
2. $\int (4x+2)\sqrt{x^2+x+1}\, dx$
3. $\int e^{\tan^{-1} x} \frac{1}{1+x^2} dx$
4. $\int \frac{dx}{x\sqrt{x^2-a^2}}$
5. $\int x\sqrt{x^2+5}\, dx$
6. $\int \frac{dx}{x^2\sqrt{1-x^2}}$
7. $\int \frac{\cos x\, dx}{(a+b\sin x)^2}$
8. $\int \frac{e^{2x}-1}{e^{2x}+1}\, dx$
9. $\int \frac{\sin(\tan^{-1} x)}{1+x^2} dx$
10. $\int \frac{(x+1)(x+\log x)^2}{x}\, dx$
11. $\int \sqrt{\frac{1+x}{1-x}}\, dx$
12. $\int \frac{3x-1}{\sqrt{3x^2-2x+7}}\, dx$
13. $\int \frac{e^x(1+x)}{\cos^2(xe^x)}\, dx$
14. $\int \frac{1}{x^2(x^4+1)^{3/4}}\, dx$
15. $\int \frac{x^2+5x+2}{x+2}\, dx$
16. $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\, dx$
17. $\int \cos^4 x\, dx$
18. $\int \sin^3 x\, dx$
19. $\int \cos^2 x\, dx$
20. $\int \frac{2x+5}{x^2+5x-7}\, dx$
21. $\int \frac{1}{\sin(x-a)\sin(x-b)}\, dx$
22. $\int \frac{dx}{\sin(x-a)\cos(x-b)}$
23. $\int \sin x \sin 2x \sin 3x\, dx$
24. $\int \frac{x+2}{(x+1)^2}\, dx$
25. $\int \frac{3x+1}{(3x^2+2x+1)^3}\, dx$

SOME STANDARD INTEGRALS

(A) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C, \ (a \neq 0)$

(B) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C, \ (|x| > |a|)$

(C) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C, \ (|x| < |a|)$

(D) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$

(E) $\int \frac{dx}{x^2-a^2} = \log\left|x+\sqrt{x^2-a^2}\right| + C$

(F) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

Proof (A) : Let us put $x = a\tan\theta$, then $dx = a\sec^2\theta d\theta$

Now $\int \frac{dx}{x^2+a^2} = \int \frac{a\sec^2\theta d\theta}{a^2\tan^2\theta + a^2} = \int \frac{a\sec^2\theta}{a^2\sec^2\theta} d\theta = \frac{1}{a}\int d\theta$

$= \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Proof (B) : $\int \frac{dx}{x^2-a^2} = \int \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$, by partial fraction

$= \frac{1}{2a}\left[\int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx\right]$

$= \frac{1}{2a}\left[\log|x-a| - \log|x+a|\right] + C$

$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

Proof (C) : First apply partial fraction and then proceed as in (B).

Proof (D) : Let $x = a\tan\theta$, then $dx = a\sec^2\theta d\theta$

Now $\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta d\theta}{\sqrt{a^2\tan^2\theta + a^2}} = \int \sec\theta d\theta$

$= \log|\sec\theta + \tan\theta| + C_1 = \log\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1$

$= \log\left|\sqrt{x^2+a^2} + x\right| - \log|a| + C_1$

$= \log\left|x+\sqrt{x^2+a^2}\right| + C$

Proof (E) : Here put $x = a\sec\theta$ and then proceed.

Proof (F) : Here put $x = a\sin\theta$ and then proceed.

SOME OTHER TYPES OF INTEGRALS

Here we will discuss the application of the above standard integrals for finding the integrals of the following types

(A) $\int \frac{dx}{ax^2+bx+c}$ (B) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ (C) $\int \frac{px+q}{ax^2+bx+c}$

(D) $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$ (E) $\int \frac{dx}{(ax+b)\sqrt{cx+d}}$

(F) $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$

(Here a, b, c, d, p, q, r are constants)

Solution : (A) We can wirte ax^2+bx+c

$$= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$= a\left\{\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right\}$$

Now we put $x+\frac{b}{2a}=t$ and then $dx=dt$

$$\therefore \int \frac{dx}{ax^2+bx+c}=\frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\left(\frac{4ac-b^2}{4a^2}\right)}=\frac{1}{a}\int \frac{dt}{t^2\pm \mathrm{K}^2}$$

where $\mathrm{K}^2=\frac{4ac-b^2}{4a^2}$

Now this can be evaluated by using standard formula.

(B) Proceeding as in (A) we can write

$\int \frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\int \frac{dt}{\sqrt{t^2\pm \mathrm{K}^2}}$ which can be evaluated by using standard formula.

(C) In this case we are to find real numbers A and B such that

$$px+q=\mathrm{A}\frac{d}{dx}\left(ax^2+bx+c\right)+\mathrm{B}=\mathrm{A}\left(2ax+b\right)+\mathrm{B}\,.$$

Now for finding A and B, we equate from both sides the coefficients of x and the constant terms. Then we can write

$$\int \frac{px+q}{ax^2+bx+c}\,dx = A\int \frac{f'(x)}{f(x)}dx + B\int \frac{1}{f(x)}dx \text{ where}$$
$$f(x) = ax^2+bx+c\,.$$

Now the right hand side can be integrated by suitable methods discussed earlier.

(D) In this case we can write

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx \text{ as } A\int \frac{f'(x)}{\sqrt{f(x)}}dx + B\int \frac{1}{\sqrt{f(x)}}\,dx$$

by the procedure discussed in (C).

(E) Let $cx+d = t^2$, then $c\,dx = 2t\,dt$

$$\text{Now } \int \frac{dx}{(ax+b)\sqrt{(cx+d)}} = \frac{2}{c}\int \frac{t\,dt}{\left(a\frac{t^2-d}{c}+b\right)t}$$
$$= 2\int \frac{dt}{at^2+(bc-ad)}$$

It can be now evaluated with the help of standard integrals.

(F) Here we take $px+q = \frac{1}{t}$, then $p\,dx = -\frac{1}{t^2}dt$. Then the given integral reduces to the form $-\int \frac{dt}{\sqrt{At^2+Bt+C}}$ which can be evaluated by the method discussed in (B).

ILLUSTRATIVE EXAMPLES

Example 1 : $\int \frac{\cos x+\sin x}{\cos x-\sin x}\,dx$

Solution : $\int \frac{\cos x+\sin x}{\cos x-\sin x}\,dx = \int \frac{1+\tan x}{1-\tan x}dx$, (Dividing both numerator and denominator by $\cos x$)

$$= \int \tan\left(\frac{\pi}{4}+x\right)dx = \log\left|\sec\left(\frac{\pi}{4}+x\right)\right| + C$$

Example 2 : $\int \frac{3x^2}{x^6+1}\,dx$

Solution : $\int \frac{3x^2}{x^6+1}\,dx = \int \frac{3x^2}{(x^3)^2+1}dx$

We take
$x^3 = t$
$\therefore\ 3x^2dx = dt$

$$= \int \frac{dt}{t^2+1}$$

$$= \tan^{-1}(t) + C = \tan^{-1}\left(x^3\right) + c$$

Examples 3 : $\int \frac{1}{2x^2+8x+20}\,dx$

Solution : $\int \frac{1}{2x^2+8x+20}\,dx$

$$= \frac{1}{2}\int \frac{1}{x^2+4x+10}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\left(x^2+4x+4\right)+6}\,dx$$

$$= \frac{1}{2}\int \frac{dx}{(x+2)^2+\left(\sqrt{6}\right)^2}$$

$$= \frac{1}{2}\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{x+2}{\sqrt{6}}\right) + C$$

Example 4 : $\int \frac{dx}{\sqrt{2x^2+3x+4}}$

Solution : $\int \frac{dx}{\sqrt{2x^2+3x+4}} = \int \frac{dx}{\sqrt{2}\sqrt{x^2+\frac{3}{2}x+2}}$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+2.\frac{3}{4}.x+\frac{9}{16}+\frac{23}{16}}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x+2}\right| + C$$

Example 5 : $\int \frac{4x+3}{3x^2+3x+1}\,dx$

Solution : We write

$$4x+3 = A\frac{d}{dx}\left(3x^2+3x+1\right)+B$$

$$\Rightarrow 4x+3 = A(6x+3)+B$$

$$\Rightarrow 4x+3 = 6Ax+3A+B$$

Equating the coefficients of x and the constant terms from both sides

$6A = 4$ and $3A + B = 3$

$\therefore A = 2/3$ and $B = 3 - 3A = 3 - 3.\frac{2}{3} = 1$

$\therefore$ The given integral is

$$\int \frac{\frac{2}{3}(6x+3)+1}{3x^2+3x+1}\,dx$$

$$= \frac{2}{3}\int \frac{6x+3}{3x^2+3x+1}\,dx + \int \frac{1}{3x^2+3x+1}\,dx$$

$$= \frac{2}{3}\log\left|3x^2+3x+1\right| + \frac{1}{3}\int \frac{1}{x^2+x+1/3}\,dx$$

$$= \frac{2}{3}\log\left|3x^2+3x+1\right| + \frac{1}{3}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{12}}\right)^2}\,dx$$

$$= \frac{2}{3}\log|3x^2+3x+1| + \frac{\sqrt{12}}{3}\tan^{-1}\left(\frac{x+1/2}{\frac{1}{\sqrt{12}}}\right) + C$$

$$= \frac{2}{3}\log|3x^2+3x+1| + \frac{2}{\sqrt{3}}\tan^{-1}\left[\sqrt{3}(2x+1)\right] + C$$

Example 6 : $\int \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$

Solution : Let us take $1+x = \frac{1}{t}$, then $dx = -\frac{1}{t^2}dt$

$$\therefore \int \frac{dx}{(1+x)\sqrt{1+2x-x^2}} = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{1+2\left(\frac{1}{t}-1\right)-\left(\frac{1}{t}-1\right)^2}}$$

$$= \int \frac{-\frac{1}{t}dt}{\sqrt{-2+\frac{4}{t}-\frac{1}{t^2}}} = -\int \frac{dt}{\sqrt{-2t^2+4t-1}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-t^2+2t-\frac{1}{2}}} = -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2-(t-1)^2}}$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{t-1}{\frac{1}{\sqrt{2}}}\right) = -\frac{1}{\sqrt{2}}\sin^{-1}\left[\sqrt{2}\left(\frac{1}{1+x}-1\right)\right]$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{-x\sqrt{2}}{x+1}\right) = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x\sqrt{2}}{x+1}\right)$$



CHECK YOUR PROGRESS - 3

Q. Integrate the followings :

1. $\int \frac{1}{4x^2+9}\,dx$
2. $\int \frac{e^x dx}{e^{2x}+6e^x+5}$
3. $\int \frac{dx}{x\left(x^5+1\right)}$
4. $\int \frac{dx}{\sqrt{1-x-x^2}}$
5. $\int \frac{2x+3}{\sqrt{x^2+x+1}}\,dx$
6. $\int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx$
7. $\int \frac{dx}{(x+2)\sqrt{x+1}}$
8. $\int \frac{dx}{(1+x)\sqrt{1-x^2}}$

1.5.2 INTEGRATION BY PARTS

An important integration formula can be derived from the derivative of the product of two functions.

We know from differential calculus that for two differentiable functions $f(x)$ and $q(x)$,

$$\frac{d}{dx}\{f(x).q(x)\} = f(x).\frac{d}{dx}\{q(x)\} + q(x).\frac{d}{dx}\{f(x)\}$$

Integrating both sides w.r.t. x we get

$$f(x).q(x) = \int \left[f(x).\frac{d}{dx}\{q(x)\} + q(x).\frac{d}{dx}\{f(x)\} \right] dx$$

$$= \int f(x)\frac{d}{dx}\{q(x)\}dx + \int q(x).\frac{d}{dx}\{f(x)\}dx$$

or $$\int f(x)\frac{d}{dx}\{q(x)\}dx = f(x).q(x) - \int q(x).\frac{d}{dx}\{f(x)\}dx \qquad ...(1)$$

Now let us take $f(x) = u$ and $\frac{d}{dx}\{q(x)\} = v$.

Then $q(x) = \int v dx$

If we substitute there in (1) then we get

$$\int uvdx = u\int vdx - \int \left\{ \frac{d}{dx}(u).\int vdx \right\} dx$$

The above formula can be stated as below—

"The integral of the product of two functions = 1st function × integral of the 2nd function – integral of [differential coefficient of 1st function × integral of the 2nd function]"

Note : While applying the rule of integration by parts for the product of two functions, it is very important to choose the first and the second function very carefully. As an aid to the learners, we can choose the first function as the function that comes first in the word "ILATE". Here the letters stand for as given below—

I – Inverse trigonometric function

L – Logarithmic function

A – Algebraic function

T – Trigonometic function

E – Exponential function.

ILLUSTRATIVE EXAMPLES

Example 1 : Evaluate $\int x^2 e^x dx$

Solution : $\int x^2 e^x dx = x^2 \int e^x dx - \int \left\{ \frac{d}{dx}(x^2) \int e^x dx \right\} dx$

$= x^2 e^x - \int 2x e^x dx$ | Here we have taken $x^2 =$ 1st function, $e^x =$ 2nd function

$= x^2 e^x - 2\left[x \int e^x dx - \int \left\{ \frac{d}{dx}(x) . \int e^x dx \right\} dx \right]$

$= x^2 e^x - 2\left\{ x e^x - \int 1 . e^x dx \right\} = x^2 e^x - 2\left(x e^2 - e^x \right) + C$

$= e^x \left(x^2 - 2x + 2 \right) + C$

Example 2 : Evaluate $\int x \sin 3x \, dx$

Solution : $\int x \sin 3x \, dx = x \int \sin 3x \, dx - \int \left\{ \frac{d}{dx}(x) . \int \sin 3x \, dx \right\} dx$

$= x\left(-\frac{\cos 3x}{3} \right) - \int 1 . \left(-\frac{\cos 3x}{3} \right) dx$

$= -\frac{x}{3} \cos 3x + \frac{1}{3} \int \cos 3x \, dx$

$= -\frac{1}{3} x \cos 3x + \frac{1}{3} \frac{\sin 3x}{3} + C$

$= -\frac{1}{3} x \cos 3x + \frac{1}{9} \sin 3x + C$

Example 3 : Evaluate $\int x \tan^{-1} x \, dx$

Solution : $\int x \tan^{-1} x \, dx$ [Here 1st function $= \tan^{-1} x$, 2nd function $= x$]

$= \tan^{-1} x \int x \, dx - \int \left\{ \frac{d}{dx} \left(\tan^{-1} x \right) . \int x \, dx \right\} dx$

$= \frac{x^2}{2} \tan^{-1} x - \int \frac{1}{1+x^2} \frac{x^2}{2} dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$

$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \left(x - \tan^{-1} x \right) + C$

$= \frac{1}{2} \left\{ \left(x^2 + 1 \right) \tan^{-1} x - x \right\} + C$

Example 4 : $\int \log x \, dx$

Solution : $\int \log x\,dx = \int \log x.1\,dx$

Now take 1st function = log x

2nd function = 1 and proceed.

Similarly in case of $\int \sin^{-1} x\,dx$ we take $\sin^{-1} x$ as first function and 1 as second function.

STANDARD INTEGRALS

A. $\int e^{ax} \sin bx\,dx = \frac{1}{a^2+b^2} e^{ax}(a\sin bx - b\cos bx)$

B. $\int e^{ax} \cos bx\,dx = \frac{1}{a^2+b^2} e^{ax}(a\cos bx + b\sin bx)$

Proof A : Let $I = \int e^{ax} \sin bx\,dx$

Then $I = e^{ax}\int \sin bx\,dx - \int \left\{\frac{d}{dx}(e^{ax})\int \sin bx\,dx\right\} dx$

$$= -\frac{1}{b} e^{ax} \cos bx - \int ae^{ax}\left(-\frac{1}{b}\cos bx\right) dx$$

$$= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b}\int e^{ax} \cos bx\,dx$$

$$= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b}\left[e^{ax}\int \cos bx\,dx - \int \left\{\frac{d}{dx}(e^{ax})\int \cos bx\,dx\right\} dx\right]$$

$$= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b}\left[e^{ax}\frac{\sin bx}{b} - \int ae^{ax}\frac{\sin bx}{b}\,dx\right]$$

$$= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b}\left[\frac{1}{b} e^{ax} \sin bx - \frac{a}{b}\int e^{ax} \sin bx\,dx\right]$$

$$= -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} I$$

$$\therefore I + \frac{a^2}{b^2} I = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b^2} e^{ax} \sin bx$$

$$\Rightarrow \frac{a^2+b^2}{b^2} I = \frac{1}{b^2}(ae^{ax} \sin bx - be^{ax} \cos bx)$$

$$\therefore I = \frac{1}{a^2+b^2} e^{ax}(a\sin bx - b\cos bx)$$

Proof B : It is left for the learners to verify the result.

SOME STANDARD FORMULAE

1. $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|$

2. $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|$

3. $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} - \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

Proof 1 : Let $I = \int 1.\sqrt{x^2+a^2}\,dx$

$$\Rightarrow I = \sqrt{x^2+a^2}\int 1.dx - \int\left\{\frac{d}{dx}\left(\sqrt{x^2+a^2}\right).\int 1dx\right\}dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{2x}{2\sqrt{x^2+a^2}}.x\,dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}dx = x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx$$

$$= x\sqrt{x^2+a^2} - \int\sqrt{x^2+a^2}\,dx + a^2\int \frac{1}{\sqrt{x^2+a^2}}dx$$

$$= x\sqrt{x^2+a^2} - I + a^2\log\left|x+\sqrt{x^2+a^2}\right|$$

$$\Rightarrow 2I = x\sqrt{x^2+a^2} + a^2\log\left|x+\sqrt{x^2+a^2}\right|$$

$$\therefore I = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|$$

Note : (1) Proofs of (2) and (3) are left for the learners

(2) Though we have not given the constant of integration in each of the above results, the learners should not forget to put it in each cases.

SOME OTHER SPECIAL FORMS

(a) $\int \sqrt{ax^2+bx+c}\,dx$ where $a \neq 0$

In this case, ax^2+bx+c is to be expressed either in the form $a\left\{(x+\alpha)^2 \pm \beta^2\right\}$ or $b\left\{\beta^2-(x+\alpha)^2\right\}$.

Now if $x+\alpha$ is substituted by *t*, then the integral will reduce to a known form discussed earlier.

(b) $\int (px+q)\sqrt{ax^2+bx+c}\,dx, \quad a \neq 0$

Here we take $px+q=\mathrm{A}\frac{d}{dx}\left(ax^2+bx+c\right)+\mathrm{B}$

Now by simple calculation we can write

$$px+q=\frac{p}{2q}(2ax+b)+\left(q-\frac{bp}{2a}\right)$$

Then the given integral can be expressed in two parts. For the first part we take $ax^2+bx+c=t$ and the second part is similar to any one form discussed earlier.

(c) $\int e^x\left\{f(x)+f'(x)\right\}dx$

Here taking $\mathrm{I}=\int e^x f(x)dx$ and applying the formula for "integration by parts" we can prove that $\int e^x\left\{f(x)+f'(x)\right\}dx=e^x f(x)$.

ILLUSTRATIVE EXAMPLES

Example 1 : Evaluate $\int e^x \cos x\,dx$

Solution : Let $\mathrm{I}=\int e^x\cos x\,dx$

$$=e^x\int\cos x\,dx-\int\left\{\frac{d}{dx}\left(e^x\right).\int\cos x\,dx\right\}dx$$

$$=e^x\sin x-\int e^x\sin x\,dx$$

$$=e^x\sin x-\left[e^x\int\sin x\,dx-\int\left\{\frac{d}{dx}\left(e^x\right).\int\sin x\,dx\right\}dx\right]$$

$$=e^x\sin x+e^x\cos x-\int e^x\cos x\,dx$$

$$=e^x(\sin x+\cos x)-\mathrm{I}$$

$\Rightarrow 2\mathrm{I}=e^x(\sin x+\cos x)$

$\therefore\ \mathrm{I}=\frac{1}{2}e^x(\sin x+\cos x)+\mathrm{C}$

Example 2 : $\int\sqrt{4-3x-2x^2}\,dx$

Solution : $\int\sqrt{4-3x-2x^2}\,dx=\sqrt{2}\int\sqrt{2-\frac{3}{2}x-x^2}\,dx$

$$=\sqrt{2}\int\sqrt{2+\frac{9}{16}+\left\{-\left(\frac{3}{4}\right)^2-2.\frac{3}{4}.x-x^2\right\}}\,dx$$

$$= \sqrt{2}\int \sqrt{\frac{41}{16} - \left(x + \frac{3}{4}\right)^2}\, dx \qquad \left|\begin{array}{l}\text{Let} \quad x + \frac{3}{4} = t \\ \text{then} \quad dx = dt\end{array}\right.$$

$$= \sqrt{2}\left[\frac{t\sqrt{\frac{41}{16} - t^2}}{2} + \frac{41/16}{2}\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{4}}\right)\right] + C$$

$$= \sqrt{2}\left[\frac{1}{2}\left(x + \frac{3}{4}\right)\sqrt{\frac{41}{16} - \left(x + \frac{3}{4}\right)^2} + \frac{41}{32}\sin^{-1}\left(\frac{x + 3/4}{\sqrt{41/4}}\right)\right] + C$$

$$= \frac{\sqrt{2}}{8}(4x + 3)\frac{\sqrt{4 - 3x - 2x^2}}{\sqrt{2}} + \frac{41\sqrt{2}}{32}\sin^{-1}\left(\frac{4x + 3}{\sqrt{41}}\right) + C$$

$$= \frac{1}{8}(4x + 3)\sqrt{4 - 3x - 2x^2} + \frac{41\sqrt{2}}{32}\sin^{-1}\left(\frac{4x + 3}{\sqrt{41}}\right) + C$$

Example 3 : $\int (3x - 2)\sqrt{1 - x + x^2}\, dx$

Solution : We can write $3x - 2 = \frac{3}{2}(2x - 1) - \frac{1}{2}$

$$\therefore I = \frac{3}{2}\int (2x - 1)\sqrt{x^2 - x + 1}\, dx - \frac{1}{2}\int \sqrt{x^2 - x + 1}\, dx$$

For the first part of the R.H.S. we take $x^2 - x + 1 = t$

Then $\int (2x - 1)\sqrt{x^2 - x + 1}\, dx = \int \sqrt{t}\, dt = \frac{2}{3}t^{3/2} = \frac{2}{3}\left(x^2 - x + 1\right)^{3/2}$

For the 2nd part

$$\int \sqrt{x^2 - x + 1}\, dx = \int \sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}\, dx$$

$$= \int \sqrt{z^2 + \left(\sqrt{3}/2\right)^2}\, dt \quad (\text{taking } x - \frac{1}{2} = z)$$

$$= \frac{z\sqrt{z^2 + 3/4}}{2} + \frac{3}{4}.\frac{1}{2}\log\left|z + \sqrt{z^2 + \frac{3}{4}}\right|$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\right)\sqrt{x^2 - x + 1} + \frac{3}{8}\log\left|\left(x - \frac{1}{2}\right) + \sqrt{x^2 - x + 1}\right|$$

$$\therefore I = \left(x^2 - x + 1\right)^{3/2} - \frac{1}{8}(2x-1)\sqrt{x^2 - x + 1} - \frac{3}{16}\log\left|\left(x - \frac{1}{2}\right) + \sqrt{x^2 - x + 1}\right| + C$$

Example 4: Evaluate $\int e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$

Solution : $\int e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$

$$= \int e^x \frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2 x/2}\, dx = \int e^x \left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right) dx - \int e^x \cot\frac{x}{2}\, dx$$

$$= -\int e^x \cot\frac{x}{2}\, dx + \frac{1}{2}\int e^x \operatorname{cosec}^2\frac{x}{2}\, dx$$

Here $f(x) = -\cot\frac{x}{2}$

$$\therefore f'(x) = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$$

$$= -e^x \cot\frac{x}{2} + c \qquad \left[\int e^x \left[f(x) + f'(x)\right] dx = e^x f(x)\right]$$



CHECK YOUR PROGRESS - 4

Q. Evaluate the followings :

1. $\int x^2 \cos x\, dx$
2. $\int x^3 \log 2x\, dx$
3. $\int x^3 \tan^{-1} x\, dx$
4. $\int x^2 e^{-x} dx$
5. $\int \sin^{-1} x\, dx$
6. $\int \cos\sqrt{x}\, dx$
7. $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$
8. $\int e^x \sin x\, dx$
9. $\int e^x \sin x \sin 2x\, dx$
10. $\int e^x (\cos x + \sin x) dx$

11. $\int e^x \sec x(1+\tan x)dx$ 12. $\int \sin(\log x)dx$

13. $\int (3x-2)\sqrt{x^2+x+1}\,dx$ 14. $\int \sqrt{7x-10-x^2}\,dx$

1.5.3 INTEGRATION BY PARTIAL FRACTIONS

From our earlier discussion we know the method of finding the integrals of the form $\int \frac{1}{ax^2+bx+c}\,dx$, while *a, b, c* are constants. But when we have to integrate a rational fraction, say of the form $\frac{f(x)}{\phi(x)}$, where the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator, the best way of effecting the integration is first to decompose the given fraction into a number of partial fractions whose sum gives the original fraction and then to integrate each fraction separately by suitable method. If in case, the degree of the polynomial in the numberator i.e. *f*(*x*), be not of a lower degree than $\phi(x)$, then we first divide $f(x)$ by $\phi(x)$ by method of ordinary division of a polynomial by another polynomial and express the given fraction to a form like $a_p x^p + a_{p-1}x^{p-1} + \ldots\ldots\ldots + a_0 + \frac{g(x)}{\phi(x)}$ where $g(x)$ is the remainder and hence of a lower degree than $\phi(x)$. After that $\frac{g(x)}{\phi(x)}$ is resolved into partial fractions and integrate the terms separately.

Here we shall avoid a detailed discussion of the various methods of partial fractions, since these topics are covered in Higher Secondary Algebra.

In our present discussion, we will study integration of rational fractions $\frac{f(x)}{\phi(x)}$ where the denominator contains

(i) real, linear and non-repeated factors
(ii) real, linear but some repeated factors
(iii) real, quadratic and non-repeated factors
(iv) real, quadratic and repeated factors

Integration of rational fractions when the denominator contains factors which are real, linear but non-repeated

The method can be understood from the following examples.

Example 1 : Integrate $\int \frac{x^2 dx}{(x-1)(x-2)(x-3)}$

Solution : Let us take

$\frac{x^2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{(x-2)} + \frac{C}{x-3}$ where A, B, C are constants to be determined.

Then $x^2 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

If $x = 1$,then $1 = A(-1)(-2) \Rightarrow A = \frac{1}{2}$

$x = 2$,then $4 = B(1)(-1) \Rightarrow B = -4$

and $x = 3$ then, $9 = C(2)(1) \Rightarrow C = \frac{9}{2}$

$$\therefore \frac{x^2}{(x-1)(x-2)(x-3)} = \frac{\frac{1}{2}}{x-1} + \frac{-4}{x-2} + \frac{\frac{9}{2}}{x-3}$$

$$\therefore \int \frac{x^2}{(x-1)(x-2)(x-3)}\, dx = \int \left\{ \frac{1}{2}\left(\frac{1}{x-1}\right) - 4\left(\frac{1}{x-2}\right) + \frac{9}{2}\left(\frac{1}{x-3}\right) \right\} dx$$

$$= \frac{1}{2}\log|x-1| - 4\log|x-2| + \frac{9}{2}\log|x-3| + c$$

Example 2: Integrate $\int \frac{2x+3}{x^3 + x^2 - 2x} dx$

Solution : Here $x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x+2)(x-1)$

Now $\frac{2x+3}{x^3 + x^2 - 2x} = \frac{2x+3}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x-1)}$ (say)

$$\therefore 2x + 3 = A(x+2)(x-1) + Bx(x-1) + Cx(x+2)$$

Let $x = 0$, then $3 = A(2)(-1) \Rightarrow A = -\frac{3}{2}$

$x = -2$, then $-1 = B(-2)(-3) \Rightarrow B = -\frac{1}{6}$

and $x = 1$, then $5 = C(1)(3) \Rightarrow C = \frac{5}{3}$

$$\therefore \int \frac{2x+3}{x^3+x^2-2x}\,dx = \int\left(-\frac{3}{2}\frac{1}{x}-\frac{1}{6}\frac{1}{x+2}+\frac{5}{3}\frac{1}{x-1}\right)dx$$

$$= -\frac{3}{2}\log|x| - \frac{1}{6}\log|x+2| + \frac{5}{3}\log|x-1| + C$$

When the denominator contains factors which are real, linear but some are repeated

Example 3: Evaluate $\int \frac{x^2 dx}{(x+1)(x+2)^2}$

Solution : Let $\frac{x^2}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$\therefore x^2 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$

$\Rightarrow 1.x^2 + 0.x + 0 = (A+B)x^2 + (4A+3B+C)x + 4A + 2B + C$

Equating the coefficients of x^2, x and of the constant terms from both sides

$A + B = 1$, $4A + 3B + C = 0$, $4A + 2B + C = 0$

Now solving for A, B and C we get

$A = 1, B = 0, C = -4$

$$\therefore \int \frac{x^2}{(x+1)(x+2)^2}\,dx = \int\left(\frac{1}{x+1} + \frac{-4}{(x+2)^2}\right)dx$$

$$= \log|x+1| + \frac{4}{x+2} + c$$

Example 4 : Evaluate $\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$

Solution : Let $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$ ... ...(i)

Then, $x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$ ... ...(ii)

If $x = 1$, then we get $B = \frac{1}{2}$

If $x = -3$, then $C = \frac{5}{8}$

Now equating the coefficients of x^2 from both sides of the identity (ii) we get

$1 = A + C \Rightarrow A = 1 - \frac{5}{8} = \frac{3}{8}$

Then from (i), $\dfrac{x^2+1}{(x-1)^2(x+3)} = \dfrac{3/8}{x-1} + \dfrac{1/2}{(x-2)^2} + \dfrac{5/8}{x+3}$

$$\therefore \int \frac{x^2+1}{(x-1)^2(x+3)}\,dx = \frac{3}{8}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{(x-1)^2}\,dx + \frac{5}{8}\int \frac{1}{x+3}\,dx$$

$$= \frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\log|x+3| + C$$

When the denominator contains factors which are real, quadratic, but non-repeated

Example 5: $\int \dfrac{x}{(x-1)(x^2+4)}\,dx$

Solution : Let $\dfrac{x}{(x-1)(x^2+4)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+4}$

$$\Rightarrow x = A(x^2+4) + (Bx+C)(x-1)$$

If $x = 1$, then $1 = 5A \Rightarrow A = 1/5$

If $x = 0$, then $0 = 4A - C \Rightarrow C = 4/5$

If $x = -1$, then $-1 = 5A + 2B - 2C$ which gives $B = -\dfrac{1}{5}$

$$\therefore \int \frac{x}{(x-1)(x^2+4)}\,dx = \int \left(\frac{1/5}{x-1} + \frac{-\frac{1}{5}x + \frac{4}{5}}{x^2+4} \right) dx$$

$$= \frac{1}{5}\int \frac{1}{x-1}\,dx - \frac{1}{5}\int \frac{x-4}{x^2+4}\,dx$$

$$= \frac{1}{5}\log|x-1| - \frac{1}{10}\int \frac{2x}{x^2+4}\,dx + \frac{4}{5}\int \frac{1}{x^2+4}\,dx$$

$$= \frac{1}{5}\log|x-1| - \frac{1}{10}\log|x^2+4| + \frac{4}{5}\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \frac{1}{5}\log|x-1| - \frac{1}{10}\log|x^2+4| + \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + C$$

When the denominatior contains factors which are real, quadratic but some are repeated.

Example 6: Evaluate $\int \frac{x^3 + x^2 + 2x + 1}{(x^2 + 1)^2}\, dx$

Solution : Let $\frac{x^3 + x^2 + 2x + 1}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{(Cx + D)}{(x^2 + 1)^2}$

Then $x^3 + x^2 + 2x + 1 = (Ax + B)(x^2 + 1) + (Cx + D)$

$\Rightarrow x^3 + x^2 + 2x + 1 = Ax^3 + Bx^2 + (A + C)x + (B + D)$

Equating the coefficients of various powers of x A=1, B=1, C=1, D=0.

$$\therefore \int \frac{x^3 + x^2 + 2x + 1}{(x^2 + 1)^2} dx = \int \left\{ \frac{x + 1}{x^2 + 1} + \frac{x}{(x^2 + 1)^2} \right\} dx$$

$$= \int \frac{x}{x^2 + 1} + \int \frac{1}{x^2 + 1} + \int \frac{x}{(x^2 + 1)^2} dx$$

$$= \frac{1}{2} \log |x^2 + 1| + \tan^{-1} x - \frac{1}{2} \frac{1}{x^2 + 1} + C$$

Example 7 : Integrate $\int \frac{x^2 + 1}{(x^2 + 2)(2x^2 + 1)}\, dx$

Solution : Let $x^2 = y$, then

$$\frac{x^2 + 1}{(x^2 + 2)(2x^2 + 1)} = \frac{y + 1}{(y + 2)(2y + 1)} = \frac{A}{y + 2} + \frac{B}{2y + 1} \text{ ... ...(i) (say)}$$

$$\therefore y + 1 = A(2y + 1) + B(y + 2)$$

For $y = -2$, $-1 = -3A$ i.e. $A = 1/3$

For $y = -1/2$, $\frac{1}{2} = B\left(\frac{3}{2}\right)$ i.e. $B = 1/3$

Then by (i) $\frac{y + 1}{(y + 2)(2y + 1)} = \frac{1/3}{y + 2} + \frac{1/3}{2y + 1}$

Then replacing y by x^2 and integerting w.r.t. x

$$\int \frac{x^2 + 1}{(x^2 + 2)(2x^2 + 1)} dx = \frac{1}{3} \left[\int \frac{1}{x^2 + 2} dx + \int \frac{1}{2x^2 + 1} dx \right]$$

$$= \frac{1}{3}\left[\int \frac{1}{x^2 + \left(\sqrt{2}\right)^2} dx + \int \frac{1}{\left(\sqrt{2}x^2\right) + (1)^2} dx\right]$$

$$= \frac{1}{3}\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{2}x}{1}\right)\right]$$

$$= \frac{1}{3\sqrt{2}}\left\{\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \tan^{-1}\left(\sqrt{2}x\right)\right\} + C$$



CHECK YOUR PROGRESS - 5

Q. Evaluate the following

1. $\int \frac{x-1}{(x-2)(x-3)} dx$
2. $\int \frac{x\,dx}{x^2 - 12x + 35}$
3. $\int \frac{3x}{x^2 - x - 2} dx$
4. $\int \frac{1}{x(x+1)^2} dx$
5. $\int \frac{x^3 - 2x + 3}{x^2 + x - 2} dx$
6. $\int \frac{x^2}{(x+1)^2(x+2)} dx$
7. $\int \frac{(x-1)(x-5)}{(x-2)(x-4)} dx$
8. $\int \frac{x}{(x-2)^2} dx$
9. $\int \frac{1}{x^3 + x} dx$
10. $\int \frac{2x^3}{(x^2+1)^2} dx$
11. $\int \frac{x^2 + 3x - 4}{x^2 - 2x - 8} dx$
12. $\int \frac{x+5}{x^3(x+1)} dx$
13. $\int \frac{1}{1 - x^3} dx$
14. $\int \frac{x^2}{(x-1)^3(x+1)} dx$
15. $\int \frac{1}{(x^2-1)^2} dx$
16. $\int \frac{x dx}{(x^2 + a^2)(x^2 + b^2)}$
17. $\int \frac{dx}{x^3 - x^2 - x + 1}$
18. $\int \frac{x^2}{x^4 + x^2 - 2} dx$
19. $\int \frac{1 - 3x^2}{3x - x^3} dx$
20. $\int \frac{x^2 dx}{(x^2+1)(x^2+4)}$
21. $\int \frac{\sin 2x\, dx}{(1 + \sin x)(2 + \sin x)}$
22. $\int \frac{\cos x\, dx}{(2 + \sin x)(3 + 4\sin x)}$
23. $\int \frac{dx}{x(x^4 + 1)}$
24. $\int \frac{\sin 2x\, dx}{(1 + \sin x)(2 + \sin x)}$
25. $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx$



1.6 LET US SUM UP

- Differentiation and integration are inverse processes of one another
- We may also define integration as a process of finding the limit of a sum [The method is discussed in the next unit].
- The set of functions that could possibly have in given functon as a derivative are called anti derivatives (or integrats or primitives) of a function.
- Based on the results of some standard derivatives, we obtain the formula of some standard simple Integration.
- There are different methods for determination of integrals.Some of the methods are (i) Integration by substitution,(ii) Integration by parts,(iii)Integration by partial fractions.
- A few important forms of integrals

  (i) $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$

  (ii) $\int \{f(x)\}^n f'(x)dx = \frac{1}{n+1}[f(x)]^{n+1} + C, \quad n \neq -1$

  (iii) $\int f'(ax+b)dx = \frac{1}{a}f(ax+b) + C$

- An important formula for product of two functions is

  $$\int uv\,dx = u\int v dx - \int \left\{\frac{d}{dx}(u).\int v dx\right\}dx$$

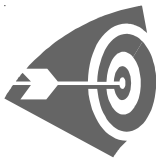
- When we have to integrate a rational fraction say $f(x)/\phi(x)$ where the degree of $f(x)$ is lower than that of $\phi(x)$, then we apply a suitable method to reduce $f(x)/\phi(x)$ into partial fractions and finally integrate each of the terms.
- When we have to integrate a rational fraction, say $f(x)/\phi(x)$, where the degree of $f(x)$ is not lower than that of $\phi(x)$, then we divide $f(x)$

by $\phi(x)$ by the method of ordinary division. The form $f(x)/\phi(x)$ then reduces to $a_p x^p + a_{p-1}x^{p-1} + \ldots\ldots\ldots + a_0 + \dfrac{g(x)}{\phi(x)}$ where the degree of $\phi(x)$ is greater than that of $g(x)$.



1.7 FURTHER READINGS

1) *Integral Calculus,* Kishan,Hari,Atlantic Publishers & Distributors.New Delhi -110027.
2) *Integral Calculus*, Dhami,H.S. New age International Publishers,New Delhi - 110002.
3) *Text book of Integral Calculus*,Sharma,A.K. Discovery Publishing House,New Delhi -110002.



1.8 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1 : (i) We know that $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}, x > 0$

and $\dfrac{d}{dx}(\log(-x)) = \dfrac{1}{-x}(-1), x < 0$

Combining above ,we get $\dfrac{d}{dx}(\log|x|) = \dfrac{1}{x}, x \neq 0$

Therefore, $\int \dfrac{1}{x}dx = \log|x|$ is anti-derivative of $\dfrac{1}{x}$.

(ii) We know that $\dfrac{d}{dx}(\cos 3x) = -3\sin 3x$

$\Rightarrow \sin 3x = -\dfrac{1}{3}\dfrac{d}{dx}(\cos 3x)$

$$\therefore \sin 3x = \frac{d}{dx}\left(-\frac{1}{3}\cos 3x\right)$$

Therefore, anti -derivative of $\sin 3x$ is $-\frac{1}{3}\cos 3x$.

(iii) We know that $\frac{d}{dx}(ax-b)^3 = 3a(ax-b)^2$

$$\Rightarrow (ax-b)^2 = \frac{1}{3a}\frac{d}{dx}(ax-b)^3$$

$$\therefore (ax-b)^2 = \frac{d}{dx}\left\{\frac{1}{3a}(ax-b)^3\right\}$$

Therefore, anti -derivative of $(ax-b)^2$ is $\frac{1}{3a}(ax-b)^3$.

Ans to Q No 2:

(i) $\int x^3\left(x^2+\frac{1}{x^2}+2x.\frac{1}{x}\right)dx = \int\left(x^5+x+2x^3\right)dx$

$$= \frac{x^6}{6}+\frac{x^2}{2}+2\frac{x^4}{4}+C$$

$$= \frac{x^6}{6}+\frac{x^2}{2}+\frac{x^4}{2}+C.$$

(ii) $\int(\cos x+\sin x)^2 dx = \int\left(\cos^2 x+\sin^2 x+2\sin x\cos x\right)dx$

$$= \int(1+\sin 2x)dx$$

$$= x-\frac{\cos 2x}{2}+C.$$

(iii) $\int\left(\frac{2\cos x}{5\sin^2 x}+\frac{1}{3\cos^2 x}\right)dx = \int\left(\frac{2}{5}\cot x\cos ecx+\frac{1}{3}\sec^2 x\right)dx$

$$= \frac{2}{5}\int\cot x\cos ecxdx+\frac{1}{3}\int\sec^2 xdx$$

$$= \frac{2}{5}(-\cos ecx)+\frac{1}{3}\tan x+C.$$

(iv) $\int\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx = \int\left(\frac{\sin^2 x}{\sin^2 x\cos^2 x}-\frac{\cos^2 x}{\sin^2 x\cos^2 x}\right)dx$

$$= \int\left(\sec^2 x-\cos ec^2 x\right)dx$$

$$= \tan x+\cot x+C$$

(v) $\int \left(e^{x\log a} + e^{a\log x} + e^{a\log a}\right)dx = \int \left(e^{\log a^x} + e^{\log x^a} + e^{\log a^a}\right)dx$

$= \int \left(a^x + x^a + a^a\right)dx$

$= \int a^x dx + \int x^a dx + \int a^a dx$

$= \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + a^a . x + C$.

(vi) $\int \sqrt{1+\cos 2x}\, dx = \int \sqrt{2\cos^2 x} dx$, $\cos 2x = 2\cos^2 x - 1$

$= \sqrt{2}\int \cos x dx$

$= \sqrt{2}\sin x + C$.

(vii) $\int \frac{(1+x)^2}{\sqrt{x}} dx = \int \frac{1+2x+x^2}{\sqrt{x}} dx$

$= \int \left(\frac{1}{\sqrt{x}} + 2\sqrt{x} + x^{\frac{3}{2}}\right)dx$

$= \int x^{-\frac{1}{2}} dx + 2\int x^{\frac{1}{2}} dx + \int x^{\frac{3}{2}} dx$

$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + 2.\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + C$

$= 2x^{\frac{1}{2}} + \frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + C$

(viii) $\int \left(\frac{x}{m} + \frac{m}{x} + x^m + m^x\right)dx = \frac{1}{m}\int x dx + m\int \frac{1}{x} dx + \int x^m dx + \int m^x dx$

$= \frac{1}{m}.\frac{x^2}{2} + m\log|x| + \frac{x^{m+1}}{m+1} + \frac{x^m}{\log m} + C$

Ans to Q No 3 : (i) Let $x^2 = t$ .Then $\int \left(x^4 + x^2 + 1\right)d\left(x^2\right) = \int \left(t^2 + t + 1\right)dt$

$= \frac{t^3}{3} + \frac{t^2}{2} + t + C$

$= \frac{x^6}{6} + \frac{x^4}{2} + x^2 + C$

(ii) $\int \frac{x^4+x^2+1}{x^2-x+1}dx = \int \frac{\left(x^2+1\right)^2-x^2}{x^2-x+1}dx$

$$= \int \frac{\left(x^2+1+x\right)\left(x^2+1-x\right)}{x^2-x+1}dx$$

$$= \int \left(x^2+x+1\right)dx = \frac{x^3}{3}+\frac{x^2}{2}+x+C.$$

(iii) $\int \frac{x^{-1/3}+\sqrt{x}+2}{\sqrt[3]{x}}\,dx = \int \frac{x^{-\frac{1}{3}}+x^{\frac{1}{2}}+2}{x^{\frac{1}{3}}}dx$

$$= \int \left(\frac{x^{-\frac{1}{3}}}{x^{\frac{1}{3}}}+\frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}}+\frac{2}{x^{\frac{1}{3}}}\right)dx$$

$$= \int \left(x^{-\frac{2}{3}}+x^{\frac{1}{2}-\frac{1}{3}}+2x^{-\frac{1}{3}}\right)dx$$

$$= \int \left(x^{-\frac{2}{3}}+x^{\frac{1}{6}}+2x^{-\frac{1}{3}}\right)dx$$

$$= \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1}+\frac{x^{\frac{1}{6}+1}}{\frac{1}{6}+1}+2\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}+C$$

$$= \frac{x^{\frac{1}{3}}}{\frac{1}{3}}+\frac{x^{\frac{7}{6}}}{\frac{7}{6}}+2\frac{x^{\frac{2}{3}}}{\frac{2}{3}}+C$$

$$= 3x^{\frac{1}{3}}+\frac{6}{7}x^{\frac{7}{6}}+3x^{\frac{2}{3}}+C.$$

Ans to Q No 4: We have $f(x) = \int f'(x)dx = \int \left(\frac{1}{x}+\frac{1}{1+x^2}\right)dx$

$$= \int \frac{1}{x}dx + \int \frac{1}{x^2+1}dx = \log|x| + \tan^{-1}x + C$$

When $x = 1$, we have $f(x) = \frac{\pi}{4}$

$$\therefore \frac{\pi}{4} = \log|1| + \tan^{-1} 1 + C$$

$$\Rightarrow \frac{\pi}{4} = \frac{\pi}{4} + C$$

$$\Rightarrow C = 0$$

$$\therefore f(x) = \log|x| + \tan^{-1} x .$$

Ans to Q No 5: Given, $\frac{dy}{dx} = 3x^2 - 12$

$$\therefore f'(x) = 3x^2 - 12$$

$$\therefore f(x) = \int f'(x)dx = \int (3x^2 - 12)dx$$

$$= 3.\frac{x^3}{3} - 12x + C$$

$$f(x) = x^3 - 12x + C \quad \ldots (1)$$

(1) passes through the point (2,-7).

$$\therefore -7 = 2^3 - 12.2 + C$$

$$\Rightarrow C = 9$$

Required equation of the curve is $f(x) = x^3 - 12x + 9$.

CHECK YOUR PROGRESS – 2

Ans to Q No 1: Let $9 - 4x^2 = t$

$$\therefore -8xdx = dt$$

$$\int \frac{x}{9 - 4x^2} dx = -\frac{1}{8}\int \frac{1}{t} dt$$

$$= -\frac{1}{8}\log|t| + C$$

$$= -\frac{1}{8}\log|9 - 4x^2| + C$$

Ans to Q No 2: Let $x^2 + x + 1 = t \quad \therefore (2x+1)dx = dt$

$$\int (4x+2)\sqrt{x^2 + x + 1}dx$$

$$= \int 2\sqrt{t}dt = 2\int \sqrt{t}dt = 2\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{4}{3}\left(x^2 + x + 1\right)^{\frac{3}{2}} + C$$

Ans to Q No 3: Let $\tan^{-1} x = t$

$$\therefore \frac{1}{1+x^2}dx = dt$$

$$\int \frac{e^{\tan^{-1}x}dx}{1+x^2} = \int e^t dt$$

$$= e^t + C$$

$$= e^{\tan^{-1}x} + C$$

Ans to Q No 4: We put $x = a\sec\theta \Rightarrow \frac{x}{a} = \sec\theta \Rightarrow \theta = \sec^{-1}\frac{x}{a}$

$$dx = a\sec\theta\tan\theta d\theta$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta\sqrt{a^2\sec^2\theta - a^2}}$$

$$= \int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta\sqrt{a^2\tan^2\theta}}$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + C$$

$$= \frac{1}{a}\sec^{-1}\frac{x}{a} + C$$

Ans to Q No 5 : Let $x^2 + 5 = t$

$$\Rightarrow 2xdx = dt$$

$$\Rightarrow xdx = \frac{1}{2}dt$$

$$\int x\sqrt{x^2+5}\,dx = \frac{1}{2}\int t^{\frac{1}{2}}dt = \frac{1}{2}\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{1}{2}\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{1}{3}t^{\frac{3}{2}} + C$$

$$= \frac{1}{3}\left(x^2+5\right)^{\frac{3}{2}} + C$$

Ans to Q No 6 : Let $x = \sin\theta \Rightarrow dx = \cos\theta d\theta$

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = \int \frac{\cos\theta d\theta}{\sin^2\theta\sqrt{1-\sin^2\theta}}$$

$$= \int \frac{\cos\theta d\theta}{\sin^2\theta\cos\theta}$$

$$= \int \cos ec^2\theta d\theta$$

$$= -\cot\theta + C$$

$$= -\cot\left(\sin^{-1}x\right) + C$$

Ans to Q No 7: Let $a + b\sin x = t \Rightarrow b\cos x dx = dt$

$$\cos x dx = \frac{1}{b}dt$$

$$\int \frac{\cos x\, dx}{\left(a+b\sin x\right)^2} = \int \frac{\frac{1}{b}dt}{t^2} = \frac{1}{b}\int \frac{dt}{t^2}$$

$$= \frac{1}{b}.\frac{t^{-2+1}}{-2+1} + C$$

$$= -\frac{1}{b}.\frac{1}{t} + C$$

$$= -\frac{1}{b}.\frac{1}{a+b\sin x} + C$$

Ans to Q No 8: Dividing numerator and denominator by e^x, we have

$$\frac{\frac{e^{2x}-1}{e^x}}{\frac{e^{2x}+1}{e^x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Let $e^x + e^{-x} = t$

$$\therefore \left(e^x - e^{-x}\right)dx = dt$$

$$\int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$$

$$= \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log\left|e^{x} + e^{-x}\right| + C$$

Ans to Q No 9: Let $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2}dx = dt$

$$\int \frac{\sin\left(\tan^{-1} x\right)}{1+x^2}dx = \int \sin t\,dt = -\cot t + C$$

$$= -\cot\left(\tan^{-1} x\right) + C$$

Ans to Q No 10 :

$$\frac{(x+1)(x+\log x)^2}{x} = \left(\frac{x+1}{x}\right)(x+\log x)^2 = \left(1+\frac{1}{x}\right)(x+\log x)^2$$

Let $x + \log x = t \qquad \therefore \left(1+\frac{1}{x}\right)dx = dt$

$$\int \left(1+\frac{1}{x}\right)(x+\log x)^2 dx = \int t^2 dt$$

$$= \frac{t^3}{3} + C$$

$$= \frac{1}{3}(x+\log x)^3 + C$$

Ans to Q No 11 : Let $I = \int \sqrt{\frac{1+x}{1-x}}dx = \int \frac{1+x}{\sqrt{1-x^2}}dx$

$$= \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x}{\sqrt{1-x^2}}dx$$

$$= \sin^{-1} x + I_1 + C$$

In I_1,we substitute $1 - x^2 = t^2 \Rightarrow -2xdx = 2tdt \Rightarrow xdx = -tdt$

$$I_1 = -\int \frac{tdt}{\sqrt{t^2}} = -\int dt = -t = -\sqrt{1-x^2}$$

$\therefore I = \sin^{-1} x - \sqrt{1-x^2} + C$.

Ans to Q No 12 : Let $3x^2 - 2x + 7 = t \Rightarrow (6x-2)dx = dt$

$$\Rightarrow 2(3x-1)dx = dt$$

$$\Rightarrow (3x-1)dx = \frac{1}{2}dt$$

$$\int \frac{3x-1}{\sqrt{3x^2-2x+7}}\,dx = \int \frac{\frac{1}{2}}{\sqrt{t}}dt = \frac{1}{2}\int t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \sqrt{t} + C$$

$$= \sqrt{3x^2-2x+7} + C$$

Ans to Q No 13 : Let $e^x x = t \Rightarrow (e^x x + e^x.1)dx = dt$

$$\Rightarrow e^x(x+1)dx = dt$$

$$\therefore \int \frac{e^x(1+x)}{\cos^2(e^x x)}dx = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\,dt$$

$$= \tan t + C$$

$$= \tan(e^x x) + C$$

Ans to Q No 14 : Multiplying and dividing by x^{-3},we have

$$\frac{x^{-3}}{x^2.x^{-3}(x^4+1)^{\frac{3}{4}}} = \frac{x^{-3}(x^4+1)^{-\frac{3}{4}}}{x^2.x^{-3}} = \frac{(x^4+1)^{-\frac{3}{4}}}{x^5(.x^4)^{-\frac{3}{4}}}$$

$$= \frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$$

$$= \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$$

Let $\frac{1}{x^4} = t \Rightarrow -\frac{4}{x^5}dx = dt \Rightarrow \frac{1}{x^5}dx = -\frac{dt}{4}$

$$\therefore \int \frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx = \int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}dx$$

$$= -\frac{1}{4}\int (1+t)^{-\frac{3}{4}}dt$$

$$= -\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right]+C$$

$$= -\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}}+C$$

$$= -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$$

Ans to Q No 15: $\int \frac{x^2+5x+2}{x+2}\,dx = \int \frac{(x+3)(x+2)}{x+2}dx = \int (x+3)dx$

$$= \frac{x^2}{2}+3x+C$$

Ans to Q No 16: $\frac{1}{\sqrt{x+a}+\sqrt{x+b}} = \frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\left(\sqrt{x+a}+\sqrt{x+b}\right)\left(\sqrt{x+a}-\sqrt{x+b}\right)}$

$$= \frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\left(\sqrt{x+a}\right)^2-\left(\sqrt{x+b}\right)^2}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)} = \frac{1}{a-b}\left(\sqrt{x+a}-\sqrt{x+b}\right)$$

$$\therefore \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx = \frac{1}{a-b}\left\{\int \sqrt{x+a}dx - \int \sqrt{x+b}dx\right\}$$

$$= \frac{1}{a-b}\left\{\int \sqrt{u}du - \int \sqrt{v}dv\right\} \text{ where } x+a=u \text{ and } x+b=v$$

$$\therefore dx = du \text{ and } dx = dv$$

$$= \frac{1}{a-b}\left\{\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{v^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right\} + C$$

$$= \frac{1}{a-b}\left\{\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{3}v^{\frac{3}{2}}\right\} + C$$

$$= \frac{2}{3(a-b)}\left\{(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right\} + C$$

Ans to Q No 17 : $\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2$

$$= \frac{1}{4}\left[1 + \cos^2 2x + 2\cos 2x\right]$$

$$= \frac{1}{4}\left[1 + \left(\frac{1+\cos 4x}{2}\right) + 2\cos 2x\right]$$

$$= \frac{1}{4}\left[1 + \frac{1}{2} + \frac{\cos 4x}{2} + 2\cos 2x\right]$$

$$= \frac{1}{4}\left[\frac{3}{2} + \frac{\cos 4x}{2} + 2\cos 2x\right]$$

$$\therefore \int \cos^4 x dx = \int\left(\frac{3}{8} + \frac{\cos 4x}{8} + \frac{\cos 2x}{2}\right)dx$$

$$= \frac{3}{8}x + \frac{\sin 4x}{32} + \frac{\sin 2x}{4} + C$$

Ans to Q No 18 : From the identity $\sin 3x = 3\sin x - 4\sin^3 x$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$\therefore \int \sin^3 x dx = \frac{3}{4}\int \sin x dx - \frac{1}{4}\int \sin 3x dx$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$$

Ans to Q No 19 : From the identity, $\cos 2x = 2\cos^2 x - 1$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\int \cos^2 x dx = \frac{1}{2}\int (1+\cos 2x)dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x dx$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

Ans to Q No 20 : Let $x^2 + 5x - 7 = t \Rightarrow (2x+5)dx = dt$

$$\int \frac{2x+5}{x^2+5x-7}\, dx = \int \frac{dt}{t} = \log t + C$$

$$= \log\left(x^2 + 5x - 7\right) + C$$

Ans to Q No 21 :

We have $\int \frac{1}{\sin(x-a)\sin(x-b)}dx = \frac{1}{\sin(a-b)}\int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}dx$

$$= \frac{1}{\sin(a-b)}\int \{\cot(x-a) - \cot(x-b)\}dx$$

$$= \frac{1}{\sin(a-b)}\{\log|\sin(x-a)| - \log|\sin(x-b)|\} + C$$

$$= \cos ec(a-b)\left\{\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|\right\} + C .$$

Ans to Q No 22 :

$$\int \frac{1}{\sin(x-a)\cos(x-b)}dx = \frac{1}{\cos(a-b)}\int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos(x-a)\cos(x-b)+\sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\cos(a-b)}\int \{\cot(x-a) + \tan(x-b)\}dx$$

$$= \frac{1}{\cos(a-b)}\left[\log|\sin(x-a)| - \log|\cos(x-b)|\right] + C$$

$$= \frac{1}{\cos(a-b)}\log\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C .$$

Ans to Q No 23 : We have

$$\int \sin x \sin 2x \sin 3x dx = \frac{1}{2}\int (2\sin 2x \sin x)\sin 3x dx$$

$$= \frac{1}{2}\int (\cos x - \cos 3x)\sin 3x dx$$

$$= \frac{1}{4}\int (2\sin 3x \cos x - 2\sin 3x \cos 3x) dx$$

$$= \frac{1}{4}\int (\sin 4x + \sin 2x - \sin 6x) dx$$

$$= \frac{1}{4}\left[-\frac{\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6} \right] + C .$$

Ans to Q No 24 : $\int \frac{x+2}{(x+1)^2} dx = \int \frac{(x+1)+1}{(x+1)^2} dx$

$$= \int \frac{1}{x+1} dx + \int \frac{1}{(x+1)^2} dx$$

$$= \int \frac{1}{z} dz + \int \frac{1}{z^2} dz \text{ put } x+1 = z \ \therefore dx = dz$$

$$= \log|z| + \frac{z^{-2+1}}{-2+1} + C$$

$$= \log|z| - \frac{1}{z} + C$$

$$= \log|x+1| - \frac{1}{x+1} + C$$

Ans to Q No 25 : Let $3x^2 + 2x + 1 = t \Rightarrow (6x+2)dx = dt$

$$\Rightarrow (3x+1)dx = \frac{1}{2} dt$$

$$\int \frac{3x+1}{(3x^2+2x+1)^3} dx = \frac{1}{2}\int \frac{dt}{t^3} = \frac{1}{2}\int t^{-3} dt$$

$$= \frac{1}{2}\frac{t^{-3+1}}{-3+1} + C$$

$$= \frac{1}{2}.\frac{t^{-2}}{-2} + C$$

$$= -\frac{1}{4}.\frac{1}{t^2} + C$$

$$= -\frac{1}{4}.\frac{1}{\left(3x^2+2x+1\right)^2} + C \,.$$

CHECK YOUR PROGRESS – 3

Ans to Q No 1 : $I = \int \frac{dx}{4x^2+9} = \frac{1}{2}\int \frac{2dx}{(2x)^2+(3)^2} = \frac{1}{6}\tan^{-1}\frac{2x}{3} + C$

Ans to Q No 2 : We have $\int \frac{e^x}{e^{2x}+6e^x+5}dx = \int \frac{e^x}{\left(e^x\right)^2+6e^x+5}dx$

Let $e^x = t$,Then $e^x dx = dt$

$$\int \frac{1}{t^2+6t+5}dt = \int \frac{1}{(t+3)^2-2^2}dt$$

$$= \frac{1}{2\times 2}\log\left|\frac{t+3-2}{t+3+2}\right| + C$$

$$= \frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right| + C \,.$$

Ans to Q No 3 : We have $\int \frac{1}{x\left(x^5+1\right)}dx = \int \frac{x^4}{x^5\left(x^5+1\right)}dx$ ...(1)

Let $x^5+1 = t$

$$5x^4 dx = dt$$

From (1), $\int \frac{1}{x\left(x^5+1\right)}dx = \frac{1}{5}\int \frac{1}{t(t-1)}dt$

$$= \frac{1}{5}\int \frac{1}{t^2-t}dt$$

$$= \frac{1}{5}\int \frac{1}{t^2-t+\frac{1}{4}-\frac{1}{4}}dt$$

$$= \frac{1}{5}\int \frac{1}{\left(t-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}dt$$

$$= \frac{1}{5} \cdot \frac{1}{2\left(\frac{1}{2}\right)} \log \left| \frac{t - \frac{1}{2} - \frac{1}{2}}{t - \frac{1}{2} + \frac{1}{2}} \right| + C$$

$$= \frac{1}{5} \log \left| \frac{t-1}{t} \right| + C$$

$$= \frac{1}{5} \log \left| \frac{x^5}{x^5 + 1} \right| + C$$

Ans to Q No 4 : We have $\int \frac{1}{\sqrt{1 - x - x^2}} dx = \int \frac{dx}{\sqrt{\left\{1 - \left(x^2 + x\right)\right\}}}$

$$= \int \frac{dx}{\sqrt{\left\{1 - \left(x^2 + 2x\frac{1}{2} + \frac{1}{4} - \frac{1}{4} \right)\right\}}}$$

$$= \int \frac{dx}{\sqrt{\left\{1 - \left(x + \frac{1}{2} \right)^2 + \frac{1}{4}\right\}}}$$

$$= \int \frac{dx}{\sqrt{\left\{\frac{5}{4} - \left(x + \frac{1}{2} \right)^2 \right\}}}$$

$$= \sin^{-1} \left\{ \frac{x + \frac{1}{2}}{\frac{1}{2}\sqrt{5}} \right\} + C$$

$$= \sin^{-1} \left(\frac{2x+1}{\sqrt{5}} \right) + C .$$

Ans to Q No 5 : We have $\frac{d}{dx}\left(x^2 + x + 1\right) = 2x + 1$

Now $2x + 3 = \left(2x + 1\right) + 2$

$$\therefore \int \frac{2x+3}{\sqrt{x^2 + x + 1}} dx = \int \frac{\left(2x+1\right)}{\sqrt{x^2 + x + 1}} dx + \int \frac{2}{\sqrt{x^2 + x + 1}} dx$$

$$= 2\sqrt{x^2+x+1} + \int \frac{2}{\sqrt{\left\{\left(x+\frac{1}{2}\right)^2 + 1 - \frac{1}{4}\right\}}} dx$$

$$= 2\sqrt{x^2+x+1} + 2\int \frac{dx}{\sqrt{\left\{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right\}}}$$

$$= 2\sqrt{x^2+x+1} + 2\sinh^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= 2\sqrt{x^2+x+1} + 2\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

Ans to Q No 6 : Let $x+2 = A\frac{d}{dx}\left(x^2+5x+6\right) + B$

$\Rightarrow x+2 = A(2x+5) + B$

Comparing the coefficients of like powers of x, we get

$1 = 2A$ and $5A + B = 2$

$\Rightarrow A = \frac{1}{2}$ and $B = -\frac{1}{2}$

$$\therefore \int \frac{x+2}{\sqrt{x^2+5x+6}} dx = \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}} dx$$

$$= \frac{1}{2}\int \frac{(2x+5)}{\sqrt{x^2+5x+6}} dx - \frac{1}{2}\int \frac{1}{\sqrt{x^2+5x+6}} dx$$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{t}} - \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx \quad \text{where } t = x^2+5x+6$$

$$= \sqrt{t} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right| + C$$

$$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right| + C$$

Ans to Q No 7 : We put $x+1=t^2$, so that $dx = 2tdt$

$$\therefore \int \frac{dx}{(x+2)\sqrt{x+1}} = \int \frac{2\,tdt}{(t^2+1)t}$$

$$= 2\int \frac{dt}{t^2+1}$$

$$= 2\tan^{-1} t + C$$

$$= 2\tan^{-1}\left[\sqrt{x+1}\right] + C$$

Ans to Q No 8 : We put $1+x=\frac{1}{t}$, so that $dx = -\frac{1}{t^2}dt$. Also, $x = \frac{1}{t} - 1$

$$\therefore \int \frac{dx}{(1+x)\sqrt{1-x^2}} = \int \frac{-\left(\frac{1}{t^2}\right)dt}{\frac{1}{t}\sqrt{\left[1-\left\{\left(\frac{1}{t}\right)-1\right\}^2\right]}}$$

$$= -\int \frac{dt}{\sqrt{\left[t^2-(1-t)^2\right]}}$$

$$= -\int \frac{dt}{\sqrt{(2t-1)}}$$

$$= -\frac{1}{2}\int (2t-1)^{-\frac{1}{2}}.2dt$$

$$= -\sqrt{2t-1} + C$$

$$= -\sqrt{\frac{2}{1+x}-1} + C$$

$$= -\sqrt{\frac{1-x}{1+x}} + C$$

CHECK YOUR PROGRESS – 4

Ans to Q No 1 : $\int x^2\cos x dx = x^2\sin x - \int 2x\sin x dx$

$$= x^2 \sin x - 2\left[x(-\cos x) - \int(-\cos x)dx\right]$$

$$= x^2 \sin x + 2x\cos x - 2\sin x$$
$$= \left(x^2 - 2\right)\sin x + 2x\cos x$$

Ans to Q No 2 : $\int x^3 \log 2x dx = \log 2x \int x^3 dx - \int \left(\frac{d}{dx} \log 2x . \int x^3 dx \right) dx$

$$= \log 2x \frac{x^4}{4} - \int \frac{1}{2x} . 2 . \frac{x^4}{4} dx$$

$$= \frac{x^4 \log 2x}{4} - \frac{1}{4} \int x^3 dx$$

$$= \frac{x^4 \log 2x}{4} - \frac{1}{4} . \frac{x^4}{4} + C$$

$$= \frac{x^4 \log 2x}{4} - \frac{x^4}{16} + C .$$

Ans to Q No 3 : $\int x^3 \tan^{-1} x dx = \frac{x^4}{4} \tan^{-1} x - \int \frac{x^4}{4} . \frac{1}{1+x^2} dx$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \left[x^2 - 1 + \frac{1}{1+x^2} \right] dx$$

$$= \frac{1}{4} \left[x^4 \tan^{-1} x - \frac{x^3}{3} + x - \tan^{-1} x \right]$$

$$= \frac{1}{4} \left[\left(x^4 - 1\right) \tan^{-1} x - \frac{x^3}{3} + x \right]$$

Ans to Q No 4 : $\int x^2 e^{-x} dx = x^2 \int e^{-x} dx - \int \left(\frac{d}{dx} x^2 \int e^{-x} dx \right) dx$

$$= x^2 \left(-e^{-x}\right) - \int 2x \left(-e^{-x}\right) dx$$
$$= -x^2 e^{-x} + 2\int x e^{-x} dx$$
$$= -x^2 e^{-x} + 2\left[x\left(-e^{-x}\right) - \int 1 . \left(-e^{-x}\right) dx \right]$$
$$= -x^2 e^{-x} + 2\left(-xe^{-x} + e^{-x}\right)$$
$$= -x^2 e^{-x} - 2xe^{-x} + 2e^{-x}$$

Ans to Q No 5 : Let $\sin^{-1} x = t$,Then $x = \sin t \Rightarrow dx = \cos t dt$

$$\therefore \int \sin^{-1} x dx = \int t \cos t dt$$
$$= t \sin t - \int 1 . (\sin t) dt$$
$$= t \sin t + \cos t + C$$

$$= x\sin^{-1}x + \sqrt{1-\sin^2 t} + C$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

Ans to Q No 6 : We substitute $x = t^2$,We get $dx = 2tdt$

$$\therefore \int \cos\sqrt{x}dx = 2\int t\cos t dt$$

$$= 2\left[t\sin t - \int \sin t dt\right]$$

$$= 2\left[t\sin t + \cos t\right] + C$$

$$= 2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + C$$

Ans to Q No 7 : We put $x = \tan\theta$,so that $dx = \sec^2\theta d\theta$

$$\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx = \int \cos^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta}\sec^2\theta d\theta$$

$$= \int \cos^{-1}(\cos 2\theta)\sec^2\theta d\theta$$

$$= 2\int \theta\sec^2\theta d\theta$$

$$= 2\left[\theta\tan\theta - \int 1.\tan\theta d\theta\right]$$

$$= 2\left[\theta\tan\theta - \log\sec\theta\right] + C$$

$$= 2\left[x\tan^{-1}x - \log\sqrt{1+x^2}\right] + C.$$

Ans to Q No 8 : Let $I = \int e^x \sin x dx = e^x(-\cos x) + \int e^x \cos x dx$

$$= -e^x\cos x + e^x\sin x - \int e^x \sin x dx$$

$$\Rightarrow I = -e^x\cos x + e^x\sin x - I$$

$$\Rightarrow 2I = e^x(\sin x - \cos x)$$

$$\therefore I = \frac{e^x}{2}(\sin x - \cos x)$$

Ans to Q No 9 : Let $I = \int e^x \sin x \sin 2x dx$

$$= \frac{1}{2}\int e^x(\cos x - \cos 2x)dx$$

$$= \frac{1}{2}\int e^x \cos x dx - \int e^x \cos 2x dx$$

$$= \frac{1}{2}e^x\frac{(\cos x + \sin x)}{1^2+1^2} - \frac{1}{2}e^x\frac{(\cos 3x + 3\sin 3x)}{1^2+3^2} + C$$

$$= \frac{1}{4}e^x(\cos x + \sin x) - \frac{1}{20}e^x(\cos 3x + 3\sin 3x) + C$$

Ans to Q No 10 : We have $\int e^x(\sin x+\cos x)dx = \int e^x \sin x dx + \int e^x \cos x dx$

$$= \sin x e^x - \int \cos x e^x dx + \int \cos x e^x dx$$

$$= \sin x e^x + C$$

Ans to Q No 11 : Let $I = \int e^x \sec x(1+\tan x)dx = \int e^x(\sec x + \sec x \tan x)dx$

Also, let $\sec x = f(x) \Rightarrow \sec x \tan x = f'(x)$

We know that, $\int e^x\{f(x)+f'(x)\}dx = e^x f(x)+C$

$\therefore I = e^x \sec x + C$.

Ans to Q No 12 : Let $\log x = t$,then $x = e^t \Rightarrow dx = e^t dt$

$$\therefore I = \int(\sin t)e^t dt = -e^t \cos t - \int e^t(-\cos t)dt$$

$$= -e^t \cos t + \int e^t \cos t dt$$

$$= -e^t \cos t + \left[e^t \sin t - \int e^t \sin t dt\right]$$

$$\Rightarrow I = -e^t \cos t + e^t \sin t - I$$

$$\Rightarrow 2I = -e^t \cos t + e^t \sin t$$

$$\Rightarrow I = \frac{e^t}{2}(\sin t - \cos t) + C$$

Hence, $\int \sin(\log x)dx = \frac{x}{2}\{\sin(\log x) - \cos(\log x)\} + C$

Ans to Q No 13 : Let $3x-2 = A\frac{d}{dx}(x^2+x+1)+B$, A and B are constants.

$$\Rightarrow 3x-2 = A(2x+1)+B$$

On comparing the coefficients,we get $3 = 2A \Rightarrow A = \frac{3}{2}$

Also, $-2 = A+B \Rightarrow B = -2-\frac{3}{2}$

$$\Rightarrow B = -\frac{7}{2}$$

$$\therefore \int(3x-2)\sqrt{x^2+x+1}\,dx = \int\left[\frac{3}{2}(2x+1)-\frac{7}{2}\right]\sqrt{x^2+x+1}dx$$

$$= \frac{3}{2}\int(2x+1)\sqrt{x^2+x+1}dx - \frac{7}{2}\int(x^2+x+1)dx$$

$$= \frac{3}{2}\int(x^2+x+1)^{\frac{1}{2}}(2x+1)dx - \frac{7}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$= \frac{3}{2}.\frac{2}{3}\left(x^2+x+1\right)^{\frac{3}{2}} - \frac{7}{2}\left[\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left\{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right\}} + \frac{1}{2}.\frac{3}{4}\sinh^{-1}\left\{\frac{\left(x+\frac{1}{2}\right)}{\sqrt{\frac{3}{4}}}\right\}\right]$$

$$= \left(x^2+x+1\right)^{\frac{3}{2}} - \frac{7}{4}\left(x+\frac{1}{2}\right)\sqrt{\left\{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right\}} - \frac{21}{16}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

Ans to Q No 14 : Do yourself.

CHECK YOUR PROGRESS – 5

Ans to Q No 1 : Let $I = \int \frac{x-1}{(x-2)(x-3)}dx$

$$\frac{x-1}{(x-2)(x-3)} = \frac{A}{x-2}+\frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$$\therefore x-1 = A(x-3)+B(x-2)$$

Putting $x=3$ on both sides we get $B=2$. Again putting $x=2$ on both sides we get $1 = A(2-3) \therefore A=-1$.

$$\therefore I = -\int\frac{dx}{x-2}+2\int\frac{dx}{x-3}$$

$$= -\log(x-2)+2\log(x-3)+C$$

Ans to Q No 2 : Let $I = \int \frac{x}{x^2-12x+35}dx = \int \frac{x}{(x-7)(x-5)}dx$

Now, $\frac{x}{(x-7)(x-5)} = \frac{A}{x-7}+\frac{B}{x-5} = \frac{A(x-5)+B(x-7)}{(x-7)(x-5)}$

$$\therefore x = A(x-5)+B(x-7)$$

Putting $x=7$ and $x=5$ successively on both sides we get, $A=\frac{7}{2}$, $B=-\frac{5}{2}$.

$$\therefore I = \frac{7}{2}\int\frac{1}{x-7}dx - \frac{5}{2}\int\frac{1}{x-5}dx$$

$$= \frac{7}{2}\log(x-7)-\frac{5}{2}\log(x-5)+C.$$

Ans to Q No 3 : Let $I = \int \frac{3xdx}{x^2-x-2} = \int \frac{3xdx}{(x-2)(x+1)}$

$$\frac{3x}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

$$\therefore 3x = A(x+1)+B(x-2)$$

Putting $x=2$ on both sides we get $6=3A \quad \therefore A=2$.

Again putting $x=-1$,we get $-3=-3B \quad \therefore B=1$

$$\therefore I = 2\int \frac{1}{x-2}dx + \int \frac{1}{x+1}dx$$

$$= 2\log(x-2)+\log(x+1)+C$$

Ans to Q No 4 : Let $\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$$= \frac{A(x+1)^2 + B(x^2+x)+Cx}{x(x+1)^2}$$

$$\therefore 1 = A(x+1)^2 + B(x^2+x)+Cx$$

Putting $x=0$ on both sides,we get $A=1$ and putting $x=-1$ on both sides we get, $1=-C$, or $C=-1$.

Equating coefficient of x^2 from both sides we get

$$A+B=0$$

or $$B=-A=-1$$

$$\therefore \int \frac{dx}{x(x+1)^2} = \int \frac{1}{x}dx - \int \frac{1}{x+1}dx - \int \frac{1}{(x+1)^2}dx$$

$$= \log x - \log(x+1) + \frac{1}{x+1} + C.$$

Ans to Q No 5 : Do yourself.

Ans to Q No 6 : Do yourself.

Ans to Q No 7 : Do yourself.

Ans to Q No 8 : Do yourself.

Ans to Q No 9 : Let $\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

Equating the co-efficients of x^2, x and constant terms,we get

$A+B=0, C=0, A=1$

On solving these equations, we obtain $A=1, B=-1, C=0$.

$$\therefore \frac{1}{x\left(x^2+1\right)} = \frac{1}{x} + \frac{-x}{x^2+1}$$

$$\Rightarrow \int \frac{1}{x\left(x^2+1\right)}dx = \int \left\{\frac{1}{x} - \frac{x}{x^2+1}\right\}dx = \log|x| - \frac{1}{2}\log\left|x^2+1\right| + C$$

Ans to Q No 10 : Do yourself.

Ans to Q No 11 : Do yourself.

Ans to Q No 12 : Do yourself.

Ans to Q No 13 : Let $I = \int \frac{1}{1-x^3}\,dx$

$$= \int \frac{1}{(1-x)\left(1+x+x^2\right)}dx$$

$$\frac{1}{(1-x)\left(1+x+x^2\right)} = \frac{A}{1-x} + \frac{Bx+C}{1+x+x^2}$$

$$\therefore 1 = A\left(1+x+x^2\right) + (Bx+C)(1-x) \quad \text{... (1)}$$

Putting $x = 1$ in (1) we get $A = \frac{1}{3}$ and putting $x = 0$,we get

$$A + C = 1 \ \therefore C = 1 - \frac{1}{3} = \frac{2}{3}$$

Equating the co-efficients of x^2 on both sides we get

$$A - B = 0 \ \therefore B = \frac{1}{3}$$

$$\therefore I = \frac{1}{3}\int \frac{1}{1-x}dx + \frac{1}{3}\int \frac{x+2}{1+x+x^2}\,dx$$

$$= \frac{1}{3}\int \frac{1}{1-x}dx + \frac{1}{3}.\frac{1}{2}\int \frac{2x+1+3}{1+x+x^2}\,dx$$

$$= \frac{1}{3}\int \frac{1}{1-x}dx + \frac{1}{6}\int \frac{2x+1}{1+x+x^2}\,dx + \frac{1}{6}.3\int \frac{1}{1+x+x^2}\,dx$$

$$= -\frac{1}{3}\log(1-x) + \frac{1}{6}\log\left(1+x+x^2\right) + \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= -\frac{1}{3}\log(1-x) + \frac{1}{6}\log\left(1+x+x^2\right) + \frac{1}{2}.\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

$$= -\frac{1}{3}\log(1-x)+\frac{1}{6}\log\left(1+x+x^2\right)+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C$$

Ans to Q No 14 : Do yourself.

Ans to Q No 15 : Do yourself.

Ans to Q No 16 : Let $I=\int\frac{xdx}{\left(x^2+a^2\right)\left(x^2+b^2\right)}$

Let $x^2=z \quad \therefore 2xdx=dz$

$$\therefore I=\frac{1}{2}\int\frac{dz}{\left(z+a^2\right)\left(z+b^2\right)}$$

Let $\frac{1}{\left(z+a^2\right)\left(z+b^2\right)}=\frac{A}{z+a^2}+\frac{B}{z+b^2}$

$$\therefore 1=A\left(z+a^2\right)+B\left(z+b^2\right) \quad \text{... (1)}$$

Now, putting $z=-a^2$ and $z=-b^2$ successively in (1), we get

$$A=\frac{1}{a^2-b^2} \quad \text{and} \quad B=-\frac{1}{a^2-b^2}$$

$$\therefore I=\frac{1}{2\left(a^2-b^2\right)}\int\frac{1}{z+a^2}dz-\frac{1}{2\left(a^2-b^2\right)}\int\frac{dz}{z+b^2}$$

$$=\frac{1}{2\left(a^2-b^2\right)}\log\left(z+a^2\right)-\frac{1}{2\left(a^2-b^2\right)}\log\left(z+b^2\right)+C$$

$$=\frac{1}{2\left(a^2-b^2\right)}\log\frac{z+a^2}{z+b^2}+C$$

$$=\frac{1}{2\left(a^2-b^2\right)}\log\frac{x^2+a^2}{x^2+b^2}+C$$

Ans to Q No 17: Let $I=\int\frac{dx}{x^3-x^2-x+1}=\int\frac{dx}{(x+1)\left(x^2-1\right)}$

Now, $\frac{1}{(x+1)(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$

$$\therefore 1=A(x-1)(x+1)+B(x+1)+C(x-1)^2 \quad \text{... (1)}$$

Putting $x=1$ in (1),we get $2B=1$ or $B=\frac{1}{2}$

Again putting $x=-1$ in (1),we get $4C=1$ or $C=\frac{1}{4}$

Equating coefficients of x^2 on both sides we get

$A+C=0$, $A=-C=-\frac{1}{4}$ or $A=-\frac{1}{4}$

$$\therefore I=-\frac{1}{4}\int\frac{dx}{x-1}+\frac{1}{2}\int\frac{dx}{(x-1)^2}+\frac{1}{4}\int\frac{dx}{x+1}$$

$$=-\frac{1}{4}\log(x-1)+\frac{1}{2}\frac{1}{x+1}+\frac{1}{4}\log(x+1)+C$$

Ans to Q No 18 : Do yourself.

Ans to Q No 19 : Do yourself.

Ans to Q No 20 : Let $x^2=y$, then $\frac{x^2}{(x^2+1)(x^2+4)}=\frac{y}{(y+1)(y+4)}$

Let $\frac{y}{(y+1)(y+4)}=\frac{A}{y+1}+\frac{B}{y+4}$ ... (1)

$\Rightarrow y=A(y+4)+B(y+1)$ ... (2)

Putting $y=-1$ and $y=-4$ successively in (2), we get $A=-\frac{1}{3}$ and $B=\frac{4}{3}$ Substituting the values of A and B in (1), we obtain

$$\frac{y}{(y+1)(y+4)}=-\frac{1}{3(y+1)}+\frac{4}{3(y+4)}$$

Replacing y by x^2, we obtain

$$\frac{x^2}{(x^2+1)(x^2+4)}=-\frac{1}{3(x^2+1)}+\frac{4}{3(x^2+4)}$$

$$\therefore\int\frac{x^2}{(x^2+1)(x^2+4)}dx=-\frac{1}{3}\int\frac{1}{(x^2+1)}dx+\frac{4}{3}\int\frac{1}{(x^2+4)}dx$$

$$=-\frac{1}{3}\tan^{-1}x+\frac{4}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

$$=-\frac{1}{3}\tan^{-1}x+\frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right)+C$$

Ans to Q No 21 : Do yourself.

Ans to Q No 22 : Let $\sin x=t$ and $\cos x dx=dt$,we get

$$I = \int \frac{dt}{(2+t)(3+4t)}$$

Let $\frac{1}{(2+t)(3+4t)} = \frac{A}{2+t} + \frac{B}{3+4t}$ ... (i)

$\Rightarrow 1 = A(3+4t) + B(2+t)$ ... (ii)

Putting $3+4t=0$ i.e $t=-\frac{3}{4}$ in (ii), we get $1 = B\left(2-\frac{3}{4}\right) \Rightarrow B = \frac{4}{5}$

Putting $2+t=0$ i.e $t=-2$ in (ii), we get $1 = A(3-8) \Rightarrow A = -\frac{1}{5}$

Substituting the values of A and B in (i), we get

$$\frac{1}{(2+t)(3+4t)} = -\frac{1}{5}\frac{1}{(2+t)} + \frac{4}{5}\frac{1}{3+4t}$$

$$\therefore I = \int \frac{1}{(2+t)(3+4t)}dt = -\frac{1}{5}\int \frac{1}{2+t}dt + \frac{4}{5}\int \frac{1}{3+4t}dt$$

$$= -\frac{1}{5}\log|2+t| + \frac{4}{5}.\frac{1}{4}\log|3+4t| + C$$

$$= -\frac{1}{5}\log|2+\sin x| + \frac{1}{5}\log|3+4\sin x| + C$$

Ans to Q No 23 : Do yourself.

Ans to Q No 24 : Do yourself.

Ans to Q No 25 : We put $x^2 = t$ We have

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(t+1)(t+2)}{(t+3)(t+4)} = 1 + \frac{(t+1)(t+2)}{(t+3)(t+4)} - 1$$

$$= 1 + \frac{(t+1)(t+2)-(t+3)(t+4)}{(t+3)(t+4)}$$

Let $\frac{(t+1)(t+2)-(t+3)(t+4)}{(t+3)(t+4)} = \frac{A}{t+3} + \frac{B}{t+4}$

$\Rightarrow (t+1)(t+2)-(t+3)(t+4) = A(t+4) + B(t+3)$

Putting $t=-3,-4$ respectively, we get $A=2, B=-6$

$$\therefore \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 + \frac{2}{t+3} - \frac{6}{t+4}$$

$$= 1 + \frac{2}{x^2+3} - \frac{6}{x^2+4}$$

$$\Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx = \int dx + 2\int \frac{1}{x^2+3}dx - 6\int \frac{1}{x^2+4}dx$$

$$= x + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + C.$$



1.9 MODEL QUESTIONS

A. Evaluate

1. $\int \frac{x^4}{x^2+1}dx$

2. $\int \left(1 + \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} + \frac{5}{x\sqrt{x^2-1}} + 3^x\right)dx$

3. $\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)dx$

4. $\int \frac{4^x + 3^x}{5^x}dx$

B. Integrate the following–

1. $\int \frac{\cos 2x + x + 1}{x^2 + 2x + \sin 2x}dx$

2. $\int \frac{1 - \sin 2x}{x + \cos^2 x}dx$

3. $\int \frac{x+1}{x(x + \log^x)}dx$

4. $\int (4x+2)\sqrt{x^2+x+1}\,dx$

5. $\int \cos^5 x\,dx$

6. $\int \sin^3 x \cos^5 x\,dx$

7. $\int \frac{1}{\sqrt{16x^2+25}}dx$

8. $\int \frac{5}{x^2-8x+25}dx$

9. $\int \frac{1}{3-2x-x^2}dx$

10. $\int \frac{1}{x(x^n+1)}dx$

11. $\int \frac{1}{\sqrt{x^2+4x+2}}dx$

12. $\int \frac{1}{\sqrt{6+x-x}}dx$

13. $\int \frac{x+2}{2x^2+6x+5}dx$

14. $\int \frac{x^2+2}{x^2-5x+6}dx$

15. $\int \frac{2x+1}{\sqrt{x^2+2x-1}}dx$

16. $\int \frac{3x+1}{\sqrt{5-2x-x^2}}dx$

17. $\int \frac{1}{5+4\cos x}dx$

18. $\int \frac{1}{5-4\sin x}dx$

C. Evaluate

1. $\int \tan^{-1} x\,dx$ 2. $\int x \sin^{-1} x\,dx$ 3. $\int x^2 e^{-x}\,dx$ 4. $\int x^2 \sin^2 x\,dx$

5. $\int \log_{10}^{x} dx$ 6. $\int \log_{e}^{x} dx$ 7. $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$ 8. $\int e^{-x}dx \cos x\,dx$

9. $\int e^{ax}dx \cos x\,dx$ 10. $\int \sqrt{1+x-2x^2}\,dx$ 11. $\int (3x-2)\sqrt{x^2-x+1}\,dx$

12. $\int 2^x \sin x\,dx$ 13. $\int \sqrt{5x^2+8x+4}\,dx$

D. Evaluate

1. $\int \frac{3x+2}{x^3-6x^2+11x-6}dx$ 2. $\int \frac{x^2+x-1}{x^2+x-6}dx$ 3. $\int \frac{3x}{(x^2-4)(x+1)}dx$

4. $\int \frac{x}{(1+x)(1+x^2)}dx$ 5. $\int \frac{2x}{(x^2+1)(x^2+3)}dx$ 6. $\int \frac{2x+1}{(x+2)(x-3)^2}dx$

7. $\int \frac{dx}{x(x^4+2)}$ 8. $\int \frac{dx}{1-x^3}$

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