UNIT 2 : DEFINITE INTEGRALS

UNIT STRUCTURE

2.1 Learning Objectives

2.2 Introduction

2.3 Definite integral– Integration as the Limit of a Sum

2.4 Fundamental Theorem of Integral Calculus

2.5 Summation of Series with the Help of Integration

2.6 Evaluation of Definite Integrals by Using Properties

2.8 Let Us Sum Up

2.9 Further Readings

2.10 Answer to Check Your Progress

2.11 Model Questions

2.1 LEARNING OBJECTIVES

After going through this unit, you will be able to understand

- the meaning of definite integral as the limit of a sum,
- the fundamental theorem of integral calculus,
- the process of evaluating definite integration,
- the method of expressing the limits of sums of certain series as definite integrals and thus evaluate them,
- the properties of definite integrals

2.2 INTRODUCTION

In the preceding unit, we discussed the development of integration as the inverse of differentiation. In this unit we will introduce integration as the limit of a sum.

2.3 DEFINITE INTEGRALS – INTEGRATION AS THE LIMIT OF A SUM

Let f be a bounded function defined in $[a, b]$. We divide the interval $[a, b]$ into a finite subintervals, not necessarily of all equal length, by

Further, we take the points $\xi_1, \xi_2, \ldots \ldots \ldots \xi_n$ at the right hand end points of the respective sub-intervals.

i.e. $\xi_1 = x_1 = x_0 + h = a + h$

$\xi_2 = x_2 = (a + h) + h = a + 2h$

and so on. Finally $\xi_n = x_n = b = a + nh$

Also from $b = a + nh$ we get $n = \dfrac{b - a}{h}$

For here we see that for $n \to \infty$, we have $h \to 0$.

Therefore, from (1) above we can write

$$\int_a^b f(x)dx = \lim_{n\to\infty} S_n$$

$$= \lim_{h\to 0}\left[hf(a+h) + hf(a+2h) + \ldots\ldots\ldots + hf(a+nh)\right]$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh)$$

Note 2: If we take the points $\xi_1, \xi_2, \ldots \ldots \ldots \xi_n$ at the left hand end points, then we get

$$\int_a^b f(x)dx = \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh)$$

ILLUSTRATIVE EXAMPLES

Example 1: Find the value of $\int_0^1 x^2 dx$ by the method of integration as the limit of a sum.

Solution : Here $a = 0$, $b = 1$ and $f(x) = x^2$

$\therefore\ b = a + nh \Rightarrow 1 = nh$

Also $f(a+rh) = (a+rh)^2 = (rh)^2$ where $r = 1,2,3,\ldots\ldots\ldots n$

Now $\int_a^b f(x)dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh)$

$$\Rightarrow \int_0^1 x^2 dx = \lim_{h\to 0} h\sum_{r=1}^{n} (rh)^2$$

$$= \lim_{h\to 0} h\left\{(h)^2 + (2h)^2 + (3h)^2 + \ldots\ldots\ldots + (nh)^2\right\}$$

$$= \lim_{h\to 0} h^3\left\{1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2\right\}$$

$$= \lim_{h\to 0} h^3\,\frac{n(n+1)(2n+1)}{6} = \lim_{h\to 0} \frac{nh(nh+h)(2nh+h)}{6}$$

Proof : As discussed earlier, we can write

$$\int_a^b f(x)dx = \lim_{\Delta x_r \to 0} \sum_{r=1}^{n} f(\xi_r)\Delta x_r \quad \text{... (1)}$$

Now $\phi(x)$ satisfies all the conditions of mean value theorem of differential calculus, and so there exists a point satisfying $x_{r-1} < \alpha_r < x_r$ such that

$$\begin{aligned}\phi(x_r) - \phi(x_{r-1}) &= (x_r - x_{r-1})\phi'(\alpha_r) \\ &= \Delta x_r . f(\alpha_r) \quad \text{...(2)}\end{aligned}$$

Since (1) holds for every choice of ξ_r, so taking $\xi_r = \alpha_r$, we have from (1) and (2)

$$\sum_{r=1}^{n} f(\xi_r)\Delta x_r = \sum_{r=1}^{n} \{\phi(x_r) - \phi(x_{r-1})\} = \phi(b) - \phi(a)$$

and from here

$$\lim_{\Delta x_r \to 0} \sum_{r=1}^{n} f(\xi_r)\Delta x_r = \int_a^b f(x)dx = \phi(b) - \phi(a)$$

i.e., $\int_a^b f(x)dx = \phi(b) - \phi(a)$ proved.

Note: If F(x) is an integral of $f(x)$, then $\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b) - F(a)$

Here we donot write the constant of integration C. Because, if instead of F(x) we take $\mathrm{F}(x) + \mathrm{C}$ as an integral of $f(x)$ for some value of C, then

$$\begin{aligned}\int_a^b f(x)dx &= \left[\mathrm{F}(x) + \mathrm{C}\right]_a^b = \{\mathrm{F}(b) + \mathrm{C}\} - \{\mathrm{F}(a) + c\} \\ &= \mathrm{F}(b) - \mathrm{F}(a)\end{aligned}$$

ILLUSTRATIVE EXAMPLES

Example 1: Evaluate $\int_0^1 (3x^2 + x - 9)dx$

Solution: Here we first evaluate $\int (3x^2 + x - 9)dx$

then put the upper and the lower limits

So $$\int_0^1 (3x^2 + x - 9)dx = \left[3.\frac{x^3}{3} + \frac{x^2}{2} - 9x\right]_0^1$$

$$= (1)^3 + \frac{(1)^2}{2} - 9(1)$$

$$= -15/2$$

$$= \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2} dx}{4\left(1+\tan^2 \frac{x}{2}\right)+10\tan x/2}$$

$$= \int_0^1 \frac{2dt}{4(1+t^2)+10t} \qquad \text{(We take here } \tan\frac{x}{2} = t\text{)}$$

$$= \int_0^1 \frac{dt}{2t^2+5t+2} = \int_0^1 \frac{dt}{2\left(t^2+\frac{5}{2}t+1\right)} = \frac{1}{2}\int_0^1 \frac{dt}{\left(t+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\left[\frac{1}{2.\frac{3}{4}}\log\frac{\left(t+\frac{5}{4}\right)-\frac{3}{4}}{\left(t+\frac{5}{4}\right)+\frac{3}{4}}\right]_0^1 = \frac{1}{3}\left[\log\left(\frac{2t+1}{2t+4}\right)\right]_0^1$$

$$= \frac{1}{3}\left(\log\frac{3}{6}-\log\frac{1}{4}\right) = \frac{1}{3}\log 2$$

2.5 SUMMATION OF SERIES WITH THE HELP OF INTEGRATION

We have already found that

$$\int_a^b f(x)dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh) \text{ where } b = a+nh$$

Now, in particular, if $a = 0, b = 1$ then we can write

$$\int_0^1 f(x)dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(rh) \qquad \text{... (i)}$$

Since $b = a+nh$ becomes $1 = nh$ i.e. $h = \frac{1}{n}$. So we get $h \to 0$ is equivalent to $n \to \infty$. Therefore from (i) we get

$$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$$

ILLUSTRATIVE EXAMPLES

Example 1: Find $\lim_{n\to\infty}\left[\frac{n}{n^2}+\frac{n}{1^2+n^2}+\frac{n}{2^2+n^2}+\ldots\ldots\ldots+\frac{n}{(n-1)^2+n^2}\right]$

$$= 2\log 2 - 1 = \log 4 - \log e = \log\left(4/e\right)$$

$$\therefore P = 4/e$$



CHECK YOUR PROGRESS - 1

Q 1 : Find the value of the following by the method of integration as the limit of a sum

(i) $\int_a^b x\,dx$ (ii) $\int_0^1 x\,dx$ (iii) $\int_a^b x^2\,dx$

(iv) $\int_0^1 x^3\,dx$ (v) $\int_0^1 (ax+b)\,dx$

Q 2 : Evaluate the integrals

(i) $\int_0^1 xe^x\,dx$ (ii) $\int_0^1 x^2e^x\,dx$ (iii) $\int_0^{\pi/2} \sin 2x\;dx$

(iv) $\int_0^1 xe^{x^2}\,dx$ (v) $\int_0^1 \sin^{-1} x\,dx$ (vi) $\int_0^{\pi/2} x\sin x\,dx$

(vii) $\int_0^{\sqrt{e}} x\log x\,dx$ (viii) $\int_0^{\pi/2} x^3\sin x\,dx$

Q 3 : Evaluate

(i) $\int_0^{\pi/2} \cos 3\theta\cos 2\theta\,d\theta$ (ii) $\int_a^b \frac{1}{x}\log x\,dx$

(iii) $\int_0^a \frac{dx}{\sqrt{\left(a^2+x^2\right)^3}}$ (iv) $\int_0^2 \sqrt{2x-2^2}\,dx$

(v) $\int_0^1 \frac{dx}{x^2+4x+5}$ (vi) $\int_1^2 \frac{dx}{\sqrt{x^2+x-2}}$ (vii) $\int_1^e \frac{dx}{x(1+\log x)}$

(viii) $\int_1^{\log 2} \frac{e^x}{1+e^x}\,dx$ (ix) $\int_2^3 \frac{dx}{(x-1)\sqrt{x^2-2x}}$ (x) $\int_1^2 \frac{dx}{(x)(1+2x)^2}$

Q 4 : Show that

(i) $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$

(ii) $\int_0^a \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2a\tan^{-1} a - \log\sqrt{1+a^2}$

(iii) $\int_1^2 (\log x)^2\,dx = 2 - 4\log 2 + 2(\log 2)^2$

Now $\int_a^b f(x)dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$... ... ...(i)

and $\int_a^b f(z)dt = [\phi(z)]_a^b = \phi(b) - \phi(a)$... ... ...(ii)

$\therefore$ From (i) and (ii), $\int_a^b f(x)dx = \int_a^b f(z)dz$

Proof (P. 2) $\int_a^b f(x)dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$ and

$$-\int_b^a f(x)dx = -[\phi(x)]_b^a = -[\phi(a) - \phi(b)] = \phi(b) - \phi(a)$$

$$\therefore \int_a^b f(x)dx = -\int_b^a f(x)dx$$

Proof (P. 3) Let $\int f(x)dx = \phi(x)$

$$\therefore \text{LHS} = \int_a^b f(x)dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$$

$$\begin{aligned}\text{RHS} &= \int_a^c f(x)dx + \int_c^b f(x)dx \\ &= [\phi(x)]_a^c + [\phi(x)]_c^b \\ &= [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)] \\ &= \phi(b) - \phi(a)\end{aligned}$$

$$\therefore \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

where $a < c < b$.

Corollary of (P.3) The above property can be generalised as below–

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots\ldots\ldots + \int_{c_{n-1}}^{c_n} f(x)dx + \int_{c_n}^b f(x)dx$$

where $a < c_1 < c_2 < \ldots\ldots\ldots < c_{n-1} < c_n < b$.

Proof (P.4) Here we put $a - x = z$ on the R.H.S.

$\therefore$ R.H.S. $= \int_0^a f(a - x)dx$

For $a - x = z$

$dx = -dz$

and $x = 0 \Rightarrow z = a$

$x = a \Rightarrow z = 0$

$$= \int_a^0 f(z)(-dz)$$

$$= -\int_a^0 f(z)dz = \int_0^a f(z)dz, \quad \text{by properly (P.2)}$$

$$= \int_0^a f(x)dx, \quad \text{by property (P.1)}$$

$$= \text{LHS}.$$

$$\therefore \int_0^a f(x)dx = \int_0^a f(a - x)dx$$

$$= -\int_a^0 f(-z)dz$$

$$= \int_0^a f(-z)dz,$$

$$= \int_0^a f(-x)dx$$

$\therefore$ (i) becomes

$$\int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx$$

$$= \int_0^a [f(x) + f(-x)]dx$$

Corollary 7 (i) If $f(-x) = f(x)$,i.e. if f is even, then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

Corollary 7 (ii) If $f(-x) = -f(x)$ i.e. if f is odd, then

$$\int_{-a}^{a} f(x)dx = 0$$

ILLUSTRATIVE EXAMPLES

Example 1. Show that $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}$

Proof : Let $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \{\cos(\pi - x)\}^2} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - I$$

$$\therefore 2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

$$= \pi \int_1^{-1} \frac{-dz}{1+z^2}$$ (Let us put $z = \cos x$)

$$= -\pi \int_1^{-1} \frac{dz}{1+z^2}$$ Then $dz = -\sin x\, dx$

$$= \pi \int_{-1}^{1} \frac{dz}{1+z^2}$$ Also for $x = 0, z = 1$ and $x = \pi,\ z = -1$

and if $x = 1$ then $\theta = \pi/4$

Then the given integral (say I) becomes

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} \sec^2\theta d\theta = \int_0^{\pi/4} \log(1+\tan\theta) d\theta$$

$$= \int_0^{\pi/4} \log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\} d\theta$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log\frac{2}{1+\tan\theta} d\theta$$

$$= \int_0^{\pi/4} \log 2 d\theta - \int_0^{\pi/4} \log(1+\tan\theta) d\theta$$

$$= \log 2 \int_0^{\pi/4} d\theta - I$$

$$\Rightarrow 2I = (\log 2)[\theta]_0^{\frac{\pi}{4}} = (\log 2)\frac{\pi}{4} = \frac{\pi}{4}\log 2$$

$$\therefore \quad I = \frac{\pi}{8}\log 2$$



CHECK YOUR PROGRESS - 2

Q. Prove the followings :

1. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx = \frac{\pi}{4}$

2. $\int_0^{\pi/2} \log\sin x dx = \int_0^{\pi/2} \log\cos x dx = \frac{\pi}{2}\log\left(\frac{1}{2}\right)$

3. $\int_0^{\pi} \frac{x dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi^2}{2ab}$

4. $\int_0^{\pi/4} \log(1+\tan\theta) d\theta = \frac{\pi}{8}\log 2$

5. $\int_{\frac{-\pi}{2}}^{\pi/2} \sin^9 x dx = 0$

6. $\int_0^{\pi} \sin^8 x \cos^7 x dx = 0$

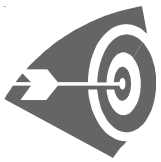
7. $\int_{-a}^{a} \phi(y^2) dy = 2\int_0^a \phi(y^2) dy$

8. $\int_0^{\pi/2} \frac{\cos x - \sin x}{\cos x \sin x + 1} dx = 0$

9. $\int_0^{\pi/2} (2\log\sin x - \log\sin 2x) dx = \frac{\pi}{2}\log\frac{1}{2}$

2) *Integral Calculus*, Dhami, H.S. New age International Publishers, New Delhi - 110002.

3) *Text book of Integral Calculus*, Sharma, A.K. Discovery Publishing House, New Delhi -110002.



2.10 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1 : (i) We have

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h)\right]$$

Here $f(x)=x$, $a=a$, $b=b$ and $h=\dfrac{b-a}{n} \Rightarrow nh=b-a$

$$\therefore \int_a^b f(x)dx = \lim_{h\to 0} h\left[a+(a+h)+(a+2h)+\cdots+(a+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[(a+a+\cdots+a)+(h+2h+3h+\cdots+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[na+h(1+2+3+\cdots+(n-1))\right]$$

$$= \lim_{h\to 0} h\left[na+h\frac{n(n-1)}{2}\right]$$

$$= \lim_{h\to 0} \left[(nh)a+\frac{nh(nh-h)}{2}\right]$$

$$= \lim_{h\to 0} \left[(b-a)a+\frac{(b-a)(b-a-h)}{2}\right]$$

$$= \left[(b-a)a+\frac{(b-a)(b-a)}{2}\right]$$

$$= (b-a)\left[a+\frac{b-a}{2}\right]$$

$$= (b-a)\left[\frac{2a+b-a}{2}\right]$$

$$= \frac{(b-a)(b+a)}{2}$$

$$= \lim_{h\to 0} h\left[\frac{ahn(n+1)}{2} + nb\right] \text{ where } \sum_{r=1}^{n} r = \frac{n(n+1)}{2} \text{ and } \sum_{r=1}^{n} 1 = n.$$

$$= \lim_{h\to 0}\left[\frac{a.nh.(nh+h)}{2} + nhb\right]$$

$$= \lim_{h\to 0}\left[\frac{a.1.(1+h)}{2} + 1.b\right]$$

$$= \frac{a(1+0)}{2} + b$$

$$= \frac{a}{2} + b$$

$$= \frac{1}{2}(a+2b)$$

Ans to Q No 2 : (i) We have $\int_0^1 xe^x dx = \left[xe^x\right]_0^1 - \int_0^1 1.e^x dx$

$$= \left(1.e^1 - 0.e^0\right) - \left[e^x\right]_0^1$$

$$= \left(e^1 - 0\right) - \left(e - e^0\right)$$

$$= 1.$$

(ii) Do Yourself.

(iii) Do Yourself.

(iv) Let $I = \int_0^1 xe^{x^2} dx$

We put $x^2 = t \Rightarrow 2xdx = dt$

As $x = 0, t = 0$ and $x = 1, t = 1$

$$\therefore I = \frac{1}{2}\int_0^1 e^t dt = \frac{1}{2}\left[e^t\right]_0^1$$

$$= \frac{1}{2}[e-1]$$

(v) Let $I = \int_0^1 \sin^{-1} xdx = \int_0^1 \sin^{-1} x.1dx$

Integrating by parts, we obtain

$$I = \left[\sin^{-1} x.x\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} xdx$$

$$= -\int \frac{dz}{\sqrt{1-z^2}}$$

$$= -\sin^{-1} z$$

$$= -\sin^{-1}\left(\frac{1}{x-1}\right)$$

$$\therefore \int_2^3 \frac{dx}{(x-1)\sqrt{x^2-2x}} = \left[-\sin^{-1}\left(\frac{1}{x-1}\right)\right]_2^3$$

$$= -\sin^{-1}\frac{1}{2} + \sin^{-1} 1$$

$$= -\frac{\pi}{6} + \frac{\pi}{2} = \frac{\pi}{3}.$$

(x) Let $\frac{1}{x(1+2x)^2} = \frac{A}{x} + \frac{B}{1+2x} + \frac{C}{(1+2x)^2}$

$$\therefore 1 = A(1+4x+4x^2) + B(x+2x^2) + Cx \quad \text{... (1)}$$

Putting $x = 0$ in (1) we get $A = 1$ and putting $x = -\frac{1}{2}$ we get $C = -2$.

Equating the coefficients of x^2 on both sides we get $4A + 2B = 0$

$$\therefore B = -2$$

$$\therefore \int_1^2 \frac{dx}{x(1+2x)^2} = \int_1^2 \left[\frac{1}{x} - 2\frac{1}{1+2x} - 2\frac{1}{(1+2x)^2}\right] dx$$

$$= \left[\log x - \log(1+2x) + \frac{1}{1+2x}\right]_1^2$$

$$= \log 2 - \log 5 + \frac{1}{5} + \log 3 - \frac{1}{3}$$

$$= \log\frac{6}{5} - \frac{2}{15}$$

Answer to Q No 4 : (i) Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x dx$

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\left[\sin x\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\left[\frac{\sin 3x}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{6}\sin\frac{3\pi}{2}$$

$$= \frac{1}{2} + \frac{1}{6}\times 1$$

$$= \frac{2}{3}$$

(v) Do Yourself.

Ans to Q No 5: (i) Here the general term i.e., r th term $= \frac{1}{n+r}$ and r varies from 1 to n .Thus we have to find $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}$.

We have $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n\left\{1+\left(\frac{r}{n}\right)\right\}}$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\left(\frac{r}{n}\right)}$$

The limits of r in this summation are 1 to n . Therefore the lower limit of integration $\lim_{n\to\infty}\frac{1}{n} = 0$, and the upper limit of integration $\lim_{n\to\infty}\frac{n}{n} = 1$.

Hence, the required limit $= \int_0^1 \frac{1}{1+x}dx = \left[\log(1+x)\right]_0^1 = \log 2$.

(ii) Here the general term i.e., r th term $= \frac{n}{n^2+r^2}$ and r varies from 1 to n .

Proceeding as in Q 4 (i), we get the required limit $= \frac{\pi}{4}$.

(iii) Here the general term $= \frac{1}{\sqrt{\left(n^2-r^2\right)}} = \frac{1}{n}.\frac{1}{\sqrt{\left\{1-\left(\frac{r}{n}\right)^2\right\}}}$, and varies from 0 to $(n-1)$.

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$= \int_0^{\frac{\pi}{2}} 1.dx$$

$$= [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0$$

$$\Rightarrow I = \frac{\pi}{4}$$

Ans to Q No 2: Let $I = \int_0^{\frac{\pi}{2}} \log \sin x dx$ ... (i)

$$I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log \cos x dx \quad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \log \sin x dx + \int_0^{\frac{\pi}{2}} \log \cos x dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log(\sin x \cos x) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{2 \sin x \cos x}{2}\right) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log \sin 2x dx - \int_0^{\frac{\pi}{2}} \log 2 dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$\Rightarrow 2I = 2\pi \int_0^{\frac{\pi}{2}} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$\Rightarrow 2I = 2\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx$$ (Dividing num. and deno.by $\cos^2 x$)

$$\Rightarrow I = \pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx$$

Let $\tan x = t$, then $\sec^2 x dx = dt$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t \to \infty$

$$\therefore I = \pi \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2}$$

$$= \frac{\pi}{b^2} \int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$$

$$= \frac{\pi}{b^2} \times \frac{1}{\left(\frac{a}{b}\right)} \left[\tan^{-1} \frac{t}{\left(\frac{a}{b}\right)} \right]_0^{\infty}$$

$$\Rightarrow I = \frac{\pi}{ab} \left[\tan^{-1} \left(\frac{bt}{a} \right) \right]_0^{\infty}$$

$$= \frac{\pi}{ab} \left[\tan^{-1} \infty - \tan^{-1} 0 \right]$$

$$= \frac{\pi}{ab} \left[\frac{\pi}{2} - 0 \right]$$

$$\Rightarrow I = \frac{\pi^2}{2ab}.$$

Ans to Q No 4: Let $I = \int_0^{\frac{\pi}{4}} \log(1 + \tan\theta) d\theta$ ... (1)

Ans to Q No 6 : Do yourself.

Ans to Q No 7 : Do yourself.

Ans to Q No 8 : Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\cos x \sin x + 1} dx$ ... (1)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - x\right) - \sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)\sin\left(\frac{\pi}{2} - x\right) + 1} dx \quad \text{since} \quad \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{\sin x \cos x + 1} dx \quad \text{... (2)}$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{0}{\sin x \cos x + 1} dx$$

$$\Rightarrow I = 0 .$$

Ans to Q No 9: Let $I = \int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log\sin x - \log(2\sin x\cos x)\}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log\sin x - \log\sin x - \log\cos x - \log 2\}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log\sin x - \log\cos x - \log 2)dx \quad \text{... (1)}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left\{\log\sin\left(\frac{\pi}{2} - x\right) - \log\cos\left(\frac{\pi}{2} - x\right) - \log 2\right\}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log\cos x - \log\sin x - \log 2)dx \quad \text{... (2)}$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} (-\log 2 - \log 2)dx$$

$$\Rightarrow 2I = \frac{\pi}{4}\left[\tan^{-1} t\right]_0^{\infty}$$

$$\Rightarrow 2I = \frac{\pi}{4}\left[\tan^{-1}\infty - \tan^{-1} 0\right]$$

$$\Rightarrow 2I = \frac{\pi}{4}\left[\frac{\pi}{2} - 0\right]$$

$$\Rightarrow I = \frac{\pi^2}{16}.$$

Ans to Q No 11: Let $I = \int_0^p \frac{\sqrt{x}}{\sqrt{x}+\sqrt{p-x}}\,dx$ ... (1)

$$\Rightarrow I = \int_0^p \frac{\sqrt{p-x}}{\sqrt{p-x}+\sqrt{x}}\,dx \quad \text{... (2)}$$

Adding (1) and (2),we get,

$$2I = \int_0^p \frac{\sqrt{p-x}+\sqrt{x}}{\sqrt{p-x}+\sqrt{x}}\,dx$$

$$\Rightarrow 2I = \int_0^p 1.dx$$

$$\Rightarrow 2I = [x]_0^p = p$$

$$\Rightarrow I = \frac{p}{2}.$$

Ans to Q No 12 : Do yourself.

Ans to Q No 13 : Let $I = \int_0^{\frac{\pi}{2}} \left(a\cos^2 x + b\sin^2 x\right)dx$ ... (1)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left\{a\cos^2\left(\frac{\pi}{2}-x\right) + b\sin^2\left(\frac{\pi}{2}-x\right)\right\}dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(a\sin^2 x + b\cos^2 x\right)dx \quad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \left\{a\left(\cos^2 x + \sin^2 x\right) + b\left(\sin^2 x + \cos^2 x\right)\right\}dx$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \cos x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \cos x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$$

We put $\cos x = t \Rightarrow -\sin x dx = dt$

when $x = 0, t = 1$ and when $x = \pi, t = -1$

$$\therefore 2I = \pi \int_1^{-1} \frac{-dt}{1 + t^2}$$

$$\Rightarrow 2I = -\pi \left[\tan^{-1} t\right]_1^{-1}$$

$$\Rightarrow 2I = -\pi \left[-\frac{\pi}{4} - \frac{\pi}{4}\right]$$

$$\Rightarrow 2I = \frac{\pi^2}{2}$$

$$\Rightarrow I = \frac{\pi^2}{4}.$$

Ans to Q No 16 : Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \cos x + \sin x} dx$ ... (i)

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \sin x + \cos x} dx \quad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{1 + \sin x + \cos x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1 + \cos x + \sin x - 1}{1 + \sin x + \cos x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \left\{1 - \frac{1}{1 + \sin x + \cos x}\right\} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{3+5\sin x}{3+5\cos x}.\frac{3+5\cos x}{3+5\sin x}\right)dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log 1 dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0.dx$$

$$\Rightarrow I = 0 .$$

Ans to Q No 18 : Let $I = \int_0^{\pi} \frac{xdx}{1+\sin x}$ ... (1)

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)dx}{1+\sin(\pi - x)}$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)dx}{1+\sin x} \quad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{\pi dx}{1+\sin x}$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{(1-\sin x)dx}{(1+\sin x)(1-\sin x)}$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1-\sin x dx}{1-\sin^2 x}$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} (\sec^2 x - \tan x \sec x)dx$$

$$\Rightarrow 2I = \pi [\tan x - \sec x]_0^{\pi}$$

$$\Rightarrow 2I = \pi.2$$

$$\Rightarrow I = \pi .$$

Ans to Q No 19 : We have $I = \int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx$

We put $x = \sin\theta$, $dx = \cos\theta d\theta$

Also, when $x = 0$, $\theta = 0$ and $x = 1$, $\theta = \frac{\pi}{2}$.

(iv) $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\ldots\ldots\ldots+\frac{1}{8n}\right]$

(v) $\lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{\left(n^2+r^2\right)^{3/2}}$

Q 4 : Evaluate (by using properties of definite integrals)

(i) $\int_0^{\pi/2}\frac{x\,dx}{\sin x+\cos x}$ (ii) $\int_0^{\pi}\log(1+\cos x)\,dx$ (iii) $\int_0^{\pi}\cos^6 x\,dx$

(iv) $\int_{-1}^{1}\frac{x^2\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ (v) $\int_{-1}^{1}\frac{x^2\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ (vi) $\int_0^{\pi/2}\sin 2x\log\tan x\,dx$

(vii) $\int_0^{\pi}x\sin^6 x\cos^4 x\,dx$ (viii) $\int_0^{\pi/2}\frac{dx}{1+\tan x}$

Q 5 : Show that

(i) $\int_0^{\pi/2}\phi(\sin 2x)\sin x\,dx=\int_0^{\pi/2}\phi(\sin 2x)\cos x\,dx$

(ii) $\int_0^{\pi}\sin^n x\cos^{2n+1}x\,dx=0$ (iii) $\int_0^{\pi}\frac{x\sin x}{1+\sin x}\,dx=\frac{\pi}{2}(\pi-2)$

(iv) $\int_0^{\pi/2}\frac{dx}{5+4\cos x}=\frac{2}{3}\tan^{-1}\left(1/3\right)$ (v) $\int_{-\pi/2}^{\pi/2}\frac{x\cos x\,dx}{2\sin^2 x+3\cos^2 x}=0$

(vi) $\int_{-1}^{1}|x|\,dx=1$ (vii) $\int_0^{\pi/2}\frac{\cos^3 x\,dx}{\sin^3 x+\cos^3 x}=\frac{\pi}{4}$

(viii) $\int_0^{\pi}x\cos^4 x\,dx=\frac{3\pi^2}{16}$

Q 6 : If $f(x)=|x-2|+|x-3|$ then show that $\int_2^4 f(x)dx=3$.

*** ***** ***