UNIT 4 : QUADRATURE OF PLANE AREAS

UNIT STRUCTURE

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4.1 LEARNING OBJECTIVES

After going through this unit, you will be able to find

- the area of a plane curve
- the areas between two given curves.

4.2 INTRODUCTION

Integration has a large number of applications in science, engineering and other branches of study. One of the important problems of integration is to evaluate the area of a region. In fact, integral calculus was developed in the attempt to calculate the area bounded by curves by dividing it into infinitesimal parts and then finding the sum of these parts. The process of finding the area bounded by curves is called "Quadrature".

4.3 QUADRATURE OF PLANE AREAS : CARTESIAN COORDINATES

Let $y = f(x)$ be a continuons function in the interval $[a, b]$. Let $y=f(x)$ be represented graphically by the curve PQ. Let AC and BD be two ordinates corresponding to the points $x = a$ and $x = b$. From the Fig. 4.1 we have $OA = a$ and $OB = b$.

$\therefore$ AB $= b - a$

Now let us divide AB into *n* equal parts (sub-intervals) where each part is equal to *h*.

$\therefore\ b = a + nh$

Let the points of division of AB beA, A_1, A_2, A_3... ,B.

Let the ordinates at the points of division of AB meet the curve PQ at C, C_1, C_2,... ... ..,D.

Then the abscissa and ordinate of each of C, C_1, C_2... ... ...D are

For Points	Abscissa	Ordinate
C	a	$f(a)$
C_1	$a+h$	$f(a+h)$
C_2	$a+2h$	$f(a+2h)$
...	...	...
Finally		
D	$a+nh$	$f(a+nh)$

Now let us construct the inner rectangles with the consecutive points below the curve PQ as shown in the Fig. 4.1. The rectangles so formed are ACP_1A_1, $A_1C_1P_2A_2$... etc.

Let S denote the actual area enclosed between the curve $y = f(x)$, the two ordinates $x = a$, $x = b$ and the *x*-axis.

If S_1 denotes the sum of the areas of all the inner rectangles, then

$$S_1 = hf(a) + hf(a+h) + \ldots\ldots\ldots + hf\left(a + \overline{n-1}h\right)$$
$$= h\left[f(a) + f(a+h) + \ldots\ldots\ldots + f\left(a + \overline{n-1}h\right)\right]$$
$$= h\sum_{r=0}^{n-1} f(a+rh)$$

We can also construct the outer rectangles above the curves. If S_2 denotes the sum of the areas of all the outer rectangles then we can write

$$S_2 = hf(a+h) + hf(a+2h) + \ldots\ldots\ldots + hf(a+nh)$$
$$= hf(a) + hf(a+h) + \ldots\ldots\ldots + hf\left(a + \overline{n-1}h\right) + hf(a+nh) - hf(a)$$
$$= h\sum_{r=0}^{n-1} f(a+rh) + hf(a+nh) - hf(a)$$
$$= h\sum_{r=0}^{n-1} f(a+rh) + hf(b) - hf(a)$$

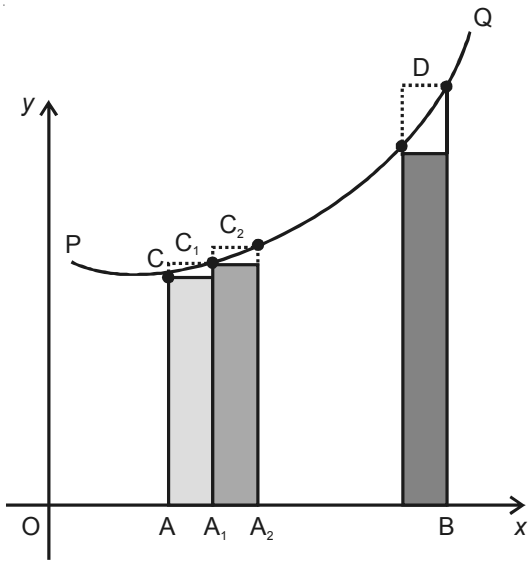


Fig. 4.1

Now it is evident that

$$S_1 < S < S_2$$

or $\sum_{r=0}^{n-1} f(a+rh) < S < \sum_{r=0}^{n-1} f(a+rh) + hf(b) - hf(a)$

If we take $n \to \infty$, then $h \to 0$ and so

$$\lim_{h\to 0} \sum_{r=0}^{n-1} f(a+rh) < S < \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh) + \lim_{h\to 0} h[f(b) - f(a)]$$

i.e. $\int_a^b f(x)dx < S < \int_a^b f(x)dx$

$\therefore\ S = \int_a^b f(x)dx$

Thus the definite integral $\int_a^b f(x)dx$ geometrically represents the area enclosed by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and the *x*-axis.

$\therefore$ Area of ABCD $= \int_a^b f(x)dx = \int_a^b y dx$

Note : It can be shown that the area bounded by the curve $x = f(y)$, *y*-axis and the abscissae $y = a$ and $y = b$ is given by $\int_a^b x dy$ or $\int_a^b f(y)dy$.

ILLUSTRATIVE EXAMPLES

Example 1: Find, by integration, the area bounded by the *x*-axis, the *y*-axis and the line $y - x = 2$.

Solution : The line $y - x = 2$ cuts the *y*-axis at B(0, 2) and the *x*-axis at A(–2, 0). Therefore we are to find the area of the shaded region AOB shown in the Fig. 4.2.

$\therefore$ Required area

$$= \int_{-2}^{0} y dx = \int_{-2}^{0} (x+2)dx = \left[\frac{x^2}{2} + 2x\right]_{-2}^{0}$$

$$= [0] - [2-4] = 2 \text{ sq.unit.}$$

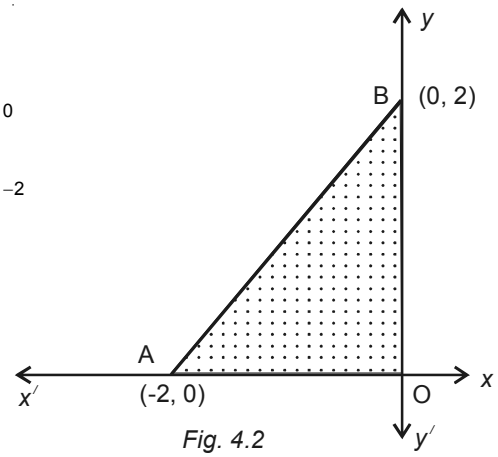


Fig. 4.2

Example 2: Find, by the method of integration, the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution : The equation of the curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

i.e. $y^2=\frac{b^2}{a^2}\left(a^2-x^2\right)$

The curve is symmetrical about both the axes. Hence the whole area of the ellepse is 4 times the area in the first quadrant. The area in the first quadrant is area of the region OAB.

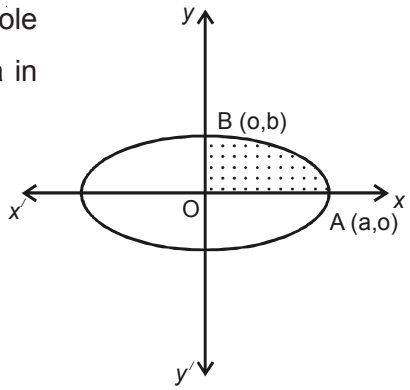


Fig. 4·3

Hence the required area

$$=4\int_0^a y\,dx$$

$$=4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$$

$$=\frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$=\frac{4b}{a}\left[\frac{a^2}{2}\sin^{-1}(1)-\frac{a^2}{2}\sin^{-1}(0)\right]=\frac{4ab}{2}.\frac{\pi}{2}=\pi ab$$

Example 3 : Find the area bounded by the curve $y=\sin x$,the x-axis between $x=0$ to $x=2\pi$.

Solution : The graph of $y=\sin x$ between $x=0$ and $x=2\pi$ is shown in the Fig. 4.4.

From the figure it is seen that between $x=0$ and $x=\pi$, the curve lies above the *x*-axis and so ordinates are all positive. But between $x=\pi$ and $x=2\pi$ the curve lies below the *x*-axis and so the ordinates are all negative. We take absolute values to find the total area.

∴ The required area

$$=\left|\text{area OAB}\right|+\left|\text{area BCD}\right|$$

$$=\left|\int_0^{\pi}\sin x\,dx\right|+\left|\int_{\pi}^{2\pi}\sin x\,dx\right|$$

$$=\left|\left[-\cos x\right]_0^{\pi}\right|+\left|\left[-\cos x\right]_{\pi}^{2\pi}\right|$$

$$=|2|+|-2| \quad =2+2=4$$

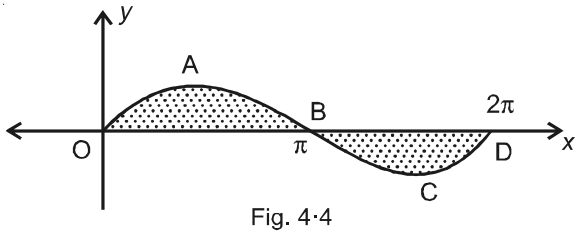


Fig. 4·4

Example 4 : Show that the area cut off a parabola by any double ordinate is two-third of the corresponding rectangles contained by that double ordinate and its distance from the vertex.

Solution : Let the parabola be $y^2 = 4ax$ and the double ordinate be $x = h$. Here ACB be the double ordinate.

$\therefore$ The required area cut off from the parabola by the double ordinate = area of AOBCA

$= 2 \times$ area of AOC (Since the parabola is symmetrical about x-axis)

$$= 2\int_{x=0}^{x=h} y\,dx$$

$$= 2\int_0^h \sqrt{4ax}\,dx$$

$$= \sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^h = \frac{8}{3}\sqrt{a}.h^{3/2} \quad \text{... ... ...(i)}$$

Here, the distance of the double ordinate ACB from the vertex $= OC = h$.

Let $AC = y_1$

$\therefore$ Coordinate of A are (h, y_1).

Since A is on the parabola, so its coordinate will satisfy the equation of the equation of the parabola $y^2 = 4ax$.

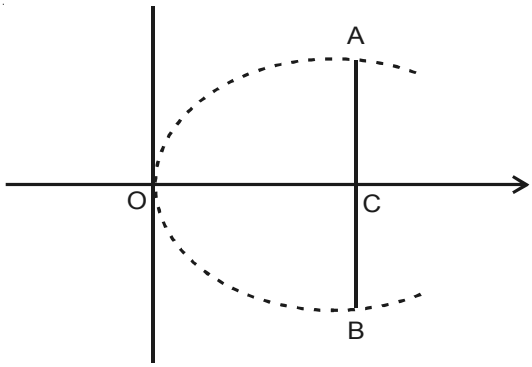


Fig. 4·5

$\therefore {y_1}^2 = 4ah \Rightarrow y_1 = 2\sqrt{ah}$

$\therefore$ Length of the double ordinate $ACB = 2y_1 = 4\sqrt{ah}$

$\therefore \frac{2}{3}\times$ area of the rectangle contained by the double ordinate ACB and its distance from the vertex

$$= \frac{2}{3} \times AB \times OC$$

$$= \frac{2}{3} \times 4\sqrt{ah} \times h = \frac{8}{3}\sqrt{a}h^{3/2} \quad \text{... ... ...(ii)}$$

Hence (i) and (ii) are same.

3.3.1 Area between two given curves and two given ordinates

Let AB and CD be two curves represented by the equations $y = f_1(x)$ and $y = f_2(x)$ respectively.

Let the two ordinates be $x = a$ and $x = b$ indicated by P_1M_1 and P_2M_2 respectively.

$\therefore OM_1 = a$ and $OM_2 = b$.

Now the area included between the two curves $y = f_1(x)$ and $y = f_2(x)$, and the two ordinates $x = a$ and $x = b$ is the shaded part in the Fig. 4.6.

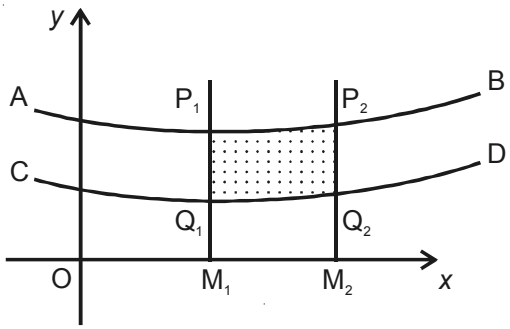


Fig. 4·6

$\therefore$ Required area = Area of $P_1M_1M_2P_2$ – area of $Q_1M_1M_2Q_2$

$$= \int_a^b f_1(x)dx - \int_a^b f_2(x)dx$$

$$= \int_a^b [f_1(x) - f_2(x)]dx$$

Example 5 : Find the area included between the curves $y^2 = 4ax$ and $y = mx$.

Solution : Here $y^2 = 4ax$ is a parabola whose vertex is at (0,0). Also $y = mx$ is a straight line passing through the origin and is inclined to x-axis at an angle θ where $\tan\theta = m$.

Now solving $y^2 = 4ax$ and $y = mx$ we get $(mx)^2 = 4ax$

$\Rightarrow x(xm^2 - 4a) = 0$

$\therefore x = 0$ or $x = \dfrac{4a}{m^2}$

$\therefore$ The two curves cut at the points

where $x = 0$ and $x = \dfrac{4a}{m^2}$.

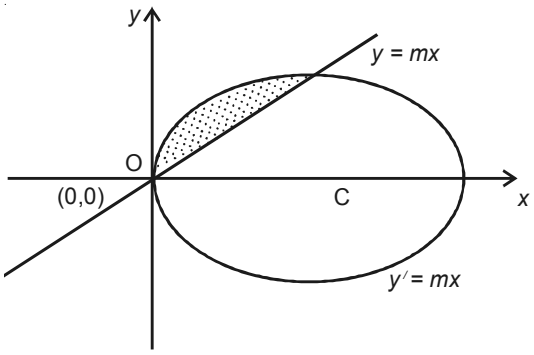


Fig. 4·7

$\therefore$ The required area is the shaded region which is included between the given curves.

$\therefore$ The required area $= \int_0^{4a/m^2} \left(\sqrt{4ax} - mx\right)dx$

$$= \left[2\sqrt{a}\,\frac{2}{3}x^{3/2}\right]^{4a/m^2} - \left[m\,\frac{1}{2}x^2\right]_0^{4a/m^2}$$

$$= \frac{4\sqrt{a}}{3}\left(\frac{4a}{m^2}\right)^{3/2} - \frac{m}{2}\left(\frac{4a}{m^2}\right)^2$$

$$= \frac{32a^2}{3m^3} - \frac{8a^2}{m^3} = \frac{8a^2}{3m^3}$$

Example 6 : Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$.

Solution : Here $x^2 + y^2 = 1$ is a circle whose centre is O (0,0) and radius is 1.

For the second circle, centre is B(1,0) and radius (BO) = 1.

Now solving the given equations we find that the two curves intersect at $A\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $E\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Now the required area is the region OEBAO, i.e. the shaded part (light shaded and dark shaded together). Here the dark shaded part (i.e. the region in the first quadrant) is symmetrical with the light shaded part (i.e. the region in the 4th quadrarnt).

$\therefore$ The required area

$= 2(\text{Area OCBAO})$

$= 2(\text{area OCAO} + \text{area CBAC})$ ... ...(i)

Area of OCAO

= area bounded by $(x-1)^2 + y^2 = 1$, x-axis and the ordinates $x = 0$ and $x = ½$.

$$= \int_0^{½} \sqrt{1-(x-1)^2}\, dx$$

Also the area of CBAC

= area bounded by $x^2 + y^2 = 1$, x-axis and the ordinates $x = ½$ and $x = 1$

$$= \int_{½}^{1} \sqrt{1-x^2}\, dx$$

$\therefore$ From (i), the total area of the region

$$= 2\left[\int_0^{½}\sqrt{1-(x-1)^2}\,dx + \int_{½}^{1}\sqrt{1-x^2}\,dx\right]$$

$$= 2\left[\frac{(x-1)\sqrt{1-(x-1)^2}}{2} + \frac{1}{2}\sin^{-1}\left(\frac{x-1}{1}\right)\right]_0^{½}$$

$$+ 2\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}(x)\right]_{½}^{1}$$

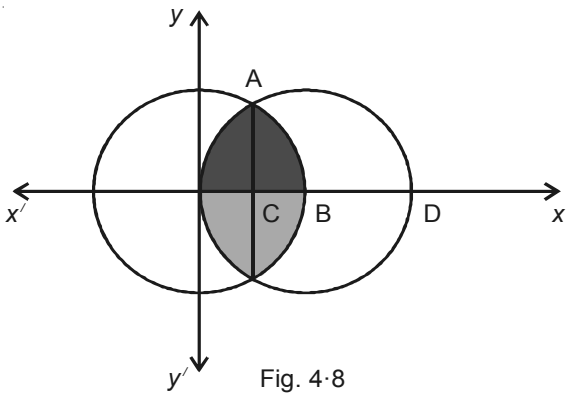


Fig. 4·8

$$= \left\{-\frac{\sqrt{3}}{4} + \sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right\}$$

$$+ \left\{\sin^{-1}(1) - \frac{\sqrt{3}}{4} - \sin^{-1}\left(\frac{1}{2}\right)\right\}$$

$$= \left\{-\frac{\sqrt{3}}{4} + \sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right\}$$

$$+ \left\{\sin^{-1}(1) - \frac{\sqrt{3}}{4} - \sin^{-1}\left(\frac{1}{2}\right)\right\}$$

$$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6}$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$



CHECK YOUR PROGRESS - 1

Q 1 : Find the area of the region bounded by the parabola $y^2 = 4ax$ and its latus rectum.

Q 2 : Find the area of the region bounded by the upper half of the circle $x^2 + y^2 = 25$, the *x*-axis and the ordinates $x = -3$ and $x = 4$.

Q 3 : Find the area of the region bounded by the curve $x = y^3$, the *y*-axis and lines $y = 3, y = 6$.

Q 4 : Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Q 5 : Find the area of the circle $x^2 + y^2 = a^2$.

Q 6 : Find the area of the segment cut off from $y^2 = 4x$ by the line $y = x$.

Q 7 : Find the area of the region included between the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Q 8 : Find the area of the region cut off from the parabola by $4y = 3x^2$ by the line $3x - 2y + 12 = 0$.

Q 9 : Find the area of the region which is common to the circle $x^2 + y^2 = 16a^2$ and the parabola $y^2 = 6ax$.

Q 10 : Show that the area of the region bounded by the curves $y^2 = 16x$ and $x = 4$ is $\frac{128}{3}$ sq. unit.

Q 11 : Find the area bounded by the curve $x^2 = 4y$ and the straight line $x - 4y + 2 = 0$.

Q 12 : Show that the area bounded by the parabola $y^2 = 4ax$ and the lines $y = 2a$ and $x = 0$ is $\frac{2}{3}a^2$ sq. unit.

Q 13 : Find the area bounded by the curves $y = x^3$ and $y = x$.

Q 14 : Find the area of the region bounded by $x^2 = 16y$, $y = 1$, $y = 4$ and the *y*-axis lying in the first quadrant.

Q 15 : Find the area common to the circle $x^2 + y^2 = 4$ and the ellipse $x^2 + 4y^2 = 9$.

Q 16 : Show that the larger of the two areas into which the circle $x^2 + y^2 = (8a)^2$ is divided by the parabola $y^2 = 12ax$ is

$$\frac{16a^2}{3}\left(8\pi - \sqrt{3}\right).$$

Q 17 : Show that the area common to the two curves $x^2 - 4ax + y^2 = 0$ and $y^2 = ax$ is $\left(3\sqrt{3} + \frac{4}{3}\pi\right)a^2$.

Q 18 : Find the area, lying above *x*-axis and included between the circle $x^2 - 8x + y^2 = 0$ and the parabola $4x = y^2$.



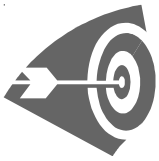
4.4 LET US SUM UP

- The area enclosed by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and the x-axis is given by $\int_a^b f(x)dx$ or $\int_a^b ydx$.
- The area bounded by the curve $x = f(y)$, y-axis and the abscissae $y = a$ and $y = b$ is given by $\int_a^b xdy$ of $\int_a^b f(y)dy$.
- The area between two curves $y = f_1(x)$ and $y = f_2(x)$ and two ordinates $x = a$ and $x = b$ is given by $\int_a^b [f_1(x) - f_2(x)]dx$.



4.5 FURTHER READINGS

1) *Integral Calculus,* Kishan, Hari, Atlantic Publishers & Distributors. New Delhi -110027.
2) *Integral Calculus*, Dhami, H.S. New age International Publishers, New Delhi - 110002.
3) *Text book of Integral Calculus*, Sharma, A.K. Discovery Publishing House, New Delhi -110002.



4.6 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1 : The latus rectum is the line perpendicular to the axis of the parabola and passing through the focus S(a,0). The parabola is symmetric about the x-axis. Limits for the upper half of the curve are $x = 0$ to $x = a$. The required area AOBSA = 2 area AOSA

$$= 2\int_0^a ydx$$

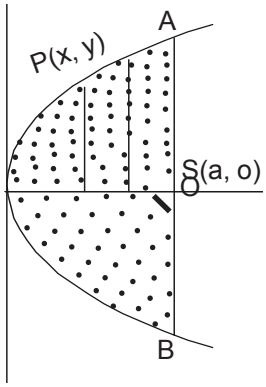


Fig. 4.9

$$= 2\int_0^a 2\sqrt{ax}\,dx$$

$$= 4\sqrt{a}\,\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^a$$

$$= \frac{8}{3}\sqrt{a}\,a^{\frac{3}{2}}$$

$$= \frac{8}{3}a^2.$$

Ans to Q No 2 : Do yourself.

Ans to Q No 3 : Do yourself.

Ans to Q No 4: The area BPAO $= \int_0^a y\,dx$

But $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $y = \frac{b}{a}\sqrt{a^2 - x^2}$

$$= \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

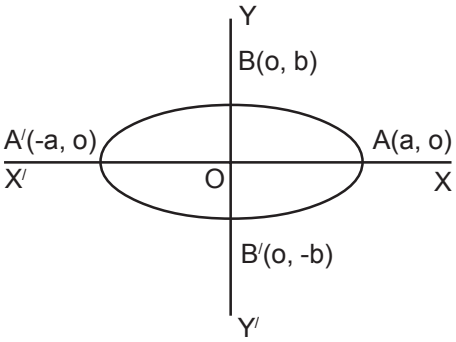


Fig. 4.10

$$= \frac{b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}.\frac{a^2}{2}\sin^{-1}1$$

$$= \frac{1}{2}ab.\frac{\pi}{2}$$

$$= \frac{\pi}{4}ab$$

$\therefore$ The whole area of the ellipse $= 4 \times \frac{\pi}{4}ab = \pi ab$.

Ans to Q No 5 : The area OAB $= \int_0^a y\,dx$

$$= \int_0^a \sqrt{a^2 - x^2}\,dx \qquad \left(y^2 = a^2 - x^2\right)$$

[$y = +\sqrt{a^2 - x^2}$, y being positive in the 1st and 2nd quadrants]

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{a^2}{2}.\frac{\pi}{2}$$

$$= \frac{\pi}{4}a^2$$

$\therefore$ area of the complete circle $= 4 \times \frac{\pi}{4}a^2 = \pi a^2$.

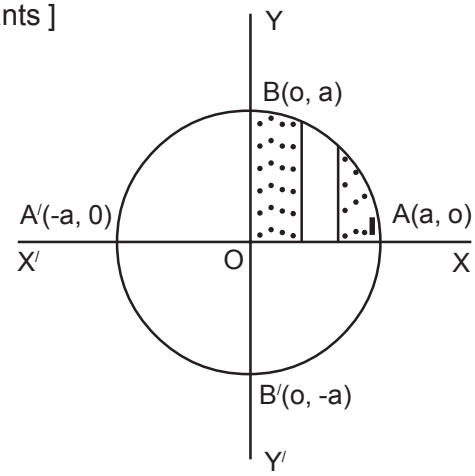


Fig. 4.11

Ans to Q No 6 : Do yourself.

Ans to Q No 7 : Solving the equations of the two given curves, we have

$$y^4 = 16a^2(4by) = 64a^2by$$

$$\therefore y\left(y^3 - 64a^2b\right) = 0$$

$$\Rightarrow y = 0, 4a^{\frac{2}{3}}b^{\frac{1}{3}}$$

when $y = 0, x = 0$

and when $y = 4a^{\frac{2}{3}}b^{\frac{1}{3}}, x = 4a^{\frac{1}{3}}b^{\frac{2}{3}}$.

A$\left(4a^{\frac{1}{3}}b^{\frac{2}{3}}, 4a^{\frac{2}{3}}b^{\frac{1}{3}}\right)$.

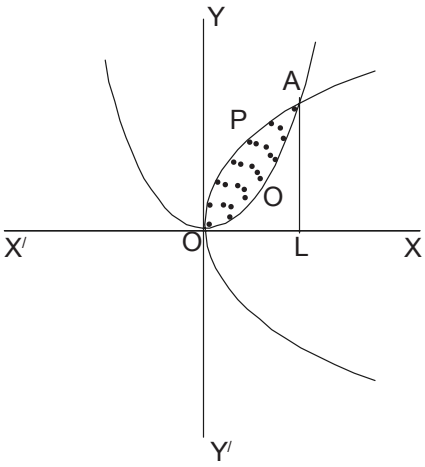


Fig. 4.12

$\therefore$ The required area = area OPAL- area OQAL.

$= \int_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} ydx$, (from the curve $y^2 = 4ax$) $- \int_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} ydx$ (from the curve $x^2 = 4by$).

$$= \int_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} \sqrt{4ax}dx - \int_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} \left(\frac{x^2}{4b}\right)dx$$

$$= 2\sqrt{a}\left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} - \frac{1}{4b}\left[\frac{x^3}{3}\right]_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}}$$

$$= 2\sqrt{a}\left[8\sqrt{a}.b\right] - \frac{1}{12b}\left(64ab^2\right)$$

$$= \frac{32}{3}ab - \frac{16}{3}ab$$

$$= \frac{16}{3}ab\,.$$

Ans to Q No 8 : Do yourself.

Ans to Q No 9 : Solving the two equations,we get $x = 2a$, $y = \pm 2\sqrt{3}a$

Required area = 2 [area OPR + area RAP]

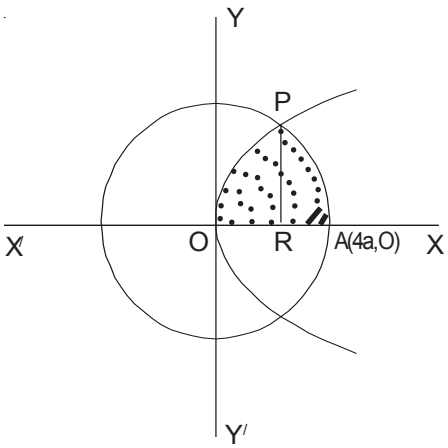


Fig. 4.13

$$= 2\int_0^{2a}\sqrt{6ax}\,dx + 2\int_{2a}^{4a}\sqrt{(4a)^2 - x^2}\,dx$$

$$= 2\sqrt{6a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{2a} + 2\left[\frac{x}{2}\sqrt{16a^2 - x^2} + \frac{16a^2}{2}\sin^{-1}\frac{x}{4a}\right]_{2a}^{4a}$$

$$= 2\sqrt{6a}.(2a)^{\frac{3}{2}} + 2\left[8a^2.\frac{\pi}{2} - a.2\sqrt{3}a - 8a^2\,\frac{\pi}{6}\right]$$

$$= \frac{4a^2}{3}\left(4\pi + \sqrt{3}\right) \text{ sq.units}$$

Ans to Q No 10 : Do yourself.

Ans to Q No 11 : The equations of the given curves are

$$x^2 = 4y \quad \text{... (1)}$$

and $$x = 4y - 2 \quad \text{...(2)}$$

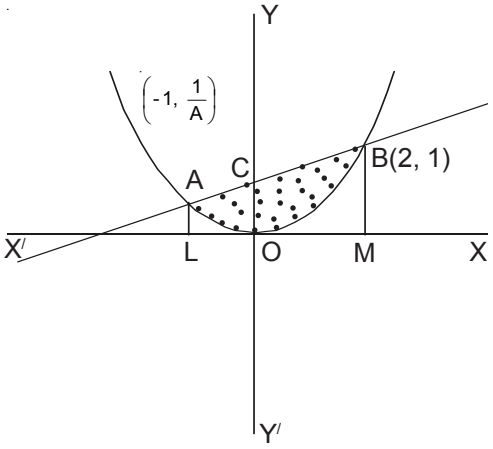


Fig. 4.14

Equation (1) represents a parabola with vertex at the origin and axis along positive direction of y-axis. Equation (2) represents a straight line which meets the coordinate axes at (-2,0) and $\left(0, \frac{1}{2}\right)$ respectively.

We solve (1) and (2) simultaneously.We obtain that the points of intersection of the given parabola and the line are B(2,1) and $A\left(-1, \frac{1}{4}\right)$.

We draw AL and BM perpendicular to x-axis.

$\therefore$ Area OBAO = Area OBCO + Area OACO ... (1)

then Area OBAO = Area OBCO - Area OMBO

$$= \int_0^2 \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx$$

$$= \left[\frac{x^2}{8} + \frac{1}{2}x - \frac{x^3}{12} \right]_0^2$$

$$= \frac{1}{4}(2 + 4) - \frac{1}{4}\left(\frac{8}{3} \right)$$

$$= \frac{5}{6}$$

Similarly, Area OACO = Area OLAC - Area OLAO

$$= \int_{-1}^0 \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx$$

$$= \left[\frac{x^2}{8} + \frac{1}{2}x - \frac{x^3}{12} \right]_{-1}^0$$

$$= \frac{1}{4}\left(\frac{1}{2} - 2 \right) - \frac{1}{12}$$

$$= \frac{1}{2} - \frac{1}{8} - \frac{1}{12}$$

$$= \frac{7}{24}$$

$\therefore$ Required area $= \frac{5}{6} + \frac{7}{24} = \frac{9}{8}$ units.

Ans to Q No 12 : The equation $y^2 = 4ax$ represents a parabola with vertex (0, 0) and axis as x-axis. The equation $y = 2a$ represents a straight line parallel to x-axis at a distance 2a from it.

$\therefore$ Required area $= \int_0^{2a} x\,dy$

$$= \int_0^{2a} \frac{y^2}{4a} dy$$

$$= \frac{1}{4a}\left[\frac{y^3}{3} \right]_0^{2a}$$

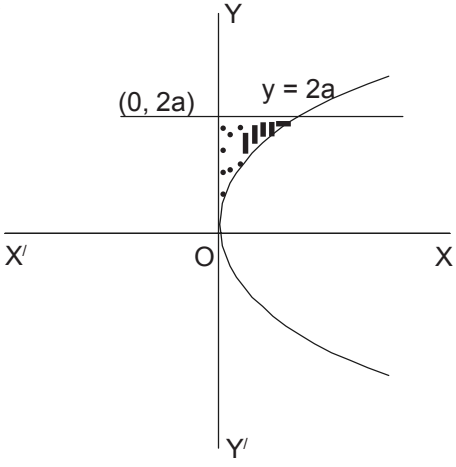


Fig. 4.15

$$= \frac{1}{4a}\left[\frac{8a^3}{3} - 0\right]$$

$$= \frac{2a^2}{3} \text{ sq. units}$$

Ans to Q No 13 : The equations of the given curves are $y = x$ ... (1)

and $y = x^3$ ... (2)

Solving (1) and (2), we find that two curves intersect at O(0,0), A(1,1) and B(-1,-1).

$\therefore$ Required area = Area BCOB + Area ODAO

Area BCOB $= \left|\int_{-1}^{0} (y_4 - y_3)dx\right|$

$= \left|\int_{-1}^{0} (x - x^3)dx\right|$ [since, $R(x, y_3)$ and $S(x, y_4)$ lie on (2) and (1)]

Area ODAO $= \int_{0}^{1} (y_2 - y_1)dx$

$= \int_{0}^{1} (x - x^3)dx$ [since, $P(x, y_1)$ and $Q(x, y_2)$ lie on (2) and (1)]

Hence, required area $= \left|\int_{-1}^{0} (x - x^3)dx\right| + \int_{0}^{1} (x - x^3)dx$

$$= \left|\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{-1}^{0}\right| + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1}$$

$$= \left|-\left(\frac{1}{2} - \frac{1}{4}\right)\right| + \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2} \text{ sq. units}$$

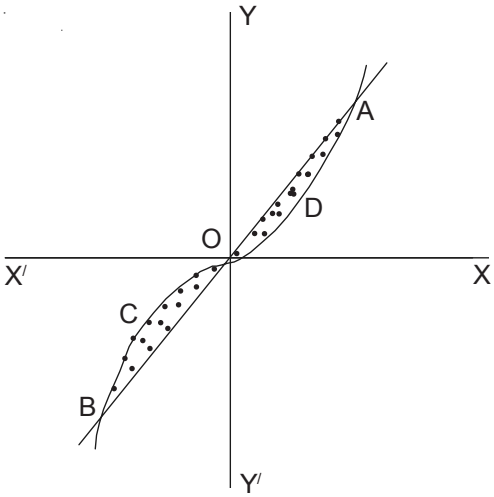


Fig. 4.16

Ans to Q No 14 : Do yourself.

Ans to Q No 15 : The equation of the circle is $x^2 + y^2 = 4$ ... (1)

and the equation of the ellipse is $x^2 + 4y^2 = 9$ ... (2)

Both the curves (1) and (2) are symmmetrical about both the axes.

Now, we solve (1) and (2) for x

$x^2 + 4(4 - x^2) = 9$ or $3x^2 = 7$ or $x^2 = \frac{7}{3}$.

$\therefore$ the x-coordinate of the point of intersection P lying in the first quadrant is $\sqrt{\frac{7}{3}}$. Also, putting $y = 0$ in $x^2 + y^2 = 4$, weget $x = 2$ at C.

$\therefore$ Required area = 4(common area lying in the first quadrant)

= 4 area OCPB

= 4 [area OBPM + area CPM]

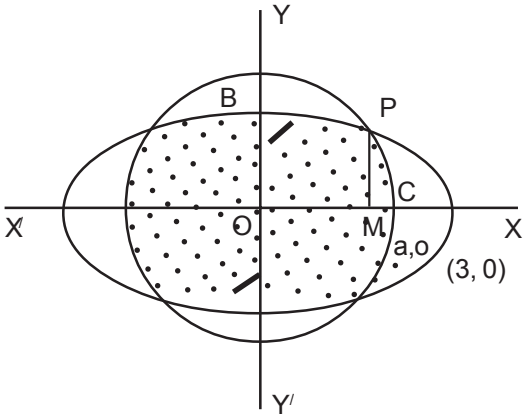


Fig. 4.17

$$= 4\left[\int_0^{\frac{7}{\sqrt{3}}} y\,dx, \text{for the ellipse} + \int_{\frac{7}{\sqrt{3}}}^{2} y\,dx, \text{for the circle}\right]$$

$$= 4\left[\int_0^{\frac{7}{\sqrt{3}}} \frac{1}{2}\sqrt{9-x^2}\,dx + \int_{\frac{7}{\sqrt{3}}}^{2} \sqrt{4-x^2}\,dx\right]$$

$$= 2\left[\frac{x\sqrt{9-x^2}}{2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^{\frac{7}{\sqrt{3}}} + 4\left[\frac{x\sqrt{4-x^2}}{2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{\frac{7}{\sqrt{3}}}^{2}$$

$$= 2\left[\frac{1}{2}\sqrt{\frac{7}{3}}\sqrt{\frac{20}{3}} + \frac{9}{2}\sin^{-1}\left\{\frac{1}{2}\sqrt{\frac{7}{3}}\right\}\right] + 4\left[2\sin^{-1}(1) - \frac{1}{2}\sqrt{\frac{7}{3}}\sqrt{\frac{5}{3}} - 2\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{7}{3}}\right)\right]$$

$$= \frac{2\sqrt{35}}{3} + 9\sin^{-1}\frac{\sqrt{7}}{3\sqrt{3}} + 4\pi - \frac{2}{3}\sqrt{35} - 8\sin^{-1}\frac{\sqrt{7}}{2\sqrt{3}}$$

$$= 4\pi + 9\sin^{-1}\frac{\sqrt{7}}{3\sqrt{3}} - 8\sin^{-1}\frac{\sqrt{7}}{2\sqrt{3}}$$

Ans to Q No 16 : $x^2 + y^2 = 64a^2$ is a circle with centre(0,0) and radius 8a and $y^2 = 12ax$ is a parabola whose vertex is at (0,0) and latus rectum 12a.Both the curves are symmetrical about x-axis.Solving the two equations the common point P is $\left(4a, 4a\sqrt{3}\right)$.We draw PM perpendicular from P to the y-axis.

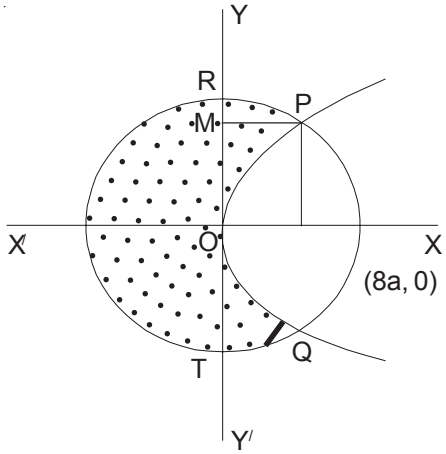


Fig. 4.18

Required area of the larger portion of the circle = Area of PRSTQOP = Area of the semi-circle RST + 2AreaOPR

$$= \frac{1}{2}.\pi.(8a)^2 + 2[\text{area OPM + area MPR }]$$

$$= \frac{1}{2}.\pi.(8a)^2 + 2\int_0^{\frac{4a}{\sqrt{3}}} xdy(\text{for } y^2 = 12ax) + 2\int_{\frac{4a}{\sqrt{3}}}^{8a} xdy(\text{for } x^2 + y^2 = 64a^2)$$

$$= \frac{1}{2}.\pi.(8a)^2 + 2\int_0^{\frac{4a}{\sqrt{3}}} \frac{y^2}{12a}dy + 2\int_{\frac{4a}{\sqrt{3}}}^{8a} \sqrt{64a^2 - y^2}\,dy$$

$$= 32\pi a^2 + \frac{1}{6a}\left[\frac{y^3}{3}\right]_0^{\frac{4a}{\sqrt{3}}} + 2\left[\frac{1}{2}y\sqrt{64a^2 - y^2} + \frac{64a^2}{2}\sin^{-1}\frac{y}{8a}\right]_{\frac{4a}{\sqrt{3}}}^{8a}$$

$$= 32\pi a^2 + \frac{1}{6a}\left[\frac{64\times 3\sqrt{3}a^3}{3}\right] + 2\left[\left\{0 - 8a^2\sqrt{2}\right\} + 32a^2\left\{\sin^{-1}1 - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\}\right]$$

$$= 32\pi a^2 + \frac{32\sqrt{3}a^3}{3} - 16a^2\sqrt{3} + \frac{32}{3}a^2\pi$$

$$= \frac{128}{3}\pi a^2 - \frac{16}{3}a^2\sqrt{3}$$

$$= \frac{16}{3}a^2\left(8\pi - \sqrt{3}\right).$$

Ans to Q No 17: $y^2 = ax$ is a parabola with vertex at the origin and axis along x-axis and latus rectum a, and $x^2 - 4ax + y^2 = 0$ is a circle with centre (2a, 0) and radius 2a.

Both these curves are symmetrical about x-axis.

Solving the equations of the two curves for x, we have

$x^2 + ax = 4ax$ or $x^2 - 3ax = 0$ or $x(x-3a) = 0$

$\therefore x = 0, 3a$

Thus, the two curves intersect at the points where $x = 0$ and $x = 3a$.

Also A is the point $(4a, 0)$.

The area common to the parabola and the circle (i.e.,the shaded area)

= 2[Area OPS + Area PSA]

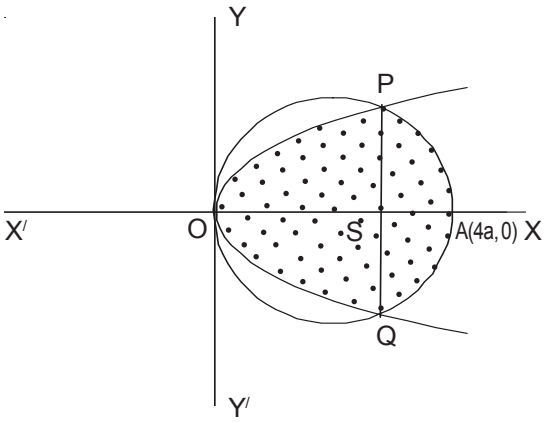


Fig. 4.19

$$= 2\left[\int_0^{3a} y\,dx, \text{from the parabola } y^2 = ax + \int_{3a}^{4a} y\,dx, \text{from the circle } x^2 + y^2 = 4ax\right]$$

$$= 2\left[\int_0^{3a} \sqrt{ax}\,dx + \int_{3a}^{4a} \sqrt{4ax - x^2}\,dx\right]$$

$$= 2\sqrt{a}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^{3a} + 2\left[\frac{1}{2}(x-2a)\sqrt{\left\{4a^2 - (x-2a)^2\right\}} + \frac{4a^2}{2}\sin^{-1}\frac{x-2a}{2a}\right]_{3a}^{4a}$$

$$= 4a^2\sqrt{3} + 2\left[\left\{0 - \frac{1}{2}a\sqrt{3}.a\right\} + 2a^2\left\{\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{6}\right)\right\}\right]$$

$$= 4\sqrt{3}a^2 - a^2\sqrt{3} + \frac{4}{3}\pi a^2$$

$$= 3\sqrt{3}a^2 + \frac{4}{3}\pi a^2$$

$$= a^2\left(3\sqrt{3} + \frac{4}{3}\pi\right).$$



4.7 MODEL QUESTIONS

Q 1 : Find the area of the quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ between the semi major and semi minor axes.

Q 2 : Find the area above the *x*-axis, included between the parabola $y^2=ax$ and the circle $x^2+y^2=2ax$.

Q 3 : Find the area common to the two curves $y^2=ax$ and $x^2+y^2=4ax$.

Q 4 : Find the area common between the curves $y^2=4ax$ and $x^2=4ay$.

Q 5 : Find the area between the parabola $y^2=8x$ and the line $x=2$.

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