UNIT 5 : RECTIFICATION OF PLANE CURVES

UNIT STRUCTURE

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5.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- define the term 'Rectification'
- find the length of a plane curve

5.2 INTRODUCTION

In previous unit, we have discussed quadratures of plane areas. In this unit we will discuss rectification of plane curves.

5.3 RECTIFICATION OF PLANE CURVES

The process of finding the length of an arc of a curve between two given points is called "rectification".

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighbouring points on the curve $y = f(x)$. Let s be the length of the arc from A to P. Also let δs be the length of the arc from P to Q.

PM and QN are perpendiculars from P and Q on *x*-axis. PS is perpendicular on QN. Then the chord PQ is joined. Now from the adjoining Fig. 5.1, we find

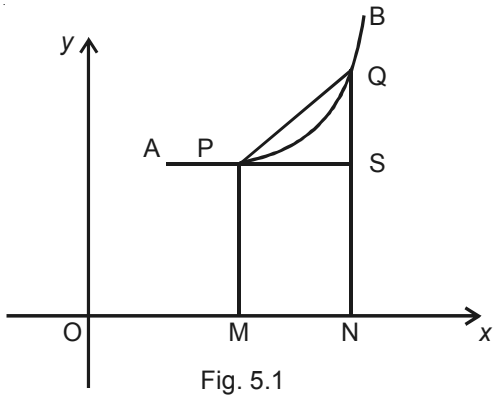


Fig. 5.1

$(\text{Chord PQ})^2 = (\text{PS})^2 + (\text{QS})^2$

$\Rightarrow (\text{chord PQ})^2 = (\delta x)^2 + (\delta y)^2$

Dividing both sides by $(\delta x)^2$, we get

$$\left(\frac{\text{Chord PQ}}{\delta x}\right)^2 = 1+\left(\frac{\delta y}{\delta x}\right)^2$$

$$\Rightarrow \left(\frac{\text{chord PQ}}{\text{arc PQ}}.\frac{\text{arc PQ}}{\delta x}\right)^2 = 1+\left(\frac{\delta y}{\delta x}\right)^2$$

$$\Rightarrow \left(\frac{\text{chord PQ}}{\text{arc PQ}}\right)^2.\left(\frac{\delta s}{\delta x}\right)^2 = 1+\left(\frac{\delta y}{\delta x}\right)^2$$

Now if $Q \to P$ along the curve, then $\delta x \to 0$ and so we get

$$\left(\frac{ds}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2 \qquad \left[\lim_{Q\to P}\left(\frac{\text{chord PQ}}{\text{arc PQ}}\right) = 1\right]$$

Hence if s increases when x increases, then we get $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

Integrating we get

$s = \int_a^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ where the lower limit a is the abscissa of the fixed point from where s is measured (here a is the abscissa of the point A).

Hence the arc length s of the curve $y = f(x)$ included between two points for which $x = a$ and $x = b$ is equal to

$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{where } b > a.$$

Note: If the curve is taken as $x = f(y)$ then the arc length between $y = c$ and $y = d$ is given by $\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy, \quad d > c$.

ILLUSTRATIVE EXAMPLES

Example 1: Find the length of the circle $x^2 + y^2 = a^2$.

Solution: Here $x^2 + y^2 = a^2$

Differentiating w.r.t. x we get,

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$\therefore$ Total length of the curve (i.e. circumference)

= 4 × length of the arc in the first quadrant.

$$= 4\int_{x=0}^{x=a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\int_0^a \sqrt{\frac{a^2}{a^2 - x^2}}\, dx$$

$$= 4a\int_0^a \frac{1}{\sqrt{a^2 - x^2}}\, dx$$

$$= 4a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4a\left[\sin^{-1}(1) - \sin^{-1}(0)\right] = 4a\left(\frac{\pi}{2} - 0\right) = 2\pi a$$

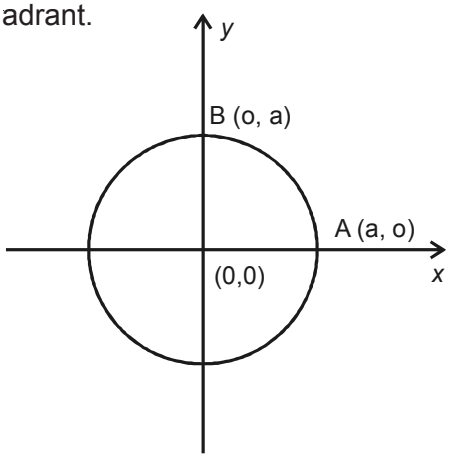


Fig. 5.2

Example 2 : Find the length of the arc of the parabola $y^2 = 4ax$ extending from the vertex to an extremity of the latus rectum.

Solution: For the parabola $y^2 = 4ax$ the coordinates of the vertex O are (0,0) and the coordinates of one extremity L of the latus rectum LSL' are (a, 2a). Now we are to find the length of the arc from O to L.

Now, $y^2 = 4ax$

Differentiating w.r.t. *x, we get*

$$2y\frac{dy}{dx} = 4a,$$

$$\therefore \frac{dy}{dx} = \frac{2a}{y},$$

or $$\frac{dx}{dy} = \frac{y}{2a} \quad ...(i)$$

If *s* denotes the length of the parabola measured of the vertex O to any point $P(x, y)$ towards the point L, then S increases as *y* increases.

Therefore $\frac{ds}{dy}$ is positive. So we can write

Now, $$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2$$

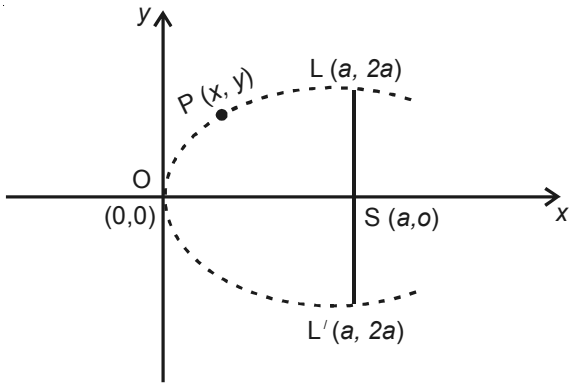


Fig. 5.3

$$= 1 + \left(\frac{y}{2a}\right)^2$$

$$= \frac{4a^2 + y^2}{4a^2}$$

$$\therefore \frac{ds}{dx} = \sqrt{\frac{4a^2 + y^2}{4a^2}} \Rightarrow ds = \frac{1}{2a}\sqrt{4a^2 + y^2}\,dy$$

$$s = \int_{y=0}^{y=2a} \frac{1}{2a}\sqrt{4a^2 + y^2}\,dy$$

$$= \frac{1}{2a}\left[\frac{y}{2}\sqrt{4a^2 + y^2} + \frac{4a^2}{2}\log\left\{y + \sqrt{4a^2 + y^2}\right\}\right]_0^{2a}$$

$$= \frac{1}{2a}\left[\frac{2a}{2}\sqrt{4a^2 + 4a^2} + 2a^2\log\left(2a + \sqrt{4a^2 + 4a^2}\right) - 2a^2\log 2a\right]$$

$$= \frac{1}{2a}\left[a.2a\sqrt{2} + 2a^2\log\left(2a + 2a\sqrt{2}\right) - 2a^2\log 2a\right]$$

$$= a\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right]$$

Example 3 : Find the whole length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution : The shape of the curve is as shown in the fig. 5.4. The curve is symmetrical about both the axes. The curve crosses *x*-axis at the points A(*a*,*o*) and C(–*a*,*o*). Also it crosses *y*-axis at the points B(*o*,*a*) and D(*o*,–*a*). From the figure it is seen that the arc of the curve in the first quadrant lies from A(*a*,*o*) to B(*o*,*a*).

Now, $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating w.r.t x,

$$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{-1/3} = -\left(\frac{y}{x}\right)^{1/3}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{y^{2/3}}{x^{2/3}} = \frac{x^{2/3} + y^{2/3}}{x^{2/3}} = \frac{a^{2/3}}{x^{2/3}}$$

$\therefore$ The whole length of the curve

$$= 4 \times \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 4\int_0^a \sqrt{\frac{a^{2/3}}{x^{2/3}}}\,dx$$

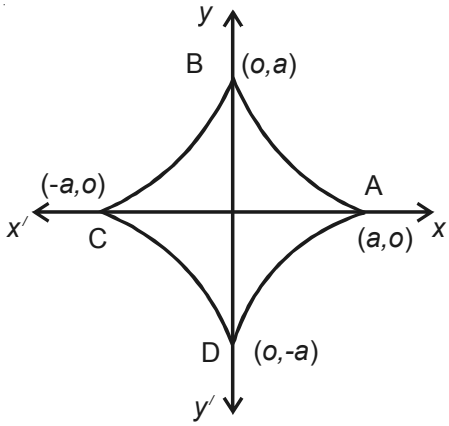


Fig. 5.4

$$= 4a^{1/3}\int_0^a x^{-1/3}dx = 4a^{1/3}\left[\frac{3}{2}x^{2/3}\right]_0^a = 6a^{1/3}.a^{2/3}$$

$$= 6a$$



CHECK YOUR PROGRESS

Q 1 : Find the perimeter of the circle $x^2 + y^2 = a^2$.

Q 2 : Find the length of the arc of the parabola $x^2 = 4ay$ cut off by its latus rectum.

Q 3 : Find the length of the arc of the parabola $y^2 = 4ax$ between the points where the line $y = 3x$ intersects the parabola.

Q 4 : Find the length of the arc of the parabola $x^2 = 4y$ from the vertex to the point where $x = 2$.

Q 5 : Find the lengths of the following curves between the points indicated with equations of the curves.

(a) $y = \log\sec x$, $x = 0$ and $x = \pi/3$

(b) $4y + \log x - 2x^2 = 0$, $x = 1$ and $x = 2$

(c) $ay^2 = x^3$, from the point (0,0) to the point (a,a)

(d) $6xy - x^4 = 3$, from $x = 1$ to $x = 2$

(e) $y = ae^x$, between the points (o,a) and (x,y).

Q 6 : Show that the length of the loop of the curve $3ay^2 = x(x-a)^2$ in $4a/\sqrt{3}$.

Q 7 : Find the total length of the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$.

Q 8 : Find the length of the curve $x^{2/3} + y^{2/3} = 4$ from $x = 0$ to $x = 8$ in the first quadrant.



5.4 LET US SUM UP

- The process of finding the length of an arc of a curve between two given points is called “rectification”.
- The arc length S of the curve $y = f(x)$ included between two points $x = a$ and $x = b$ is given by $S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx,\ b > a$.
- If the curve is taken as $x = f(y)$ then the arc length between $y = c$ and $y = d$ is given by $S = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy,\ d > c$.



5.5 FURTHER READINGS

1) *Integral Calculus,* Kishan, Hari, Atlantic Publishers & Distributors. New Delhi -110027.

2) *Integral Calculus*, Dhami, H.S. New age International Publishers, New Delhi - 110002.

3) *Text book of Integral Calculus*, Sharma, A.K. Discovery Publishing House, New Delhi -110002.



5.6 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS

Ans to Q No 1 : The equation of the curve is $x^2 + y^2 = a^2$ ... (1)

Here the curve is a circle with centre (0,0) and radius a. Also, it is symmetrical about both the axes. So, the required perimeter will be

four times the arc length lying in the first quadrant i.e.,between $x = 0$ to $x = a$.

Differentiating (1) w.r.t x, we get $2x + 2y\dfrac{dy}{dx} = 0$

or $\dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right) = -\dfrac{x}{\sqrt{a^2 - x^2}}$ from (1)

$\therefore$ the required perimeter = 4 $\times$ {length of the arc in the first quadrant from (0, a) to (a, 0)}

$$= 4\int_{x=0}^{a}\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$$

$$= 4\int_{x=0}^{a}\sqrt{\left[1+\frac{x^2}{a^2-x^2}\right]}dx$$

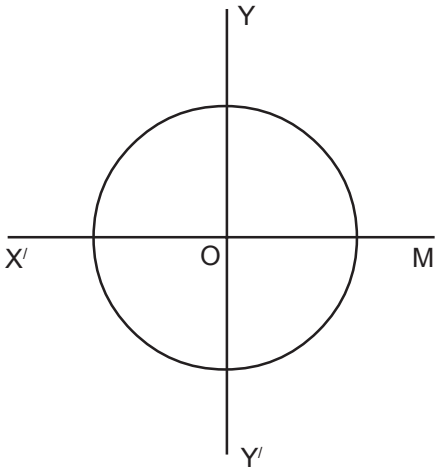
$$= 4a\int_{0}^{a}\frac{dx}{\sqrt{a^2-x^2}}dx$$

$$= 4a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4a\left[\sin^{-1}1 - \sin^{-1}0\right]$$

$$= 4a\left[\frac{\pi}{2} - 0\right]$$

$$= 2\pi a.$$



Ans to Q No 2 : Do yourself.

Ans to Q No 3 : We have $y^2 = 4ax$ ... (1)

and $y = 3x$ ... (2)

Substituting for x in (1) from (2), we get

$$y^2 = 4a\left(\frac{y}{3}\right)$$

or $3y^2 - 4ay = 0$

or $y(3y - 4a) = 0$

or $y = 0, \dfrac{4a}{3}$

Hence the points of intersection of the parabola and the given line are (0, 0) and $\left(\frac{4a}{9}, \frac{4a}{3}\right)$.

Also, differentiating (1) w.r.t. x, we get $2y\frac{dy}{dx} = 4a$.

$$\therefore \frac{dy}{dx} = \frac{2a}{y}$$

$$\text{or } \frac{dx}{dy} = \frac{y}{2a}$$

If S denotes the arc length of the parabola measured from the point (0, 0) to the point $\left(\frac{4a}{9}, \frac{4a}{3}\right)$, then

$$S = \int_0^{\frac{4a}{3}} \sqrt{\left[1 + \left(\frac{dx}{dt}\right)^2\right]} dy$$

$$= \int_0^{\frac{4a}{3}} \sqrt{\left[1 + \frac{y^2}{4a^2}\right]} dy$$

$$= \frac{1}{2a} \int_0^{\frac{4a}{3}} \sqrt{\left(y^2 + 4a^2\right)} dy$$

$$= \frac{1}{2a}\left[\frac{y}{2}\sqrt{\left(y^2 + 4a^2\right)} + \frac{4a^2}{2}\log\left\{y + \sqrt{\left(y^2 + 4a^2\right)}\right\}\right]_0^{\frac{4a}{3}}$$

$$= \frac{1}{2a}\left[\frac{1}{2}\frac{4}{3}a\sqrt{\left(\frac{16}{9}a^2 + 4a^2\right)} + 2a^2\log\left\{\frac{4}{3}a^2 + \frac{2}{3}a\sqrt{13}\right\} - 2a^2\log(2a)\right]$$

$$= a\left[\frac{2}{9}\sqrt{13} + \log\left\{\frac{2}{3} + \frac{1}{3}\sqrt{13}\right\}\right]$$

Ans to Q No 4 : Do yourself.

Ans to Q No 5 : (a) The given curve is $y = \log\sec x$ ... (1)

Differentiating (1) w.r.t, x, we get

$$\frac{dy}{dx} = \frac{1}{\sec x}\sec x \tan x = \tan x$$

Now, $\left(\frac{ds}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2 = 1+\tan^2 x = \sec^2 x$ ... (2)

If the arc length S of the given curve is measured from $x = 0$ in the direction of x increasing, we have

$\frac{dS}{dx} = \sec x$ or $dS == \sec x dx$

Therefore if S_1 denotes the arc length from $x = 0$ to $x = \frac{\pi}{3}$, then

$$\int_0^{S_1} dS = \int_0^{\frac{\pi}{3}} \sec x dx = \left[\log(\sec x + \tan x)\right]_0^{\frac{\pi}{3}}$$

or $S_1 = \left[\log\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) - \log 1\right] = \log\left(2+\sqrt{3}\right).$

(b) The given curve is $4y+\log x-2x^2 = 0$

or $y = \frac{1}{2}x^2 - \frac{1}{4}\log x$ ... (1)

Differentiating (1) w.r,t. x, we get

$$\frac{dy}{dx} = x - \frac{1}{4x} = \frac{4x^2-1}{4x}$$

$\therefore$ the required length of the curve $= \int_1^2 \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} dx$

$$= \int_1^2 \sqrt{\left\{1+\frac{\left(4x^2-1\right)}{16x^2}\right\}} dx$$

$$= \int_1^2 \frac{4x^2+1}{4x} dx$$

$$= \int_1^2 \left(x+\frac{1}{4x}\right) dx$$

$$= \left[\frac{x^2}{2} + \frac{\log x}{4}\right]_1^2$$

$$= \frac{3}{2} + \frac{1}{4}\log 2 .$$

(c) The given curve is $ay^2 = x^3$ ... (1)

It is symmetrical about the x-axis.We have to find the length of the arc from

$x = 0$ to $x = a$ in the first quadrant.

Differentiating (1) w.r.t. x, we get $2ay\left(\frac{dy}{dx}\right) = 3x^2$

$$\therefore \frac{dy}{dx} = \frac{3x^2}{2ay} = \frac{3x^2}{2a\left(\frac{x^3}{a}\right)^{\frac{1}{2}}}$$, substituting for y from (1)

$$= \frac{3}{2}\left(\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)$$

$$\therefore \text{required length} = \int_0^a \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$$

$$= \int_0^a \sqrt{\left\{1+\frac{9x}{4a}\right\}}dx$$

$$= \frac{1}{2\sqrt{a}}\int_0^a (9x+4a)^{\frac{1}{2}}dx$$

$$= \frac{1}{2\sqrt{a}}\left[\frac{2}{3}..(9x+4a)^{\frac{3}{2}}.\frac{1}{9}\right]_0^a$$

$$= \frac{1}{2\sqrt{a}}\left[(13a)^{\frac{3}{2}} - (4a)^{\frac{3}{2}}\right]$$

$$= \frac{a}{27}\left[13\sqrt{13} - 8\right].$$

(d) Do yourself.

(e) Do yourself.

Ans to Q No 6 : The given curve is symmetrical about the x-axis.

At $y = 0$, we have $x = 0$ and $x = a$.

so, for the loop, x varies from 0 to a.

The equation of the given curve is $3ay^2 = x(x-a)^2$.

Taking logarthm, we have $\log 3a + 2\log y = \log x + 2\log(x-a)$

Now differentiating w.r.t x, we get $\frac{2}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{2}{x-a} = \frac{3x-a}{x(x-a)}$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{(3x-a)^2}{x^2(x-a)^2} \cdot \frac{y^2}{4}$$

$$= \frac{(3x-a)^2 x(x-a)^2}{x^2(x-a)^2 .12a}$$

$$= \frac{(3x-a)^2}{12ax}$$

$$\therefore \frac{ds}{dx} = \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{1+\frac{(3x-a)^2}{12ax}} = \frac{3x+a}{\sqrt{12ax}}$$

$\therefore$ the required length of the whole loop $= 2\int_0^a \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx$

$$= 2\int_0^a \frac{3x+a}{\sqrt{12ax}}dx$$

$$= \frac{1}{\sqrt{3a}}\int_0^a \left(3x^{\frac{1}{2}} + ax^{-\frac{1}{2}}\right)dx$$

$$= \frac{1}{\sqrt{3a}}\left[3.\frac{2}{3}x^{\frac{3}{2}} + 2ax^{\frac{1}{2}}\right]_0^a$$

$$= \frac{1}{\sqrt{3a}}\left[4a^{\frac{3}{2}}\right]$$

$$= \frac{4a}{\sqrt{3}}.$$

Ans to Q No 7 : Do yourself.



5.7 MODEL QUESTIONS

Q 1 : Show that the length of the curve $y = \log\sec x$ between the points and $x = \pi/3$.

Q 2 : Find the perimeter of the circle $x^2 + y^2 = a^2$.

Q 3 : Find the perimeter of the hypo-cycloid $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$.

Q 4 : Prove that the length of the loop of the curve $3ay^2 = x(x-a)^2$ is $4a\sqrt{3}$.

Q 5 : Find the length of the arc of the parabola $x^2 = 4y$ from the vertex to the point where $x = 2$.

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