UNIT 6 : VOLUME OF REVOLUTION AND SURFACE OF REVOLUTION

UNIT STRUCTURE

6.1 Learning Objectives

6.2 Introduction

6.3 Volume of Solids of Revolution

6.4 Surface of Solids of Revolution

6.5 Let Us Sum Up

6.6 Further Readings

6.7 Answers to Check Your Progress

6.8 Model Questions

6.1 LEARNING OBJECTIVES

After going through this unit, you will be able to-

- the volume of a solid of revolution
- the surface area of a solid of revolution

6.2 INTRODUCTION

In previous units,we have discussed quadratures and rectification of plane curves.In this unit we will focus on volume of solids of revolution as well as surface of solids of revolution.

6.3 VOLUMES OF SOLIDS OF REVOLUTION

Definition : Solid of revolution : If a plane curve or a plane area is revolved about a fixed line in its own plane then the body so generated by the revolution is called a solid of revolution.

Surface of revolution : If a plane curve is revolved about a fixed line in its own plane, then the surface so generated by the revolution of the curve is called a surface of revolution.

Axis of revolution : The fixed straight line about which the area or the curve revolves to form a solid or a surface is called the axis of revolution or axis of rotation.

For example, we may take a right angled triangle ABC where $\angle B = 90^0$. Now if ΔABC revolves about AB then it will generate a right circular cone. The area of the triangle will generate the volume and the hypotenuse AC will generate the curved surface of the cone.

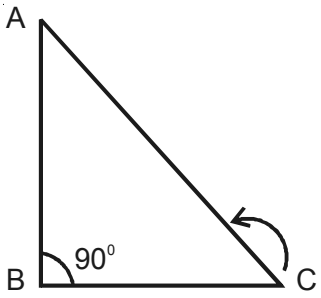


Fig. 6.1

The volume of the solid generated by the revolution of the area bounded by the curve $y = f(x)$, *x*-axis and the ordinates about the axis.

Let AB be the curve given by the equation $y = f(x)$. Let AL and BM be the two given ordinates $x = a$ and $x = b$. Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be any two neighbouring points on the curve. PC and QD be the perpendiculars drawn from P and Q respectively on *x*-axis. Also PS is perpendicular on QD and QR is perpendicular on CP produced.

Let the volumes of the solids generated by the revolution of the areas ALCPA and ALDQPA about *x*-axis be v and $v + \delta v$ respectively. Then δv is the volume of the solid generated by the revolution of the area PCDQP about *x*-axis.

Now $PC = y$, $QD = y + \delta y$ and $CD = \delta x$

$\therefore$ Volume generated by revolving the area $PCDS = \pi y^2 \delta x$ and the volume generated by revolving the area $RCDQ = \pi(y + \delta y)^2 \delta x$.

Also the volume generated by revolving the area $PCDQP = \delta v$, where δv lies between $\pi y^2 \delta x$ and $\pi(y + \delta y)^2 \delta x$.

i.e. $\pi y^2 \delta x \le \delta v \le \pi(y + \delta y)^2 \delta x$

$$\Rightarrow \pi y^2 \le \frac{\delta v}{\delta x} \le \pi(y + \delta y)^2$$

But if $Q \to P$ along the curve, then $\delta x \to o$ and also $\delta y \to o$ and then we get

$$\lim_{\delta x \to 0} \frac{\delta v}{\delta x} = \pi y^2$$

i.e. $$\frac{dv}{dx} = \pi y^2,$$

or $$dv = \pi y^2 dx$$

Integrating we get

$$\int_{x=a}^{b} dv = \int_{x=a}^{b} \pi y^2 dx$$

$\Rightarrow [v]_a^b = \int_a^b \pi y^2 dx$

$\Rightarrow$ (volume v at $x = b$) – (volume v at $x = a$)= $\int_a^b \pi y^2 dx$

$\Rightarrow$ (volume v at $x = b$) – 0 $= \int_a^b \pi y^2 dx$

$\Rightarrow$ Volume generated by the area ALMBA $= \int_a^b \pi y^2 dx$

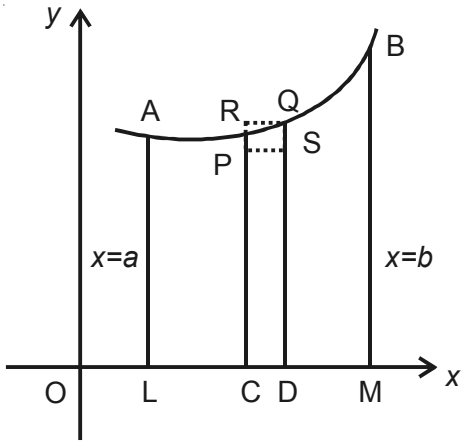


Fig. 6.2

$\therefore$ The volume of the solid generated by the revolution of the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$ about x axis is given by $\int_a^b \pi y^2 dx$.

ILLUSTRATIVE EXAMPLES

Example 1: Find the volume of a sphere of radius a.

Solution : Let $x^2 + y^2 = a^2$ be a circle whose centre is O(0,0) and radius is $OA = a$. Here OAB is the first quadrant of the circle. If this portion (or quadrant) of the circle is rotated about x-axis then it will form a hemisphere.

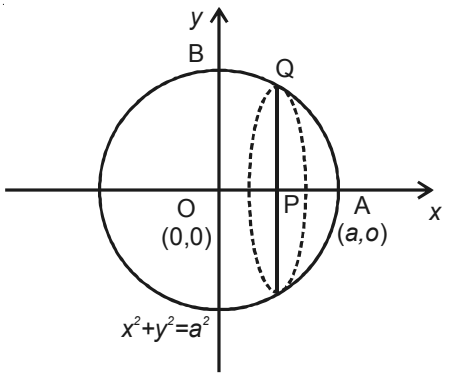


Fig. 6.3

$\therefore$ The volume of the hemisphere $= \int_0^a \pi y^2 dx$

$\therefore$ Required volume of the sphere (V)

$$= 2\int_0^a \pi y^2 dx = 2\pi \int_0^a (a^2 - x^2) dx = 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left(a^3 - \frac{a^3}{3} \right) = 2\pi . \frac{2a^3}{3} = \frac{4}{3}\pi a^3$$

Example 2 : Find the volume of the right circular cone obtained by the revolution of a right angled triangle about a side which contains the right angle.

Solution : Here OAB is the right angled triangle where $\angle A = 90^0$. If it is rotated about the side OA (i.e. about the *x*-axis) then the triangle will generate a right circular cone.

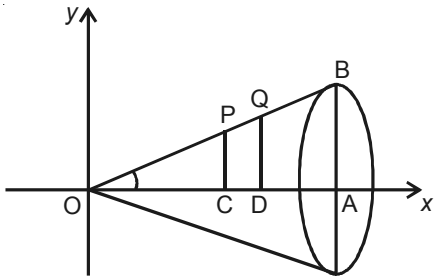


Fig. 6.4

Let $P(x, y)$ be a point on OB and let $\angle BOA = \alpha$.

The from ΔPOC, we get

$$\tan\alpha = \frac{PC}{OC} = \frac{y}{x}$$

$$\therefore y = x\tan\alpha \quad ...(i)$$

which is the equation of the line OB (OB is the hypotenuse of ΔOAB)

Now the area of the triangle is bounded by the line OB, *x*-axis and AB. If this area is rotated about *x*-axis then it will generate the right circular cone. Let OA=*h* be the height of the cone and AB=*a* be the radius of the circular base.

$$\therefore \tan\alpha = \frac{a}{h} \quad ...(ii)$$

Putting the value of $\tan\alpha$ from (i) in (ii) we get $y = x\left(\frac{a}{h}\right)$ which is the equation of the line OB expressed in terms of *a* and *h*.∴ Volume of the right circular cone (V)

$$= \int_0^h \pi y^2 dx = \int_0^h \pi \frac{x^2 a^2}{h^2} dx = \frac{\pi a^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$= \frac{\pi a^2}{h^2}\frac{h^3}{3} = \frac{1}{3}\pi a^2 h$$

Example 3 : $y^2 = 4ax$ is a parabola. The latus rectum cuts off a part of the parabola and the remaining part revolves about the *y* axis at the vertex. Find the total volume thus generated.

Solution : The parabola $y^2 = 4ax$ is cut off by the latus rectum LSL'. The part left is LOL' which revolves about the *y*-axis where O is the vertex. The revolution will generate two volumes– one above *x*-axis and the other below *x*-axis where both the parts are identical. The area LMO that revolves is enclosed between the arc LO, *y*-axis and LM.

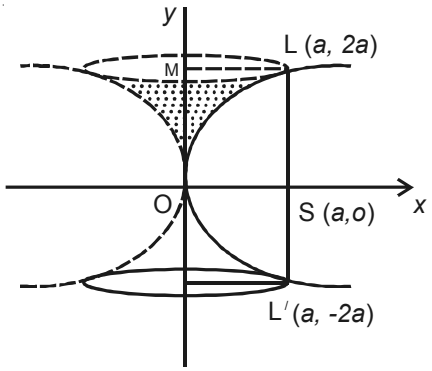


Fig. 6.5

$\therefore$ Total required volume = 2 x volume generatedby revolving the area OLM abouty y-axis.

$$= 2\int_0^{2a} \pi x^2 dy = 2\pi \int_0^{2a} \left(\frac{y^2}{4a}\right)^2 dy$$

$$= 2\pi . \frac{1}{16a^2}\left[\frac{y^5}{5}\right]_0^{2a} = 2\pi . \frac{1}{16a^2} . \frac{32a^5}{5}$$

$$= \frac{4}{5}\pi a^3 .$$



CHECK YOUR PROGRESS - 1

Q 1 : Find the volume of the solid formed by the rotation of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the major axis.

Q 2 : Find the volume of the solid formed by the rotation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its minor axis.

Q 3 : Find the volume of the solid generated by the revolution of the curve $y = \sin x$ from the points where $x = 0$ to $x = 2\pi$ about the $x-$axis.

Q 4 : Find the volume of a hemisphere.

Q 5 : Find the volume of the paraboloid which is generated by the revolution about the *x*-axis of the parabola $y^2 = 4ax$ from $x = 0$ to $x = h$.

Q 6 : The astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is revolved about the axis of x. Find the volume of the solid thus generated.

Q 7 : Find the volume of the solid generated by revolving about x-axis, the area bounded by the curve $y = 5x - x^2$, the line $x = 5$ and the y-axis.

Q 8 : Find the volume of the solid generated due to the revolution of $y^2 = 5x$ about the x axis between the ordinates $x = 0$ and $x = 9$.

6.4 SURFACE OF SOLIDS OF REVOLUTION

Suppose $f(x)$ be a continuous function of x where $f(x)$ does not change sign in the interval $a \le x \le b$. Now we are to find the area of the surface of the solid generated by revolving about the x-axis the area bounded by the curve $y = f(x)$ and the ordinates $x = a$ and $x = b$.

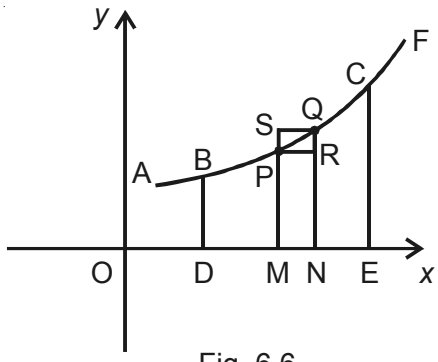


Fig. 6.6

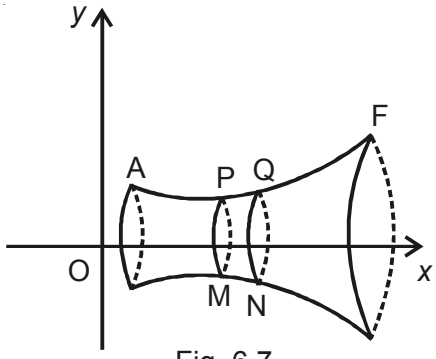


Fig. 6.7

Let the portion of the arc of the curve $y = f(x)$ which is included between the ordinates $x = a$ and $x = b$ be BC. Here the straight lines given by $x = a$ and $x = b$ are BD and CE respectively.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighbouring points on $\overset{\frown}{BC}$. PM and QN are drawn perpendiculars on the x-axis PR is perpendicular on QN. QS is drawn perpendicular on extended MP.

Let length of the arc $AP = S$ and the length of the arc $PQ = \delta S$. Let A and $A + \delta A$ denote the areas of the surfaces of the solids generated by the revolution of the arc BP and arc BQ respectively about the x-axis. Then the area of the surface generated by arc PQ revolving about x-axis is δA.

Also, the lines PR and QS will generate two cylinders when they revolve about *x*-axis and the areas of the curved surfaces of these cylinders are $2\pi y\delta S$ and $2\pi(y+\delta y).\delta S$, (since Q is a point lying very very close to P, therefore PR, PQ and SQ are almost equal)

Therefore the area of the surface generated by revolving PQ about *x*-axis lies between $2\pi y\delta S$ and $2\pi(y+\delta y)\delta S$.

$$\therefore 2\pi y\delta s \le \delta A \le 2\pi(y+\delta y)\delta S$$

$$\Rightarrow 2\pi y \le \frac{\delta A}{\delta S} \le 2\pi(y+\delta y)$$

Now when $Q \to P$ along the curve then $\delta x \to 0$ and $\delta y \to 0$. Therefore

$$2\pi y \le \lim_{\delta S\to 0}\left(\frac{\delta A}{\delta S}\right) \le \lim_{\delta y\to 0} 2\pi(y+\delta y)$$

$$\therefore \frac{dA}{dS} = 2\pi y$$

$$\Rightarrow dA = 2\pi y dS$$

Integrating, $\int_a^b dA = \int_a^b 2\pi y dS$

$$\Rightarrow [A]_a^b = 2\pi\int_a^b y ds$$

$\Rightarrow$ (Value of A when $x=b$) – (value of A when $x=a$) $= 2\pi\int_a^b y ds$

$\Rightarrow$ Area of the surface generated by revolving the arc BC about *x*-axis $= 2\pi\int_{x=a}^{x=b} y ds$.

But we know that $\left(\frac{dS}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2$

$$\therefore dS = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Hence the area of the surface generated by revolving the arc BC about *x*-axis $= 2\pi\int_a^b y dS$

$$= 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

ILLUSTRATIVE EXAMPLES

Example 1 : Find the area of the curved surface of a hemisphere of radius *a.*

Solution : Let $x^2 + y^2 = a^2$ be a circle whose centre is (0,0) and radius is *a*. Now if the arc AB of the circle (i.e. the quadrant AB) revolves about the radius OA (i.e. *x*-axis) then it will generate a hemisphere as shown in the Fig. 6.8.

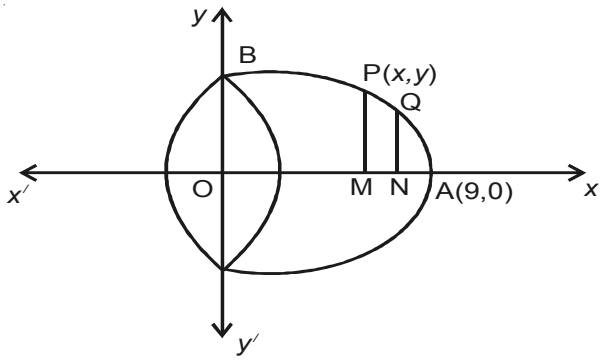


Fig. 6.8

Differentiating $x^2 + y^2 = a^2$

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore \frac{dS}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{y^2}} = \sqrt{\frac{x^2+y^2}{y^2}} = \sqrt{a^2/y^2} = a/y$$

$$\therefore ydS = adx$$

$\therefore$ Required surface area $= 2\pi \int_{x=0}^{a} ydS = 2\pi \int_{x=0}^{a} adx$

$$= 2\pi[ax]_0^a = 2\pi a^2$$

Example 2: Find the area of the surface formed by revolution about $x-$axis of the portion of arc of the parabola $y^2 = 4ax$ between the vertex and one extremity of the latus rectum.

Solution: Here the parabola is $y^2 = 4ax$

$$\therefore 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \frac{dS}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{2a}{y}\right)^2} = \sqrt{\frac{y^2+4a^2}{y^2}}$$

$$= \sqrt{\frac{4ax + 4a^2}{y^2}} = \frac{2}{y}\sqrt{a(x+a)}$$

Now the coordinates of the vertex and one extremity of the latus rectum are (0,0) and (a, $2a$) respectively.

$\therefore$ Required surface area

$$= \int_{x=0}^{x=a} 2\pi y \frac{dS}{dx} dx = \int_0^a 2\pi . y . \frac{2}{y}\sqrt{a(x+a)}dx$$

$$= 4\pi\sqrt{a}\int_0^a (x+a)^{1/2}\, dx = 4\pi\sqrt{a}\left[\frac{2}{3}(x+a)^{3/2}\right]_0^a$$

$$= 4\pi\sqrt{a}.\frac{2}{3}\left[(2a)^{3/2} - (a)^{3/2}\right] = \frac{8\pi\sqrt{a}}{3}a^{3/2}\left(2\sqrt{2}-1\right)$$

$$= \frac{8}{3}\pi a^2\left(2\sqrt{2}-1\right)$$

Example 3 : Find the surface of the solid which is generated by the revolution of the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$ about its major axis.

Solution : The ellipse is $\frac{x^2}{36} + \frac{y^2}{16} = 1$... ... ...(i)

$\therefore a = 6$ and $b = 4$

Differentiating (i) w.r.t. x, $\frac{2x}{36} + \frac{2y}{16}\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -\frac{4x}{9y}$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \frac{16x^2}{81y^2}} = \frac{\sqrt{81y^2 + 16x^2}}{9y} = \frac{1}{9y}\sqrt{81.\frac{4}{9}(36 - x^2) + 16x^2}$$

$$= \frac{1}{9y}\sqrt{36(36 - x^2) + 16x^2} = \frac{1}{9y}\sqrt{(36)^2 - 20x^2} \text{ ... ...(ii)}$$

Now the given ellipse is symmetrical about both the axes. For the arc of the ellipse which is lying in the first quadrant x varies from 0 to 6.

$\therefore$ The required area of the surface

$$= 2 \times \int_{x=0}^{6} 2\pi y \frac{ds}{dx} dx = 2\int_0^6 2\pi y . \frac{1}{9y}\sqrt{(36)^2 - 20x^2}\, dx$$

$$= \frac{4\pi}{9}\int_0^6 \sqrt{(36)^2 - 20x^2}\,dx = \frac{4\pi}{9}\sqrt{20}\int_0^6 \sqrt{\frac{1296}{20} - x^2}\,dx$$

$$= \frac{8\sqrt{5}\pi}{9}\int_0^6 \sqrt{\left(\frac{18}{\sqrt{5}}\right)^2 - x^2}\,dx$$

$$= \frac{8\sqrt{5}\pi}{9}\left[\frac{x}{2}\sqrt{\left(\frac{18}{\sqrt{5}}\right)^2 - x^2} + \frac{1}{2}.\frac{324}{5}\sin^{-1}\left(\frac{x\sqrt{5}}{18}\right)\right]_0^6$$

$$= \frac{8\sqrt{5}\pi}{9}\left[\left(3\sqrt{\frac{324}{5} - 36} + \frac{162}{5}\sin^{-1}\frac{6\sqrt{5}}{18}\right) - 0\right]$$

$$= \frac{8\sqrt{5}\pi}{9}\left(3\sqrt{\frac{144}{5}} + \frac{162}{5}\sin^{-1}\frac{\sqrt{5}}{3}\right)$$

$$= \frac{8\sqrt{5}\pi}{9}\left[\frac{36}{\sqrt{5}} + \frac{162}{5}\sin^{-1}\left(\frac{\sqrt{5}}{3}\right)\right]$$

$$= 8\sqrt{5}\pi\left[\frac{4}{\sqrt{5}} + \frac{18}{5}\sin^{-1}\left(\frac{\sqrt{5}}{3}\right)\right]$$

$$= 16\pi\left[2 + \frac{9}{\sqrt{5}}\sin^{-1}\left(\frac{\sqrt{5}}{3}\right)\right]$$



CHECK YOUR PROGRESS - 2

Q 1 : Find the surface of a sphere of radius *a*.

Q 2 : Find the curved surface of the solid which is generated by the revolution about the *x*-axis of the area bounded by the *x*-axis of the parabola $y^2 = 4ax$ and the ordinate $x = h$.

Q 3 : Find the surface area of the solid generated by revolution of $y^2 = 4ax$ about *x*-axis between the extremities of the latus rectum.

Q 4 : Show that the surface area of the spherical portion of a sphere contained between two planes parallel to each other at a distance of *h* apart is $2\pi ah$.

Q 5 : Find the surface of the solid generated by the revolution of the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$ about its major axis.

Q 6 : Find the area of the surface generated when the arc of $y=x^2$ is revolved about *y*-axis between the point $x=0$ and $x=\sqrt{2}$.

Q 7 : Find the surface area of a sphere generated by the circle $x^2+y^2=a^2$ revolving about the *x*-axis.

Q 8 : Find the surface area which is generated due to the revolution of the arc of the parabola $y^2=12x$ between $x=0$ and $x=3$ about its axis.

Q 9 : Find the area of the surface of revolution generated by revolving about the *x*-axis the arc of $y=\frac{1}{3}x^3$ from $x=0$ to $x=3$.



6.5 LET US SUM UP

- The volume of the solid generated by the revolution about *x*-axis of the area bounded by the curve $y=f(x)$, *x*-axis and the ordinates $x=a$ and $x=b$ is given by $V=\int_a^b \pi y^2 dx$.
- The area of the surface of the solid which is generated by revolving the arc of $y=f(x)$ between $x=a$ and $x=b$ about *x*-axis is given by $2\pi\int_a^b y ds$ or $2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$.

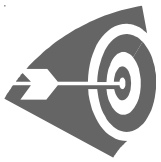


6.6 FURTHER READINGS

1) *Integral Calculus,* Kishan, Hari, Atlantic Publishers & Distributors. New Delhi -110027.

2) *Integral Calculus*, Dhami, H.S. New age International Publishers, New Delhi - 110002.

3) *Text book of Integral Calculus*, Sharma, A.K. Discovery Publishing House, New Delhi -110002.



6.7 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1 : The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The ellipse is symmetrical about the x-axis and the y-axis and for the portion of the curve lying in the first quadrant x varies from 0 to a.

Therefore the required volume of the solid formed $=2\int_0^a y^2dx$

$$=2\pi\int_0^a\frac{b^2}{a^2}\left(a^2-x^2\right)dx$$

from(1)

$$=2\pi\frac{b^2}{a^2}\left[a^2x-\frac{1}{3}x^3\right]_0^a$$

$$=2\pi\frac{b^2}{a^2}\left[a^3-\frac{1}{3}a^3\right]$$

$$=\frac{4}{3}\pi ab^3$$

Ans to Q No 2 : Do yourself.

Ans to Q No 3 : Do youself.

Ans to Q No 4 : The hemisphere is generated by the revolution of a quadrant of the circle $x^2+y^2=a^2$ about x-axis.The limits for the volume will be from 0 to a.

$\therefore$ the required volume of the hemisphere $=\int_0^a\pi y^2dx$

$$=\int_0^a\pi\left(a^2-x^2\right)dx$$

$$= \pi\left[a^2 x - \frac{1}{3}x^3 \right]_0^a$$

$$= \pi\left[a^3 - \frac{1}{3}a^3 \right]$$

$$= \frac{2}{3}\pi a^3 .$$

Ans to Q No 5 : The given parabola is $y^2 = 4ax$

The paraboloid is formed by the revolution of the area ODC about x-axis.

Also for the area ODC, x varies from $x = 0$ to $x = h$.

$\therefore$ the required volume $= \int_0^h \pi y^2 dx$

$$= \int_0^h \pi(4ax)dx$$

$$= 4\pi a\left[\frac{x^2}{2} \right]_0^h$$

$$= 2\pi a h^2$$

Ans to Q No 6 : The given curve is $x^{2/3} + y^{2/3} = a^{2/3}$ ... (1)

The curve is symmetrical about both the axes. The coordinates of B are (0, a) and those of A are (a, 0).

$\therefore$ the required volume $= 2\int_0^a \pi y^2 dx$, by symmetry

$$= 2\pi\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}} \right)^3 dx$$

[from (1), $y^{\frac{2}{3}} = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}} \right)$ so that $y^2 = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}} \right)^3$]

$$= 2\pi\int_0^{\frac{\pi}{2}} a^2 \cos^6\theta . 3a\sin^2\theta\cos\theta d\theta$$

[putting $x = a\sin^3\theta$ so that $dx = 3a\sin^2\theta\cos\theta d\theta$]

$$= 6\pi a^2 \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^7\theta d\theta$$

$$= 6\pi a^2 \frac{1.6.4.2}{9.7.5.3.1} = \frac{32\pi a^3}{105}.$$

Ans to Q No 7 : Do yourself.

Ans to Q No 8 : Do yourself.

CHECK YOUR PROGRESS – 2

Ans to Q No 1 : Suppose the sphere is generated by the revolution of a semi- circle of radius a about its bounding diameter (x-axis).

Let the equation of the circle be $x^2 + y^2 = a^2$, ... (1)

the centre being the origin.

Also, for the semi-circle, x varies from -a to a.

$\therefore$ the required surface $= 2\pi \int_{-a}^{a} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= 2\pi \int_{-a}^{a} y . \frac{a}{y} dx$$

$$= 2\pi \int_{-a}^{a} a dx$$

$$= 2\pi a \left[x\right]_{-a}^{a}$$

$$= 2\pi a (a + a)$$

$$= 4\pi a^2 .$$

Ans to Q No 2: The given parabola is $y^2 = 4ax$ ... (1)

Differentiating (1) w.r.t. x, we get $2y\left(\frac{dy}{dx}\right) = 4a$

or $\frac{dy}{dx} = \frac{2a}{y}$

$$\therefore \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\frac{y^2 + 4a^2}{y^2}} = \frac{\sqrt{y^2 + 4a^2}}{y}$$

$$= \frac{\sqrt{4ax + 4a^2}}{y}, \text{ from (1)}$$

$$= \frac{2\sqrt{a}\sqrt{x+a}}{y}$$

$\therefore$ the required curved surface $= \int_0^h 2\pi y \frac{ds}{dx} dx = 2\pi \int_0^h y . \frac{2\sqrt{a}\sqrt{x+a}}{y} dx$

$$= 4\pi\sqrt{a} \int_0^h (x+a)^{\frac{1}{2}} dx$$

$$= 4\pi\sqrt{a} \left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^h$$

$$= \frac{8\pi}{3} \sqrt{a} \left[(h+a)^{\frac{3}{2}} - a^{\frac{3}{2}} \right].$$

Ans to Q No 3 : Do yourself.

Ans to Q No 4 : Let the sphere be generated by the revolution about the x-axis of the circle $x^2 + y^2 = a^2$ ... (1)

Let the two parallel planes bounding the spherical zone be formed by the revolution of the lines $x = b$ and $x = b + h$.

Then the required surface is generated by the revolution of the arc AC about x-axis.

$\therefore$ the required surface $= 2\pi \int_b^{b+h} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

$$= 2\pi \int_b^{b+h} y . \frac{a}{y} dx$$

$$= 2\pi \int_b^{b+h} a dx$$

$$= 2\pi a [x]_b^{b+h}$$

$$= 2\pi a (b + h - h)$$

$$= 2\pi ab$$

Ans to Q No 5 : The given ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$ ... (1)

Comparing it with the standard form of the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,we get a=4, b=2 and the major axis is along the x-axis.So we have to revolve the curve (1) about x-axis.
Differentiating (1) w.r.t. x, we get

$$2x+8y\frac{dy}{dx}=0$$

or $$\frac{dy}{dx}=-\frac{x}{4y}$$

$$\therefore \frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\sqrt{\left\{1+\frac{x^2}{16y^2}\right\}}=\sqrt{\frac{16y^2+x^2}{16y^2}}$$

$$=\frac{\sqrt{\{(64-4x^2)+x^2\}}}{4y}$$

$$=\frac{\sqrt{64-3x^2}}{4y} \quad \text{... (2)}$$

Now the ellipse (1) is symmetrical about both axes and for the arc of the ellipse lying in the first quadrant x varies from 0 to 4.

$\therefore$ the required surface = 2 $\times$ surface generated by the revolution of the arc in the first quadrant.

$$=2\int_0^4 2\pi y\frac{ds}{dx}dx$$

$$=4\pi\int_0^4 y.\frac{\sqrt{64-3x^2}}{4y}dx \text{ ,from (2)}$$

$$=\pi\int_0^4\sqrt{64-3x^2}\,dx$$

$$=\pi\sqrt{3}\int_0^4\sqrt{\left\{\left(\frac{8}{\sqrt{3}}\right)^2-x^2\right\}}dx$$

$$=\pi\sqrt{3}\left[\frac{x}{2}\sqrt{\left\{\left(\frac{8}{\sqrt{3}}\right)^2-x^2\right\}}-x^2+\frac{1}{2}.\frac{64}{3}\sin^{-1}\left(\frac{x\sqrt{3}}{2}\right)\right]_0^1$$

$$= \pi\sqrt{3}\left[2\sqrt{\left(\frac{64}{3}-16\right)}+\frac{32}{3}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \pi\sqrt{3}\left[\frac{8}{\sqrt{3}}+\frac{32}{3}.\frac{\pi}{3}\right]$$

$$= 8\pi\left[1+\frac{4\pi}{3\sqrt{3}}\right]$$

Ans to Q No 6 : Do yourself.

Ans to Q No 7 : Do yourself.

Ans to Q No 8 : Do yourself.

Ans to Q No 9 : Do yourself.



6.8 MODEL QUESTIONS

Q 1 : Find the volume of the solid generated by revolving about the *y*-axis the plane area bounded by $y = 4x^2$, $x = 0$ and $y = 16$.

Q 2 : Find the volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and its latus rectum about *x*-axis.

Q 3 : Find the volume generated by revolving the area cut off from the parabola $y = 4x - x^2$ by the *x*-axis about the line $y = 6$.

Q 4 : Find the volume of the solid formed by revolving about the *x*-axis, the area bounded by $\sqrt{x}+\sqrt{y}=\sqrt{a}$, *x*-axis and *y*-axis.

Q 5 : Find the volume of the solid generated by the revolution of the curve $y^2 = x^2(2-x)$ about the *x*-axis.

Q 6 : Find the area of the surface of revolution generated by revolving about the *y*-axis the arc of $x = y^3$ from $y = 0$ to $y = 1$.

Q 7 : Find the area of the surface of revolution generated by revolving a loop of the curve $x^2\left(a^2 - x^2\right) = 8a^2y^2$ about the *x*-axis.

Q 8 : Find the area of the surface generated by revolving the line $y = mx$ from $x = 0$ to $x = 2$ about x-axis.

Q 9 : The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the area of the curved surface of the reel so formed.

Q 10 : Find the surface area of the solid formed by the revolution of the curve $y^2(a + x) = x^2(a - x)$ about x-axis.

*** ***** ***