UNIT 10 : FIRST ORDER AND HIGHER DEGREE DIFFERENTIAL EQUATIONS

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10.1 LEARNING OBJECTIVES

After going through this unit, you will able to know

- differential equation of first order and higher degree
- equation solvable for p
- equation solvable for x
- equation solvable for y
- clairaut's form.

10.2 INTRODUCTION

In unit 9, we have discussed the methods of solving linear equations in which dependent variable and its derivatives are of first order and first degree. In this unit, we shall consider non linear equations in which dependent variable and its derivatives are of first order and of higher degree. There is no specific method to solve all such equations. Here,some typical types of equations are considered to describe the methods of solution of such equations.

10.3 EQUATION OF FIRST ORDER AND HIGHER DEGREE

Differential equations of the first order but of a higher degree can be solved by one or more of the following four methods :

(a) Equations solvable for p , $\left(p = \frac{dy}{dx} \right)$

(b) Equations solvable for y : $y = f(x, p)$

(c) Equations solvable for x : $x = f(y, p)$

(d) Clairaut's equation : $y = px + f(p)$

10.3.1 Equation solvable for p

Consider a differential equation of first order but of the nth degree :

$p^n + f_1(x,y)p^{n-1} + f_2(x,y)p^{n-2} + ... + f_{n-1}(x,y)p + f_n(x,y) = 0$ .....(1)

which is solvable for p .

Splitting up the left side into n linear factors, we have

$[p - \phi_1(x,y)][p - \phi_2(x,y)]..[p - \phi_n(x,y)] = 0$

Equating each factor to zero,we have

$p = \phi_1(x,y)$, $p = \phi_2(x,y)$,..., $p = \phi_n(x,y)$.

Each of these n equations is of first order and first degree, which can be solved by a method discussed in unit 5.

Suppose $F_1(x,y,c) = 0, F_2(x,y,c) = 0,..., F_n(x,y,c) = 0$ are solutions of these n equations. These n solutions constitute the general solution of (1). The general solution can also be written as $F_1(x,y,c).F_2(x,y,c)...F_n(x,y,c) = 0$.

Example 1 : Solve $p^2 + p\left(e^x + e^{-x}\right) + 1 = 0$

Solution : From the given equation

$$p^2 + p\left(e^x + e^{-x}\right) + 1 = 0$$

or $p\left(p + e^x\right) + e^{-x}\left(p + e^x\right) = 0$

or $\left(p + e^x\right)\left(p + e^{-x}\right) = 0$

we have $p + e^x = 0$, or, $p + e^{-x} = 0$

solving, we have $y + e^x + c_1 = 0$,or , $y - e^{-x} + c_2 = 0$

The given equation is of first order and hence its general solution contains only one arbitrary constant. So, we replace c_1 and c_2 by single constant c. Since each of these two solutions is a solution of the given equation, the required solution is $\left(y + e^{x} + c\right)\left(y - e^{-x} + c\right) = 0$.

Example 2 : Solve, $p^2 + 2py\cot x = y^2$

Solution: Solving for p, we get

$$p = \frac{1}{2}\left(-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}\right)$$

$$= -y\cot x \pm y\cos ecx \quad \text{... (1)}$$

From (1), $\frac{dy}{dx} = -y\cot x + y\cos ecx$

or $\frac{dy}{dx} = y\left(\cos ecx - \cot x\right)$

or $\frac{dy}{y} = \left(\cos ecx - \cot x\right)dx$

Integrating, we have

$$\log y = \log\tan\frac{x}{2} - \log\sin x + \log c$$

or $y = c\left(\frac{\tan\frac{x}{2}}{\sin x}\right) = \frac{c}{\left(2\cos^2\frac{x}{2}\right)}$

or $y\left(1 + \cos x\right) = c$ ... (2)

Again from (1), $\frac{dy}{dx} = -y\left(\cot x + \cos ecx\right)$

or $\frac{dy}{y} = -\left(\cot x + \cos ecx\right)dx$

Integrating, we have $\log y = -\log\sin x - \log\tan\frac{x}{2} + \log c$

or $y = \frac{c}{2\sin^2\frac{x}{2}}$

or $y\left(1 - \cos x\right) = c$ ... (3)

combining (2) and (3), we get

$y(1 \pm \cos x) = c$ is the required solution.

Example 3 : Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

Solution : The given equation is $p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$

$$\text{or } p^2 - p\left(\frac{x}{y} - \frac{y}{x}\right) - 1 = 0$$

$$\text{or } p\left(p - \frac{x}{y}\right) + \frac{y}{x}\left(p - \frac{x}{y}\right) = 0$$

$$\text{or } \left(p - \frac{x}{y}\right)\left(p + \frac{y}{x}\right) = 0$$

$$\text{or } p = \frac{x}{y}, p = -\frac{y}{x}$$

Solving $p = \frac{x}{y} \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

Integrating, we get, $x^2 - y^2 = c$

Again solving $p = -\frac{y}{x} \Rightarrow \frac{dy}{dx} + \frac{y}{x} = 0$

$$\text{or } \frac{dy}{y} + \frac{dx}{x} = 0$$

Integrating, we get $xy = c$

So, $x^2 - y^2 = c$ and $xy = c$ constitute the required solution.

Example 4 : Solve $p^3 + 3xp^2 - y^3p^2 - 3xy^3p = 0$

Solution : The given equation is

$$p^3 + 3xp^2 - y^3p^2 - 3xy^3p = 0$$
$$\text{or } p^2(p + 3x) - py^3(p + 3x) = 0$$
$$\text{or } (p^2 - py^3)(p + 3x) = 0$$
$$\text{or } p(p - y^3)(p + 3x) = 0$$

(i) If $p = 0$, then $\frac{dy}{dx} = 0$ or $dy = 0$, i.e., $y = c$.

(ii) If $p - y^3 = 0$, then $\frac{dy}{dx} = y^3$ or $\frac{dy}{y^3} = dx$

Integrating we get, $-\frac{1}{2y^2} = x + c$

or $2xy^2 + 2cy^2 + 1 = 0$

(iii) If $p + 3x = 0$ then $\frac{dy}{dx} + 3x = 0$ or $dy + 3xdx = 0$

Integrating we get, $y + \frac{3}{2}x^2 = c$

or $2y + 3x^2 - 2c = 0$

Hence the solution of the given differential equation is

$$(y - c)(2xy^2 + 2cy^2 + 1)(2y + 3x^2 - 2c) = 0 .$$



CHECK YOUR PROGRESS - 1

Q 1 : Solve the following differential equations :

(i) $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

(ii) $p^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0$

(iii) $p^2 - px + 1 = 0$

(iv) $x^2p^2 + xyp - 6y^2 = 0$

10.3.2 Equation solvable for x

Suppose an equation of first order and higher degree can be transformed into the form : $x = f(y, p)$ ... (1)

Differentiating (1) w.r.t 'y', it can be transformed again to the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dx}\right)$$

Suppose it is possible to solve the new differential equation in y and p.

Let its solution be $F(y, p, c) = 0$.

Eliminating p from this equation with the help of the given equation, we obtain the required solution.

Example 1 : Solve $x = 4(p + p^2)$

Solution : Given, $x = 4(p + p^2)$ ... (1)

Differentiating (1) w.r.t y , we have

$$\frac{1}{p} = 4(1+2p)\frac{dp}{dy}$$

or $dy = 4p(1+2p)dp$

Integrating, we get

$$y = 4\int\left(p+2p^2\right)dp = \frac{4}{2}p^2 + \frac{8}{3}p^3 + c$$

$y = 2p^2 + \frac{8}{3}p^3 + c$ which is the required solution.

Example 2 : Solve $4yp^2 - 2px + y = 0$

Solution : The given equation can be written as

$$x = 2py + \frac{y}{2p} \quad \text{... (1)}$$

Differentiating (1) w.r.t y yields

$$\frac{1}{p} = 2\frac{dp}{dy}.y + 2p + \frac{2p - y2\frac{dp}{dy}}{4p^2}$$

$$= 2\frac{dp}{dy}.y + 2p + \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy}$$

or $0 = 2p + 2y\frac{dp}{dy} - \frac{1}{2}\frac{1}{p} - \frac{y}{2p^2}\frac{dp}{dy}$

$$= 4p^2\left(\frac{1}{2p} + \frac{y}{2p^2}\frac{dp}{dy}\right) - \left(\frac{1}{2p} + \frac{y}{2p^2}\frac{dp}{dy}\right)$$

$$= \left(\frac{1}{2p} + \frac{y}{2p^2}\frac{dp}{dy}\right)\left(4p^2 - 1\right)$$

Solving, we get $1 + \frac{y}{p}\frac{dp}{dy} = 0$

or $\frac{dy}{y} + \frac{dp}{p} = 0$

Integrating, we have $yp = c$.

Subtituting $p = \frac{c}{y}$ in the given equation, we have

$$4y.\frac{c^2}{y^2} - 2.\frac{c}{y}.x + y = 0$$

or $4c^2 - 2cx + y^2 = 0$ which is the required general solution.

Example 3 : Solve $y = 2px + y^2 p^3$

Solution : The given equation can be written as

$$x = \frac{y}{2p} - \frac{y^2 p^2}{2} \quad \text{... (1)}$$

Differentiating (1) w.r.t y, we have

$$\frac{1}{p} = \frac{2p - 2y\frac{dp}{dy}}{4p^2} - \frac{2yp^2 + y^2 2p\frac{dp}{dy}}{2}$$

$$= \frac{1}{2}\frac{p - y\frac{dp}{dy}}{p^2} - yp^2 - y^2 p\frac{dp}{dy}$$

or $0 = \frac{1}{2p} - \frac{y\frac{dp}{dy}}{2p^2} - yp^2 - y^2 p\frac{dp}{dy} - \frac{1}{p}$

or $0 = \frac{1}{2p^2} y\frac{dp}{dy} + yp^2 + y^2 p\frac{dp}{dy} + \frac{1}{2p}$

or $0 = yp\left(y\frac{dp}{dy} + p\right) + \frac{1}{2p^2}\left(y\frac{dp}{dy} + p\right)$

or $0 = \left(y\frac{dp}{dy} + p\right)\left(yp + \frac{1}{2p^2}\right)$

Solving, $y\frac{dp}{dy} + p = 0$, separating variables, we have

$$\log py = \log c$$

or, $py = c$

or $p = \frac{c}{y}$

Eliminating p from the given equation, we have

$$y = \frac{2cx}{y} + y^2 . \frac{c^3}{y^3}$$

or $y^2 = 2cx + c^3$ which is required solution.



CHECK YOUR PROGRESS - 2

Q 1 : Solve the following differential equations :

(i) $y^2 \log y = xyp + p^2$

(ii) $p = \tan\left(x - \frac{p}{1+p^2}\right)$ (iii) $p^3 - 4xyp + 8y^2 = 0$

10.3.3 Equation solvable for y

Suppose a differential equation of first order and higher degree takes the form $y = f(x, p)$ ... (1)
Differentiating (1) w.r.t x, we have

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right)$$

If it is possible to solve this equation and $F(x, p, c) = 0$ ... (2) be a solution, then on elimination of p between (1) and (2), we get the required solution. If the elimination of p is not possible then solving (1) and (2) for x and y ,we obtain $x = F_1(p, c)$ and $y = F_2(p, c)$ as the solution.

Example 1 : Solve $y = px + ap(1-p)$

Solution : Given, $y = px + ap(1-p)$ ... (1)
Differentiating (1) w.r.t. x, we get

$$p = \frac{dp}{dx}x + p + a\frac{dp}{dx} - 2p\frac{dp}{dx}a$$

or $\frac{dp}{dx}(x + a - 2ap) = 0$

So, $\frac{dp}{dx} = 0 \Rightarrow p = \text{constant} = c$

and $p = \frac{1}{2a}(x + a)$

Eliminating p, we have $y = cx + ac(1-c)$ which is the general solution.

and $y = \frac{1}{2a}(x + a)x + \frac{a}{2a}(x + a)\left(1 - \frac{x + a}{2a}\right)$

$$= \frac{1}{2a}(x+a)\left\{x+a-\frac{ax+a^2}{2a}\right\}$$

$$= \frac{1}{2a}(x+a)^2 - \frac{1}{4a}(x+a)^2$$

$$= \frac{1}{4a}(x+a)^2$$

$$4ay = (x+a)^2$$

Example 2 : Solve $y - 2px = \tan^{-1}(xp^2)$

Solution : Given $y - 2px = \tan^{-1}(xp^2)$ ... (1)

Differentiating (1) w.r.t x, we get

$$p = 2\frac{dp}{dx}x + 2p + \frac{1}{1+x^2p^4}\left(p^2 + x2p\frac{dp}{dx}\right)$$

or $0 = \left(p + 2x\frac{dp}{dx}\right) + \frac{p}{1+x^2p^4}\left(p + 2x\frac{dp}{dx}\right)$

or $0 = \left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1+x^2p^4}\right)$

This gives $p + 2x\frac{dp}{dx} = 0$ and p is given by $p^4x^2 + p + 1 = 0$

$$\Rightarrow \frac{dp}{p} + \frac{1}{2}\frac{dx}{x} = 0$$

Integrating, $\log p + \frac{1}{2}\log x = \log c$

or $px^{\frac{1}{2}} = c$

or $p = \frac{c}{\sqrt{x}}$

Eliminating p from the given equation, the required solution is

$$y - 2c\sqrt{x} = \tan^{-1} c^2 .$$

Example 3 : Solve $y = 2px + p^n$

Solution : Given, $y = 2px + p^n$ ... (1)

Differentiating (1) w.r.t x, we get

$$p = 2\frac{dp}{dx}x + 2p + np^{n-1}\frac{dp}{dx}$$

or $0 = 2\frac{dp}{dx}x + p + np^{n-1}\frac{dp}{dx}$

or $0 = 2x + p\frac{dx}{dp} + np^{n-1}$

or $\frac{2x}{p} + \frac{dx}{dp} = -np^{n-2}$

Here I.F $= e^{2\log p} = p^2$ and solution is $x.p^2 = -\int np^n dp$

or $x.p^2 = -n\frac{p^{n+1}}{n+1} + c$

or $x = -\frac{n}{n+1}p^{n-1} + cp^{-2}$

Also substituting for x in the given equation, we have

$$y = -\frac{2n}{n+1}p^n + 2cp^{-1} + p^n$$

$$= \frac{2c}{p} + \frac{1-n}{1+n}p^n$$

The required solution is given by

$x = \frac{c}{p^2} - \frac{n}{n+1}p^{n-1}$ and $y = \frac{2c}{p} + \frac{1-n}{1+n}p^n$.



CHECK YOUR PROGRESS - 3

Q 1 : Solve :

(i) $y + px = x^4 p^2$

(ii) $y = x + a\tan^{-1} p$

(iii) $y = p\tan p + \log(\cos p)$

10.4 CLAIRAUT'S FORM

An equation of the form $y = px + f(p)$ ... (i) is known as Clairaut's equation.

Differentiating (1) w.r.t x, we have

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or $0 = \left(x + f'(p)\right)\frac{dp}{dx}$

$\therefore \quad \frac{dp}{dx} = 0$ or $x + f'(p) = 0$ ... (ii)

From $\frac{dp}{dx} = 0$, we have $p = c$ ... (iii)

Eliminating p, from (i) and (iii), we get $y = cx + f(c)$ ... (iv)

which is the general solution of (i).



Alexis Claude Clairaut (13 May 1713 – 17 May 1765) was a prominent French mathematician.

Thus we see that the general solution of the Clairaut's equation is obtained by replacing p by c.

Note : Clairaut's equation is named after the French mathematician is Alexis Claude Clairaut (1713-1765).

Example 1: Solve the following differential equations :

(a) $y = px + \frac{a}{p}$

Solution : The given equation is in Clairaut's form. So replacing p by c its solution is $y = cx + \frac{a}{c}$.

(b) $p = \log(px - y)$

Solution : Given, $p = \log(px - y)$

or $e^p = px - y$

or $y = px - e^p$, which is Clairaut's form. So replacing p by c, its solution is $y = pc - e^c$.

(c) $p = \tan(px - y)$

Solution : Given, $p = \tan(px - y)$

or $\tan^{-1} p = px - y$

or $y = px - \tan^{-1} p$ which is Clairaut's form. So replacing p by c, its solution is $y = Cx - \tan^{-1} C$.

(d) $y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) = 4\left(\frac{dx}{dy}\right)^2$

Solution : The given equation can be written as

$$y^2 + x^2 p^2 - 2xy\,\frac{1}{p} = \frac{4}{p^2}$$

or $(y - px)^2 = \dfrac{4}{p^2}$

or $y = px \pm \left(\dfrac{2}{p}\right)$ which is Clairaut's form. So replacing p by c, its solution is $y = Cx \pm \left(\dfrac{2}{C}\right)$

or $(y - Cx)^2 = \left(\dfrac{2}{C}\right)^2$

or $y^2 + x^2C^2 - 2xyC = \dfrac{4}{C^2}$.



CHECK YOUR PROGRESS - 4

Q 1 : Solve the following differential equations :

(i) $y = px + p - p^3$

(ii) $y = xp + \sqrt{1 + p^2}$

(iii) $y = xp + ap(1 - p)$

(iv) $y = xp + \sin^{-1} p$

(v) $y = xp + e^p$

(vi) $y = xp + p^2$

(vii) $y = xp + \dfrac{p}{p-1}$

(viii) $y = px + \sqrt{1 - p^2}$

10.4.1 Equations reducible to Clairaut's form

Many differential equations of the first order but of higher degree can be reduced to Clairaut's form with the help of suitable substitutions. However, there is no specific method of selecting such substitutions. It depends upon practice and experience.

Example 1 : Transform the equation $x^2(y-px)=p^2y$ into Clairaut's form and then solve it.

Solution : Substituting $x^2=u$ i.e., $2xdx=du$

and $y^2=v$ i.e., $2ydy=dv$

So, $\frac{dv}{du}=\frac{\frac{dv}{dy}}{\frac{du}{dx}}.\frac{dy}{dx}=\frac{y}{x}\frac{dy}{dx}$

Hence, $p=\frac{dy}{dx}=\frac{x}{y}\frac{dv}{du}=\sqrt{\frac{u}{v}}\frac{dv}{du}$

The given equation transforms to

$$u\left(\sqrt{v}-\sqrt{\frac{u}{v}}\sqrt{u}\frac{du}{dv}\right)=\frac{u}{v}\left(\frac{dv}{du}\right)^2.\sqrt{v}$$

or $u\left(\sqrt{v}-\frac{u}{\sqrt{v}}\frac{du}{dv}\right)=\frac{u}{\sqrt{v}}\left(\frac{dv}{du}\right)^2$

or $u\sqrt{v}.\frac{\sqrt{v}}{u}-\frac{u^2}{\sqrt{v}}\frac{\sqrt{v}}{u}\frac{dv}{du}=\left(\frac{dv}{du}\right)^2$

or $v-u\frac{dv}{du}=\left(\frac{dv}{du}\right)^2$

or $v=u\frac{dv}{du}+\left(\frac{dv}{du}\right)^2$, which is Clairaut's form.

The general solution is given by replacing $\frac{dv}{du}$ by a constant C.

Hence its general solution is

$$v=uC+C^2$$

i.e., $\sqrt{y}=\sqrt{x}C+C^2$.

Example 2 : Solve $y^2(y-px)=x^4p^2$, transforming to Clairaut's form with the help of substitutions $u=\frac{1}{x}$ and $v=\frac{1}{y}$.

Solution : From $u=\frac{1}{x}$, we get $\frac{du}{dx}=-\frac{1}{x^2}$ and

from $v=\frac{1}{y}$, we get $\frac{dv}{dy}=-\frac{1}{y^2}$.

or $\frac{dy}{dx}=\frac{y^2}{x^2}\frac{dv}{du}$

The given equation transforms to

$$\frac{1}{v^2}\left(\frac{1}{v}-\frac{1}{u}\frac{u^2}{v^2}\frac{dv}{du}\right)=\frac{1}{u^4}\frac{u^4}{v^4}\left(\frac{dv}{du}\right)^2$$

or $$v^2\left(\frac{1}{v}-\frac{u}{v^2}\frac{dv}{du}\right)=\left(\frac{dv}{du}\right)^2$$

or $$v-u\frac{dv}{du}=\left(\frac{dv}{du}\right)^2$$

or $v=u\frac{dv}{du}+\left(\frac{dv}{du}\right)^2$, which is Clairaut's form.

The general solution is given by replacing $\frac{dv}{du}$ by a constant C.

Hence its general solution is

$$v=uC+C^2$$

$$\text{or}\frac{1}{y}=\frac{1}{x}C+C^2$$

or $x=Cy+C^2xy$.

Example 3 : Solve $(px-y)(py+x)=2p$

Solution : Substituting $x^2=u$ i.e., $2xdx=du$

and $y^2=v$ i.e., $2ydy=dv$

We get

$$\frac{dv}{du}=\frac{\frac{dv}{dy}}{\frac{du}{dx}}.\frac{dy}{dx}=\frac{y}{x}\frac{dy}{dx}$$

Hence, $p=\frac{dy}{dx}=\frac{x}{y}\frac{dv}{du}=\sqrt{\frac{u}{v}}\frac{dv}{du}$

The given equation becomes

$$\left(\sqrt{\frac{u}{v}}.\frac{dv}{du}.\sqrt{u}-\sqrt{v}\right)\left(\sqrt{\frac{u}{v}}\frac{dv}{du}\sqrt{v}+\sqrt{u}\right)=2\sqrt{\frac{u}{v}}\frac{dv}{du}$$

or $$\left(\frac{u}{\sqrt{v}}\frac{dv}{du}-\sqrt{v}\right)\left(\sqrt{u}\frac{dv}{du}+\sqrt{u}\right)=2\sqrt{\frac{u}{v}}\frac{dv}{du}$$

or $$u\frac{dv}{du}-v=\frac{2\frac{dv}{du}}{1+\frac{dv}{du}}$$

or $$v=u\frac{dv}{du}-\frac{2\frac{dv}{du}}{1+\frac{dv}{du}}$$, which is of Clairaut's form.

So, its general solution is

$$v=uC-\frac{2C}{1+C}$$

or $$y^2=x^2C-\frac{2C}{1+C}.$$



CHECK YOUR PROGRESS - 5

Q 1: Reduce to Clairaut's form with the help of given substitution and solve.

(i) $xp^2-2yp+x+2y=0\ \left(u=x^2,v=y-x\right)$

(ii) $\left(px-y\right)\left(py+x\right)=a^2p\ \left(u=x^2,v=y^2\right)$

(iii) $x^2\left(y-px\right)=p^2y\ \ \left(u=x^2,v=y^2\right)$

(iv) $x^2p^2+yp\left(2x+y\right)+y^2=0\ \left(v=xy,u=y\right)$

(v) $xp^2-2yp+x+2y=0\ \ \left(u=x^2,v=y-x\right)$



10.5 LET US SUM UP

- Differential equations of the first order but of a higher degree can be solved by one or more of the following four methods :

(a) Equations solvable for p , $p = \frac{dy}{dx}$

(b) Equations solvable for y : $y = f(x, p)$

(c) Equations solvable for x : $x = f(y, p)$

(d) Clairaut's equation : $y = px + f(p)$

- Consider a differential equation of first order but of the nth degree :

$p^n + f_1(x, y)p^{n-1} + f_2(x, y)p^{n-2} + ... + f_{n-1}(x, y)p + f_n(x, y) = 0$

which is solvable for p .

Splitting up the left side into n linear factors,we have

$[p - \phi_1(x, y)][p - \phi_2(x, y)]...[p - \phi_n(x, y)] = 0$

Equating each factors to zero,we have

$p = \phi_1(x, y)$, $p = \phi_2(x, y)$,..., $p = \phi_n(x, y)$.

Suppose $F_1(x, y, c) = 0, F_2(x, y, c) = 0,..., F_n(x, y, c) = 0$ are solutions of these n equations.These n solutions constitute the general solution .The general solution can also be written as

$F_1(x, y, c).F_2(x, y, c)...F_n(x, y, c) = 0$.

- Suppose an equation of first order and higher degree can be transformed into the form : $x = f(y, p)$ ... (1)

Differentiating (1) w.r.t 'y', it can be transformed again to the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dx}\right)$$

Suppose it is possible to solve the new differential equation in y and p .

Let its solution be $F(y, p, c) = 0$.

Eliminating p from this equation with the help of the given equation, we obtain the required solution.

- Suppose a differential equation of first order and higher degree takes the form $y = f(x, p)$ ... (1)

Differentiating (1) w.r.t x , we have

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right)$$

If it is possible to solve this equation and $F(x, p, c) = 0$ ... (2) be a solution then on elimination of p between (1) and (2), we get the

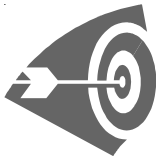
required solution.If the elimination of p is not possible then solving (1) and (2) for x and y, we obtain $x = F_1(p,c)$ and $y = F_2(p,c)$ as the solution.

- The differential equation of the form $y = px + f(p)$ is known as Clairaut's equation. The general solution of Clairaut's equation is obtained by replacing p by c. Some complex differential equations can be reduced to Clairaut's form with the help of proper substitutions.



10.6 FURTHER READINGS

1) Tenenbaum, Morris and Pollard, Harry. Ordinary Differential Equations, Dover Publications, Inc, N.Y 11501.
2) Differential Equations : An Introduction with Mathematica. Clay C.Ross, second Edition, Springer.
3) Differential Equations : Linear, Nonlinear, ordinary, partial. A.C. King, J. Billinggham and S. R. Otto, Cambridge-2003.
4) Differential equations with Application and Programs. S. Balachandra Rao and H. R. Anuradha, University Press (India) Limited 1996.



10.7 ANSWERS TO CHECK YOUR PROGRESS

Check your progress -1

Ans to Q No 1: (i) The given equation can be re-written as

$$p\left(p^2 + 2xp - y^2 p - 2xy^2\right) = 0$$

or $p\{p(p+2x) - y^2(p+2x)\} = 0$

or $p(p+2x)(p-y^2) = 0$

or $p = 0, p + 2x = 0, p - y^2 = 0$

Integrating, $y = C$, $y + x^2 = C$ and $x + \frac{1}{y} = C$

Hence, the combined solution is

$$\left(y - C\right)\left(y + x^2 - C\right)\left(x + \frac{1}{y} - C\right) = 0.$$

(ii) Given, $(py)^2 - 2(py)x\tan^2\alpha + \left(y^2\sec^2\alpha - x^2\tan^2\alpha\right) = 0$

$$\therefore py = \frac{2x\tan^2\alpha \pm \sqrt{\left\{4x^2\tan^4\alpha - 4\left(y^2\sec^2\alpha - x^2\tan^2\alpha\right)\right\}}}{2}$$

or $py = x\tan^2\alpha \pm \sqrt{\left\{x^2\tan^2\alpha\left(\tan^2\alpha + 1\right) - y^2\sec^2\alpha\right\}}$

or $y\frac{dy}{dx} = x\tan^2\alpha \pm \sec\alpha\sqrt{x^2\tan^2\alpha - y^2}$

or $ydy - x\tan^2\alpha dx = \pm\sec\alpha\sqrt{\left(x^2\tan^2\alpha - y^2\right)}dx$

or $\pm\frac{x\tan^2\alpha dx - ydy}{\sqrt{x^2\tan^2\alpha - y^2}} = -\sec\alpha dx$

Integrating, $\pm\sqrt{\left(x^2\tan^2\alpha - y^2\right)} = C - x\sec\alpha$

Squaring, we get

$x^2\tan^2\alpha - y^2 = C^2 - 2Cx\sec\alpha + x^2\sec^2\alpha$

or $x^2 + y^2 - 2Cx\sec\alpha + C^2 = 0$.

(iii) Do Yourself

(iv) Do yourself.

Check your progress -2

Ans to Q No 2: (i) Solving for x, $x = \frac{y\log y}{p} - \frac{p}{y}$ ... (1)

Differentiating (1) w.r.t. y **,** we get

$$\frac{1}{p} = \left(\log y + y.\frac{1}{y}\right)\frac{1}{p} - y\log y.\frac{1}{p^2}\frac{dp}{dy} - \left[-\frac{1}{y^2}.p + \frac{1}{y}\frac{dp}{dy}\right]$$

or $\frac{1}{p} = \frac{\log y}{p} + \frac{1}{p} + \frac{p}{y^2} - \frac{dp}{dy}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right)$

or $0 = \frac{p}{y}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right) - \frac{dp}{dy}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right)$

or $0 = \left(\frac{y \log y}{p^2} + \frac{1}{y}\right)\left(\frac{p}{y} - \frac{dp}{dy}\right)$

Omitting the first factor, we have $\frac{p}{y} - \frac{dp}{dy} = 0$

or $\frac{dp}{p} = \frac{dy}{y}$

Integrating, $\log p = \log y + \log C$

or $p = Cy$ ... (2)

Putting the value of p given by (2) in (1), we get

$x = \frac{\log y}{C} - C$

or $\log y = Cx + C^2$.

(ii) Solving for x, we have $x = \tan^{-1} p + \frac{p}{1+p^2}$ ... (1)

Differentiating both sides w.r.t. y,

$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{1+p^2} \cdot \frac{dp}{dy} + \frac{\left(1+p^2\right) - 2p^2}{\left(1+p^2\right)^2} \cdot \frac{dp}{dy}$

or $\frac{1}{p} = \frac{2\left(1+p^2\right) - 2p^2}{\left(1+p^2\right)^2} \cdot \frac{dp}{dy}$

or $dy = \frac{2p}{\left(1+p^2\right)^2} dp$

Integrating, $y = C - \frac{1}{1+p^2}$ ... (2)

Equations (1) and (2) together constitute the general solution.

(iii) Given , $p^3 - 4xyp + 8y^2 = 0$ where $p = \frac{dy}{dx}$ ... (1)

Solving (1) for x, $x = \left(\frac{1}{4y}\right)p^2 + \left(\frac{1}{p}\right)(2y)$ ... (2)

Differentiating (2) w.r.t. y, we get

$\frac{1}{p} = -\frac{p^2}{4y^2} + \frac{2p}{4y}\frac{dp}{dy} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$

or $\frac{p^2}{4y^2} - \frac{1}{p} - \frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right) = 0$

or $\left(\frac{p^2}{4y^2}-\frac{1}{p}\right)-\frac{2y}{p}\frac{dp}{dy}\left(\frac{p^2}{4y^2}-\frac{1}{p}\right)=0$

or $\left(\frac{p^2}{4y^2}-\frac{1}{p}\right)\left(1-\frac{2y}{p}\frac{dp}{dy}\right)=0$ ... (3)

Neglecting the first factor which doesnot involve $\frac{dp}{dy}$, (3) reduces to

$1-\frac{2y}{p}\frac{dp}{dy}=0$

or $\left(\frac{2}{p}\right)dp=\left(\frac{1}{y}\right)dy$

Integrating, $2\log p=\log y+\log C'$

or $p^2=C'y$ ... (4)

From (1), $p\left(p^2-4xy\right)=-8y^2$

Squaring, $p^2\left(p^2-4xy\right)^2=64y^4$

or $C'y\left(C'y-4xy\right)^2=64y^4$, using (4)

or $4Cy\left(4Cy-4xy\right)^2=64y^4$ where $C'=4C$

or $C\left(C-x\right)^2=y$.

Check your progress -3

Ans to Q No 1: (i) Given, $y=-px+x^4p^2$ ... (1)

Differentiating both sides w.r.t. x,

$\frac{dy}{dx}=p=-p-x\frac{dp}{dx}+4x^3p^2+2x^4p\frac{dp}{dx}$

or $2p+x\frac{dp}{dx}-2px^3\left(2p+x\frac{dp}{dx}\right)=0$

or $\left(2p+x\frac{dp}{dx}\right)\left(1-2px^3\right)=0$

or $2p+x\frac{dp}{dx}=0$, discarding the factor $\left(1-2px^3\right)$

or $\frac{dp}{p}+2\frac{dx}{x}=0$

Integrating, $\log p + 2\log x = \log C$

or $\log px^2 = \log C$

or $px^2 = C$

or $p = \dfrac{C}{x^2}$

Putting this value of p in(1), we have $y = -\dfrac{C}{x} + C^2$ which is the required solution.

(ii) Given, $y = x + a\tan^{-1} p$ ... (1)

Differentiating both sides w.r.t. x, we get

$$p = 1 + \frac{a}{1+p^2}\frac{dp}{dx}$$

or $p - 1 = \dfrac{a}{1+p^2}\dfrac{dp}{dx}$

or $dx = \dfrac{a}{(p-1)(1+p^2)}dp$

or $dx = \dfrac{a}{2}\left[\dfrac{1}{p-1} - \dfrac{1+p}{1+p^2}\right]dp$, by partial fraction

or $dx = \dfrac{a}{2}\left[\dfrac{1}{p-1} - \dfrac{1}{1+p^2} - \dfrac{p}{1+p^2}\right]dp$

Integrating, $x = \dfrac{a}{2}\left[\log(p-1) - \tan^{-1} p - \dfrac{1}{2}\log(1+p^2) + \log C\right]$

or $x = \dfrac{a}{2}\left[\log\left\{C(p-1)(1+p^2)^{-\frac{1}{2}}\right\} - \tan^{-1} p\right]$ ... (2)

Putting the value of x given by (2) in (1), we have

$$y = \frac{a}{2}\left[\log\left\{C(p-1)(1+p^2)^{-\frac{1}{2}}\right\} + \tan^{-1} p\right] \quad \text{... (3)}$$

(2) and (3) together form the required solution in parametric form, p being a parameter.

(iii) Given, $y = p\tan p + \log(\cos p)$ ... (1)

Differentiating both sides w.r.t. x, we get

$$p = \left[\tan p + p\sec^2 p + \left\{\frac{1}{\cos p}\right\}(-\sin p)\right]\left(\frac{dp}{dx}\right)$$

or $p = \left[\tan p + p\sec^2 p - \tan p\right]\left(\frac{dp}{dx}\right)$

or $p = p\sec^2 p\left(\frac{dp}{dx}\right)$

or $dx = \sec^2 p\,dp$

Integrating, $x = \tan p + C$ ... (2)

(1) and (2) form the solution in parametric form, p being a parameter.

Check your progress - 4

(i) Given, $y = px + p - p^3$

This is of the Clairaut's form.

Hence putting C for p, the required solution is $y = Cx + C - C^3$

(ii) Given, $y = xp + \sqrt{1 + p^2}$

This is of the Clairaut's form.

Hence putting C for p, the required solution is $y = Cx + \sqrt{1 + C^2}$.

(iii) Given, $y = xp + ap(1 - p)$

This is of the Clairaut's form.

Hence putting C for p, the required solution is $y = Cx + aC(1 - C)$.

(iv) Given, $y = xp + \sin^{-1} p$

This is of the Clairaut's form.

Hence putting C for p, the required solution is $y = Cx + \sin^{-1} C$.

(v) Given, $y = xp + e^p$

This is of the Clairaut's form.

Hence putting C for p, the required solution is $y = Cx + e^C$.

(vi) Given, $y = xp + p^2$

This is of the Clairaut's form.

Hence putting C for p, the required solution is $y = Cx + C^2$.

(vii) Given, $y = px + \frac{p}{p-1}$.

This is of the Clairaut's form.

Hence putting C for p ,the required solution is $y = Cx + \frac{C}{C-1}$.

(viii) Given, $y = px + \sqrt{1 - p^2}$

This is of the Clairaut's form.

Hence putting C for p ,the required solution is $y = Cx + \sqrt{1 - C^2}$.

Check your progress - 5

Ans to Q No 1: (i) Given, $xp^2 - 2yp + x + 2y = 0$ ... (1)

Also, given $y - x = v$ and $x^2 = u$ ... (2)

Differentiating w.r.t. x

$\frac{dy}{dx} - 1 = \frac{dv}{dx}$ and $2x = \frac{du}{dx}$

or $p - 1 = \frac{dv}{du}$ and $2x = \frac{du}{dx}$

$\therefore \frac{dv}{du} = \frac{\frac{dv}{dx}}{\frac{du}{dx}} = \frac{p-1}{2x}$

or $P = \frac{p-1}{2x}$

or $p = 2xP + 1$

Put this in the given equation. Then, we get

$x(2xP+1)^2 - 2y(2xP+1) + x + 2y = 0$

or $2x^2P^2 - 2(y-x)P + 1 = 0$

or $2uP^2 - 2vP + 1 = 0$, using (2)

or $v = uP + \frac{1}{2P}$ which is of Clairaut's form.

So, its general solution is $v = uC + \frac{1}{2C}$

or $y - x = x^2C + \frac{1}{2C}$

or $2C^2x^2 - 2C(y-x) + 1 = 0$.

(ii) Given, $(px - y)(py + x) = a^2p$

or $p^2xy + p(x^2 - y^2 - a^2) - xy = 0$ ... (1)

Substituting $x^2 = u$ i.e., $2xdx = du$ and $y^2 = v$ i.e., $2ydy = dv$

So, $\frac{dv}{du} = \frac{\frac{dv}{dx}}{\frac{du}{dx}} \cdot \frac{dy}{dx} = \frac{y}{x}\frac{dy}{dx}$

$$\frac{y}{x}\frac{dy}{dx} = \frac{dv}{du} \Rightarrow \quad p = \frac{x}{y}\frac{dv}{du} \quad \text{... (2)}$$

Putting the value of p in (2) from (1), we have

$$xy\frac{x^2}{y^2}\left(\frac{dv}{du}\right)^2 + \frac{x}{y}\left(\frac{dv}{du}\right)\left(x^2 - y^2 - a^2\right) - xy = 0$$

or $x^2\left(\frac{dv}{du}\right)^2 + \frac{dv}{du}\left(x^2 - y^2 - a^2\right) - y^2 = 0$

If we put $\frac{dv}{du} = P$, and $x^2 = u \quad y^2 = v$

$uP^2 + \left(u - v - a^2\right)P - v = 0$

or $uP\left(P+1\right) - v\left(P+1\right) - Pa^2 = 0$

or $v = uP - \frac{P}{P+1}a^2$, which is of Clairaut's form.

So, its general solution is $v = uC - \frac{C}{C+1}a^2$

or $y^2 = x^2C - \frac{C}{C+1}a^2$.

(iii) Given, $x^2\left(y - px\right) = p^2y$

or $y - px = \frac{yp^2}{x^2}$

or $y^2 = pxy + \left(\frac{py}{x}\right)^2$

or $y^2 = \left(\frac{py}{x}\right)x^2 + \left(\frac{py}{x}\right)^2$ ... (1)

Substituting $x^2 = u$ i.e., $2xdx = du$ and $y^2 = v$ i.e., $2ydy = dv$ ... (2)

So, $\frac{dv}{du} = \frac{\frac{dv}{dx}}{\frac{du}{dx}} \cdot \frac{dy}{dx} = \frac{y}{x}\frac{dy}{dx}$

or $P = \frac{y}{x}p$ ... (3)

Using (2) and (3), (1) becomes $v = Pu + P^2$ which is of Clairaut's form.

So, its general solution is $v = Cu + C^2$

(iv) Given equation is $x^2p^2 + yp(2x + y) + y^2 = 0$ ... (1)

Given $y = u$ and $xy = v$ ... (2)

Differentiation (2), $dy = du$ and $xdy + ydx = dv$

$$\therefore \frac{xdy + ydx}{dy} = \frac{dv}{du}$$

or $$x + y\frac{dx}{dy} = \frac{dv}{du}$$

or $$x + \frac{y}{p} = P$$

or $$\frac{y}{p} = P - x$$

or $$p = \frac{y}{P - x}$$ where $p = \frac{dy}{dx}$, $P = \frac{dv}{du}$.

Putting $p = \frac{y}{P - x}$ in (1), we have

$$\frac{x^2y^2}{(P - x)^2} + \frac{y^2}{P - x}(2x + y) + y^2 = 0$$

or $x^2 + (P - x)(2x + y) + (P - x)^2 = 0$

or $Py - xy + P^2 = 0$

or $v = uP + P^2$, using (2), ... (3), which is of Clairaut's form.

So, its general solution is $v = uC + C^2$

or $xy = yC + C^2$.

(v) Given equation is $xp^2 - 2yp + x + 2y = 0$ ... (1)

Also, given $y - x = v$ and $x^2 = u$ ... (2)

Differentiating (2), $dy - dx = dv$ and $2xdx = du$

$$\therefore \frac{dy - dx}{2xdx} = \frac{dv}{du}$$

or $$\frac{dy}{dx} - 1 = 2x\frac{dv}{du}$$

or $p - 1 = 2xP$, where $p = \frac{dy}{dx}$, $P = \frac{dv}{du}$

or $p = 1 + 2xP$

Putting $p = 1 + 2xP$ in (1), we get

$x(1+2xP)^2 - 2(1+2xP) + x + 2y = 0$

or $2x^2P^2 - 2(y-x)P + 1 = 0$

or $2uP^2 - 2vP + 1 = 0$, using (2)

or $v = uP + \frac{1}{2P}$, which is of Clairaut's form.

So, its general solution is $v = uC + \frac{1}{2C}$

or $y - x = Cx^2 + \frac{1}{2C}$.



10.8 MODEL QUESTIONS

Q 1 : Solve the following differential equations :

(a) $xyp^2 + (x^2 + xy + y^2)p + x^2 + xy = 0$

(b) $p^3(x+2y) + 3p^2(x+y) + p(y+2x) = 0$

(c) $4y^2p^2 + 2pxy(3x+1) + 3x^3 = 0$

(d) $y = x\left\{p + (1+p^2)^{\frac{1}{2}}\right\}$

(e) $p^2 + 2xp - 3x^2 = 0$

(f) $p^2 - p(e^x + e^{-x}) + 1 = 0$

(g) $yp^2 - (x-y)p - x = 0$

(h) $xyp^2 - (x^2 - y^2)p - xy = 0$

(i) $p(p+y) = x(x+y)$

(j) $y = x\left(p + \sqrt{1+p^2}\right)$

(k) $xyp^2 - (x^2 + y^2)p + xy = 0$

Q 2 : Solve the following differential equations :

(a) $4(xp^2 + yp) = y^4$

(b) $xp^3 = a + bp$

(c) $y = 3px + 4p^3$

(d) $\sqrt{1+p^2}\,x + p = \sqrt{1+p^2}\,a$

(e) $p^3 y + 2px - y = 0$

(f) $x - yp - ap^2 = 0$

Q 3 : Solve the following differential equations :

(a) $y - px + x - \left(\frac{y}{b}\right) = a$

(b) $e^{p-y} = p^2 - 1$

(c) $p - y = \log\left(p^2 - 1\right)$

(d) $4p^3 + 3xp = yp$

(e) $y = p\sin p + \cos p$

(f) $y = xp^2 + p$

(g) $x - yp = ap^2$

(h) $xp^2 - 2yp + ax = 0$

Q 4 : Solve the following differential equations :

(a) $px - y + p^3 = \frac{m^3}{p^3}$

(b) $p^2\left(x^2 - a^2\right) - 2pxy + y^2 + a^4 = 0$

(c) $y^2 - 2pxy + p^2\left(x^2 - 1\right) = m^2$ s

(d) $\left(x^2 - 1\right)p^2 - 2xyp + y^2 = 1$

(e) $xp^2 - yp + a = 0$

(f) $4yp^2 + 2xp - y = 0$

(g) $y = xp + p^3$

Q 5 : Solve the following differential equations :

(a) $(px - y)(x - py) = 2p\ \left(u = x^2, v = y^2\right)$

(b) $y = 2px + 6y^2p^2\ (y^3 = v\ )$

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