UNIT 9 : EXACT DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

UNIT STUCTURE

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9.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- Exact differential equation
- Equations reducible to Exact differential equation.

9.2 INTRODUCTION

In previous unit,we have discussed ordinary differential equation of first order and first degree. We have also discussed the different methods of solving differential equations which are of the first order and first degree. This unit will focus on exact differential equations of first order and first degree as well as equations reducible to exact differential equation of first order and first degree.

9.3 EXACT DIFFERENTIAL EQUATION

A differential equation $M(x,y)dx+N(x,y)dy=0$ is said be an exact equation if left hand expression of this equation is total differential of a function $u(x,y)$, say, that is if $M(x,y)dx+N(x,y)dy=0$ be exact, then $M(x,y)dx+N(x,y)dy=du(x,y)$ and solution of this equation is $u(x,y)=$ constant.

So, an exact differential equation can also be defined as an equation obtainable by differentiating an equation in x and y ,without any other powers.

The necessary and sufficient condition for $M(x,y)dx+N(x,y)dy=0$ to be an exact differential equation :

Theorem : The necessary and sufficient condition for $M(x,y)dx+N(x,y)dy=0$ to be an exact is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Proof : Suppose $M(x,y)dx+N(x,y)dy=0$ is exact. Then there is a function $u(x,y)$ such that

$$du = Mdx + Ndy$$

$$\text{or } \frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy = Mdx + Ndy$$

Equating the coefficients of dx and dy , we have

$M=\frac{\partial u}{\partial x}$ and $N=\frac{\partial u}{\partial y}$

So, $\frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\partial x}$ and $\frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}$

Assuming the condition for equality of $\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}$,

we have $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Conversely suppose $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Let $\int Mdx=V$, then $\frac{\partial V}{\partial x}=M$ and $\frac{\partial M}{\partial y}=\frac{\partial^2 V}{\partial y\partial x}=\frac{\partial^2 V}{\partial x\partial y}=\frac{\partial N}{\partial x}$.

Then $\frac{\partial}{\partial x}\left(N-\frac{\partial V}{\partial y}\right)=0\Rightarrow N-\frac{\partial V}{\partial y}$ is independent of x.

So, $\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right)$. Hence $N=\frac{\partial V}{\partial y}+\phi'(y)$.

Therefore $Mdx+Ndy=\left(\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy\right)+\phi'(y)dy$

$$= dV + d(\phi(y))$$

$$= dU \text{ where } U = V + \phi(y)$$

So, $M(x, y)dx + N(x, y)dy = 0$ is exact and its solution is $V + \phi(y) = \text{constant}$.

or $\int\limits_{y=\text{constant}} Mdx + \int\left(N - \frac{\partial V}{\partial y}\right)dy = \text{constant}$.

Note : If $M(x, y)dx + N(x, y)dy = 0$ be exact, then its solution is

$$\int\limits_{y=\text{constant}} Mdx + \int(N - \text{term containing } x)dy = C$$

Example 1 : Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

Solution : Here $M = x^2 - 4xy - 2y^2$ and $N = y^2 - 4xy - 2x^2$

So, $\frac{\partial M}{\partial y} = -4x - 4y$ and $\frac{\partial N}{\partial x} = -4y - 4x$

i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So, the given equation is exact.

The solution of the equation is

$\int\limits_{y=\text{constant}} (x^2 - 4xy - 2y^2)dx + \int y^2 dy = 0$ (From N delete the terms containing x)

or $\frac{x^3}{3} - 2x^2y - 2y^2x + \frac{y^3}{3} = C$.

Example 2 : Solve $\frac{dy}{dx} + \frac{ax + hy + g}{hx + by + f} = 0$

Solution : The given equation can be written as

$$(ax + hy + g)dx + (hx + by + f)dy = 0$$

Here $M = ax + hy + g$ and $N = hx + by + f$

So, $\frac{\partial M}{\partial y} = h = \frac{\partial N}{\partial x}$

Hece the equation is exact.

The solution is

$$\int\limits_{y=\text{constant}} Mdx + \int(\text{terms free from } x \text{ in N})dy = C$$

or $\int(ax + hy + g)dx + \int(by + f)dy = C$

or $\frac{ax^2}{2}+hyx+gx+\frac{by^2}{2}+fy=C$

or $ax^2+2hxy+by^2+2gx+2fy=2C$, which is the required solution.

Example 3 : Solve $y\sin 2xdx-\left(1+y+\cos^2 x\right)dy=0$

Solution : Here $M=y\sin 2x$ and $N=-1-y-\cos^2 x$

So, $\frac{\partial M}{\partial y}=\sin 2x=\frac{\partial N}{\partial x}$

Hence the given equation is exact.

The solution is

$$\int_{y=\text{constant}} y\sin 2xdx+\int_{(\text{terms free from } x \text{ in N})}(-1-y)dy=C$$

or $-\frac{1}{2}y\cos 2x-y-\frac{y^2}{2}=C$

or $y\cos 2x+2y+y^2+2C=0$, which is the required solution.



CHECK YOUR PROGRESS - 1

Q 1 : Solve the following differential equations :

(a) $\left(y^2e^{xy^2}+4x^3\right)dx+\left(2xye^{xy^2}-3y^2\right)dy=0$

(b) $\left\{y\left(1+\frac{1}{x}\right)+\cos y\right\}dx+\left(x+\log x-x\sin y\right)dy=0$

(c) $\left[\cos x\tan y+\cos(x+y)\right]dx+\left[\sin x\sec^2 y+\cos(x+y)\right]dy=0$

(d) $\left(5x^4+3x^2y^2-2xy^3\right)dx+\left(2x^3y-3x^2y^2-5y^4\right)dy=0$

9.4 EQUATIONS REDUCIBLE TO EXACT EQUATIONS

If an equation $M(x,y)dx+N(x,y)dy=0$ becomes an exact when multiplied by a function $\mu(x,y)$ then $\mu(x,y)$ is called an integrating factor (I.F).

For example, $xdy-ydx=0$ is not an exact equation,

since $\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(-y)=-1$ but $\frac{\partial N}{\partial x}=\frac{\partial x}{\partial x}=1$.

Multiplying by a function $\mu(x,y)$ it can be made an exact equation and it can be done in many ways.

(i) $\mu_1(x,y)=\frac{1}{x^2}$ is an I.F since $\frac{1}{x^2}(xdy-ydx)=d\left(\frac{y}{x}\right)$

(ii) $\mu_2(x,y)=\frac{1}{y^2}$ is an I.F since $\frac{1}{y^2}(xdy-ydx)=-d\left(\frac{x}{y}\right)$

(iii) $\mu_3(x,y)=\frac{1}{xy}$ is an I.F since $\frac{1}{xy}(xdy-ydx)=d\left[\log\frac{y}{x}\right]$

(iv) $\mu_4(x,y)=\frac{1}{x^2+y^2}$ is an I.F since $\frac{1}{x^2+y^2}(xdy-ydx)$

$$=\frac{\frac{xdy-ydx}{x^2}}{1+\left(\frac{y}{x}\right)^2}$$

$$=\frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}=d\left(\tan^{-1}\frac{y}{x}\right)$$

Proposition : The number of integrating factors is infinite.

Proof : Suppose $\mu(x,y)$ is an integrating factor of $Mdx+Ndy=0$.

So, there is a $u(x,y)$ such that $\mu Mdx+\mu Ndy=du$.

Multiplying both sides by any function $f(u)$,we have

$(\mu f(u))Mdx+(\mu f(u))Ndy=f(u)du=\phi'(u)du$,say

Then RHS $=d\phi(x,y)$, a total differential.

So, $(\mu f(u))[Mdx+Ndy]$ is a total differential.

Hence, $\mu f(u)$is an I.F of $Mdx+Ndy=0$ .Since f is an arbitrary function,the I.F, $\mu f(u)$ can be chosen in an infinite ways.

To solve a differential equation $Mdx+Ndy=0$,difficulty lies in finding an I.F to make it an exact equation.Here we discuss some methods of finding an I.F.

I. Method of Inspection to find an I.F.

To find I.F by inspection, terms of the given equation are to be arranged so as to obtain some total differentials. For example

(i) $d(xy) = ydx + xdy$ (ii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iii) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$ (iv) $d\left(\frac{1}{xy}\right) = \frac{-(xdy + ydx)}{x^2y^2}$

(v) $\frac{1}{2}d(x^2 + y^2) = xdx + ydy$ (vi) $\frac{1}{2}d(\log(x^2 + y^2)) = \frac{xdx + ydy}{x^2 + y^2}$

(vii) $d\left(\log\frac{x}{y}\right) = \frac{ydx - xdy}{xy}$ (viii) $d\left(\log\frac{y}{x}\right) = \frac{xdy - ydx}{xy}$

(ix) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$ (x) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$

Example 1 : Solve (i) $y(2xy + e^x)dx = e^x dy$

(ii) $xdx + ydy = a(x^2 + y^2)dy$

(iii) $(2xy^2 - y)dx + xdy = 0$

(iv) $xdy - ydx = \sqrt{x^2 + y^2}\,dx$

(v) $xdx = y(x^2 + y^2 - 1)dy$

(vi) $ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y^2\right)dy$

(vii) $ydx - xdy + \log x dx = 0$

Solution : (i) $ye^x dx - e^x dy + 2xy^2 dx = 0$

Dividing by y^2, we have

$$\frac{ye^x dx - e^x dy}{y^2} + 2xdx = 0$$

or $d\left(\frac{e^x}{y}\right) + d(x^2) = 0$

Integrating, the requird solution is

$$\frac{e^x}{y} + x^2 = C$$

(ii) $xdx + ydy = a\left(x^2 + y^2\right)dy$

or $$\frac{2xdx+2ydy}{x^2+y^2}=2ady$$

or $$\frac{d\left(x^2 + y^2\right)}{x^2 + y^2} = 2ady$$

Integrating, the requird solution is

$$\log\left(x^2 + y^2\right) = 2ay + C$$

(iii) $2xy^2dx + \left(xdy - ydx\right) = 0$

or $2xdx + \frac{xdy - ydx}{y^2} = 0$

or $d\left(x^2\right) - d\left(\frac{x}{y}\right) = 0$

Integrating, the requird solution is

$$x^2 - \frac{x}{y} = C$$

or $x^2y - x = Cy.$

(iv) $xdy - ydx = \sqrt{x^2 + y^2}dx$

or $$\frac{xdy \quad - \quad ydx}{\sqrt{x^{\,2} \, + \, y^{\,2}}} \; = \; dx$$

or $$\frac{\frac{xdy - ydx}{x^2}}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \frac{1}{x}dx$$

Integrating, the requird solution is

$$\int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \int \frac{1}{x}dx$$

or $$\log\left(\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}\right) = \log x + \log C$$

or $\log\left(\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}\right)=\log(Cx)$

or $\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}=Cx$

or $y+\sqrt{x^2+y^2}=Cx^2$.

(v) $xdx=y\left(x^2+y^2-1\right)dy$

or $xdx+ydy=y\left(x^2+y^2\right)dy$

or $\frac{d\left(x^2+y^2\right)}{x^2+y^2}=2ydy$

Integrating, the requird solution is

$\log\left(x^2+y^2\right)=y^2+C$

(vi) $ye^{\frac{x}{y}}dx=\left(xe^{\frac{x}{y}}+y^2\right)dy$

or $e^{\frac{x}{y}}\left(ydx-xdy\right)=y^2dy$

or $e^{\frac{x}{y}}\frac{ydx-xdy}{y^2}=dy$

or $e^{\frac{x}{y}}d\left(\frac{x}{y}\right)=dy$

Integrating, the requird solution is

$e^{\frac{x}{y}}=y+C$

(vii) $ydx-xdy+\log xdx=0$

or $-\frac{xdy-ydx}{x^2}+\frac{\log x}{x^2}dx=0$

or $-d\left(\frac{y}{x}\right)-\log x.\frac{1}{x^2}dx=0$

Integrating, the requird solution is

$-\frac{y}{x}-\left[-\log x.\frac{1}{x}-\int\frac{1}{x}.\frac{-1}{x}dx\right]=C$

or $y + \log x + 1 = Cx$

or $y + \log x + 1 - Cx = 0$.

II. Solution of homogeneous equation $Mdx + Ndy = 0$.

Proposition : If $Mdx + Ndy = 0$ be a homogeneous equation of degree r, then $\frac{1}{Mx + Ny}$ is an integrating factor provided $Mx + Ny \neq 0$.

Proof : Let $M = x^r \phi\left(\frac{y}{x}\right)$ and $N = x^r \psi\left(\frac{y}{x}\right)$. Multiplying by $\frac{1}{Mx + Ny}$ and substituting for M and N, we have

$$\frac{x^r \phi\left(\frac{y}{x}\right)}{x^{r+1}\phi\left(\frac{y}{x}\right) + yx^r \psi\left(\frac{y}{x}\right)} dx + \frac{x^r \psi\left(\frac{y}{x}\right)}{x^{r+1}\phi\left(\frac{y}{x}\right) + x^r y \psi\left(\frac{y}{x}\right)} dy = 0$$

It is enough to check the exactness of this equation

$$\frac{\partial}{\partial y}\left(\frac{x^r \phi\left(\frac{y}{x}\right)}{x^{r+1}\phi\left(\frac{y}{x}\right) + yx^r \psi\left(\frac{y}{x}\right)}\right)$$

$$= \frac{\phi'\left(\frac{y}{x}\right).\frac{1}{x}\left\{x\phi\left(\frac{y}{x}\right) + y\psi\left(\frac{y}{x}\right)\right\} - \phi\left(\frac{y}{x}\right)\left\{x\phi'\left(\frac{y}{x}\right).\frac{1}{x} + \psi\left(\frac{y}{x}\right) + \frac{y}{x}\psi'\left(\frac{y}{x}\right)\right\}}{\left(x\phi\left(\frac{y}{x}\right) + y\psi\left(\frac{y}{x}\right)\right)^2}$$

$$= \frac{\phi'\phi + \frac{y}{x}\phi'\psi - \phi'\phi - \phi\psi - \frac{y}{x}\phi\psi'}{(x\phi + y\psi)^2}$$

$$= \frac{\frac{y}{x}(\phi'\psi - \phi\psi') - \phi\psi}{(x\phi + y\psi)^2} \quad \text{... (1)}$$

Similarly, $\frac{\partial}{\partial x}\left(\frac{\psi\left(\frac{y}{x}\right)}{x\phi\left(\frac{y}{x}\right) + y\psi\left(\frac{y}{x}\right)}\right)$

$$= \frac{\psi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)\left\{x\phi\left(\frac{y}{x}\right)+y\psi\left(\frac{y}{x}\right)\right\}-\psi\left(\frac{y}{x}\right)\left\{\phi\left(\frac{y}{x}\right)+x\phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)+y\phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)\right\}}{\left(x\phi\left(\frac{y}{x}\right)+y\psi\left(\frac{y}{x}\right)\right)^2}$$

$$= \frac{-\frac{y}{x}\phi\psi' - \frac{y^2}{x^2}\psi\psi' - \phi\psi + \psi\phi'.\frac{y}{x} + \frac{y^2}{x^2}\psi\psi'}{\left(x\phi\left(\frac{y}{x}\right)+y\psi\left(\frac{y}{x}\right)\right)^2}$$

$$= \frac{\frac{y}{x}\left(\phi'\psi - \psi'\phi\right) - \phi\psi}{\left(x\phi + y\psi\right)^2} \quad \text{... (2)}$$

From (1) and (2), we see that the condition of exactness is satisfied. Hence $\frac{1}{Mx + Ny}$ is an integrating factor.

Example 2 : Solve $\left(x^2y - 2xy^2\right)dx - \left(x^3 - 3x^2y\right)dy = 0$

Solution : This equation is homogeneous in x and y.

$$\text{I.F} = \frac{1}{Mx + Ny} = \frac{1}{\left(x^2y - 2xy^2\right)x + \left(-x^3 + 3x^2y\right)y}$$

$$= \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2}$$

$$= \frac{1}{x^2y^2}$$

Multiplying the given equation by $\frac{1}{x^2y^2}$, we have

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(-\frac{x}{y^2} + \frac{3}{y}\right)dy = 0$$

which is an exact equation since $\frac{\partial}{\partial y}\left(\frac{1}{y} - \frac{2}{x}\right) = -\frac{1}{y^2}$ and $\frac{\partial}{\partial x}\left(-\frac{x}{y^2} + \frac{3}{y}\right) = -\frac{1}{y^2}$.

The solution is

$$\int_{y=\text{constant}} \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int_{(\text{terms independent of x in coefficient of dy})} \frac{3}{y}dy = C$$

or $\frac{x}{y} - 2\log x + 3\log y = C$.

Example 3 : Solve $y^2 dx + \left(x^2 - xy - y^2\right)dy = 0$

Solution : It is a homogeneous equation.

$$\text{I.F} = \frac{1}{Mx + Ny} = \frac{1}{xy^2 + x^2 y - xy^2 - y^3}$$

$$= \frac{1}{y\left(x^2 - y^2\right)}$$

Multiplying the equation by $\frac{1}{y\left(x^2 - y^2\right)}$, we have

$$\frac{y}{x^2 - y^2}dx + \left(\frac{1}{y} - \frac{x}{x^2 - y^2}\right)dy = 0$$

Now,

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2 - y^2}\right) = -\frac{1}{2}\log\left(x^2 - y^2\right), \frac{\partial}{\partial x}\left(\frac{1}{y} - \frac{x}{x^2 - y^2}\right) = -\frac{1}{2}\log\left(x^2 - y^2\right)$$

$\therefore$ the equation is exact.

The solution is

$$\int_{y=\text{constant}} \frac{y}{x^2 - y^2}dx + \int_{\text{terms free from } x \text{ in coeff.of } dy} \frac{1}{y}dy = C$$

or $\frac{y}{2y}\log\frac{x-y}{x+y} + \log y = \log C$

or $\log y\sqrt{\frac{x-y}{x+y}} = \log C$

or $y\sqrt{\frac{x-y}{x+y}} = C$

or $y^2\left(x-y\right) = C^2\left(x+y\right)$.

III (A): Integrating factor for an equation of type

$f_1(xy)ydx + f_2(xy)xdy = 0$ is $\frac{1}{xyf_1(xy) - xyf_2(xy)}$ if the denominator is not zero.

Proof : Multiplying the given equation by $\frac{1}{xyf_1(xy) - xyf_2(xy)}$, we have

$$\frac{yf_1(xy)dx}{xy\{f_1(xy)-f_2(xy)\}}+\frac{xf_2(xy)dx}{xy\{f_1(xy)-f_2(xy)\}}=0$$

We check that this is an exact equation.

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(\frac{f_2}{x(f_1-f_2)}\right)=\frac{x^2f_1'(f_1-f_2)-f_1\{x^2(f_1'-f_2')\}}{x^2(f_1-f_2)^2}$$

$$=\frac{f_1f_2'-f_1'f_2}{(f_1-f_2)^2}$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{f_1}{y(f_1-f_2)}\right)=\frac{y^2f_2'(f_1-f_2)-f_2\{y^2(f_1'-f_2')\}}{y^2(f_1-f_2)^2}$$

$$=\frac{f_1f_2'-f_1'f_2}{(f_1-f_2)^2}$$

So, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Hence the given equation is an exact equation and it can be solved by the method already discussed above.

III (B) : With the help of the equation

$$\int\frac{f(xy)+F(xy)}{f(xy)-F(xy)}\frac{d(xy)}{xy}+\log\frac{x}{y}=C$$

Show that $\frac{1}{xy\{f(xy)-F(xy)\}}$ is an integrating factor of $f(xy)ydx+F(xy)xdy=0$. Hence solve

$(x^2y^2+xy+1)ydx-x(x^2y^2-xy+1)dy=0$

It is enough to show that

$\frac{yf(xy)dx+xF(xy)dy}{xy\{f(xy)-F(xy)\}}$ is a total differential.

$$\frac{yf(xy)dx+xF(xy)dy}{xy\{f(xy)-F(xy)\}}$$

$$=\frac{\frac{1}{2}\left\{(xyf(xy)+xyF(xy))\left(\frac{dx}{x}+\frac{dy}{y}\right)+(xyf(xy)-xyF(xy))\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}}{xyf(xy)-xyF(xy)}$$

$$=\frac{1}{2}\frac{xyf(xy)+xyF(xy)}{xyf(xy)-xyF(xy)}\frac{d(xy)}{xy}+\frac{1}{2}\frac{ydx-xdy}{xy}$$

$$= \frac{1}{2}\left\{\frac{f(xy)+F(xy)}{f(xy)-F(xy)}\frac{d(xy)}{xy}+\frac{d\left(\frac{x}{y}\right)}{\left(\frac{x}{y}\right)}\right\}$$

So, $\int \frac{f(xy)ydx+F(xy)xdy}{xyf(xy)-xyF(xy)} = \frac{1}{2}\left[\int \frac{f(xy)+F(xy)}{f(xy)-F(xy)}\frac{d(xy)}{xy}+\log\frac{x}{y}\right]$

$$= \frac{1}{2}C$$

So, $\frac{f(xy)ydx+F(xy)xdy}{xyf(xy)-xyF(xy)} = d\left(\frac{1}{2}C\right)$, a total differential.

Hence $\frac{yf(xy)dx+xF(xy)dy}{xy\{f(xy)-F(xy)\}}=0$ is an exact equation and

$\frac{1}{xy\{f(xy)-F(xy)\}}$ is an integrating factor of $f(xy)ydx+F(xy)xdy=0$.

Second part : Applying the preceding result, the integrating factor of $(x^2y^2+xy+1)ydx-x(x^2y^2-xy+1)dy=0$ is

$$\frac{1}{xy\{x^2y^2+xy+1+x^2y^2-xy+1\}} = \frac{1}{xy(2x^2y^2+2)}$$

$$\therefore \quad \frac{x^2y^2+xy+1}{2x(x^2y^2+1)}dx+\frac{xy-x^2y^2-1}{2y(x^2y^2+1)}dy=0$$

or $\left(\frac{y}{2(1+x^2y^2)}+\frac{1}{2x}\right)dx+\left(\frac{x}{2(1+x^2y^2)}-\frac{1}{2y}\right)dy=0$

So, its solution is

$$\frac{1}{2y}\int\frac{ydx}{1+x^2y^2}+\frac{1}{2}\int\frac{dx}{x}-\frac{1}{2}\int\frac{dy}{y}=\log C$$

or $\frac{1}{2}\tan^{-1}(xy)+\frac{1}{2}\log\left(\frac{x}{y}\right)=\log C$

or $C^2y = xe^{\tan^{-1}(xy)}$, the required solution.

Example 4 : Solve $(1+xy)ydx+(1-xy)xdy=0$

Solution : It is of the form $f_1(xy)ydx+f_2(xy)xdy=0$.

$$\text{I.F} = \frac{1}{Mx - Ny} = \frac{1}{xy(1+xy) - xy(1-xy)} = \frac{1}{2x^2y^2}$$

Multiplying the given equation by $\frac{1}{2x^2y^2}$, we have

$$\frac{1+xy}{2x^2y}dx + \frac{1-xy}{2xy^2}dy = 0$$

This is an exact equation and its solution is

$$\int_{y=\text{constant}} \left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \int -\frac{1}{2y}dy = \log C$$

or $-\frac{1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = \log C$

or $-\frac{1}{xy} = \log\left(\frac{C^2y}{x}\right)$

or $\frac{C^2y}{x} = e^{-\frac{1}{xy}}$

or $C^2y = xe^{-\frac{1}{xy}}$.

Example 5 : Solve $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

Solution : It is of the form $f_1(xy)ydx + f_2(xy)xdy = 0$.

So, I.F = $\frac{1}{xy\{f(xy) - F(xy)\}} = \frac{1}{xy(xy + 2x^2y^2) - xy(xy - x^2y^2)}$

$$= \frac{1}{3x^3y^3}$$

Multipling the given equation throughout by the I.F,we get

$$\frac{y(1+2xy)}{x^2y^2}dx + \frac{x(1-xy)}{x^2y^2}dy = 0$$

or $\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$

The equation is exact and its solution is

$$\int_{y=\text{constant}} \left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \int -\frac{1}{y}dy = \log C$$

or $-\frac{1}{xy}+2\log x-\log y=\log C$

or $-\frac{1}{xy}=\log\frac{Cy}{x^2}$

or $\frac{Cy}{x^2}=e^{-\frac{1}{xy}}$

or $Cy=x^2e^{-\frac{1}{xy}}$

IV (A) If $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x)$ be a function of x or constant, then show that $e^{\int f(x)dx}$ is an integrating factor of $Mdx+Ndy=0$.

Proof : It is enough to show that $\left(Me^{\int f(x)dx}\right)dx+\left(Ne^{\int f(x)dx}\right)dy=0$ is an exact equation, equivalently left side in a total differential.

$$\frac{\partial}{\partial y}\left(Me^{\int f(x)dx}\right)=e^{\int f(x)dx}\frac{\partial M}{\partial y}$$

and $\frac{\partial}{\partial x}\left(Ne^{\int f(x)dx}\right)=e^{\int f(x)dx}\frac{\partial N}{\partial x}+Ne^{\int f(x)dx}.f(x)$

$$=e^{\int f(x)dx}\left(\frac{\partial N}{\partial x}+\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$$, replacing $Nf(x)$ by $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}$

$$=e^{\int f(x)dx}\frac{\partial M}{\partial y}$$

So, $\frac{\partial}{\partial y}\left(Me^{\int f(x)dx}\right)=\frac{\partial}{\partial x}\left(Ne^{\int f(x)dx}\right)$. Hence $e^{\int f(x)dx}$ is an integrating factor of $Mdx+Ndy=0$.

IV (B). If $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=f(y)$ is a function of y then $e^{\int f(y)dy}$ is an integrating factor of $Mdx+Ndy=0$.

Proof : $\frac{\partial}{\partial y}\left(e^{\int f(y)dy}M\right)=e^{\int f(y)dy}.f(y)M+e^{\int f(y)dy}\frac{\partial M}{\partial y}$

$$=e^{\int f(y)dy}\left\{\frac{\partial M}{\partial y}+f(y)M\right\}$$

$$=e^{\int f(y)dy}\left\{\frac{\partial M}{\partial y}+\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right\}$$

$$=e^{\int f(y)dy}.\frac{\partial N}{\partial x}$$

$$=\frac{\partial\left\{e^{\int f(y)dy}.N\right\}}{\partial x}$$

So, $Mdx+Ndy=0$ is an exact equation and $e^{\int f(y)dy}$ is an integrating factor of $Mdx+Ndy=0$.

Example 6 : Solve : (i) $\left(xy^2-e^{\frac{1}{x^3}}\right)dx-x^2ydy=0$

(ii) $\left(xy^3+y\right)dx+2\left(x^2y^2+x+y^4\right)dy=0$

Solution : (i) $M=xy^2-e^{\frac{1}{x^3}}$ and $N=-x^2y$

$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2xy+2xy}{-x^2y}=-\frac{4}{x}$, function of x.

I.F $=e^{-\int\frac{4}{x}dx}=e^{-4\log x}=x^{-4}$

Multiplying both sides by x^{-4}, we have

$\left(x^{-3}y^2-x^{-4}e^{\frac{1}{x^3}}\right)dx-x^{-2}ydy=0$, an exact equation.

Required solution is

$$y^2\int x^{-3}dx-\int e^{\frac{1}{x^3}}.x^{-4}dx+0=C$$

or $\frac{y^2x^{-2}}{-2}+\frac{1}{3}\int e^u du=C$ [Putting $u=x^{-3}$, $du=-3x^{-4}dx$]

or $-\frac{1}{2}\frac{y^2}{x^2}+\frac{1}{3}e^{x^{-3}}=C$.

(ii) Here, $M=xy^3+y$ and $N=2\left(x^2y^2+x+y^4\right)$

So, $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{y\left(xy^2+1\right)}\left(4xy^2+2-3xy^2-1\right)=\frac{1}{y}=f(y)$ is a function of y.

So, I.F $=e^{\int f(y)dy}=e^{\log y}=y$

Multiplying the equation by y, we have

$\left(xy^4+y^2\right)dx+2\left(x^2y^3+xy+y^5\right)dy=0$, which is an exact equation.

The solution of the equation is

$y^4\int xdx+y^2\int dx+2\int y^5dy=C$

or $\frac{1}{2}x^2y^4+xy^2+\frac{1}{3}y^6=C$.

IV(c) Show that the equation $\left(Pdx+Qdy\right)e^{\int f(x)dx}=0$ satisfies the conditions of exactness if $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}+Qf(x)$.

From this show that I.F can be determined if $\frac{1}{Q}\left[\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right]$ is a function of x. Hence solve $\left(x^3+xy^4\right)dx+2y^3dy=0$.

Let $M=Pe^{\int f(x)dx}$ and $N=Qe^{\int f(x)dx}$

$\frac{\partial M}{\partial y}=\frac{\partial P}{\partial y}e^{\int f(x)dx}$ and $\frac{\partial N}{\partial x}=Qe^{\int f(x)dx}.f(x)+\frac{\partial Q}{\partial x}e^{\int f(x)dx}$

So, condition of exactness is satisfied if

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

or $\frac{\partial P}{\partial y}e^{\int f(x)dx}=Qe^{\int f(x)dx}.f(x)+\frac{\partial Q}{\partial x}e^{\int f(x)dx}$

or $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}+Qf(x)$

For second part, we see that $\left(Pdx+Qdy\right)e^{\int f(x)dx}$ is exact, equivalentley $e^{\int f(x)dx}$ is an I.F of $Pdx+Qdy=0$ if

$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}+Qf(x)$

i.e., $\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=f(x)$, a function of x.

To solve : $\left(x^3+xy^4\right)dx+2y^3dy=0$

Here $P=x^3+xy^4$ and $Q=2y^3$.

So, $\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=\frac{1}{2y^3}\left(4xy^3-0\right)=2x=f(x)$

I.F $=e^{\int f(x)dx}=e^{\int 2xdx}=e^{x^2}$

Multiplying both sides of the equation by e^{x^2}, we have

$e^{x^2}\left(x^3+xy^4\right)dx+e^{x^2}.2y^3dy=0$

The equation is exact.

The solution of the equation is

$\int e^{x^2}.x^2.xdx+y^4\int e^{x^2}xdx=C$

or $\frac{1}{2}\int e^u udu+\frac{1}{2}y^4\int e^u du=C \qquad \left(u=x^2\right)$

or $\frac{1}{2}\left[ue^u-\int e^u du+y^4e^u\right]=C$

or $ue^u-e^u+y^4e^u=2C$

or $e^{x^2}\left(x^2-1+y^4\right)=k$.

(V) For the equation of the type

$x^ay^b\left(mydx+nxdy\right)+x^cy^d\left(pydx+qxdy\right)=0$,

the integrating factor is x^hy^k.

Multiplying by x^hy^k throughout we have

$x^{a+h}y^{b+k}\left(mydx+nxdy\right)+x^{c+h}y^{d+k}\left(pydx+qxdy\right)=0$ which is of the form

$Mdx+Ndy=0$ with

$$M=mx^{a+h}y^{b+k+1}+px^{c+h}y^{d+k+1}$$

and $N=nx^{a+h+1}y^{b+k}+qx^{c+h+1}y^{d+k}$.

From the condition of exactness, we have

$$mx^{a+h}.(b+k+1)y^{b+k}+px^{c+h}.(d+k+1)y^{d+k}$$
$$=n(a+h+1)x^{a+h}y^{b+k}+q(c+h+1)x^{c+h}y^{d+k}$$

Comparing coefficients, we have

$m(b+k+1)=n(a+h+1)$ and $p(d+k+1)=q(c+h+1)$

or $\frac{b+k+1}{n}=\frac{a+h+1}{m}$ and $\frac{d+k+1}{q}=\frac{c+h+1}{p}$.

Solving these two equations, values of h and k can be obtained so as to obtain the integrating factor.

Example 7 : Solve $y(xy+2x^2y^2)dx+x(xy-x^2y^2)dy=0$

Solution : Writing the equation in the standard form

$x^a y^b(mydx+nxdy)+x^c y^d(pydx+qxdy)=0$, we have

$xy(ydx+xdy)+x^2y^2(2ydx-xdy)=0$.

Let $x^h y^k$ be an integrating factor, then

$\frac{1+h+1}{1}=\frac{1+k+1}{1}$ and $\frac{2+h+1}{2}=\frac{2+k+1}{-1}$

or $h=k$ and $-3-h=6+2k$.

Solving, we have $h=k=-3$. So, integrating factor is $\frac{1}{x^3y^3}$.

$\therefore$ the equation becomes

$\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy=0$, which is an exact equation.

Its solution is

$$\frac{1}{y}\left(-\frac{1}{x}\right)+2\log x-\log y=C$$

or $2\log x-\log y-\frac{1}{xy}=C$.

Example 8 : Solve $x(4ydx+2xdy)+y^3(3ydx+5xdy)=0$

Solution : Let $x^h y^k$ be an integrating factor of the given equation. So,

$$\frac{1+h+1}{4}=\frac{k+1}{2} \text{ and } \frac{h+1}{3}=\frac{3+k+1}{5}$$

or $2k+2=2+h$ and $12+3k=5h+5$

or $h=2k$ and $5h-3k=7$

Solving, $h=2$ and $k=1$.So, x^2y is an integrating factor of the equation.

Multiplying the equation by x^2y throughout, we have,

$(4x^3y^2+3x^2y^5)dx+(2x^4y+5x^3y^4)dy=0$, which is an exact equation.

The solution of this equation is

$$4y^2\int x^3dx+3y^5\int x^2dx+0=C$$

or $y^2x^4+x^3y^5=C$.



CHECK YOUR PROGRESS - 2

Q 1 : Solve the following differential equations :

(i) $ydx - xdy + \left(1 + x^2\right)dx + x^2 \sin ydy = 0$

(ii) $\left(x^2 y - 2xy^2\right)dx - \left(x^3 - 3x^2 y\right)dy = 0$

(iii) $\left(2x^2 y^2 + y\right)dx - \left(x^3 y - 3x\right)dy = 0$

(iv) $\left(xy\sin xy + \cos xy\right)ydx + \left(xy\sin xy - \cos xy\right)xdy = 0$

(v) $\left(x^2 + y^2 + x\right)dx + xydy = 0$



9.5 LET US SUM UP

- The necessary and sufficient condition for $M(x,y)dx + N(x,y)dy = 0$ to be an exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
- If $M(x,y)dx + N(x,y)dy = 0$ be exact, then its solution is

$$\int_{y=\text{constant}} Mdx + \int \left(N - \text{term containing } x\right)dy = C$$

- A function $\mu(x,y) \neq 0$ which is used to convert a differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ (which is not exact) into an exact equation is called Integrating factor (I.F) of the equation $M(x,y)dx + N(x,y)dy = 0$.



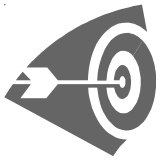
9.6 FURTHER READINGS

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2) Differential Equations : An Introduction with Mathematica. Clay C. Ross, second Edition, Springer.

3) Differential Equations : Linear, Nonlinear, ordinary, partial. A.C. King, J. Billinggham and S.R. Otto, Cambridge-2003.

4) Differential equations with Application and Programs. S. Balachandra Rao and H.R. Anuradha, University Press(India) Limited 1996.



9.7 ANSWERS TO CHECK YOUR PROGRESS

Check your progress -1

Ans to Q No 1: (a) Comparing the given equation with $Mdx + Ndy = 0$, we get

$M = y^2 e^{xy^2} + 4x^3$ and $N = 2xye^{xy^2} - 3y^2$

Also, $\frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2.2xye^{xy^2}$

and $\frac{\partial N}{\partial x} = 2ye^{xy^2} + y^2.2xye^{xy^2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence, the given equation is exact and so its solution is

$$\int_{y=\text{constant}} Mdx + \int (\text{terms free from } x \text{ in N})dy = C$$

or $\int \left(y^2 e^{xy^2} + 4x^3\right)dx + \int \left(-3y^2\right)dy = C$

or $y^2 \times \frac{1}{y^2} \times e^{xy^2} + 4 \times \frac{1}{4} \times x^4 - 3 \times \frac{y^3}{3} = C$

or $e^{xy^2} + x^4 - y^3 = C$.

(b) Do yourself.

(c) Here $M = \cos x \tan y + \cos(x + y)$

$N = \sin x \sec^2 y + \cos(x + y)$

$\therefore \frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x + y) = \frac{\partial N}{\partial x}$

Thus the given equation is exact and its solution is

$$\int_{y=\text{constant}} Mdx + \int (\text{terms free from } x \text{ in N})dy = C$$

or $\int [\cos x \tan y + \cos(x+y)]dx = C$

or $\sin x \tan y + \sin(x+y) = C$.

(d) Do Yourself

Required solution is

$x^5 + x^3y^2 - x^2y^3 - y^5 = C$.

Check your progress - 2

Ans to Q No 1: (i) Dividing the given equation by x^2, we get

$$\frac{ydx - xdy}{x^2} + \left(\frac{1+x^2}{x^2}\right)dx + \sin y dy = 0$$

or $-\frac{xdy - ydx}{x^2} + \left(\frac{1}{x^2} + 1\right)dx + \sin y dy = 0$

or $-d\left(\frac{y}{x}\right) + \left(1 + \frac{1}{x^2}\right)dx + \sin y dy = 0$

or $-\frac{y}{x} + x - \left(\frac{1}{x}\right) - \cos y = c$

or $-y + x^2 - 1 - x\cos y = cx$, which is the required solution.

(ii) The given equation is homogeneous in x and y with

$M = x^2y - 2xy^2$ and $N = -x^3 + 3x^2y$

Now, $Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$

I.F $= \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}$

Multiplying throughout by $\frac{1}{x^2y^2}$, the given equation becomes

$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$ which is exact.

The required solution is

$$\int_{y=\text{constant}} \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}dy = C$$

or $\frac{x}{y} - 2\log x + 3\log y = C$.

(iii) Given $\left(2x^2y^2 + y\right)dx + \left(3x - x^3y\right)dy = 0$ ... (1)

Multiplying both sides of (1) by I.F $x^h y^k$,we get

$\left(2x^{h+2}y^{k+2} + x^h y^{k+1}\right)dx + \left(3x^{h+1}y^k - x^{h+3}y^{k+1}\right)dy = 0$ ... (2)

which must be exact. Comparing (2) with $Mdx + Ndy = 0$, we have

$M = 2x^{h+2}y^{k+2} + x^h y^{k+1}$ and $N = 3x^{h+1}y^k - x^{h+3}y^{k+1}$

For (2) to be exact, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow 2(k+2)x^{h+2}y^{k+1} + (k+1)x^h y^k = 3(h+1)x^h y^k - (h+3)x^{h+2}y^{k+1}$

$\Rightarrow 2(k+2) = -(h+3)$ and $(k+1) = 3(h+1)$

$\Rightarrow h + 2k = -7$ and $3h - k = -2$

$\Rightarrow h = -\frac{11}{7}$ and $k = -\frac{19}{7}$

Hence an I.F of (1) is $x^{-\frac{11}{7}}y^{-\frac{19}{7}}$. Multiplying (1) by $x^{-\frac{11}{7}}y^{-\frac{19}{7}}$, we have

$$\left(2x^{\frac{3}{7}}y^{-\frac{5}{7}} + x^{-\frac{11}{7}}y^{-\frac{12}{7}}\right)dx + \left(3x^{-\frac{4}{7}}y^{-\frac{19}{7}} - x^{\frac{10}{7}}y^{-\frac{12}{7}}\right)dy = 0$$

which must be exact. Hence, as usual the required solution is

$$\frac{2x^{\frac{10}{7}}y^{-\frac{5}{7}}}{\frac{10}{7}} + \frac{x^{-\frac{4}{7}}y^{-\frac{12}{7}}}{-\frac{4}{7}} = \frac{7C}{20}$$

or $4x^{\frac{10}{7}}y^{-\frac{5}{7}} - 5x^{-\frac{4}{7}}y^{-\frac{12}{7}} = C$

(iv) Do yourself.

Required solution is

$$\frac{x}{y}\sec xy = C$$

(v) Do yourself.

Required solution is

$3x^4 + 6x^2y^2 + 4x^3 = C$.



9.8 MODEL QUESTIONS

Q 1 : Solve the following differential equations :

(a) $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0$

(b) $xdy+ydx+\frac{xdy-ydx}{x^2+y^2}=0$

(c) $\left(a^2-2xy-y^2\right)dx-\left(x+y\right)^2dy=0$

(d) $\left(2ax+by\right)ydx+\left(ax+2by\right)xdy=0$

Q 2 : Solve the following differential equations :

(a) $\left(x^2-3xy+2y^2\right)dx+x\left(3x-2y\right)dy=0$

(b) $\left(x^2-2xy+5y^2\right)dx+\left(2y^2-7xy+x^2\right)dy=0$

(c) $\left(2x-y\right)e^{\frac{y}{x}}dx+\left(y+xe^{\frac{y}{x}}\right)dy=0$

(d) $y-x\frac{dy}{dx}=x+y\frac{dy}{dx}$

(e) $\left(2x^3-y^3\right)dy-y^2xdx=0$

(f) $\left(x^4+y^4\right)dx-xy^3dy=0$

(g) $y^2dx+\left(x^2-xy-y^2\right)dy=0$

(h) $\left(y^4-2x^3y\right)dx+\left(x^4-2xy^3\right)dy=0$

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