UNIT 4: CIRCLE AND ELLIPSE DRAWING ALGORITHM

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4.1 LEARNING OBJECTIVES

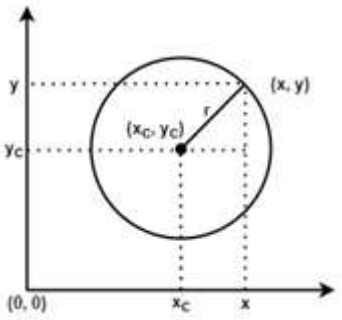
After going through this unit, you will be able to:

- describe about the properties of circle and ellipse
- identify the circle equation need to plot points on circle path.
- Recall the midpoint circle drawing algorithm
- Explain the role of decision parameter in circle drawing algorithm.
- Indicate the Bresenham's circle drawing algorithm.
- define the equations of an ellipse.

4.2 INTRODUCTION

The circle is a common picture element used in drawing pictures and graphs. A full circle or arcs of a circle can be generated through a procedure in a computer graphics system.

4.3 PROPERTIES OF A CIRCLE



A circle can be defined as a set of points that are all at a given distance r from a given point (x_c, y_c) as shown in figure 4.1. The point is known as the center of the circle and the equal distance of each point from the center to the boundary is known as the radius (r) of the circle. The complete length of the circle arc or the boundary of the circle is known as the circumference. The angle formed at the center is measured to be 360^0. A straight line from the circle boundary to the opposite boundary across the center is called the diameter. The diameter of a circle is always measured as two times the circle radius, i.e. diameter d = 2 * radius(r). The relation between radius (r) and the center (c) can be expressed in terms of Pythagorean Theorem in Cartesian coordinate as

$$(x - x_c)^2 + (y - y_c)^2 = r^2 \quad \text{.....(1)}$$

Using the equation (1), it is possible to calculate the coordinate positions of circle circumference by stepping along the x-axis in unit steps from (x_c- r) to (x_c + r) and calculate the corresponding y coordinate for each pixel using the following equation (2):

$$y = y_c \pm \sqrt{r^2 - (x_c - x)^2} \quad \text{.....(2)}$$

This process of circle generation requires considerable computation at each step and the spacing between the plotted pixels is not uniform. This non-uniformity can reduce by calculating the circle boundary points using polar coordinates r and θ. The expression used to calculate the circle boundary pixel position is:

$$\left.\begin{aligned} x &= x_c + r\cos\theta \\ y &= y_c + r\sin\theta \end{aligned}\right\} \quad \text{.....(3)}$$

Another method to reduce the computation steps is the consideration of the symmetry of a circle. The shape of a circle is similar in each quadrant and each octant. We can generate the circle section in the second quadrant of the xy plane by noting that the two circle sections are symmetric to the y axis. The circle sections in the third and fourth quadrant can obtain from sections in the first and second quadrants considering symmetry to the x-axis. Circle sections in adjacent octants are symmetric about the 450 lines dividing the two octants.

Calculation of pixel positions along a circle boundary or circumference using (1) and (3) still requires a remarkable computation time. The Cartesian equation (1) is required to perform multiplication and square root operations, while equation (3) requires multiplications and trigonometric calculations. An efficient algorithm is generally based on the incremental calculation of decision parameters. One efficient algorithm to generate a circle is the Midpoint Circle Algorithm.

4.4 MIDPOINT CIRCLE DRAWING ALGORITHM

As seen in the raster line algorithm, the sampling is done at unit intervals and the closest pixel position to the specified circle path is determined at each step. Considering the radius r and the center (x_c, y_c), let us set the algorithm to calculate the circle boundary pixel positions centered at the coordinate origin (0, 0). Then each calculated position(x, y) is moved to its proper screen position by adding x_c to x and y_c to y. Along the circle section from x=0 to x=y in the first quadrant, the slope of the curve varies from 0 to -1. Therefore we can take unit steps in the positive x-direction over this octant and use a decision parameter to determine which of the two possible y positions is closer to the circle path at each step. Positions in the other seven octants can obtain by symmetry.

For applying the midpoint circle algorithm, the definition of circle function can be express as:

$$f_{circle}(x, y) = x^2 + y^2 - r^2 \quad \text{.....(4)}$$

Any point on the circle circumference with radius r satisfies the equation $f_{circle}(x, y) = 0$. For an interior point, the circle function gives a negative value. If the pixel position is outside of the circle boundary, the circle function gives a positive value. Thus checking the sign of the circle function, the position of (x, y) point can be determined. The following figure 4.2 shows the midpoint between two candidate pixels at sampling position $x_k + 1$. Assume that the currently plotted pixel is (x_k, y_k). We need to determine the next pixel position, whether the pixel position $(x_k + 1, y_k)$ or $(x_k + 1, y_k - 1)$ is closer to the circle path. The circle function from the equation (4), we can have the decision parameter to determine the midpoint between these two-pixel positions.

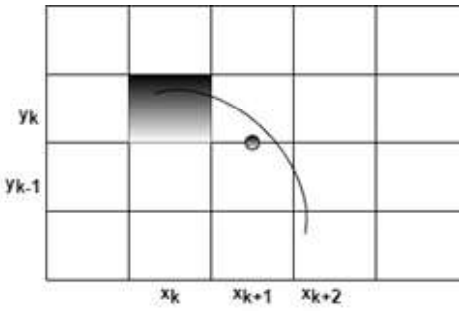


Figure 4.2 Midpoint between candidate pixels at sampling position x_k+ 1 along the circular path

$$p_k = f_{circle}(x_k + 1, y_k - \frac{1}{2})$$

$$=(x_k + 1)^2 + (y_k - \frac{1}{2})^2 - r^2 \quad \text{.....(5)}$$

If pk< 0, the midpoint is inside the circle. And the pixel on scan lineyk is closer to the circle boundary.

Otherwise, the mid position is outside or on the circle boundary, and select the pixel position y_k- 1.

Successive decision parameters can be obtained using incremental calculations. We obtain a recursive expression for the next decision parameter by evaluating the circle function at sampling position $x_{k+1} + 1 = x_k + 2$:

$$p_{k+1} = f_{circle}(x_{k+1}+1, y_{k+1}-\frac{1}{2})$$

$$=[(x_k+1)+1]^2+(y_{k+1}-\frac{1}{2})^2-r^2$$

$$= p_k + 2(x_k+1)+(y^2{}_{k+1}-y^2{}_k)-(y_{k+1}-y_k)+1 \quad \ldots\ldots(6)$$

Where y_{k+1} is either y_k or y_k-1 depending on the sign of p_k.

If p_k is negative, then y_{k+1} = yk, and from the equation (6), we can calculate the successive p_{k+1}:

$$p_{k+1} = p_k + 2x_{k+1} + 1 \quad \ldots\ldots(7)$$

Otherwise, $y_{k+1} = y_k - 1$, and

$$p_{k+1} = p_k + 2x_{k+1} + 1 - 2y_{k+1} \quad \ldots\ldots(8)$$

Where
$$2x_{k+1} = 2x_k + 2$$
$$2y_{k+1} = 2y_k - 2$$

Initially, if the starting coordinate is (0, r), then these two terms have the value 0 and 2r respectively. Each successive value is obtained by adding 2 to the previous value of 2x and subtracting 2 from the previous value of 2y. The initial decision parameter is obtained by evaluating the circle function for the starting coordinate position (x_0, y_0) = (0, r):

$$p_0 = f_{circle}(0+1, r-1/2)$$

$$=1^2+(r-1/2)^2-r^2$$

$$=\frac{5}{4}-r \quad \ldots\ldots(9)$$

If the radius r is specified as an integer, then the value of p_0 can be calculated as:

$$p_0 = 1 - r \quad \ldots\ldots(10)$$

As in Bresenham's line drawing algorithm, the midpoint circle drawing algorithm calculates the pixel positions on circle boundary using integer addition and subtractions. The summary of the midpoint circle algorithm is as follows:

4.4.1 Steps In Midpoint Circle Algorithm:

1. Consider a circle with radius r, and center coordinate (x_c, y_c). Let the first point on the circumference of the circle centered at the coordinate origin (0, 0) is (x_0, y_0) = (0, r)
2. Calculate the initial decision parameter p_0 = 5/4 -1
3. At each x_k position, k starts from 0, perform the following operations :

   3.1 ifp_k< 0, the next point on the circle circumference will be (x_{k+1}, y_k) and $p_{k+1} = p_k + 2x_{k+1} + 1$

   Otherwise, the next point on the circle path is (x_k + 1, y_k - 1) and

   $$p_{k+1} = p_k + 2x_{k+1} + 1 - 2y_{k+1}$$

   Where
   $$2x_{k+1} = 2x_k + 2$$
   $$2y_{k+1} = 2y_k - 2$$
4. Need to determine the symmetry points for the other remaining octants.
5. Move each calculated pixel position (x, y) onto the circle boundary centered on (xc, yc) and plot the coordinate values: x = x + x_c and y = y + y_c
6. Repeat the steps from 3 to 5 until x $\leq$ y.

Example 1: Show the pixel pints on the circle boundary in the first quadrant, centered at the coordinate origin (0, 0) and the radius r = 10.

Solution: initial decision parameter p_0 = 1 - r = 1 - 10 = 9

Let the initial point of the circle is (x_0, y_0) = (0, 10)

Therefore the term $2x_0$ = 0 and $2y_0$ = 20

So the successive pixel coordinates along the circle path are:

k	P_k	(x_{k+1}, y_{k+1})	$2x_{k+1}$	$2y_{k+1}$
0	-9	(1, 10)	2	20
1	-6	(2, 10)	4	20
2	-1	(3, 10)	6	20
3	6	(4, 9)	8	18
4	-3	(5, 9)	10	18
5	8	(6, 8)	12	16
6	5	(7, 7)	14	4

Example 2: Find the points on a circle on of its octants with the circle centered at (5, 5) and has a radius of 8 units.

Solution: Assume that the circle is centered at the origin and proceed to solve. After finding the points add center (5, 5) to each point.

The initial point $(x_0, y_0) = (0, 8)$

$P_0 = 1 - r = 1 - 8 = -7$

k	P_k	at origin (0, 0) (x_{k+1}, y_{k+1})	At origin (5, 5) (x_{k+1}, y_{k+1})
0	-7	(1, 8)	(6, 13)
1	-4	(2, 8)	(7, 13)
2	1	(3, 7)	(8, 12)
3	-6	(4, 7)	(9, 12)
4	3	(5, 6)	(10, 11)
5	2	(6, 5)	(11, 10)



CHECK YOUR PROGRESS

Q 1: What is the equation for a circle?

A. $y = mx + c$

B. $y^2 = r^2 - x^2$

C. $rx^2y^2 = rx^2ry^2 - ry^2x^2$

D. None of the above

Q 2: Which of the following are the methods to define a circle?

A. Using the polynomial method.

B. Using polar coordinates methods.

C. Both A and B.

D. None of the above

4.5 BRESENHAM'S CIRCLE DRAWINGALGORITHM

Bresenham's circle drawing algorithm is successfully used to draw the circle points on its perimeter for one octant, for a given value of the circle radius (c_r) and the center (x_c, y_c) of the circle. For the first octant, the circle points will be generated from 900 to 450 as shown in the following

figure 4.3. The circle points coordinates will be transfer in the positive x (+x) axis and in the negative y (-y) axis.Bresenham's algorithm chooses the coordinate point for the circle boundary, which is closer to the circle path. In the following diagram: consider the circle radius is r.

dA represents the distance between the centre of the circle and the point A (x_k + 1, y_k), and ?Adenoted the distance of the point A from the circle boundary.

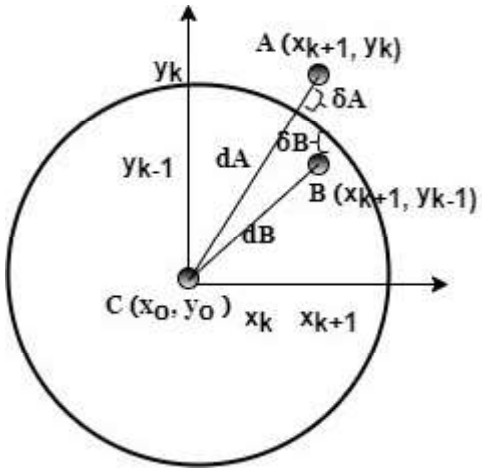


Figure 4.3: Bresemham's circle drawing algorithm

ϑB represents the distance between the centre of the circle and the point B(x_k + 1, y_k - 1), and ϑBdenoted the distance of the point B from the circle boundary.

Hence dA = r + ϑAanddB = r - ϑB

Radius (r) = dB + ϑB, now, the distance dAand dB can be computed using the following expressions:

$$dA = \sqrt{(x_k + 1 - 0)^2 + (y_k - 0)^2}$$
$$= \sqrt{(x_k + 1)^2 + (y_k)^2}$$

$$dB = \sqrt{(x_k + 1 - 0)^2 + (y_k - 1 - 0)^2}$$
$$= \sqrt{(x_k + 1)^2 + (y_k - 1)^2}$$

The ϑA is a positive value, and it can be calculated from ϑA = dA - r, similarly ϑB is a negative number, which can be calculated from the

equation ϑB = dB - r.

Now, to avoid the complex calculations containing square root, introduce two new variables as follows:

$$\delta A' = dA^2 - r^2$$
$$\delta B' = dB^2 - r^2$$

The decision parameter D can be achieved from the equation $D = \delta A' + \delta B'$, to determine the coordinate position which is closer to the circle boundary. For a positive value of D, when $\delta A' > \delta B'$, choose the coordinate point B as the next circle path point. For a negative value of D, when $\delta A' < \delta B'$, choose the coordinate point A as the next circle path point.

The decision parameter for the current pixel coordinate can be calculated as:

$$\begin{aligned} D_k &= \delta A' + \delta B' \\ &= (dA^2 - r^2) + (dB^2 - r^2) \\ &= \left[(x_k + 1)^2 - r^2 + {y_k}^2\right] + \left[(x_k + 1)^2 - r^2 + (y_k - 1)^2\right] \\ &= 2(x_k + 1)^2 + (y_k - 1)^2 + {y_k}^2 - 2r^2 \end{aligned} \quad \ldots\ldots(1)$$

The decision parameter D_{k+1} for the next pixel coordinate can be calculated from the equation (1) as following, where $x_{k+1} = x_k + 1$ and $y_{k+1} = y_k - 1$ or y_k depending on the decision parameter of the previousstep:

$$D_{k+1} = 2(x_{k+1} + 1)^2 + (y_{k+1} - 1)^2 + {y_{k+1}}^2 - 2r^2 \quad \ldots\ldots(2)$$

Now, to get the value of in terms of the previous parameter, perform subtraction operation between the equation (1) and (2).

$$D_{k+1} - D_k = 2(x_{k+1} + 1)^2 + (y_{k+1} - 1)^2 + {y_{k+1}}^2 - 2r^2 - \left[2(x_k + 1)^2 + (y_k - 1)^2 + {y_k}^2 - 2r^2\right]$$

$$\begin{aligned} \therefore D_{k+1} &= D_k + 2(x_k + 1 + 1)^2 - 2(x_k + 1)^2 + (y_{k+1} - 1)^2 - (y_k - 1)^2 + {y_{k+1}}^2 - {y_k}^2 - 2r^2 + 2r^2, \text{ here } x_{k+1} = x_k + 1 \\ &= D_k + 2(x_k + 2)^2 - 2(x_k + 1)^2 + (y_{k+1} - 1)^2 - (y_k - 1)^2 + {y_{k+1}}^2 - {y_k}^2 \\ &= D_k + 4x_k + 2({y_{k+1}}^2 - {y_k}^2) - 2(y_{k+1} - y_k)^2 + 6 \end{aligned} \quad \ldots\ldots(3)$$

If $D_k < 0$, *then* $y_{k+1} = y_k$, the equation (3) becomes

$$D_{k+1} = D_k + 4x_k + 6 \quad \ldots\ldots(4)$$

if $D_k \geq 0$, *then* $y_{k+1} = y_k - 1$, the equation (3) becomes

$$\begin{aligned} D_{k+1} &= D_k + 4x_k + 2((y_k - 1)^2 - y_k^2) - 2(y_k - 1 - y_k)^2 + 6 \\ &= D_k + 4x_k + 2(y_k^2 + 1 - 2y_k - y_k^2) + 2 + 6 \\ &= D_k + 4x_k - 4y_k + 10 \quad \text{.....(5)} \end{aligned}$$

When the value of the decision parameter $D_k < 0$, then the next pixel coordinate on the circle boundary will be $(x_k + 1, y_k)$, the decision parameter can be calculated using the equation (4).

When the value of the decision parameter $D_k \geq 0$, then the next pixel coordinate on the circle boundary will be $(x_k + 1, y_k - 1)$, the decision parameter can be calculated using the equation (5).

The initial decision parameter can be obtained from the equation (1), by placing $x_0 = 0$ *and* $y_0 = r$ the starting point of the circle path as follows:

$$\begin{aligned} D_0 &= 2(x_k + 1)^2 + (y_k - 1)^2 + y_k^2 - 2r^2 \\ &= 2(0 + 1)^2 + (r - 1)^2 + r^2 - 2r^2 \\ &= 2 + r^2 - 2r + r^2 - 2r^2 + 1 \\ &= 3 - 2r \end{aligned}$$

The initial decision parameter D_0 always gives an integral value.

4.5.1 Steps In Bresenham's Circle Drawing Algorithms:

The steps required to consider during the implementation of Bresenham's circle drawing algorithm are:

1. Determine the radius r of the circle from the given information.
2. Assume that the circle center lies on the origin (0, 0) of the given coordinate.
3. Plot the first point at $P(0, r)$, where $x = x_0$ *and* $y = r$.
4. Compute the value of the initial decision parameter by using the equation $D_0 = 3 - 2r$.
5. Repeat the steps 5 and 6 until $x \leq y$

i. *if* $d_k < 0$ *then*

$$d_{k+1} = d_k + 4x_k + 6 \text{ and}$$
$$x_{k+1} = x_k + 1$$
$$y_{k+1} = y_k$$

ii. *if* $d_k \geq 0$ *then*

$$d_{k+1} = d_k + 4(x_k - y_k) + 10 \text{ and}$$
$$x_{k+1} = x_k + 1$$
$$y_{k+1} = y_k - 1$$

6. Plot the new pixel (x_{k+1}, y_{k+1}).
7. Determine the pixel coordinates for the symmetric points of other octants also.
8. **Special Case:** if the center of the circle does not lie on the coordinate origin (x_0, y_0), assume that the center lies on some other coordinate (x_c, y_c), then in step 6, the coordinate of the plotted pixel will be $(x_{k+1} + x_c, y_{k+1} + y_c)$

Example 3: Calculate the circle path points for the circle with the radius r = 10 unit.

Solution: Here the center of the circle is not given, so let us assume that the center of the circle lies at coordinate origin (x_0, y_0).

Therefore, the initial decision parameter $D_0 = 3 - 2r = 3 - 2.10 = -17$

The first point on the circle path in the first octant will be

$(0, r) = (0, 10)$.

Since $d_0 < 0$, the x coordinate will increase by 1 unit and the y coordinate will remain same. So the next pixel coordinate on the circle path will be

$$x_{k+1} = 0 + 1 = 1 \quad \text{and}$$
$$y_{k+1} = y_k = 10$$
$$\therefore (x_{k+1}, y_{k+1}) = (1, 10)$$

Since the initial decision parameter has a negative value, so the $d_{k+1} = d_1$ can be computed using the equation (4).

$$d_1 = d_0 + 4x_0 + 6$$

Therefore, $= -17 + 6 \qquad \because x_0 = 0$

$$= -11$$

Since $d_1 < 0$, The x coordinate will increase by 1 unit and the y coordinate will remain same. So the next pixel coordinate on the circle path will be

$$x_{k+1} = 1 + 1 = 2 \quad \text{and}$$
$$y_{k+1} = y_k = 10$$
$$\therefore (x_{k+1}, y_{k+1}) = (2, 10)$$

Now the decision parameter for the next pixel will be:

Since the decision parameter d_1 has a negative value, so the $d_{k+1} = d_2$ can be computed using the equation (4).

$$d_2 = d_1 + 4x_1 + 6$$
$$= -11 + 4 + 6 \qquad \because x_1 = 1$$
$$= -1$$

Since $d_2 < 0$, The x coordinate will increase by 1 unit and the y coordinate will remain same. So the next pixel coordinate on the circle path will be

$$x_{k+1} = 2 + 1 = 3 \quad \text{and}$$
$$y_{k+1} = y_k = 10$$
$$\therefore (x_{k+1}, y_{k+1}) = (3, 10)$$

Again the decision parameter for the next pixel will be:

Since the decision parameter d_2 has a negative value, so the $d_{k+1} = d_3$ can be computed using the equation (4).

$$d_3 = d_2 + 4x_2 + 6$$
$$= -1 + 4.2 + 6 \qquad \because x_2 = 2$$
$$= 13$$

Since $d_3 > 0$, The x coordinate will increase by 1 unit and the y coordinate will decrease by 1 unit. So the next pixel coordinate on the circle path will be

Iteration no	x_{k+1}	y_{k+1}	d_k
0	0	10	-17
1	1	10	-11
2	2	10	-1
3	3	10	13
4	4	09	-5
5	5	09	17
6	6	08	11
7	7	07	13

Table 5: Shown he values of the example 5

$$x_{k+1} = 3+1 = 4 \quad \text{and}$$
$$y_{k+1} = y_k - 1 = 10 - 1 = 9$$
$$\therefore (x_{k+1}, y_{k+1}) = (4, 9)$$

This process will continue until $x \le y$. The output pixel coordinates for the first octant is shown in the table 5.

Example 4: Given the centre point coordinates (10, 10) and radius as 10, generate all the points to form a circle.

Solution:Given- that the Centre Coordinates of Circle (X_0, Y_0) = (10, 10) and the Radius of Circle = 10

Now consider the starting point coordinates (X_0, Y0) as-

- $X_0 = 0$
- $Y_0 = R = 10$

Calculate the value of initial decision parameter P0 as-

$P_0 = 3 - 2 \times R$

$P_0 = 3 - 2 \times 10$

$P_0 = -17$

As $P_{initial} < 0$, therefore,

- $X_{k+1} = X_k + 1 = 0 + 1 = 1$
- $Y_{k+1} = Y_k = 10$
- $P_{k+1} = P_k + 4_x X_{k+1} + 6 = -17 + (4 \times 1) + 6 = -7$

This step is applicable here as the given centre point coordinates is (10, 10).

$X_{plot} = X_c + X_0 = 1 + 10 = 11$

$Y_{plot} = Y_c + Y_0 = 10 + 10 = 20$

Continue the process until $X_{plot} => Y_{plot}$ as follows-

Algorithm calculates all the points of octant-1 of the circle and terminates.

P_k	P_{k+1}	(X_{k+1}, Y_{k+1})	(X_{plot}, Y_{plot})
		(0, 10)	(10, 20)
-17	-7	(1, 10)	(11, 20)
-7	7	(2, 10)	(12, 20)
7	-7	(3, 9)	(13, 19)
-7	15	(4, 9)	(14, 19)
15	13	(5, 8)	(15, 18)
13	19	(6, 7)	(16, 17)
Algorithm Terminates			

EXERCISE

Question1: Show the set of points of a circle boundary in first quadrant using Midpoint circle drawing algorithm with the center coordinate (3, 3) and the radius r= 12.

Question2: Show the set of points of a circle boundary in first quadrant using Midpoint circle drawing algorithm with the center coordinate (0, 0) and the radius r= 7.

4.6 ELLIPSE GENERATING ALGORITHM

Ellipse can be defined as a set of points such that the sum of the distances from two fixed positions known as foci is the same for all points. If the distances between the point p = (x, y) to the foci are d1 and d2 then the general equation of an ellipse can be stated as:

d1 + d2 =constant …… (24)

If the coordinates of the foci are f1(x_1, x_2) and f2(x_2, y_2), then the equation becomes

$$\sqrt{(x-x_1)^2+(y-y_1)^2}+\sqrt{x-x_2)^2+(y-y_2)^2}=cons\tan t.....(25)$$

The major axis of an ellipse is the straight line from one end of the ellipse boundary to the other across the foci. A straight line from one end of the ellipse at the shortest diameter part to the other end, intersecting the major axis at the middle is known as the minor axis. The intersecting point is known as the ellipse center between the two foci as shown figure 4.4.

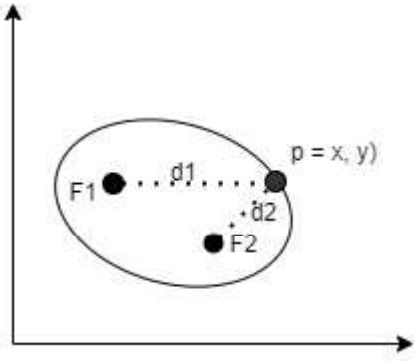


Figure 4.4Ellipse at foci F1 and F2

The ellipse equation can be expressed in terms of major and minor axes. It becomes more simplified if the axes' orientation is aligned with the coordinate axes. The following figure 4.5, it has been shown a standard position of an ellipse; where the major and minor axes are aligned parallel to the x and y axes. The semi-major and semi-minor axes are represented by the labels rx and ry. The ellipse equation can be written in terms of center coordinates (xc, yc), semi-major and minor axes is as follows:

$$\left(\frac{x-x_c}{r_x}\right)^2+\left(\frac{y-y_c}{r_y}\right)^2=1 \quad \text{.....(24)}$$

Using polar coordinates r and θ, the equation of an ellipse can be expressed as:

$$\left.\begin{aligned} x &= x_c + r_x\cos\theta \\ y &= y_c + r_y\sin\theta \end{aligned}\right\} \text{.....(25)}$$

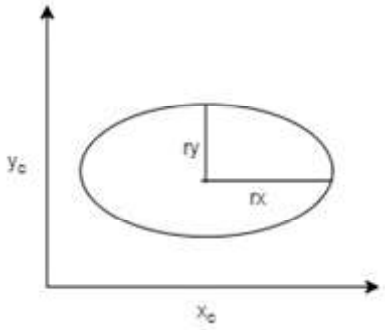


Figure 4.5Ellipse centered at (xc, yc), with semi major axis rx and semi minor axis ry.



CHECK YOUR PROGRESS

iii. Which of the following properties is followed by the ellipse?

A. 4 - symmetry property
B. 8 - symmetry property
C. 6 - symmetry property
D. All of the above

iv. The value of the decision parameter determines whether the mid-point lays ___ boundary and the then position of the mid-point help in drawing the ellipse.

A. At boundaries
B. Inside, outside, or on the ellipse
C. Only inside the boundaries
D. Only outside the boundaries

Using symmetry concepts, the steps required in computations can be reduced. An ellipse does not show the symmetric property in between octants. It shows symmetric property only on quadrants. So during the computation of boundary path pixel coordinates, need to calculate the pixel positions along the elliptical arc across one quadrant. Then the position of the pixels in the other quadrant can be calculated using symmetry.

4.6.1 Midpoint Ellipse Generating Algorithm

The goal of this algorithm is to display an ellipse. With the given parameters r_x and r_y and center coordinates (x_c, y_c) we have to determine the pixel coordinates (x, y) for an ellipse in a standard position centered at the coordinate origin and finally shift to the points such that the ellipse centered at (x_c, y_c). To display the ellipse in a non-standard position, the ellipse has to rotate about its center coordinates to re-orient the major and minor axes. Here we will consider the only standard position to display the ellipse.

The Midpoint ellipse algorithm is applied throughout the first quadrant in two parts. Following figure 4.6 shows the division of the first quadrant according to the slope of an ellipse with $r_x < r_y$. We have to process this quadrant by taking unit steps in the x - the direction where the slope of the curve has a magnitude less than 1 and taking unit steps in the y-direction where the slope has a magnitude greater than 1. In figure 4.7, we can start our plotting from the point $(0, r_y)$ and step clockwise direction along the elliptical path in the first quadrant, shifting from unit steps in x to unit steps in

y when the slope becomes less than -1.

Now consider the ellipse function derived from the equation (24) with $(x_c, y_c) = (0, 0)$ as:

$$f_{ellipse}(x,y) = r^2_y x^2 + r^2_x y^2 - r^2_x r^2_y \quad \text{.....(26)}$$

This has the following properties:

$$f_{ellipse}(x,y)\begin{cases} < 0, if\ (x,y)\ is\ inside\ the\ ellipse\ boundary \\ = 0, if\ (x,y)\ is\ on\ the\ ellipse\ boundary \\ > 0, if\ (x,y)\ is\ outside\ the\ ellipse\ boundary \end{cases} \quad \text{.....(27)}$$

Thus the ellipse function can be used as a decision parameter in the midpoint algorithm to generate an ellipse. At each sampling position, the next pixel position along the ellipse boundary can be determined using the sign of the decision parameter or the ellipse function, evaluated at the midpoint between the two candidate pixels.

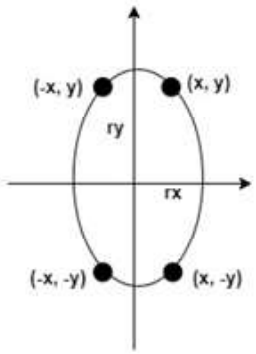


Figure 4.6Symmetry of an ellipse

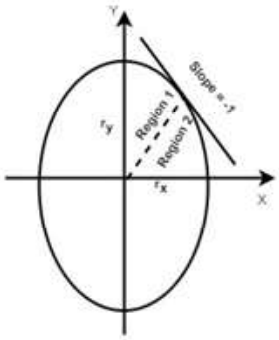


Figure 4.7Ellipse processing regions

Starting at $(0, r_y)$ we take unit steps in the x-direction until we reach the boundary between the region 1 and 2 as shown in the following figure 4.7. Then we have to shift the unit steps in the y-direction over the remainder of the curve in the first quadrant. At each step, we need to test the value of the slope of the curve. The ellipse slope can be calculated as:

$$\frac{dy}{dx} = -\frac{2r^2_y x}{2r^2_x y} \quad \text{.....(28)}$$

At the boundary between region 1 and region 2, dy/dx = - 1 and

$$2r^2_y x = 2r^2_x y$$

Therefore we have to move out from region 1 when,

$$2r^2_y x \geq 2r^2_x y$$

Figure 4.8 shows the midpoint between the two candidate pixels at sampling position x_k+1 in the first region. Assume that the point selected in the previous step is (x_k, y_k), and now we have to determine the next position along the ellipse path by evaluating the decision parameter at this midpoint.

$$p1_k = f_{ellipse}(x_k + 1, y_k - \tfrac{1}{2})$$
$$= r^2_y (x_k + 1)^2 + r^2_x (y_k - \tfrac{1}{2})^2 - r^2_x r^2_y \quad \text{.....(29)}$$

If $p1_k < 0$, the midpoint is inside the ellipse boundary and the pixel on scan line y_k is closer to the ellipse boundary. Otherwise, the midpoint will remain on or outside the ellipse and the point on scan line y_k - 1 will be closer to the ellipse boundary.

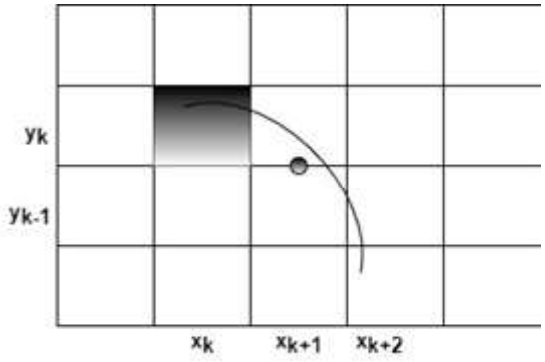


Figure 4.8Midpoint between candidate pixels at xk+1 position along an ellipse line path

At the next sampling position ($x_{k+1} + 1 = x_k + 2$), the decision parameter for region 1 is evaluated as:

$$p1_{k+1} = f_{ellipse}(x_{k+1} + 1, y_{k+1} - \tfrac{1}{2})$$
$$= r^2_y [(x_k + 1) + 1]^2 + r^2_x (y_{k+1} - \tfrac{1}{2})^2 - r^2_x r^2_y$$
$$= p1_k + 2r^2_y (x_k + 1) + r^2_y + r^2_x \left[\left(y_{k+1} - \tfrac{1}{2} \right)^2 - \left(y_k - \tfrac{1}{2} \right)^2 \right]$$

Where y_{k+1} is either y_k or $y_k - 1$, depending on the sign of decision parameter $p1_k$.

If $p1_k < 0$, then y_{k+1} is y_k, so the next point along the ellipse centered on (0, 0) is (x_{k+1}, yk)

$$p1_{k+1} = p1_k + 2r^2_y x_{k+1} + r^2_y$$

Otherwise, the next point along the ellipse is $(x_k + 1, y_k - 1)$

$$p1_{k+1} = p1_k + 2r^2_y x_{k+1} - 2r^2_x y_{k+1} + r^2_y$$

Decision parameters are incremented by the following amounts:

$$increment = \begin{cases} 2r^2_y x_{k+1} + r^2_y, & if\ p1_k < 0 \\ 2r^2_y x_{k+1} + r^2_y - 2r^2_x y_{k+1}, & if\ p1_k \geq 0 \end{cases}$$

In region 1, the initial value of the decision parameter is obtained by evaluating the ellipse function at the start position $(x_0, y_0) = (0, r_y)$;

$$\begin{aligned} p1_0 &= f_{ellipse}(1, r_y - 1/2) \\ &= r^2_y + r^2_x (r_y - 1/2)^2 - r^2_x r^2_y \\ &= r^2_y - r^2_x r_y + 1/4\, r^2_x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30) \end{aligned}$$

In region 2, we sample at unit steps in the negative y-direction, and the midpoint is now taken between horizontal pixels at each step. For this region the decision parameter is evaluated as:

$$\begin{aligned} p2_k &= f_{ellipse}(x_k + 1/2, y_k - 1) \\ &= r^2_y (x_k + 1/2)^2 + r^2_x (y_k - 1)^2 - r^2_x r^2_y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32) \end{aligned}$$

If $p2_k > 0$, the mid position is outside the ellipse boundary and we have to select the pixel x_k.

If $p2_k \leq 0$, the mid position is inside or on the ellipse boundary and we have to select the pixel $x_k + 1$.

4.6.2 Steps Involved In Midpoint Ellipse Algorithm

1. Input r_x, r_y, and the ellipse center (x_c, y_c) and obtain the first point on an ellipse centered on the origin as $(x_0, y_0)= (0, r_y)$.
2. Calculate the initial value of the decision parameter in region 1 as:

$$p1_0 = r^2_y - r^2_x r_y + \frac{1}{4} r^2_x$$

3. At each xk position in region 1, starting at k=0, perform the following test:

   If $p1_k < 0$, then y_{k+1} is y_k, so the next point along the ellipse centered on (0, 0) is (x_{k+1}, y_k)

$$p1_{k+1} = p1_k + 2r^2_y x_{k+1} + r^2_y$$

   Otherwise, the next point along the ellipse is $(x_k + 1, y_k - 1)$

$$p1_{k+1} = p1_k + 2r^2_y x_{k+1} - 2r^2_x y_{k+1} + r^2_y$$

   And continue until $2r^2_y x \geq 2r^2_x y$

4. In region 2, calculate the initial value of the decision parameter using the last point (x_0, y_0) calculated in region 1 as::

$$p2_0 = r^2_y (x_0 + \frac{1}{2})^2 + r^2_x (y_0 - 1)^2 - r^2_x r^2_y$$

5. At each yk position in region 2, starting at k=0, perform the following test:

$$p2_{k+1} = p2_k + 2r^2_y x_{k+1} - 2r^2_x y_{k+1} + r^2_x$$

   Using the same incremental calculations for x and y as in region1.
6. Determine the symmetry points in the other three quadrants.
7. Move each calculated pixel position (x, y) onto the elliptical path centered on (x_c, y_c) and plot the coordinate values: $x = x + x_c$ $y = y + y_c$
8. Repeat the steps for region 1 until $2r^2_y x \geq 2r^2_x y$

Example 5: Using mid pointellipse algorithm, solve for $r_x = 8$ and $r_y = 6$.

Solution: Given that the $r_x = 8$ and $r_y = 6$.

Assume that the ellipse centre lies on(x_c, y_c)

For region 1:

The first point on an ellipse centered on the origin (x_0, y_0) = (0, r_y) = (0, 6)

The initial value of the decision parameter is as:

$$d_0 = r_y^2 - r_x^2 r_y + \frac{1}{4} r_x^2 = 6^2 - 8^2.6 + \frac{1}{4}.8^2 = -332$$

For i=0, d_0= -332 < 0, then by the midpoint ellipse algorithm, the next point along the ellipse centered on (0, 0) is

$$(x_i + 1, y_i) = (x_0 + 1, y_0) = (1, 6)$$

$$d_1 = d_0 + 2r_y^2 x_1 + r_y^2 = -332 + 2.6^2.1 + 6^2 = -332 + 72 + 36 = -224$$

For i=1, d_1= -224< 0, the next point along the ellipse centered on (0, 0) is

$$(x_i + 1, y_i) = (x_2, y_2) = (x_1 + 1, y_1) = (2, 6) \text{ and}$$

$$d_2 = d_1 + 2r_y^2 x_2 + r_y^2 = -224 + 2.6^2.2 + 6^2 = -224 + 144 + 36 = -44$$

For i=2, d_2= - 44< 0, the next point along the ellipse centered on (0, 0) is

$$(x_i + 1, y_i) = (x_3, y_3) = (x_2 + 1, y_2) = (3, 6) \text{ and}$$

$$d_3 = d_2 + 2r_y^2 x_3 + r_y^2 = -44 + 2.6^2.3 + 6^2 = -44 + 216 + 36 = 208$$

For i=23, d_3 = 208 > 0, the next point along the ellipse centered on (0, 0) is

$$(x_i + 1, y_i - 1) = (x_4, y_4) = (x_3 + 1, y_3 - 1) = (4, 5) \text{ and}$$

$$d_4 = d_3 + 2r_y^2 x_4 - 2r_x^2 y_4 + r_y^2 = 208 + 2.6^2.4 - 2.8^2.5 + 6^2 = 208 + 288 - 640 + 36 = -108$$

Similarly, the successive decision parameter values and positions along the ellipse path are calculated uing these algorithm are:

i	d_i	(x_{i+1}, y_{i+1})
0	-332	(1, 6)
1	-224	(2, 6)
2	-44	(3, 6)
3	208	(4, 5)
4	-108	(5, 5)
5	288	(6, 4)
6	244	(7, 3)

For region 2:

The first point on an ellipse centered on the origin $(x_0, y_0) = (0, r_y) = (7, 3)$

The initial value of the decision parameter is as:

$$d_0 = r_y^2\left(x_0 + \frac{1}{2}\right)^2 + r_x^2(y_0 - 1)^2 - r_x^2 r_y^2$$

$$= 6^2\left(7 + \frac{1}{2}\right)^2 + 8^2(3-1)^2 - 8^2 6^2$$

$$= -23$$

For i=0, d_0= -23< 0, then by the midpoint ellipse algorithm, the next point along the ellipse centered on (0, 0) is

$$(x_i + 1, y_i - 1) = (x_1, y_1) = (x_0 + 1, y_0 - 1) = (8, 2)$$

$$d_1 = d_0 + 2r_y^2 x_1 - 2r_x^2 y_1 + r_x^2 = 361$$

For i=1, d_1= 361> 0, then by the midpoint ellipse algorithm, the next point along the ellipse centered on (0, 0) is

$$(x_i, y_i - 1) = (x_2, y_2) = (x_1, y_1 - 1) = (8, 1)$$

$$d_2 = d_1 - 2r_x^2 y_2 + r_x^2 = 297$$

For i=2, d_2= 297> 0, then by the midpoint ellipse algorithm, the next point along the ellipse centered on (0, 0) is

$$(x_i, y_i - 1) = (x_3, y_3) = (x_2, y_2 - 1) = (8, 0)$$

$$d_3 = d_2 - 2r_x^2 y_3 + r_x^2 = 361$$

Similarly, the successive decision parameter values and positions along the ellipse path are calculated uing these algorithm are:

i	d_i	(x_{i+1}, y_{i+1})
0	-23	(8, 2)
1	361	(8, 1)
2	297	(8, 0)



EXERCISE

Question3: Using midpoint ellipse algorithm, draw the boundary points of an ellipse with $r_x = 8$ and $r_y = 6$.

Question4: Using midpoint ellipse algorithm, draw the boundary points of an ellipse with $r_x = 8$ and $r_y = 6$.



4.7 LET US SUM UP

- In a graphics package, the functions which are used to describe the various picture components are known as graphics output primitives or simply primitives.
- One of the most used and efficient circles generating algorithms is the Midpoint Circle algorithm.
- An ellipse can be defined as a set of points located at a distance from two fixed points, where the sum of the distances from each point to the boundary points is equal. The specific points are known as the foci of the ellipse.
- The major axis of an ellipse connects both the end of the ellipse across the foci.
- The minor is a straight line segment drawn at the middle of the narrowest part of an ellipse by intersecting the major axes. The intersecting point is known as the center of the ellipse.
- The circle can divide into eight symmetric segments.
- An ellipse can divide into four symmetric segments.



4.8 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1:

i. B. y2 = r2 - x2

ii. C. Both A and B.

iii. A. 4 - symmetry property

iv. B. Inside, outside, or on the ellipse



4.9 MODEL QUESTIONS

Short Questions (Answer each question in around 150 words).

a. What do you mean by graphics output primitives?

b. Explain the midpoint circle algorithm.

c. How is the decision parameter used in the midpoint circle algorithm, explain.

d. Define the term ellipse? Write any two properties of an ellipse.

e. Define a circle. Also, write at least four properties of a circle.

f. Explain the midpoint ellipse algorithm.

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